A Viscoplastic Model Including Non-linear Isotropic and Kinematic Hardening

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Notation:

ε	Total strain tensor
ε^e	Elastic strain tensor
ε^{vp}	Viscoplastic strain tensor
$\boldsymbol{\zeta}$	Kinematic hardening strain tensor
ξ	Isotropic hardening strain variable (equivalent plastic strain)
$\boldsymbol{\sigma}$	Stress tensor
p	Pressure
$\mathbf{s}$	Deviatoric part of the stress tensor
$\mathbf{q}$	Kinematic hardening stress tensor (conjugate of $\boldsymbol{\zeta}$)
q	Isotropic hardening stress variable (conjugate of ξ)
Ψ	Free energy function
W	Elastic energy potential
K	Hardening potential
Ω	Viscoplastic potential
Φ	Yield function
γ	Viscoplastic multiplier
$\mathbf{n}$	Unit normal to the yield surface

Model parameters:

K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law
H	Linear isotropic hardening coefficient
K_H	Linear kinematic hardening coefficient
A	Non-linear kinematic hardening parameter
η	Viscosity
m	Exponent of the non-linear viscous law

Chapter 1

Mechanical Model

In this work the formulation of a general elasto-viscoplastic constitutive model is consistently derived within a thermodynamic framework. The constitutive behavior has been defined by a elasto-plastic free energy function. Plastic response has been modeled considering a J2 viscoplastic constitutive model, including non-linear isotropic and kinematic hardening. Time integration and linearization of the constitutive model is also introduced.

Finally, a number of numerical tests will show the mechanical response of a specimen submitted to a loading-unloading cycle.

1.1 Constitutive equations

In the following section, the constitutive equations of the elasto-viscoplastic model are introduced.

Figure (1.1) shows the rheological model considered for the mechanical behavior. Let's assume an *additive decomposition* of the total strain tensor $\boldsymbol{\varepsilon}$ into its *elastic* and *plastic* parts $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^{vp}$, respectively, that is

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp} \tag{1.1}$$

Let us now introduce the *Helmholtz free energy function* (per unit reference volume) $\Psi\left(\boldsymbol{\varepsilon}^e, \boldsymbol{\zeta}, \xi\right)$ as the sum of the following contributions

$$\Psi = \Psi\left(\boldsymbol{\varepsilon}^e, \boldsymbol{\zeta}, \xi\right) = W\left(\boldsymbol{\varepsilon}^e\right) + K\left(\boldsymbol{\zeta}, \xi\right) \tag{1.2}$$

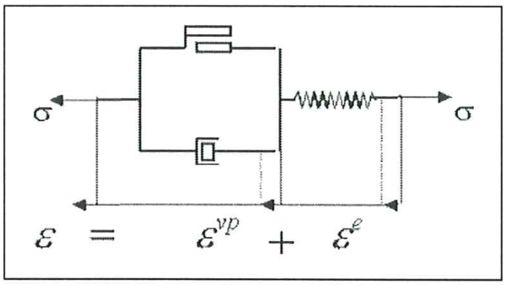


Figure 1.1: Mechanical rheological model

where $W(\boldsymbol{\varepsilon}^e)$ is the elastic stored energy, $K(\boldsymbol{\zeta},\xi)$ is the plastic hardening potential and $\boldsymbol{\zeta}$ and ξ are the kinematic and isotropic hardening variables, respectively. The expressions chosen here for these terms are the following

$$W(\boldsymbol{\varepsilon}^e) = \frac{1}{2}K\, tr^2(\boldsymbol{\varepsilon}^e) + G\, dev^2(\boldsymbol{\varepsilon}^e) \qquad (1.3a)$$

$$K(\boldsymbol{\zeta},\xi) = (\sigma_\infty - \sigma_o)\left[\xi - \frac{1-exp(-\delta\xi)}{\delta}\right] + \frac{1}{2}H\,\xi^2 + \frac{1}{3}K_H\,\|\boldsymbol{\zeta}\|^2 \qquad (1.3b)$$

where K is the bulk modulus, G the shear modulus, σ_o the initial flow stress, σ_∞ the saturation hardening limit, H the linear isotropic hardening coefficient and finally K_H the kinematic hardening coefficient.

Using the Legendre transformation is possible to express the rate of the free energy function as

$$\dot{\Psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - D_{mech} \qquad (1.4)$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and D_{mech} is the mechanical dissipation. If we differentiate the free energy function with respect to the state variables we obtain

$$D_{mech} = \left(\boldsymbol{\sigma} - \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^e}\right) : \dot{\boldsymbol{\varepsilon}} + \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^{vp} - \frac{\partial\Psi}{\partial\boldsymbol{\zeta}} : \dot{\boldsymbol{\zeta}} - \frac{\partial\Psi}{\partial\xi} : \dot{\xi} \geq 0 \qquad (1.5)$$

Applying Coleman's method [Chiumenti-98], we obtain the definition of the stress tensor as

$$\boldsymbol{\sigma} = \frac{\partial\Psi(\boldsymbol{\varepsilon}^e,\boldsymbol{\zeta},\xi)}{\partial\boldsymbol{\varepsilon}^e} = K\, tr(\boldsymbol{\varepsilon})\,\mathbf{1} + 2G\, dev(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \qquad (1.6a)$$

that can be splitted into its spheric and deviatoric parts as

$$\boldsymbol{\sigma} = p\mathbf{1} + \mathbf{s} \rightarrow \begin{cases} p = \frac{1}{3} tr(\boldsymbol{\sigma}) \\ \mathbf{s} = dev(\boldsymbol{\sigma}) = \boldsymbol{\sigma} - p\mathbf{1} \end{cases} \tag{1.7}$$

where p is the pressure and $\mathbf{s}$ is the stress deviator, respectively given by

$$p = K\, tr(\boldsymbol{\varepsilon}^e) = K\, tr(\boldsymbol{\varepsilon}) \tag{1.8a}$$

$$\mathbf{s} = 2G\, dev(\boldsymbol{\varepsilon}^e) = 2G\, dev(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{1.8b}$$

Finally, the mechanical dissipation results in

$$D_{mech} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{vp} + \mathbf{q} : \dot{\boldsymbol{\zeta}} + q : \dot{\xi} \geq 0 \tag{1.9}$$

where $\mathbf{q}$ and q are the state variable conjugate to the kinematic and isotropic hardening variables $\boldsymbol{\zeta}$ and ξ, respectively, defined as

$$\mathbf{q} = -\frac{\partial \Psi}{\partial \boldsymbol{\zeta}} = -\frac{2}{3} K_H \boldsymbol{\zeta} \tag{1.10a}$$

$$q = -\frac{\partial \Psi}{\partial \xi} = -K_\xi \tag{1.10b}$$

where

$$K_\xi = (\sigma_\infty - \sigma_o)\left[1 - exp(-\delta\xi)\right] + H\xi \tag{1.11}$$

1.2 Evolution laws

Let us define the following visco-plastic potential $\Omega(\boldsymbol{\sigma}, \mathbf{q}, q)$ as

$$\Omega(\boldsymbol{\sigma}, \mathbf{q}, q) = \frac{\eta}{m+1}\left\langle \frac{\Phi(\boldsymbol{\sigma}, \mathbf{q}, q)}{\eta} \right\rangle^{m+1} \tag{1.12}$$

where η is the viscosity and m is the exponent of the viscous law.

Function $\Phi(\mathbf{s}, \mathbf{q}, q)$ can be particularized to deal with the von Mises yield criterion combined with an isotropic and kinematic hardening as

$$\Phi(\mathbf{s}, \mathbf{q}, q, \Theta) = \|\mathbf{s} - \mathbf{q}\| - \sqrt{\frac{2}{3}}(\sigma_o - q) \leq 0 \tag{1.13}$$

where σ_o is the flow stress.

The evolution laws for the internal variables introduced in the previous section, can be obtained as

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^{vp} &= \frac{\partial \Omega}{\partial \mathbf{s}} = \left\langle \frac{\Phi}{\eta} \right\rangle^m \frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n} \\ \dot{\boldsymbol{\zeta}} &= \frac{\partial \Omega}{\partial \mathbf{q}} = \left\langle \frac{\Phi}{\eta} \right\rangle^m \frac{\partial \Phi}{\partial \mathbf{q}} = -\dot{\gamma}\,\mathbf{n} \\ \dot{\xi} &= \frac{\partial \Omega}{\partial q} = \left\langle \frac{\Phi}{\eta} \right\rangle^m \frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}} \end{aligned} \tag{1.14}$$

where $\mathbf{n}$ is the unit normal to the yield surface and $\dot{\gamma}$ is the viscoplastic multiplier, respectively given by

$$\begin{aligned} \dot{\gamma} &= \left\langle \frac{\Phi}{\eta} \right\rangle^m = \|\dot{\boldsymbol{\varepsilon}}^{vp}\| \\ \mathbf{n} &= \frac{\partial \Phi}{\partial \mathbf{s}} = \frac{\mathbf{s} - \mathbf{q}}{\|\mathbf{s} - \mathbf{q}\|} \end{aligned} \tag{1.15}$$

An extension of the model to take into account non-linear kinematic hardening is proposed in the form

$$\dot{\boldsymbol{\zeta}} = -\dot{\gamma}\;(\mathbf{n} + A\,\boldsymbol{\zeta}) = -\dot{\boldsymbol{\varepsilon}}^{vp} - A\;\|\dot{\boldsymbol{\varepsilon}}^{vp}\|\;\boldsymbol{\zeta} \tag{1.16}$$

It is interesting to observe that if we introduce the constitutive law (1.10a) into the previous equation (1.16) the result is

$$\dot{\mathbf{q}} = A\;(\mathrm{q}_\infty\,\dot{\boldsymbol{\varepsilon}}^{vp} - \mathbf{q}\;\|\dot{\boldsymbol{\varepsilon}}^{vp}\|) \tag{1.17}$$

where $\mathrm{q}_\infty = \dfrac{2}{3}\dfrac{K_H}{A}$ is the saturation value of the kinematic hardening law. In fact, in case of one-dimension analysis it results $\dot{\boldsymbol{\varepsilon}}^{vp} = \|\dot{\boldsymbol{\varepsilon}}^{vp}\| = \dot{\varepsilon}^{vp}$ so that

$$\dot{\mathrm{q}} = A\,\dot{\varepsilon}^{vp}\,(\mathrm{q}_\infty - \mathrm{q}) \tag{1.18}$$

This expession can be integrated and gives as a result the following saturation law

$$\mathrm{q} = \mathrm{q}_\infty\,[1 - \exp\left(-A\,\varepsilon^{vp}\right)] \tag{1.19}$$

In the following table the constitutive model proposed is summarized:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$
Constitutive laws	$\mathbf{s} = 2G\,dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp})$ $\mathbf{q} = -\frac{2}{3}K_H\,\boldsymbol{\zeta}$ $q = -(\sigma_\infty - \sigma_o)\,[1 - \exp\left(-\delta\,\xi\right)] - H\xi$
Plastic potential	$\Omega = \dfrac{\eta}{n+1}\left\langle\dfrac{\Phi}{\eta}\right\rangle^{n+1}$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^{vp} = \dfrac{\partial\Omega}{\partial\mathbf{s}} = \left\langle\dfrac{\Phi}{\eta}\right\rangle^{m}\dfrac{\partial\Phi}{\partial\mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\boldsymbol{\zeta}} = \begin{cases} -\dot{\gamma}\,(\mathbf{n}+A\,\boldsymbol{\zeta}) \\ or \\ -\dot{\boldsymbol{\varepsilon}}^p - A\,\|\dot{\boldsymbol{\varepsilon}}^{vp}\|\,\boldsymbol{\zeta} \end{cases}$ $\dot{\xi} = \dfrac{\partial\Omega}{\partial q} = \left\langle\dfrac{\Phi}{\eta}\right\rangle^{m}\dfrac{\partial\Phi}{\partial q} = \dot{\gamma}\sqrt{\dfrac{2}{3}}$ $\dot{\gamma} = \left\langle\dfrac{\Phi}{\eta}\right\rangle^{m} = \|\dot{\boldsymbol{\varepsilon}}^{vp}\|$
Visco-elastic domain	$J_2\,(\boldsymbol{\sigma}) < \sigma_o + R + \sigma_v$ $\begin{cases} J_2\,(\boldsymbol{\sigma} - \mathbf{q}) = \sqrt{\dfrac{3}{2}}\,\|\mathbf{s} - \mathbf{q}\| \\ R = (\sigma_\infty - \sigma_o)\,[1 - \exp\left(-\delta\,\xi\right)] + H\xi \\ \sigma_v = \eta\sqrt{\dfrac{3}{2}}\,\dot{\gamma}^{\frac{1}{m}} \end{cases}$

Chapter 2

Time Integration of the Mechanical Model

In this section the time integration of the proposed model is presented. First, the elasto-plastic operator split method will be introduced, followed by the time integration algorithm.

2.1 Elasto-plastic operator split

In this section a product formula algorithm emanating from a standard elastic-plastic operator split of the elasto-plastic constitutive equations is presented [Krieg & Krieg-77], [Simo-94]. The basic idea consists of a two-steps-algorithm to be applied to the evolution equations as follows:

1. an *elastic trial predictor*, obtained by freezing the plastic flow during the time step, followed by,

2. a *plastic corrector* that performs the closest-point-projection of the trial state onto the yield surface.

Using the compact notation $\mathbf{\Sigma} = [\mathbf{s}, \mathbf{q}, q]$, $\mathbf{E}^{vp} = [\boldsymbol{\varepsilon}^{vp}, \boldsymbol{\zeta}, \xi]$ and $\mathbf{C} = \left[2G\,\mathbf{I}, \frac{2}{3}K_H\,\mathbf{I}, H\right]$ the additive split results in

$$
\begin{array}{ccccc}
\text{Total} & = & \text{Elastic predictor} & + & \text{Plastic corrector} \\
\dot{\mathbf{E}}^{vp} = \gamma^p\, \nabla\Phi\,(\mathbf{\Sigma}) & = & \dot{\mathbf{E}}^{vp} = [\mathbf{0}, \mathbf{0}, 0] & + & \dot{\mathbf{E}}^{vp} = \dot{\gamma}\, \nabla\Phi\,(\mathbf{\Sigma})
\end{array}
$$

Applying an implicit *backward-Euler* difference scheme, it is possible to define a *trial elastic state* given by

$$\mathbf{E}_{n+1}^{vp\ trial} = \mathbf{E}_n^{vp}$$

so that the associated trial stress field results in

$$\boldsymbol{\Sigma}_{n+1}^{trial} = \mathbf{C}\cdot\left(\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^{vptrial}\right) = \mathbf{C}\cdot\left(\mathbf{E}_{n+1} - \mathbf{E}_n^{vp}\right)$$

At this stage the trial state could be inside or outside the elastic domain E_σ , it means that the loading function $\Phi(\boldsymbol{\Sigma})$ must be checked. It is possible to demonstrate that if $\Phi\left(\boldsymbol{\Sigma}_{n+1}^{trial}\right)$ is convex, then $\Phi\left(\boldsymbol{\Sigma}_{n+1}^{trial}\right) \geq \Phi\left(\boldsymbol{\Sigma}_{n+1}\right)$ so that

- if $\Phi\left(\boldsymbol{\Sigma}_{n+1}^{trial}\right) \leq 0 \rightarrow \Phi\left(\boldsymbol{\Sigma}_{n+1}\right) \leq 0$ and $\dot{\gamma} = 0$ so the process is elastic and the trial state is the final state

  $$\mathbf{E}_{n+1}^{vp} = \mathbf{E}_{n+1}^{vptrial} = \mathbf{E}_n^{vp}$$

  and

  $$\boldsymbol{\Sigma}_{n+1} = \boldsymbol{\Sigma}_{n+1}^{trial} = \mathbf{C}\cdot\left(\mathbf{E}_{n+1} - \mathbf{E}_n^{vp}\right)$$

- If, in the other hand, $\Phi\left(\boldsymbol{\Sigma}_{n+1}^{trial}\right) > 0 \rightarrow \Phi\left(\boldsymbol{\Sigma}_{n+1}\right) \geq 0$ and $\dot{\gamma} > 0$ the *plastic corrector* must be applied by a *return mapping algorithm.* According to the product formula, the resulting final state becomes in

  $$\mathbf{E}_{n+1}^{vp} = \mathbf{E}_{n+1}^{vptrial} + \dot{\gamma}\ \nabla\Phi\left(\boldsymbol{\Sigma}_{n+1}\right)$$

  and according to the *closest-point-projection* algorithm [Wilkins-64],[Krieg & Key-76], the stress fields transform in

  $$\boldsymbol{\Sigma}_{n+1} = \mathbf{G}\cdot\left(\mathbf{E}_{n+1} - \mathbf{E}_{n+1}^{vp}\right) = \boldsymbol{\Sigma}_{n+1}^{trial} - \dot{\gamma}\ \mathbf{C}\cdot\nabla\Phi\left(\boldsymbol{\Sigma}_{n+1}\right)$$

2.2 Time integration algorithm

At time t_n in a typical time increment $[t_n, t_{n+1}]$, the configuration $\mathbf{u}_n$ and the internal variables $\{\boldsymbol{\varepsilon}_n^{vp}, \zeta_n, \xi_n\}$ are given. In this phase of the product formula, one solves for the current configuration $\mathbf{u}_{n+1}$ via an iterative procedure in which the current iterate is assumed given. The computation of the new iteration involves the evaluation of the current stress fields $\bar{\boldsymbol{\sigma}}_{n+1}$ and the internal variables $\left\{\boldsymbol{\varepsilon}_{n+1}^{vp}, \zeta_{n+1}, \xi_{n+1}\right\}$ at time t_{n+1}. Given a finite element discretization, this update is performed at each quadrature point and proceeds as follows [Simo-94], [Chiumenti-98].

2.2.1 Trial state (kinematics)

According to the elasto-plastic operator split,

$$\begin{aligned}
\boldsymbol{\varepsilon}^{vp}_{n+1} &= \boldsymbol{\varepsilon}^{vp^{trial}}_{n+1} + \gamma_{n+1}\mathbf{n}_{n+1} && (2.4a)\\
\boldsymbol{\zeta}_{n+1} &= \boldsymbol{\zeta}^{trial}_{n+1} - \gamma_{n+1}\left(\mathbf{n}_{n+1} + A\,\boldsymbol{\zeta}_{n+1}\right) && (2.4b)\\
\xi_{n+1} &= \xi^{trial}_{n+1} + \gamma_{n+1}\sqrt{\frac{2}{3}} && (2.4c)
\end{aligned}$$

where $\gamma_{n+1} = \dot{\gamma}_{n+1}\,\Delta t$ and the *elastic trial predictor* for the plastic variables is computed as

$$\begin{aligned}
\boldsymbol{\varepsilon}^{vp^{trial}}_{n+1} &= \boldsymbol{\varepsilon}^{vp}_{n} && (2.5a)\\
\boldsymbol{\zeta}^{trial}_{n+1} &= \boldsymbol{\zeta}_{n} && (2.5b)\\
\xi^{trial}_{n+1} &= \xi_{n} && (2.5c)
\end{aligned}$$

2.2.2 Trial (generalized) stresses

Using the trial values obtained for the internal variables it is now possible to compute the trial generalized stresses as follows

$$\begin{aligned}
p^{trial}_{n+1} &= K\,tr\left(\boldsymbol{\varepsilon}_{n+1}\right) && (2.6a)\\
\mathbf{s}^{trial}_{n+1} &= 2G\left[dev\left(\boldsymbol{\varepsilon}_{n+1}\right) - \boldsymbol{\varepsilon}^{vp^{trial}}_{n+1}\right] && (2.6b)\\
\mathbf{q}^{trial}_{n+1} &= -\frac{2}{3}K_H\,\boldsymbol{\zeta}^{trial}_{n+1} && (2.6c)\\
q^{trial}_{n+1} &= -K_\xi\,\left(\xi^{trial}_{n+1}\right) && (2.6d)
\end{aligned}$$

where $K_\xi\,\left(\xi^{trial}_{n+1}\right)$ is given by

$$K_\xi\,\left(\xi^{trial}_{n+1}\right) = \left(\sigma_\infty - \sigma_o\right)\left[1 - exp\left(-\delta\,\xi^{trial}_{n+1}\right)\right] + H\,\xi^{trial}_{n+1} \qquad (2.7)$$

2.2.3 Trial yield function

The trial yield function is computed as

$$\Phi^{trial}_{n+1} = \left\|\boldsymbol{\beta}^{trial}_{n+1}\right\| - \sqrt{\frac{2}{3}}\left(\sigma_o - q^{trial}_{n+1}\right) \qquad (2.8)$$

where

$$\boldsymbol{\beta}_{n+1}^{trial} = \mathbf{s}_{n+1}^{trial} - \mathbf{q}_{n+1}^{trial} \tag{2.9}$$

is the so called back-stress tensor.

If $\Phi_{n+1}^{trial} \leq 0$ then the trial state is the final intermediate state,

$$\begin{aligned} \boldsymbol{\varepsilon}_{n+1}^{vp} &= \boldsymbol{\varepsilon}_{n+1}^{vp^{trial}} && (2.10a) \\ \boldsymbol{\zeta}_{n+1} &= \boldsymbol{\zeta}_{n+1}^{trial} && (2.10b) \\ \xi_{n+1} &= \xi_{n+1}^{trial} && (2.10c) \end{aligned}$$

otherwise, if $\Phi_{n+1}^{trial} > 0$ the plastic corrector must be applied. In this case it is not possible to use a standard radial return mapping, in fact in case of non-linear kinematic hardening the unit normal to the yield surface $\mathbf{n}_{n+1}$ is not constant.

2.2.4 Return mapping

The unit normal to the yield surface is defined as

$$\mathbf{n}_{n+1} = \frac{\mathbf{s}_{n+1} - \mathbf{q}_{n+1}}{\|\mathbf{s}_{n+1} - \mathbf{q}_{n+1}\|} = \frac{\boldsymbol{\beta}_{n+1}}{\|\boldsymbol{\beta}_{n+1}\|} \tag{2.11}$$

where tensor $\boldsymbol{\beta}_{n+1} = \mathbf{s}_{n+1} - \mathbf{q}_{n+1}$ is unknown but it can be expressed in terms of the plastic multiplier γ_{n+1}. In fact, $\mathbf{s}_{n+1}$ and $\mathbf{q}_{n+1}$ are respectively given by

$$\begin{aligned} \mathbf{s}_{n+1} &= \mathbf{s}_{n+1}^{trial} - 2G\,\gamma_{n+1}\,\mathbf{n}_{n+1} && (2.12a) \\ \mathbf{q}_{n+1} &= -\frac{2}{3}K_H\,\boldsymbol{\zeta}_{n+1} && (2.12b) \end{aligned}$$

where the $\boldsymbol{\zeta}_{n+1}$ can be expressed as

$$\begin{aligned} \boldsymbol{\zeta}_{n+1} &= \frac{1}{1 + A\,\gamma_{n+1}}\left(\boldsymbol{\zeta}_n - \gamma_{n+1}\,\mathbf{n}_{n+1}\right) && (2.13a) \\ &= \varpi_a\left(\boldsymbol{\zeta}_n - \gamma_{n+1}\,\mathbf{n}_{n+1}\right) && (2.13b) \end{aligned}$$

so that

$$\mathbf{q}_{n+1} = \varpi_a\,\mathbf{q}_{n+1}^{trial} + \varpi_b\,\gamma_{n+1}\,\mathbf{n}_{n+1} \tag{2.14}$$

and finally

$$\boldsymbol{\beta}_{n+1} = \bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right) - \varpi_c \mathbf{n}_{n+1} \tag{2.15}$$

where $\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right)$ is a modified version of the trial back-stress tensor, given by

$$\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right) = \mathbf{s}^{trial}_{n+1} - \varpi_a\, \mathbf{q}^{trial}_{n+1} \tag{2.16}$$

being

$$\varpi_a = \frac{1}{1 + A\,\gamma_{n+1}} \tag{2.17a}$$

$$\varpi_b = \frac{2}{3} K_H\, \varpi_a \tag{2.17b}$$

$$\varpi_c = \left(2G + \varpi_b\right)\, \gamma_{n+1} \tag{2.17c}$$

Developing expression (2.15) it is possible to obtain

$$\left(1 + \frac{\varpi_c}{\left\|\boldsymbol{\beta}_{n+1}\right\|}\right) \boldsymbol{\beta}_{n+1} = \bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right) \tag{2.18}$$

and if we take the norm of above expression

$$\left\|\boldsymbol{\beta}_{n+1}\right\| = \left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right)\right\| - \varpi_c \tag{2.19}$$

and the result is

$$\mathbf{n}_{n+1} = \mathbf{n}^{trial}_{n+1}\left(\gamma_{n+1}\right) = \frac{\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right)}{\left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right)\right\|} \tag{2.20}$$

Note that in case of linear kinematic hardening the standard hypothesis used in isothermal J2-plastic algorithm is recovered

$$\mathbf{n}_{n+1} = \mathbf{n}^{trial}_{n+1} = \frac{\boldsymbol{\beta}^{trial}_{n+1}}{\left\|\boldsymbol{\beta}^{trial}_{n+1}\right\|} = \frac{\mathbf{s}^{trial}_{n+1} - \mathbf{q}^{trial}_{n+1}}{\left\|\mathbf{s}^{trial}_{n+1} - \mathbf{q}^{trial}_{n+1}\right\|} \tag{2.21}$$

Next step is the evaluation of the yield function at time t_{n+1} as

$$\Phi_{n+1} = \left\|\boldsymbol{\beta}_{n+1}\right\| - \sqrt{\frac{2}{3}}\left(\sigma_o - q_{n+1}\right) \tag{2.22a}$$

$$= \left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\left(\gamma_{n+1}\right)\right\| - \varpi_c - \sqrt{\frac{2}{3}}\left(\sigma_o - q_{n+1}\right) \tag{2.22b}$$

To obtain the final values of the plastic multiplier γ_{n+1} the following non-linear equation must be computed

$$\dot{\gamma}_{n+1} = \left\langle \frac{\Phi_{n+1}\left(\gamma_{n+1}\right)}{\eta} \right\rangle^{m} \tag{2.23}$$

that is the same of the zero of equation

$$g_{n+1}\left(\gamma_{n+1}\right) = 0 \tag{2.24}$$

where function $g_{n+1}\left(\gamma_{n+1}\right)$ is given by

$$g_{n+1}\left(\gamma_{n+1}\right) = \Phi_{n+1}\left(\gamma_{n+1}\right) - \eta\, \left(\frac{\gamma_{n+1}}{\Delta t}\right)^{\frac{1}{m}} \tag{2.25}$$

A local Newton-Raphson algorithm can be used to find the solution, so that the linearization of above equation results in

$$g_{n+1} + \left.\frac{\partial g}{\partial \gamma}\right|_{n+1} d\gamma_{n+1} = 0 \tag{2.26}$$

with initial condition $\gamma_{n+1} = 0$.

Term $\left.\dfrac{\partial g}{\partial \gamma}\right|_{n+1}$ is computed as

$$\left.\frac{\partial g}{\partial \gamma}\right|_{n+1} = \left.\frac{\partial \Phi}{\partial \gamma}\right|_{n+1} - \frac{1}{m}\frac{\eta}{\Delta t}\left(\frac{\gamma_{n+1}}{\Delta t}\right)^{\frac{1}{m}-1} \tag{2.27}$$

where

$$\left.\frac{\partial \Phi}{\partial \gamma}\right|_{n+1} = \left.\frac{\partial \left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\right\|}{\partial \gamma}\right|_{n+1} - \left.\frac{\partial \varpi_c}{\partial \gamma}\right|_{n+1} + \frac{2}{3}\left.\frac{\partial q}{\partial \xi}\right|_{n+1} \tag{2.28}$$

being

$$\left.\frac{\partial \left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\right\|}{\partial \gamma}\right|_{n+1} = \mathbf{n}^{trial}_{n+1} : \left.\frac{\partial \bar{\boldsymbol{\beta}}^{trial}_{n+1}}{\partial \gamma}\right|_{n+1} = -A\,\varpi_a\,\varpi_b\,\varpi_n \tag{2.29a}$$

$$\left.\frac{\partial \varpi_c}{\partial \gamma}\right|_{n+1} = 2G + \varpi_b - \gamma_{n+1}\,A\,\varpi_a\,\varpi_b \tag{2.29b}$$

$$\left.\frac{\partial q}{\partial \xi}\right|_{n+1} = -K_{\xi\xi} \tag{2.29c}$$

where

$$\varpi_n = \mathbf{n}_{n+1}^{trial} : \boldsymbol{\zeta}_n \tag{2.30a}$$

$$K_{\xi\xi} = (\sigma_\infty - \sigma_o)\left[\delta \exp\left(-\delta\ \xi_{n+1}\right)\right] + H \tag{2.30b}$$

The iterative process induced by the local Newton-Raphson algorithm is the following:

$\Delta\gamma^{(o)}$	$=$	0
loop		
$d\gamma^{(k)}$	$=$	$-\dfrac{g_{n+1}^{(k)}}{\left.\dfrac{\partial g}{\partial \gamma}\right\|_{n+1}^{(k)}}$
$\Delta\gamma^{(k+1)}$	$=$	$\Delta\gamma^{(k)} + d\gamma^{(k)}$
end loop		
γ_{n+1}	$=$	$\Delta\gamma$

2.2.5 Update database and compute stresses

First, let us update the plastic variables as

$$\boldsymbol{\varepsilon}_{n+1}^{p} = \boldsymbol{\varepsilon}_{n+1}^{p\,trial} + \gamma_{n+1}\mathbf{n}_{n+1} \tag{2.31a}$$

$$\boldsymbol{\zeta}_{n+1} = \boldsymbol{\zeta}_{n+1}^{trial} - \gamma_{n+1}\left(\mathbf{n}_{n+1} + A\,\boldsymbol{\zeta}_{n+1}\right) \tag{2.31b}$$

$$\xi_{n+1} = \xi_{n+1}^{trial} + \gamma_{n+1}\sqrt{\frac{2}{3}} \tag{2.31c}$$

The deviatoric stress fields and the hardening stress tensors are computed as follows

$$\mathbf{s}_{n+1} = \mathbf{s}_{n+1}^{trial} - 2G\,\gamma_{n+1}\mathbf{n}_{n+1} \tag{2.32a}$$

$$\mathbf{q}_{n+1} = -\frac{2}{3}K_H\ \boldsymbol{\zeta}_{n+1} \tag{2.32b}$$

$$q_{n+1} = -K_\xi\ \left(\xi_{n+1}\right) \tag{2.32c}$$

so that the total stress tensor can be evaluated as

$$\boldsymbol{\sigma}_{n+1} = p_{n+1}\mathbf{1} + \mathbf{s}_{n+1} \tag{2.33}$$

where the pressure is given by

$$p_{n+1} = K\ tr\left(\boldsymbol{\varepsilon}_{n+1}\right) \tag{2.34}$$

2.3 Linearization & tangent operator

In this section the linearization of the problem and the following tangent matrix is introduced.

The constitutive tangent matrix is defined as

$$\mathbf{C}^{ep}_{n+1}=\frac{d\boldsymbol{\sigma}_{n+1}}{d\boldsymbol{\varepsilon}_{n+1}} \tag{2.35}$$

where $d\boldsymbol{\sigma}_{n+1}$ is given by

$$d\boldsymbol{\sigma}_{n+1} = \mathbf{C}^e : d\boldsymbol{\varepsilon}_{n+1} - 2G\left(\gamma_{n+1}\, d\mathbf{n}_{n+1} + d\gamma_{n+1}\, \mathbf{n}_{n+1}\right) \tag{2.36}$$

being $\mathbf{C}^e$ the elastic matrix defined as

$$\mathbf{C}^e = K\,\mathbf{1}\otimes\mathbf{1} + 2G\left(\mathbf{I}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right) \tag{2.37}$$

where $\mathbf{1}=\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ the rank-two symmetric unit tensor.

The linearization of the unit normal to the yield surface $d\mathbf{n}_{n+1}$ is given by

$$d\mathbf{n}_{n+1} = \frac{1}{\left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\right\|}\left(\mathbf{I}-\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right) : d\bar{\boldsymbol{\beta}}^{trial}_{n+1} \tag{2.38}$$

where $d\bar{\boldsymbol{\beta}}^{trial}_{n+1}$ results in

$$d\bar{\boldsymbol{\beta}}^{trial}_{n+1} = 2G\left(\mathbf{I}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right) : d\boldsymbol{\varepsilon}_{n+1} - A\,\varpi_a\,\varpi_b\,\boldsymbol{\zeta}_n d\gamma_{n+1} \tag{2.39}$$

so that

$$d\mathbf{n}_{n+1} = \frac{2G}{\left\|\bar{\boldsymbol{\beta}}^{trial}_{n+1}\right\|}\left(\mathbf{I}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}-\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right) : d\boldsymbol{\varepsilon}_{n+1} \tag{2.40}$$

$$-A\,\varpi_a\,\varpi_b\left(\mathbf{I}-\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right) : \boldsymbol{\zeta}_n d\gamma_{n+1} \tag{2.41}$$

To compute $d\gamma_{n+1}$ let us consider the following equation that must be verified

$$dg_{n+1}\left(\boldsymbol{\varepsilon}_{n+1},\gamma_{n+1}\right) = 0 \tag{2.42}$$

where function g_{n+1} was gine in (2.24), so that

$$\left.\frac{\partial g}{\partial \boldsymbol{\varepsilon}}\right|_{n+1} : d\boldsymbol{\varepsilon}_{n+1} + \left.\frac{\partial g}{\partial \gamma}\right|_{n+1} d\gamma_{n+1} = 0 \tag{2.43}$$

The first term $\left.\dfrac{\partial g}{\partial \varepsilon}\right|_{n+1}$ results in

$$\left.\frac{\partial g}{\partial \varepsilon}\right|_{n+1} = 2G\,\mathbf{n}_{n+1} \tag{2.44}$$

while to compute the second term $\left.\dfrac{\partial g}{\partial \gamma}\right|_{n+1}$, you can refer to equation (2.27) so that

$$d\gamma_{n+1} = \frac{2G}{\varpi_d}\mathbf{n}_{n+1} : d\boldsymbol{\varepsilon}_{n+1} \tag{2.45}$$

where

$$\varpi_d = -\left.\frac{\partial g}{\partial \gamma}\right|_{n+1} \tag{2.46}$$

and the final result is

$$\begin{aligned} d\mathbf{n}_{n+1} \;=\; & \frac{2G}{\left\|\boldsymbol{\beta}_{n+1}^{trial}\right\|}\left[\left(\mathbf{I}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}-\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right)\right. \\ & \left.-\frac{A\,\varpi_a\,\varpi_b}{\varpi_d}\left(\boldsymbol{\zeta}_n\otimes\mathbf{n}_{n+1}-\varpi_n\,\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right)\right] : d\boldsymbol{\varepsilon}_{n+1} \end{aligned} \tag{2.47}$$

Using equations (2.45) and (2.47) it is possible to evaluate the constitutive tangent matrix as

$$\mathbf{C}_{n+1}^{ep} = \delta_1\mathbf{I}+\delta_2\left(\mathbf{1}\otimes\mathbf{1}\right)+\delta_3\left(\mathbf{n}_{n+1}\otimes\mathbf{n}_{n+1}\right)+\delta_4\left(\boldsymbol{\zeta}_n\otimes\mathbf{n}_{n+1}\right) \tag{2.48}$$

where the coefficients δ_k, $(k = 1, 4)$ are given by

$$\delta_1 = 2G\left(1-\varpi_g\right) \tag{2.49a}$$

$$\delta_2 = K - \frac{\delta_1}{3} \tag{2.49b}$$

$$\delta_3 = 2G\left(\varpi_g - \frac{2G + A\,\varpi_a\,\varpi_b\varpi_g\varpi_n}{\varpi_d}\right) \tag{2.49c}$$

$$\delta_4 = 2G\frac{A\,\varpi_a\,\varpi_b\varpi_g}{\varpi_d} \tag{2.49d}$$

Chapter 3

Numerical Tests

The constitutive model proposed has been implemented in the finite element code COMET. In the following table it is possible to find the correspondence between the name of the material properties used in the program and the notation used here:

Model parameters:

K	Bulk modulus	KBULK
G	Shear modulus	SHEAR
σ_o	Initial flow stress	YEINI
σ_∞	Isotropic hardening saturation flow stress	YEFIN
δ	Exponent of the isotropic hardening saturation law	YEPOW
H	Linear isotropic hardening coefficient	LINHR
K_H	Linear kinematic hardening coefficient	KHARD
A	Non-linear kinematic hardening parameter	NLKHD
η	Viscosity	VISCO
m	Exponent of the non-linear viscous law	EXPVI

In the next sections a number of numerical tests will be presented showing the mechanical response of different simplified constitutive models up to the final visco-plastic one that include non-linear isotropic and kinematic hardening.

Figure (3.1) shows the loading-unloading function used in the displacement driven tests. Finally, in the next table it is possible to find the values of the material properties used in the simulations:

KBULK	=	83333.3	[MPa]
SHEAR	=	38461.5	[MPa]
YEINI	=	150.0	[MPa]
YEFIN	=	$\begin{cases} 150.0 \text{ Model H1} \\ 180.0 \text{ Others} \end{cases}$	[MPa]
YEPOW	=	7.0	–
LINHR	=	100.0	[MPa]
KHARD	=	500.0	[MPa]
NLKHD	=	50.0	–
VISCO	=	$\begin{cases} 1.0\text{E+6 Model V1} \\ 100.0 \text{ Others} \end{cases}$	[MPa]
EXPVI	=	0.128	–

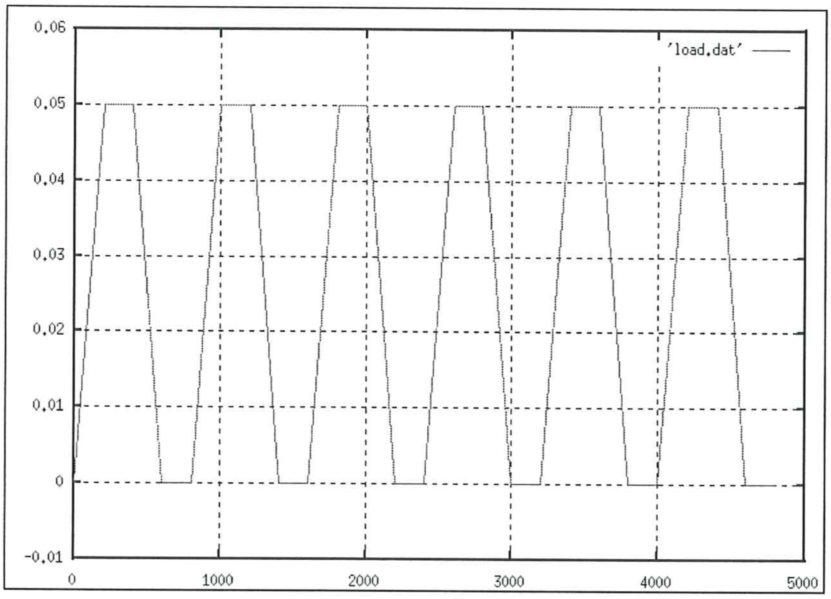


Figure 3.1: Load amplitude versus time.

3.1 Model - P1

Model characterization:

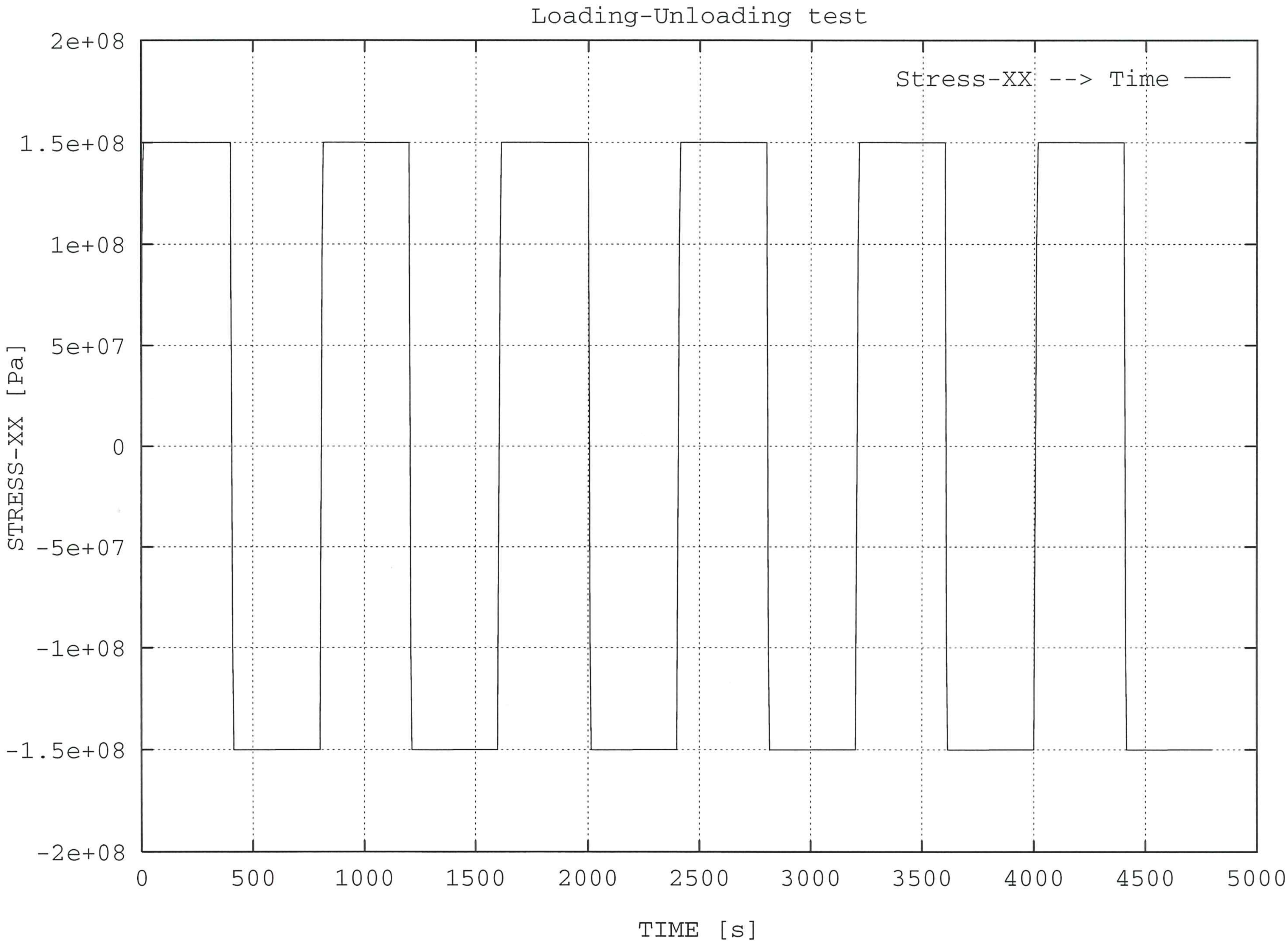
Linear isotropic hardening	OFF
Isotropic hardening saturation law	OFF
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	OFF
Non-linear viscous law	OFF

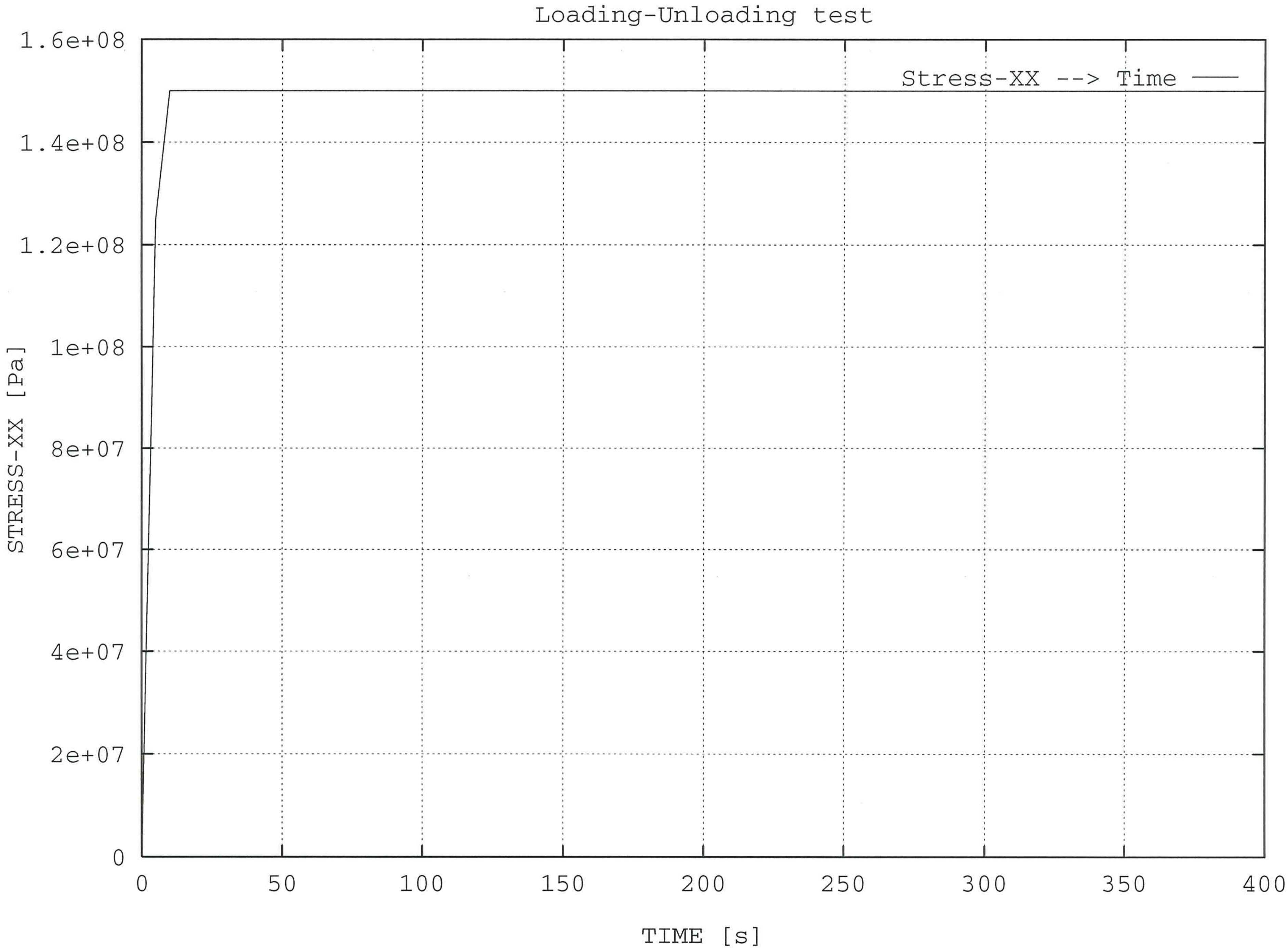
Material properties to be input:

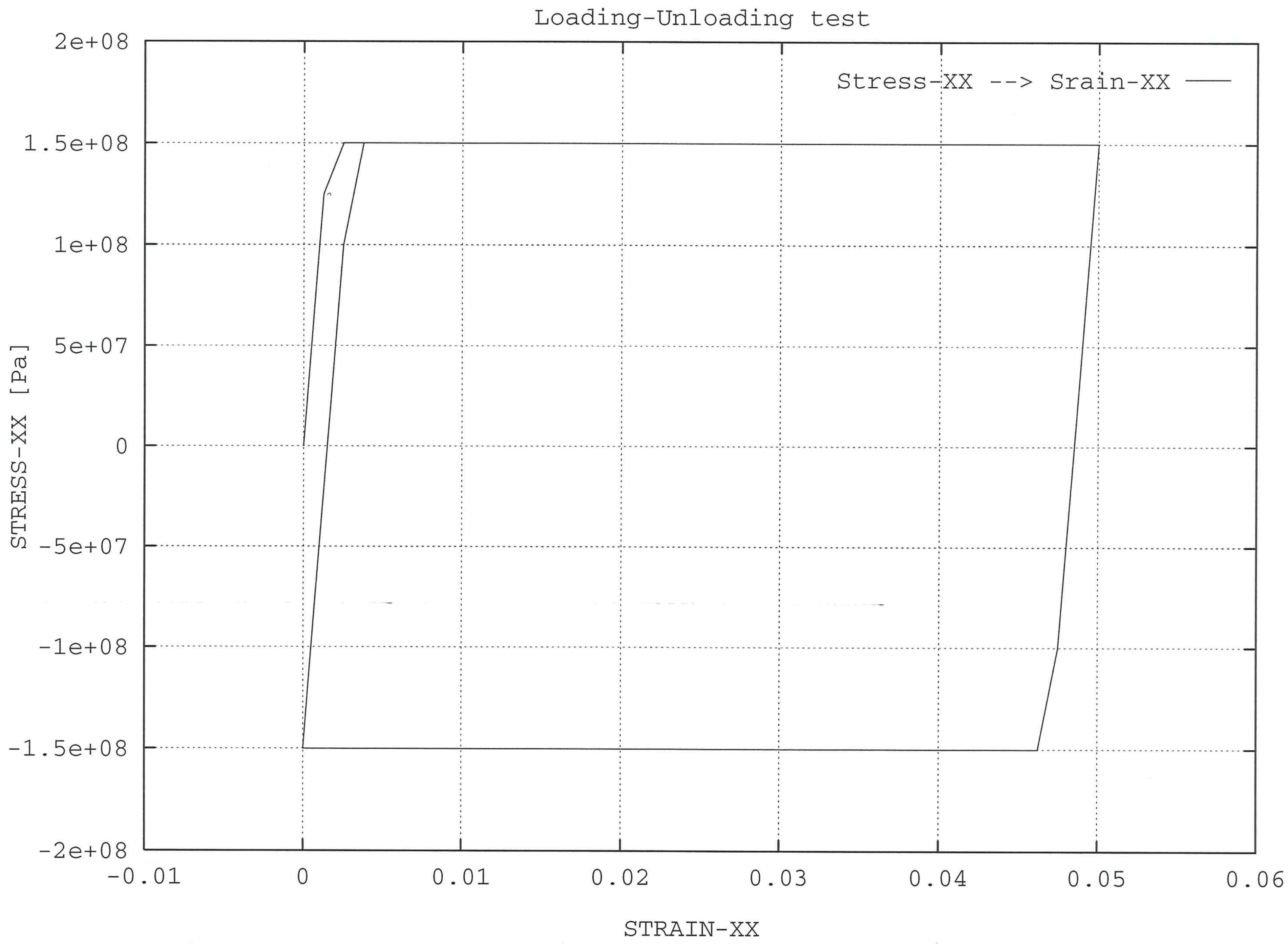
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress

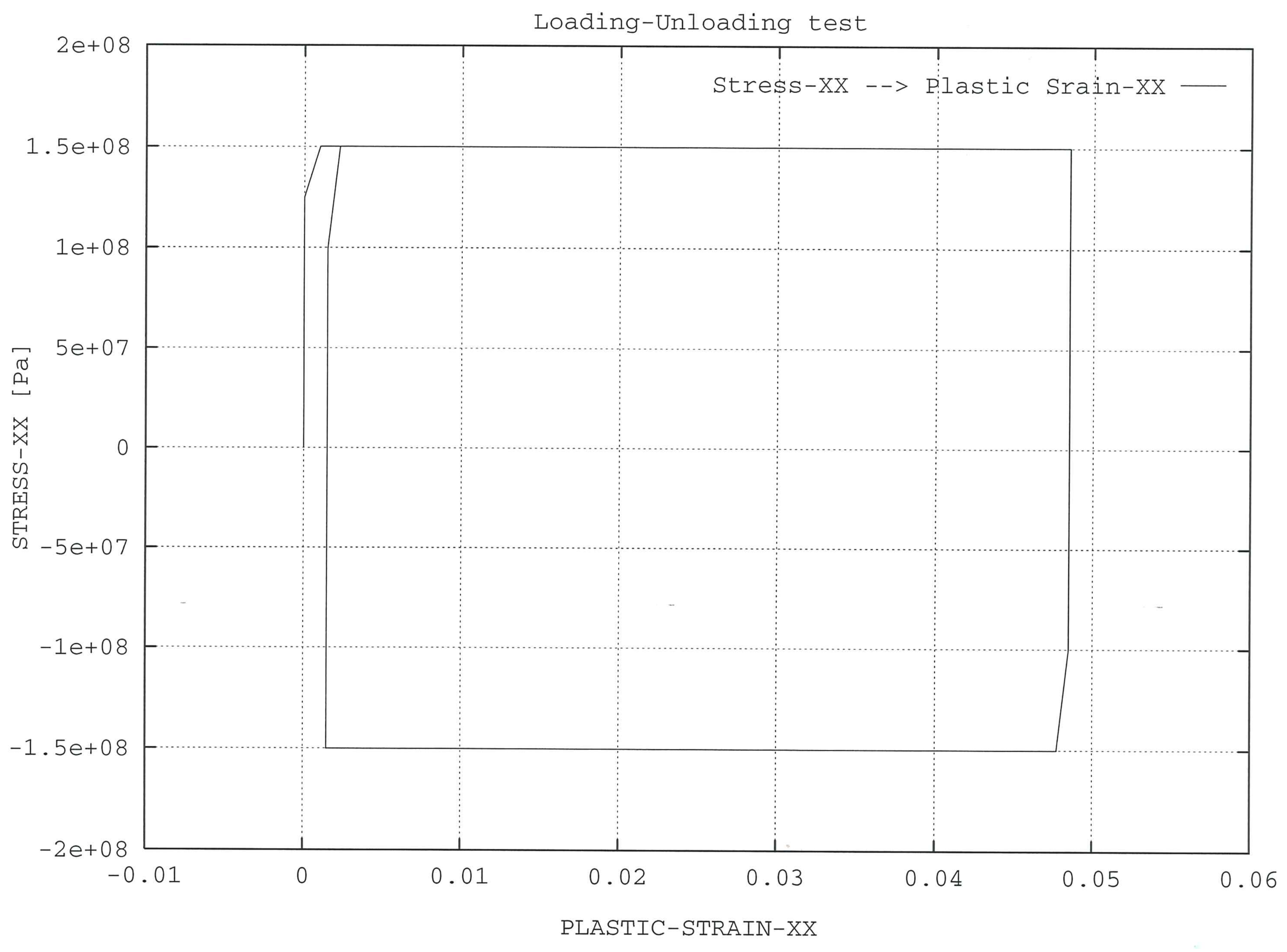
Constitutive model:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathrm{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$
Plastic potential	$\Phi = \lVert\mathrm{s}\rVert - \sqrt{\frac{2}{3}}\sigma_o$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial\Phi}{\partial \mathrm{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\gamma} = \lVert\dot{\boldsymbol{\varepsilon}}^p\rVert$
Elastic domain	$J_2(\boldsymbol{\sigma}) < \sigma_o$ $J_2(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}}\,\lVert\mathrm{s}\rVert$


Loading-Unloading test
Stress-XX --> Time
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
TIME [s]
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
50
100
150
200
250
300
350
400


Loading-Unloading test
Stress-XX --> Srain-XX
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.2 Model - H1

Model characterization:

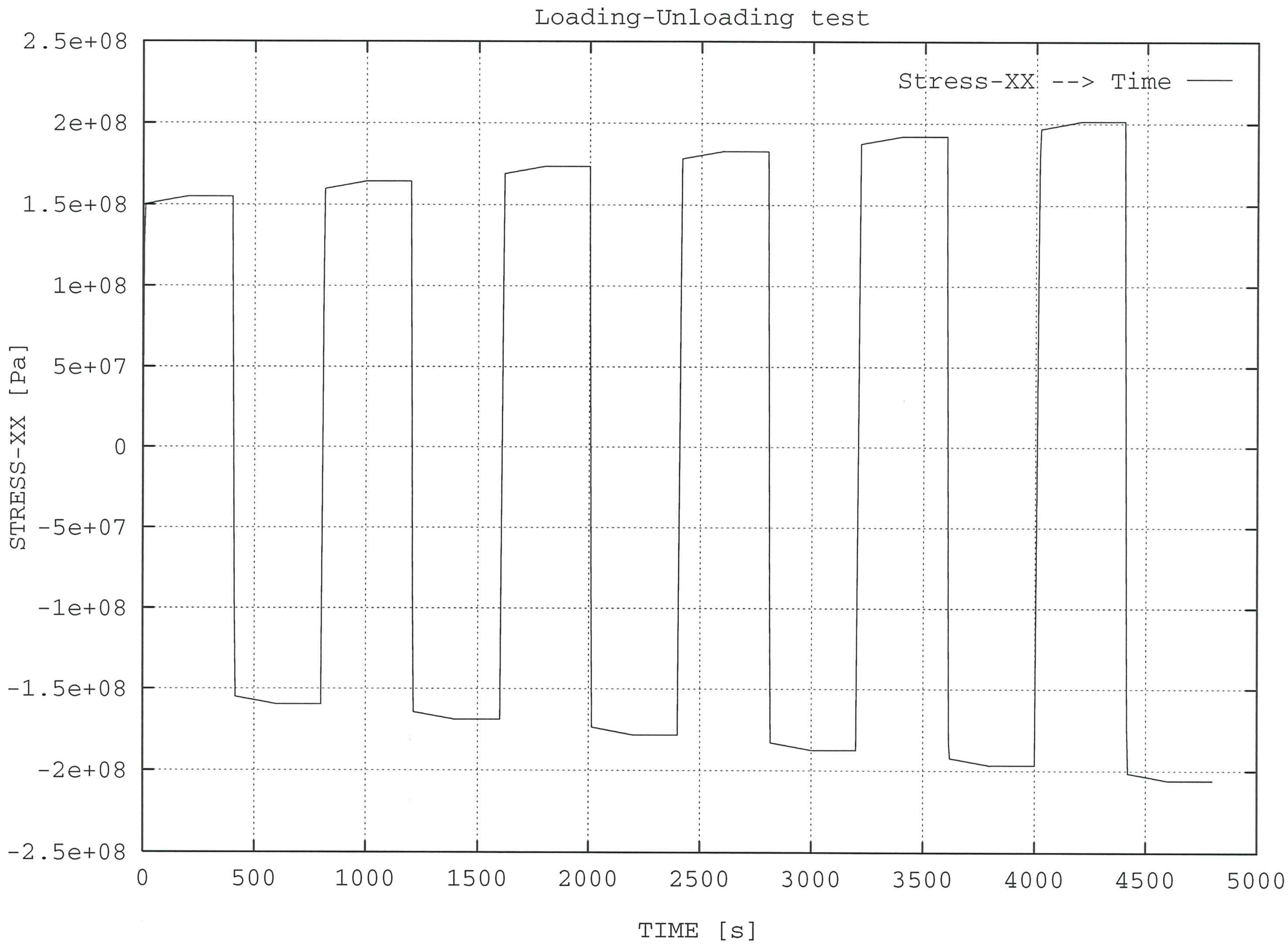
Linear isotropic hardening	ON
Isotropic hardening saturation law	OFF
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	OFF
Non-linear viscous law	OFF

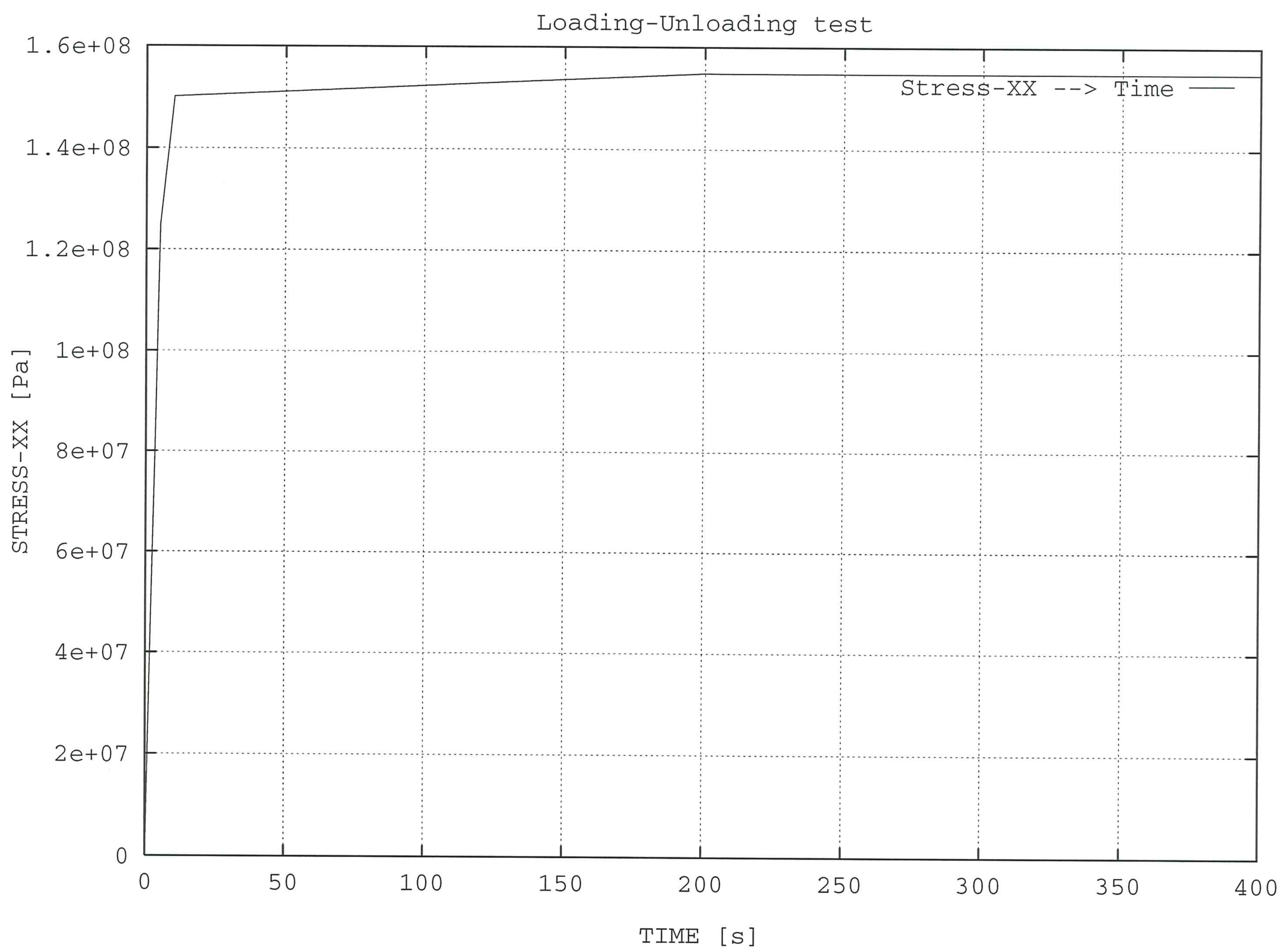
Material properties to be input:

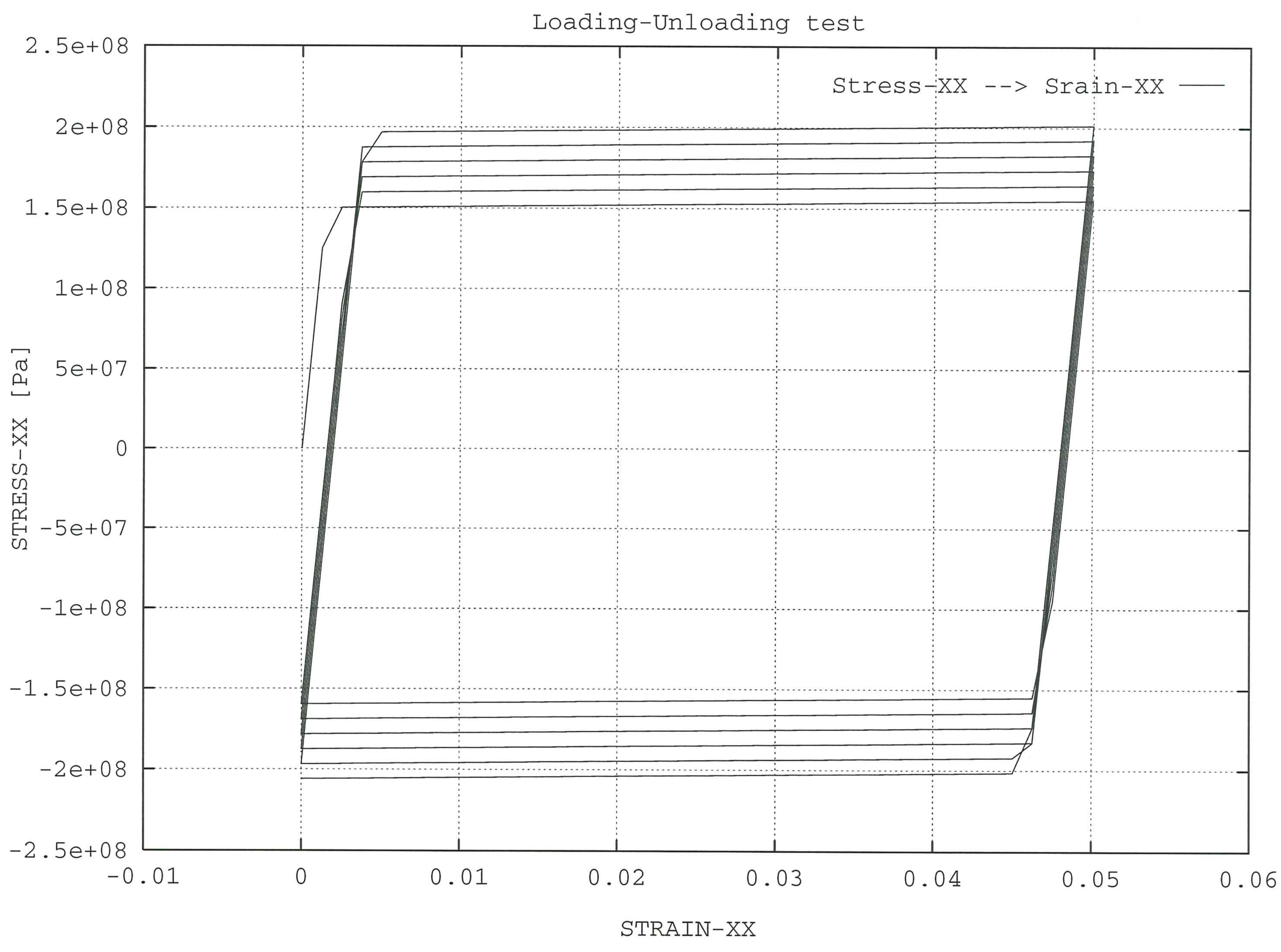
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
H	Linear isotropic hardening coefficient

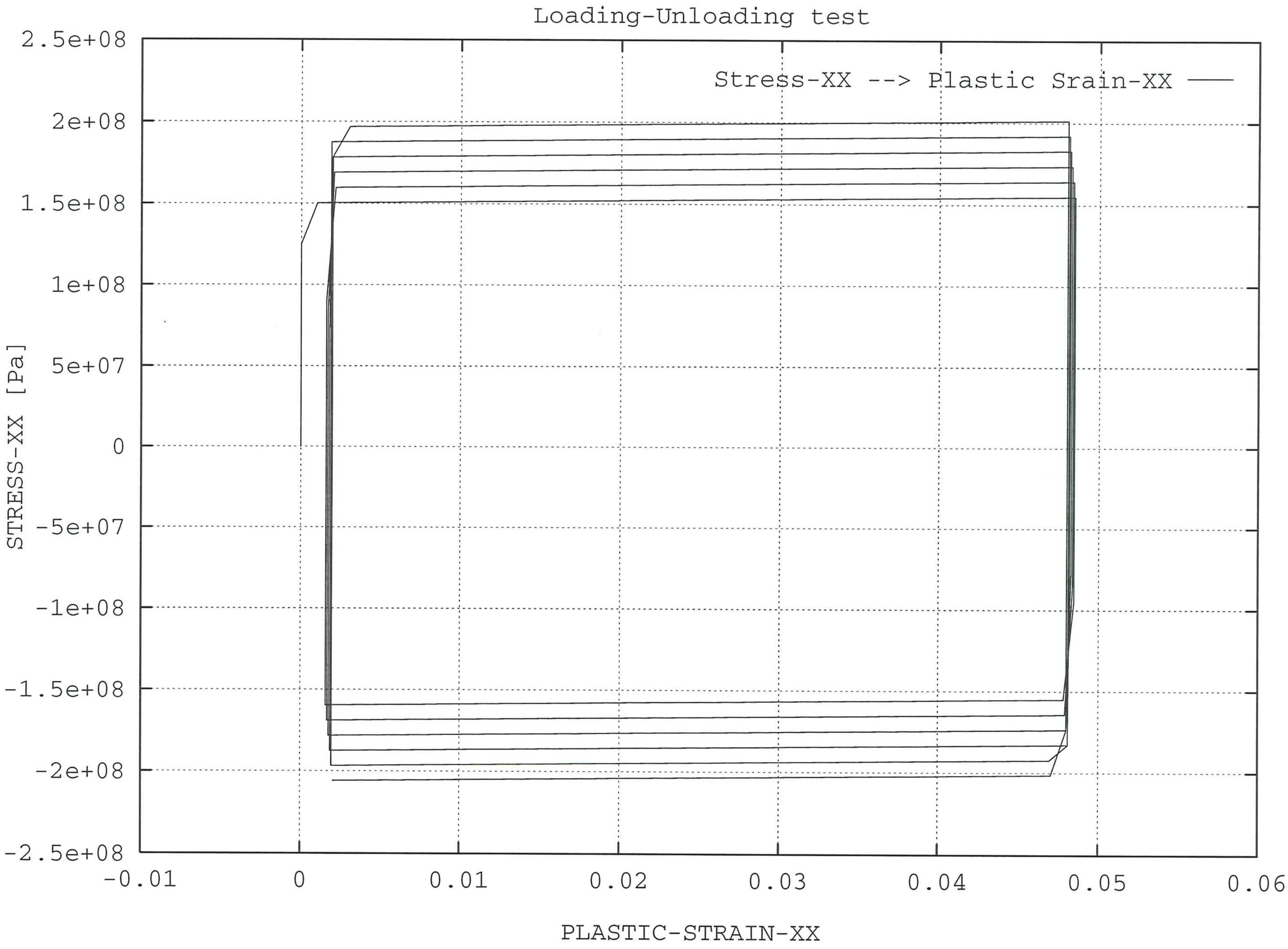
Constitutive model:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $q = -H\xi$
Plastic potential	$\Phi = \lVert \mathbf{s} \rVert - \sqrt{\frac{2}{3}}\,(\sigma_o - q)$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\xi} = \dot{\gamma}\,\frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}}$ $\dot{\gamma} = \lVert \dot{\boldsymbol{\varepsilon}}^p \rVert$
Elastic domain	$J_2\,(\boldsymbol{\sigma}) < \sigma_o + R$ $\begin{cases} J_2\,(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}}\,\lVert \mathbf{s} \rVert \\ R = H\xi \end{cases}$


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX



Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.3 Model - H2

Model characterization:

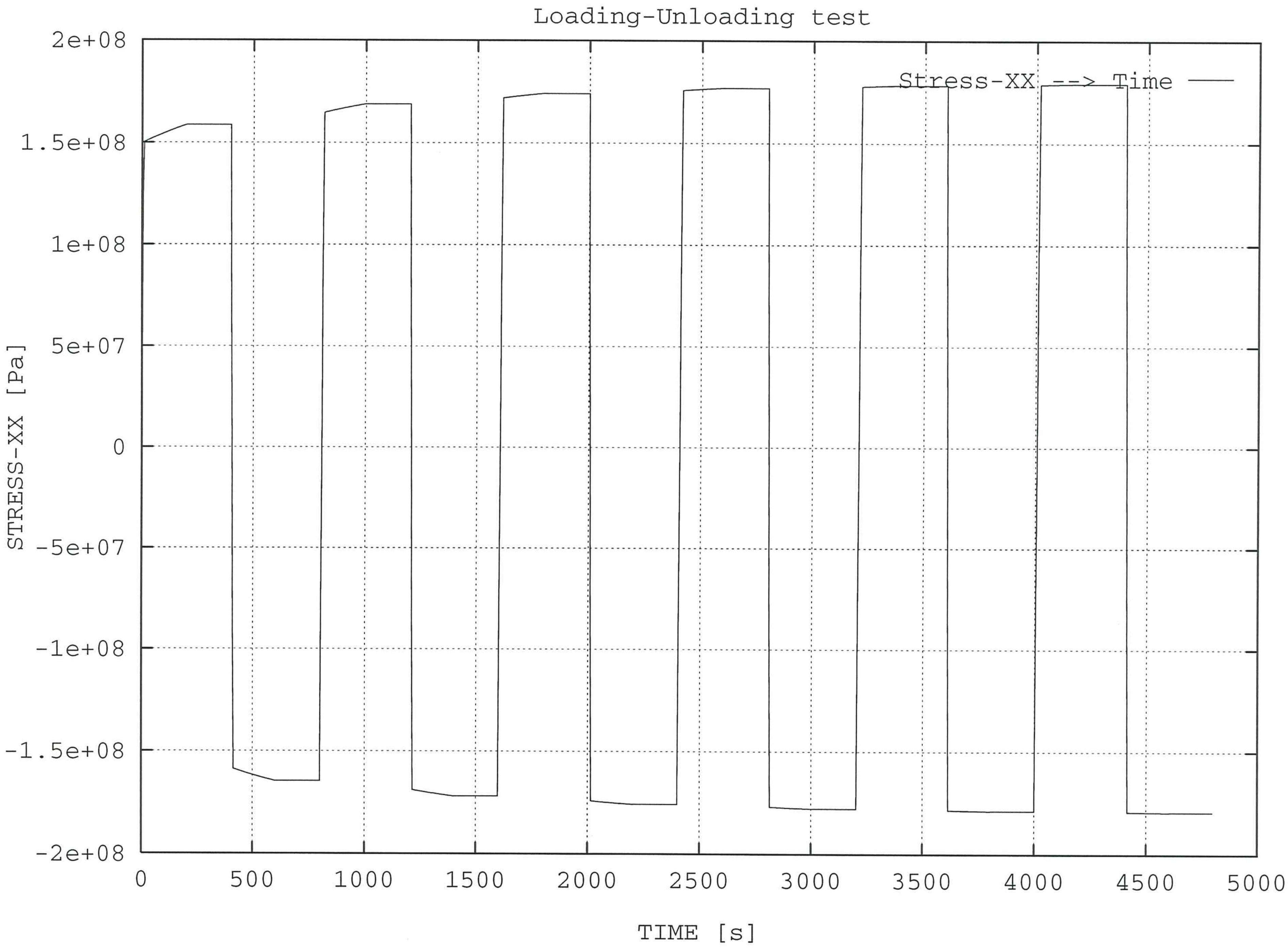
Linear isotropic hardening	OFF
Isotropic hardening saturation law	ON
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	OFF
Non-linear viscous law	OFF

Material properties to be input:

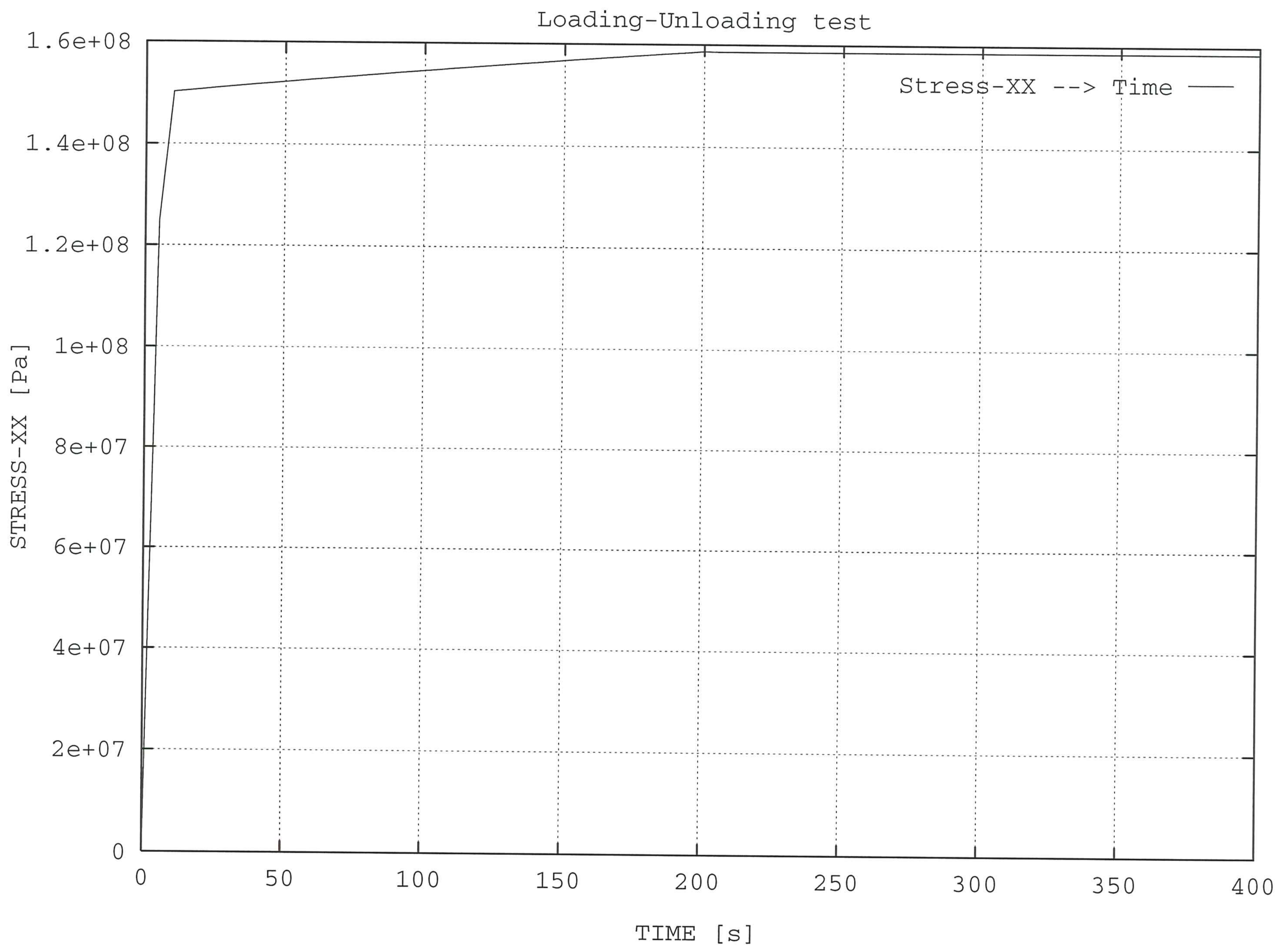
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law

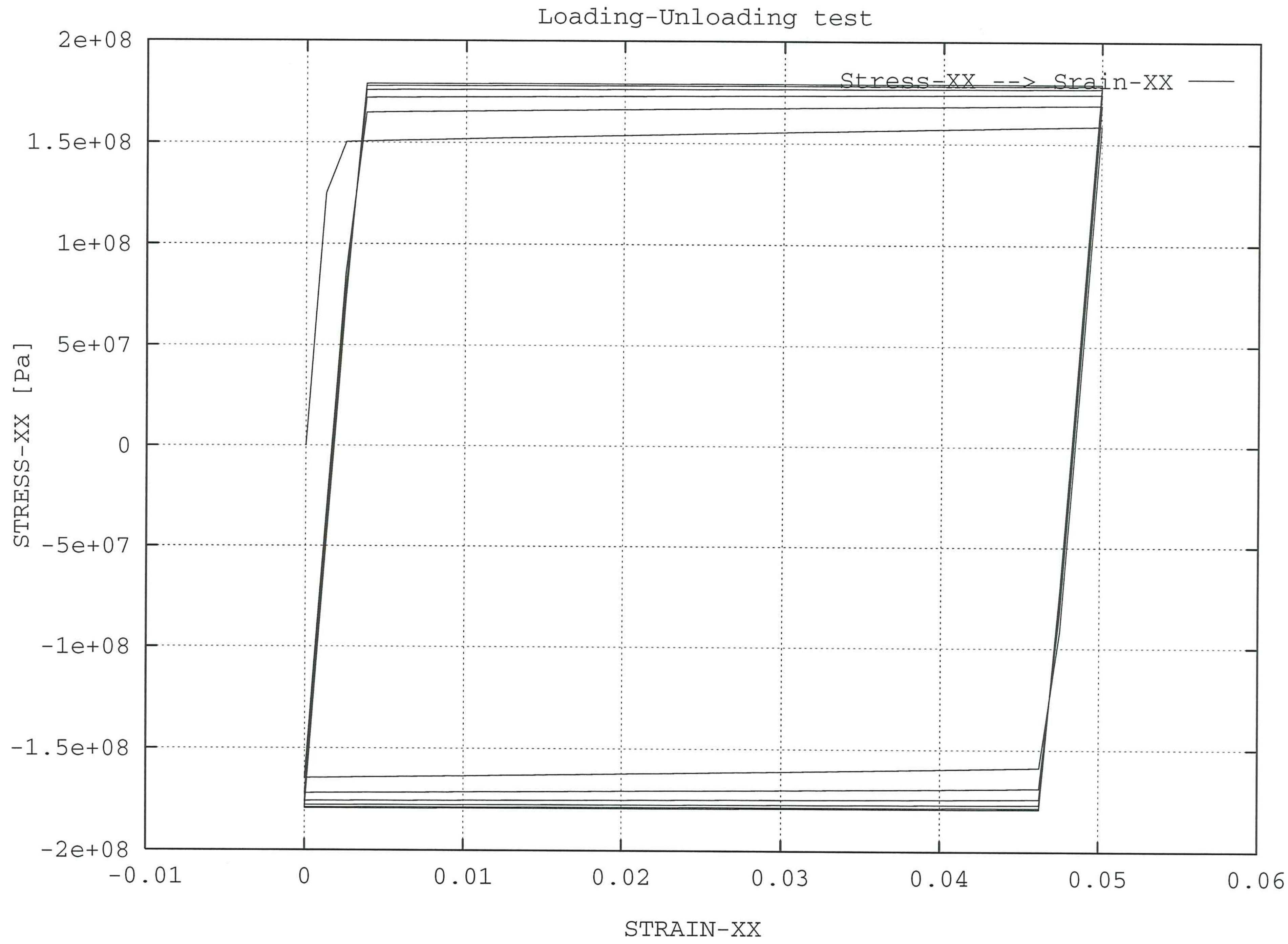
Constitutive model:

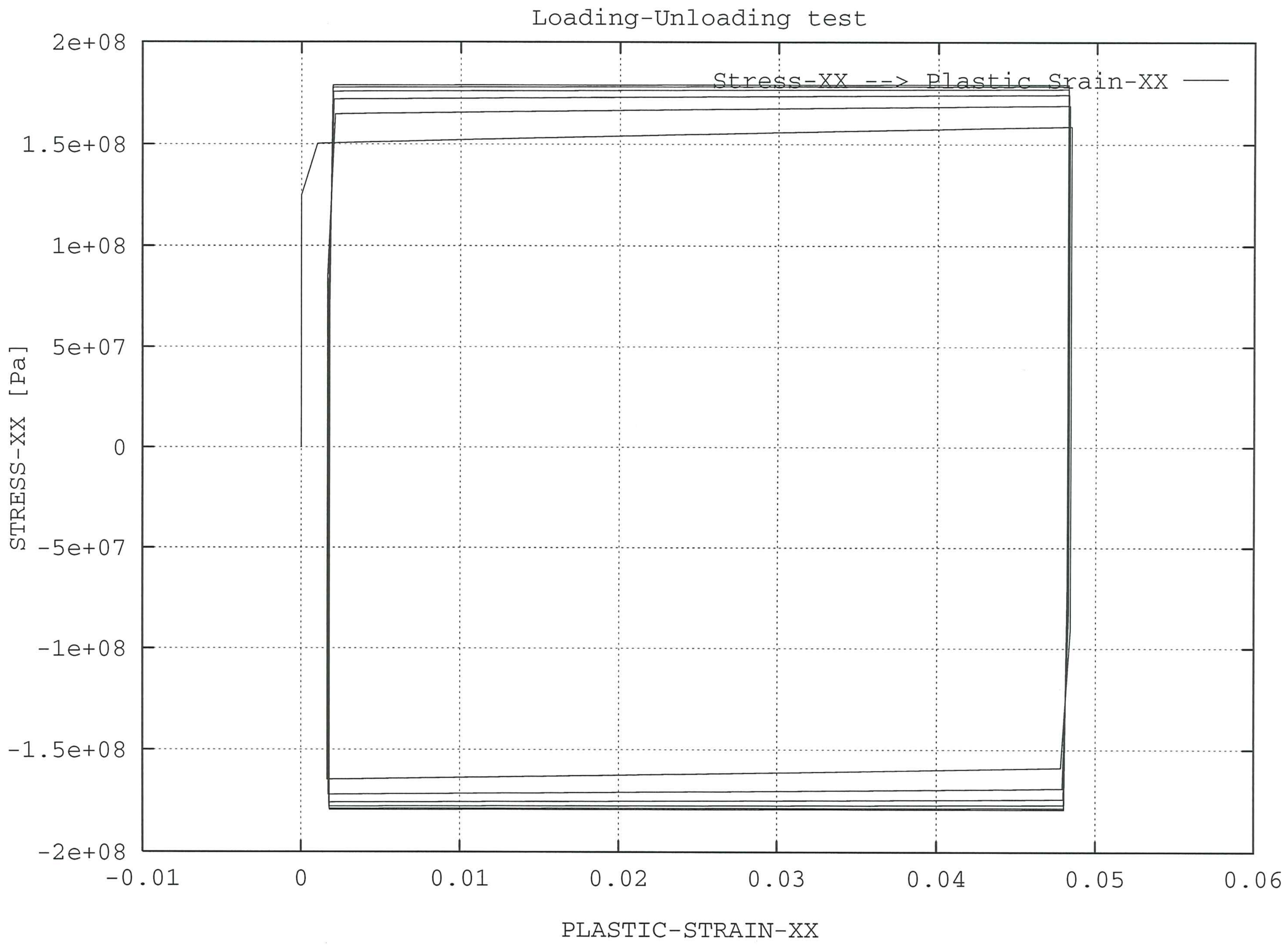
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $q = -\left(\sigma_\infty - \sigma_o\right)\left[1 - \exp\left(-\delta\,\xi\right)\right]$
Plastic potential	$\Phi = \lVert \mathbf{s} \rVert - \sqrt{\frac{2}{3}}\left(\sigma_o - q\right)$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\xi} = \dot{\gamma}\,\frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}}$ $\dot{\gamma} = \lVert \dot{\boldsymbol{\varepsilon}}^p \rVert$
Elastic domain	$J_2\left(\boldsymbol{\sigma}\right) < \sigma_o + R$ $\begin{cases} J_2\left(\boldsymbol{\sigma}\right) = \sqrt{\frac{3}{2}}\,\lVert \mathbf{s} \rVert \\ R = \left(\sigma_\infty - \sigma_o\right)\left[1 - \exp\left(-\delta\,\xi\right)\right] \end{cases}$



Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
TIME [s]
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
PLASTIC-STRAIN-XX
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06

3.4 Model - H3

Model characterization:

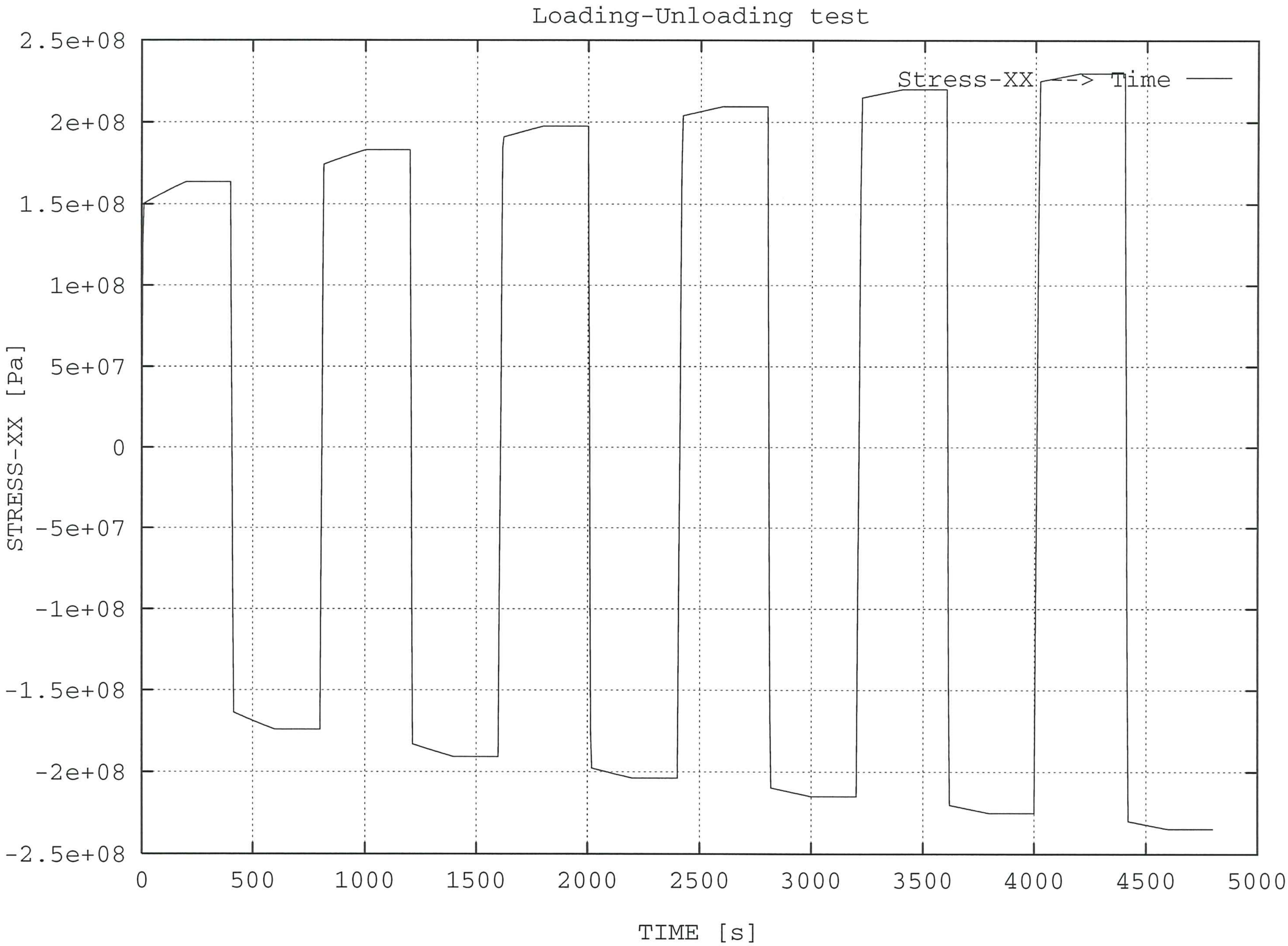
Linear isotropic hardening	ON
Isotropic hardening saturation law	ON
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	OFF
Non-linear viscous law	OFF

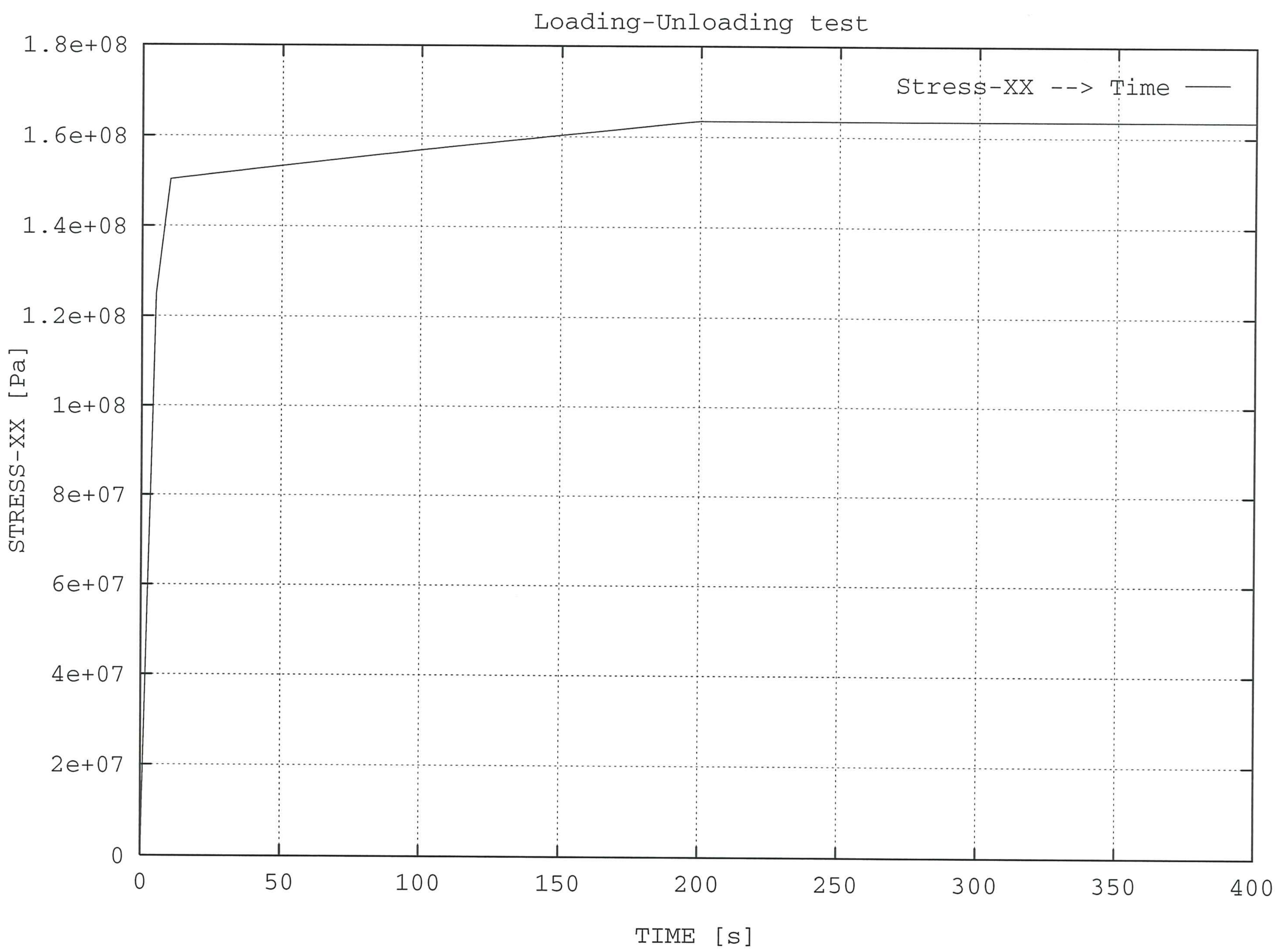
Material properties to be input:

K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law
H	Linear isotropic hardening coefficient

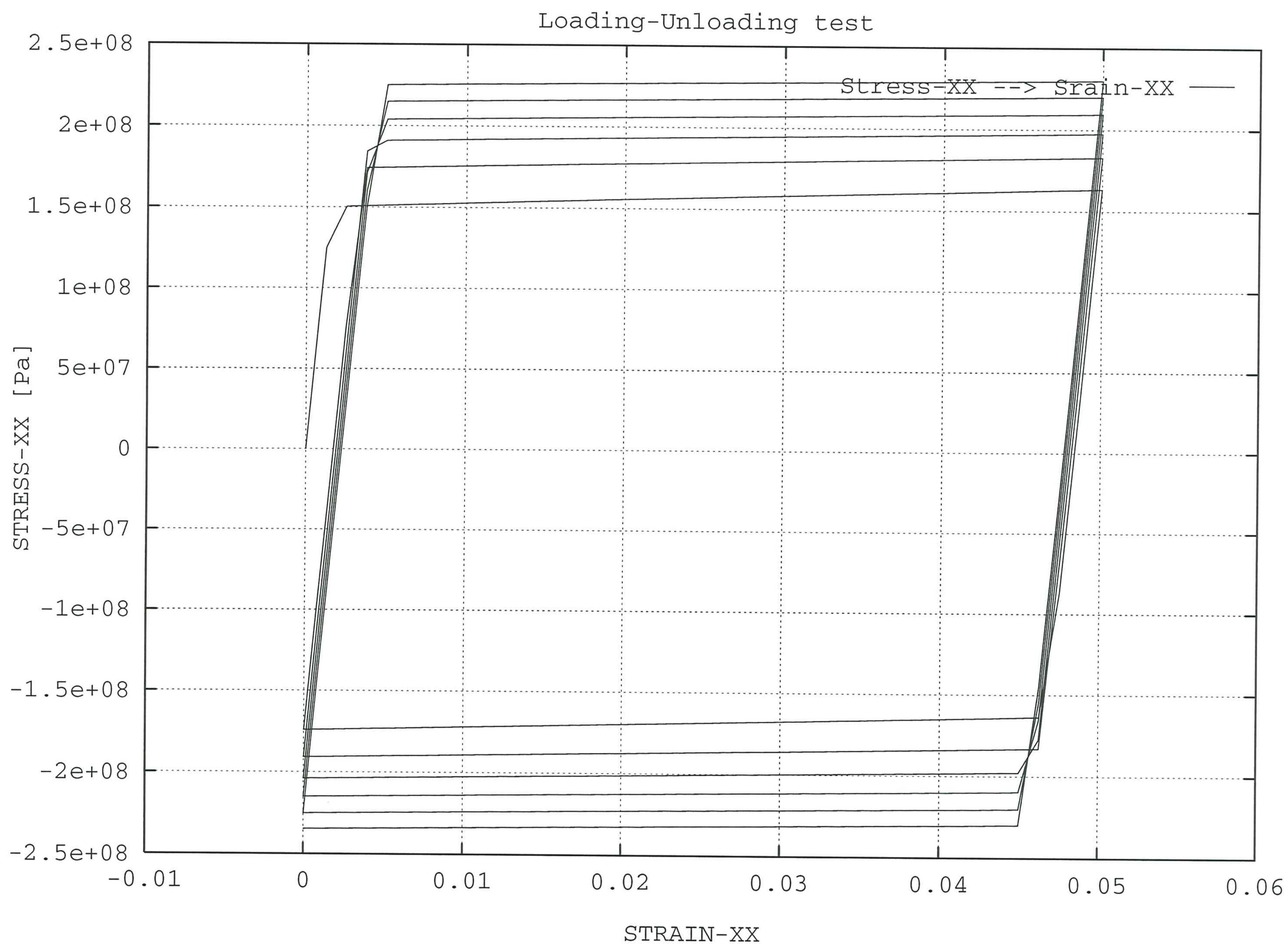
Constitutive model:

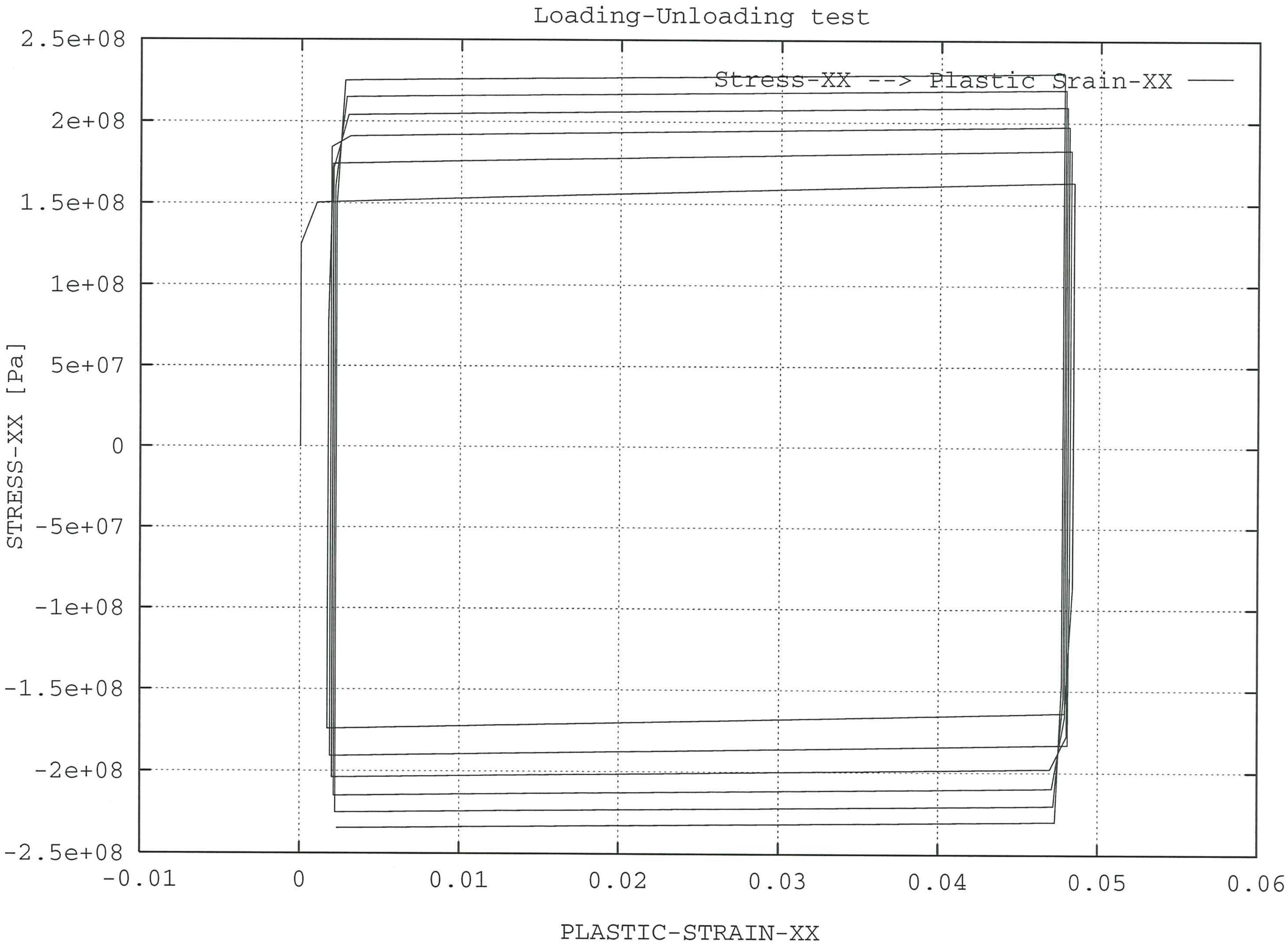
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $q = -(\sigma_\infty - \sigma_o)\,[1 - \exp(-\delta\,\xi)] - H\xi$
Plastic potential	$\Phi = \Vert\mathbf{s}\Vert - \sqrt{\frac{2}{3}}\,(\sigma_o - q)$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\xi} = \dot{\gamma}\,\frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}}$ $\dot{\gamma} = \Vert\dot{\boldsymbol{\varepsilon}}^p\Vert$
Elastic domain	$J_2(\boldsymbol{\sigma}) < \sigma_o + R$ $\begin{cases} J_2(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}}\,\Vert\mathbf{s}\Vert \\ R = (\sigma_\infty - \sigma_o)\,[1 - \exp(-\delta\,\xi)] + H\xi \end{cases}$


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]



Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
1.8e+08
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
STRAIN-XX
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.5 Model - K1

Model characterization:

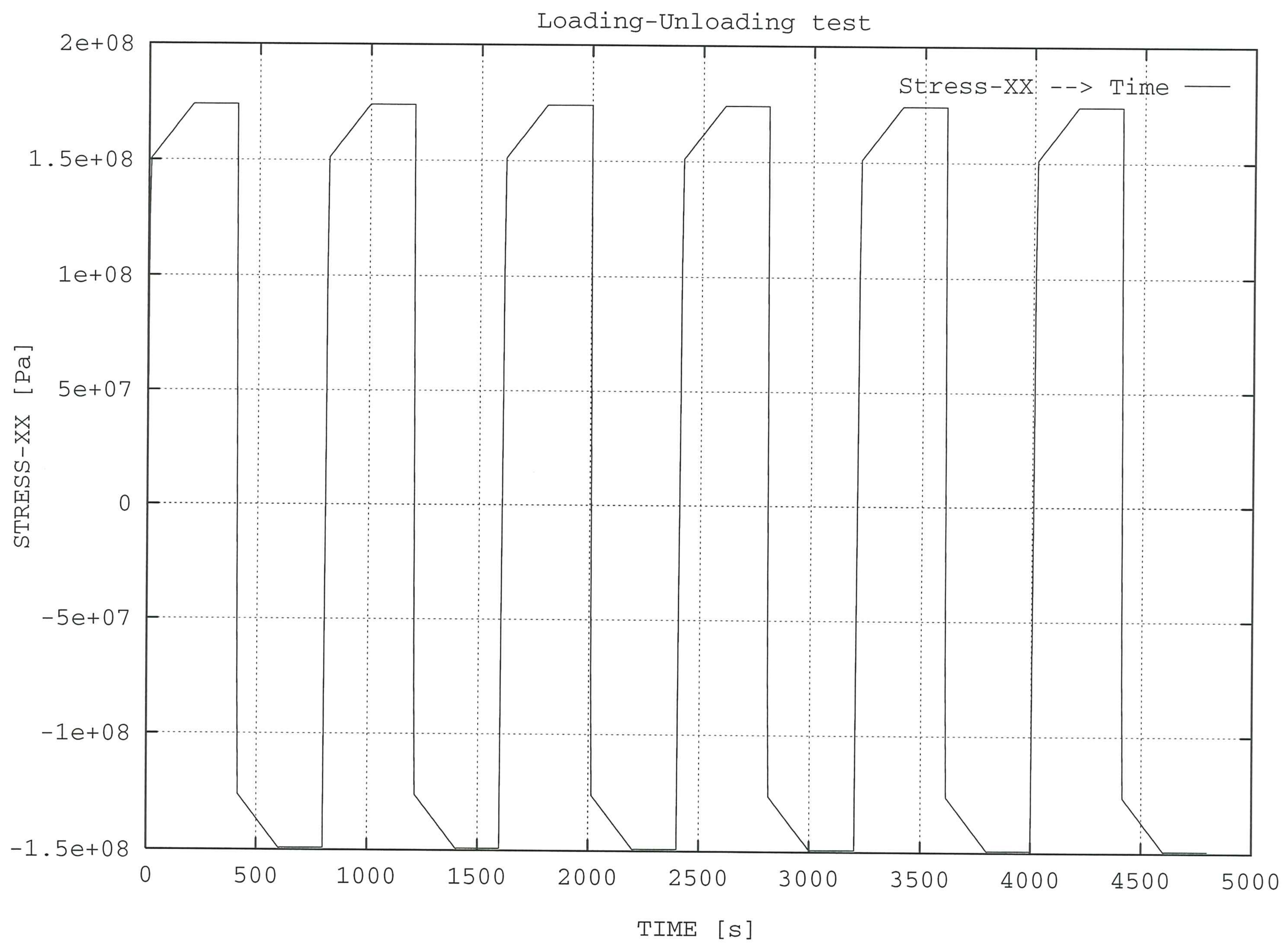
Linear isotropic hardening	OFF
Isotropic hardening saturation law	OFF
Kinematic hardening	ON
Non-linear kinematic law	OFF
Viscosity	OFF
Non-linear viscous law	OFF

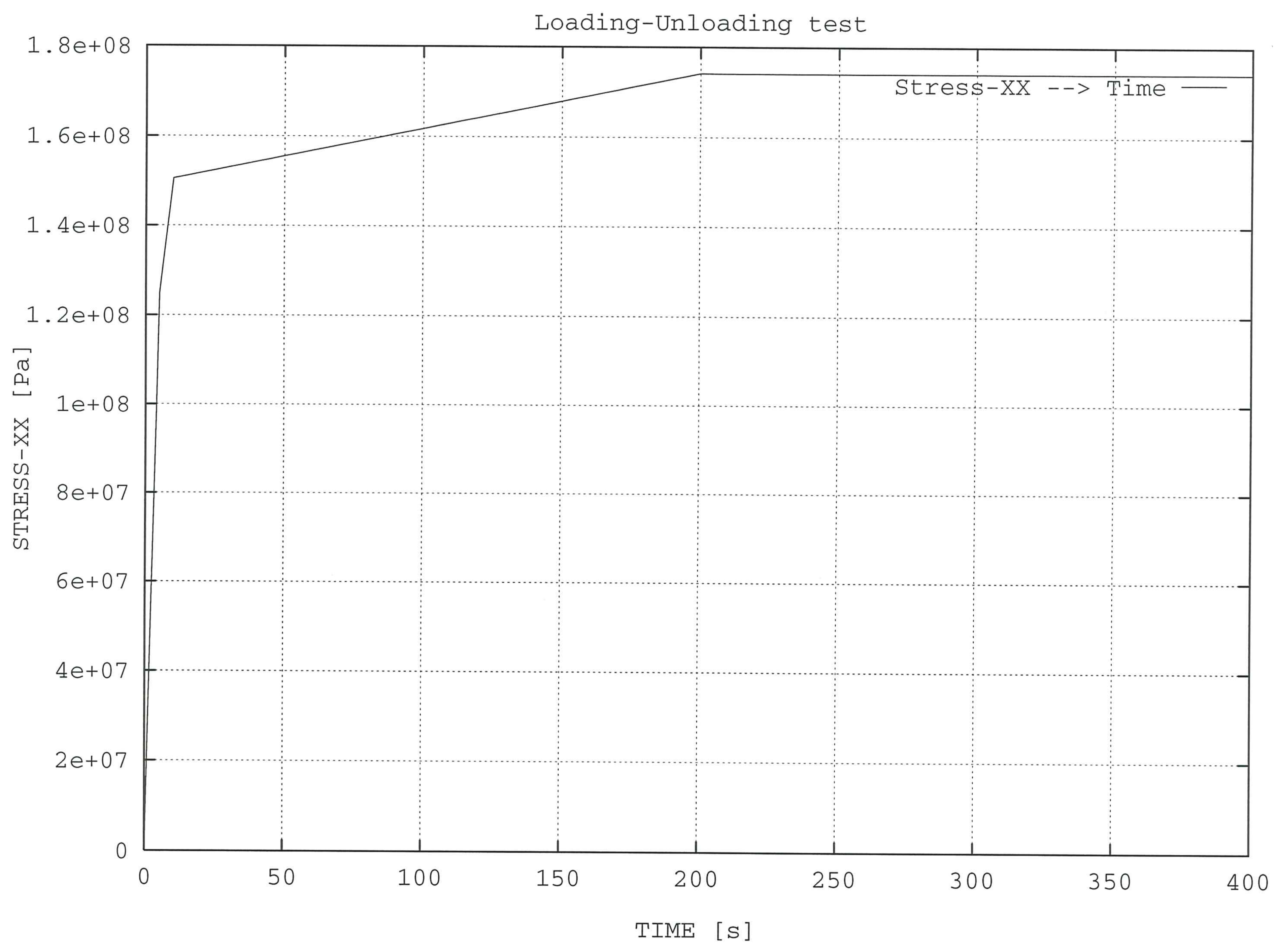
Material properties to be input:

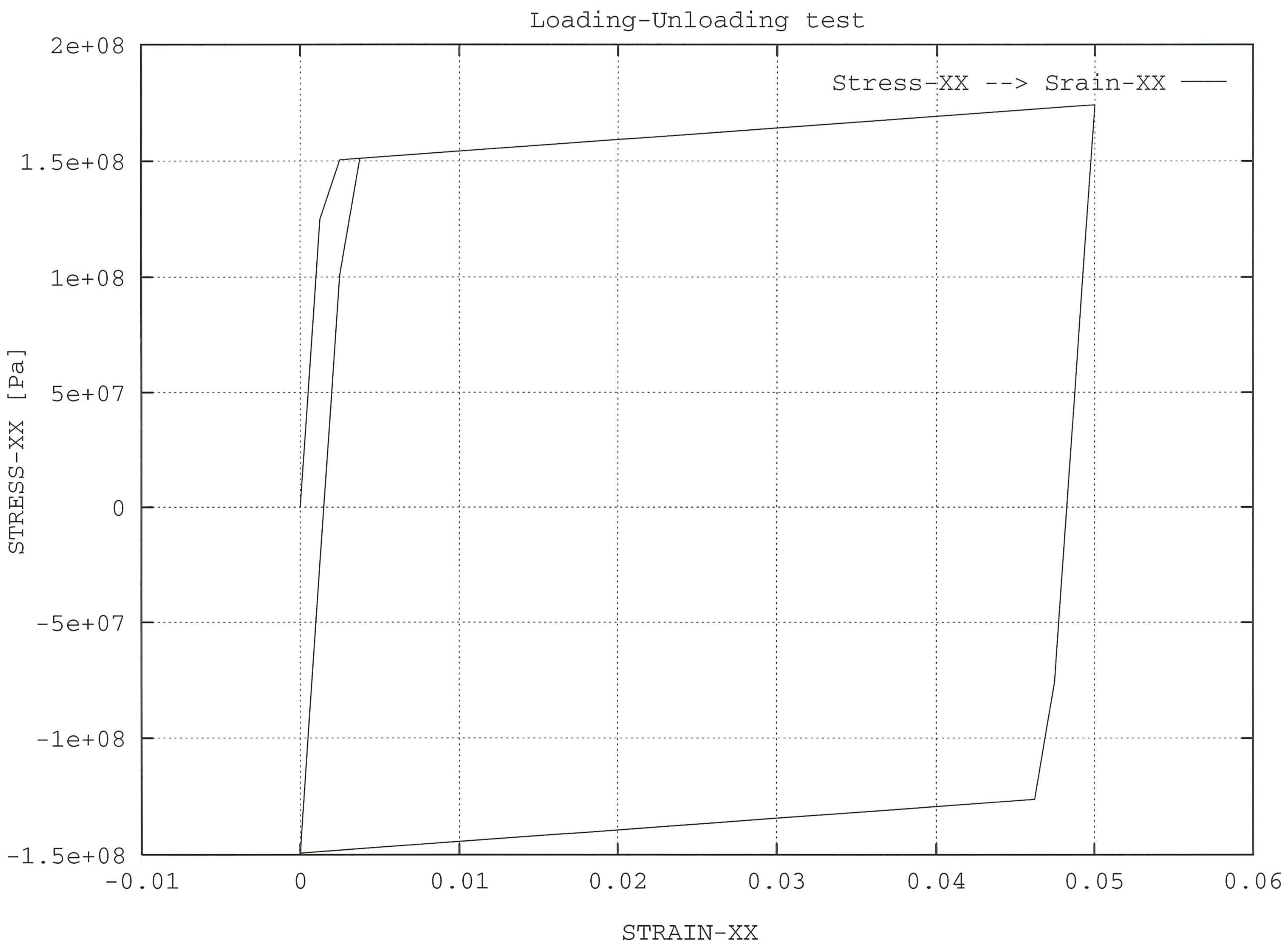
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
K_H	Linear kinematic hardening coefficient

Constitutive model:

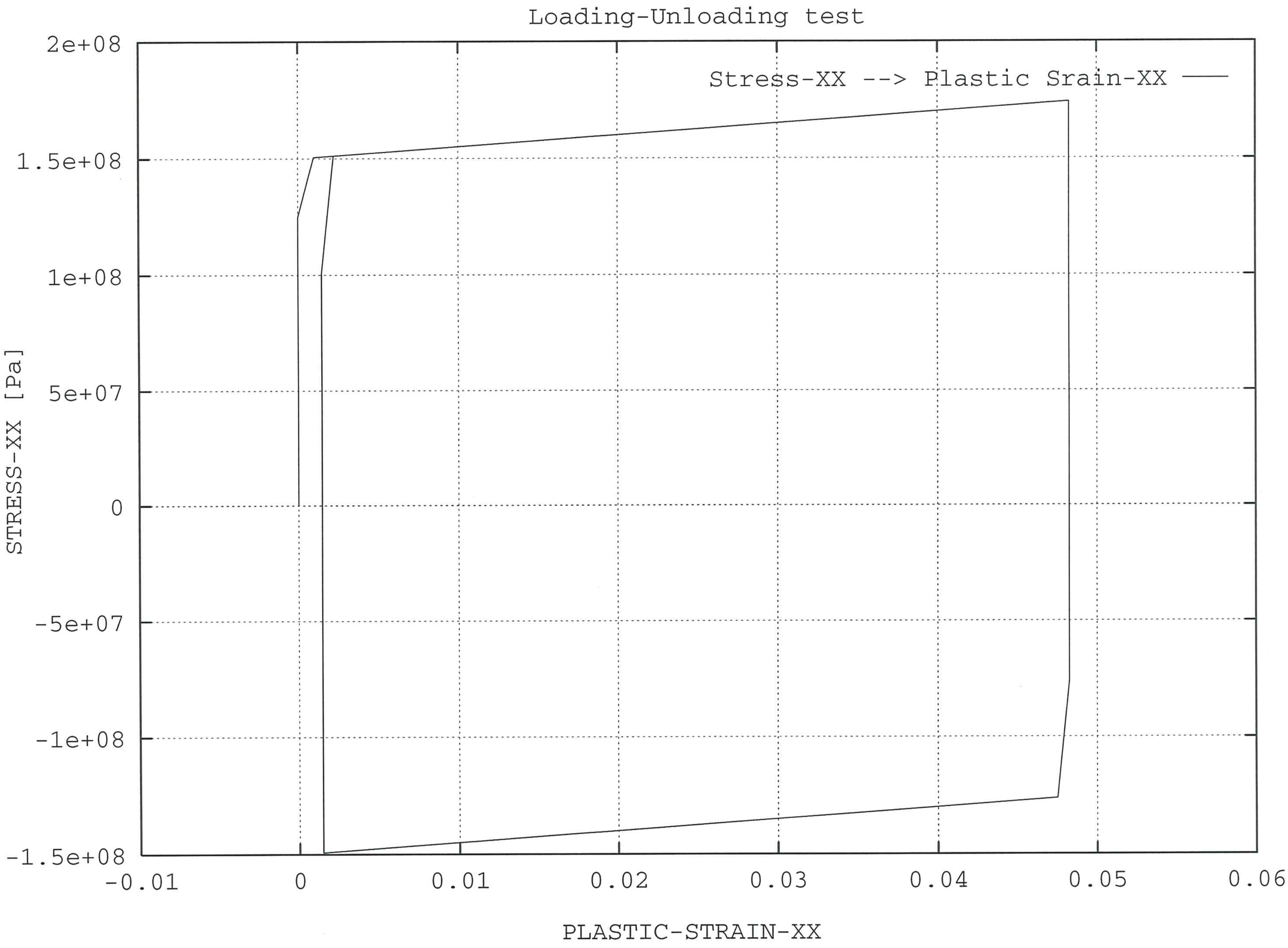
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $\mathbf{q} = -\frac{2}{3} K_H\, \boldsymbol{\zeta}$
Plastic potential	$\Phi = \|\mathbf{s} - \mathbf{q}\| - \sqrt{\frac{2}{3}}\sigma_o$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\boldsymbol{\zeta}} = \dot{\gamma}\frac{\partial \Phi}{\partial \mathbf{q}} = \begin{cases} -\dot{\gamma}\,\mathbf{n} \\ or \\ -\dot{\boldsymbol{\varepsilon}}^p \end{cases}$ $\dot{\gamma} = \|\dot{\boldsymbol{\varepsilon}}^p\|$
Elastic domain	$J_2\,(\boldsymbol{\sigma} - \mathbf{q}) < \sigma_o$ $J_2\,(\boldsymbol{\sigma} - \mathbf{q}) = \sqrt{\frac{3}{2}}\,\|\mathbf{s} - \mathbf{q}\|$


Loading-Unloading test
Stress-XX --> Time
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
1.8e+08
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]



Loading-Unloading test
Stress-XX --> Srain-XX
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX



Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.6 Model - K2

Model characterization:

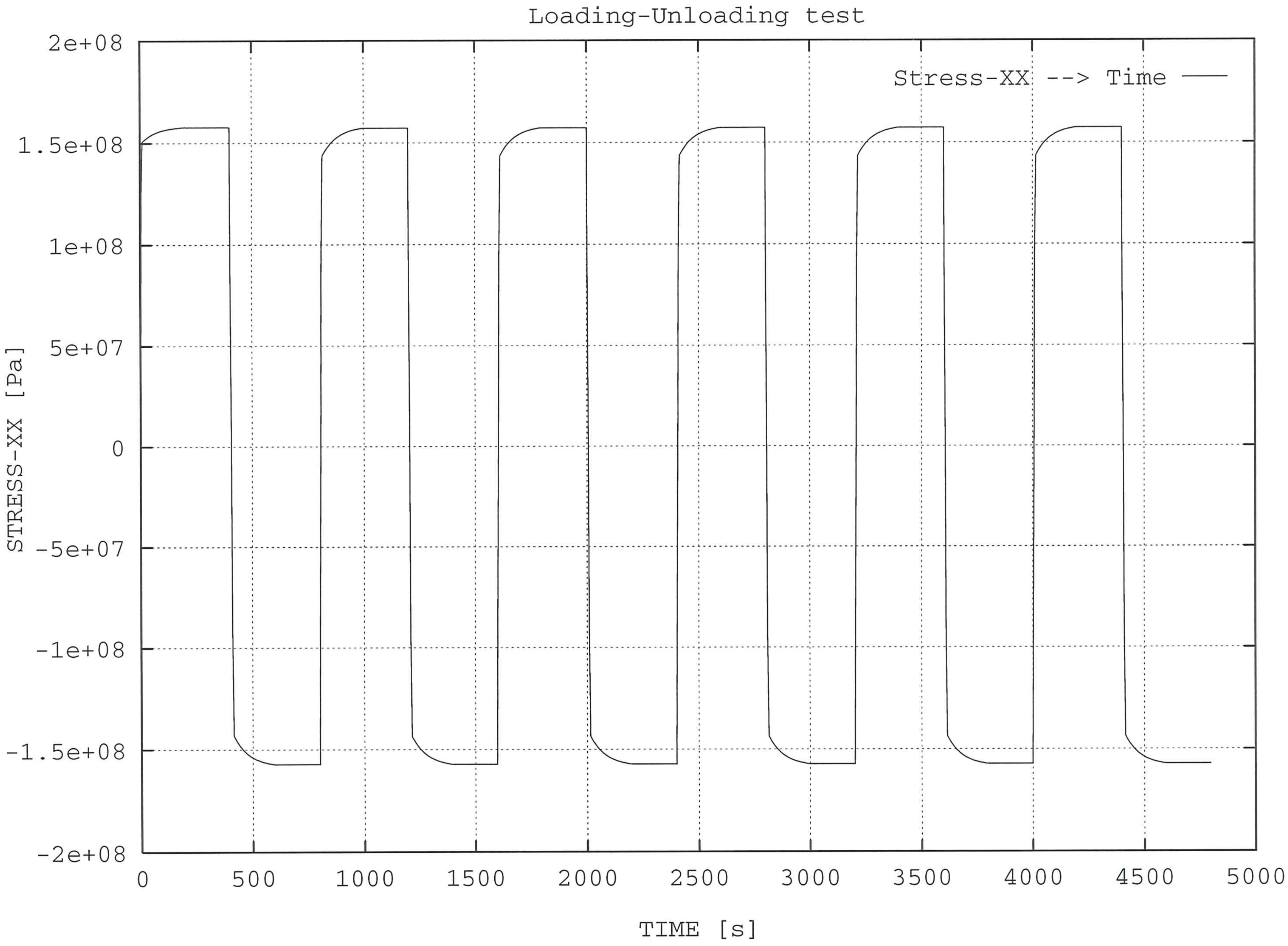
Linear isotropic hardening	OFF
Isotropic hardening saturation law	OFF
Kinematic hardening	ON
Non-linear kinematic law	ON
Viscosity	OFF
Non-linear viscous law	OFF

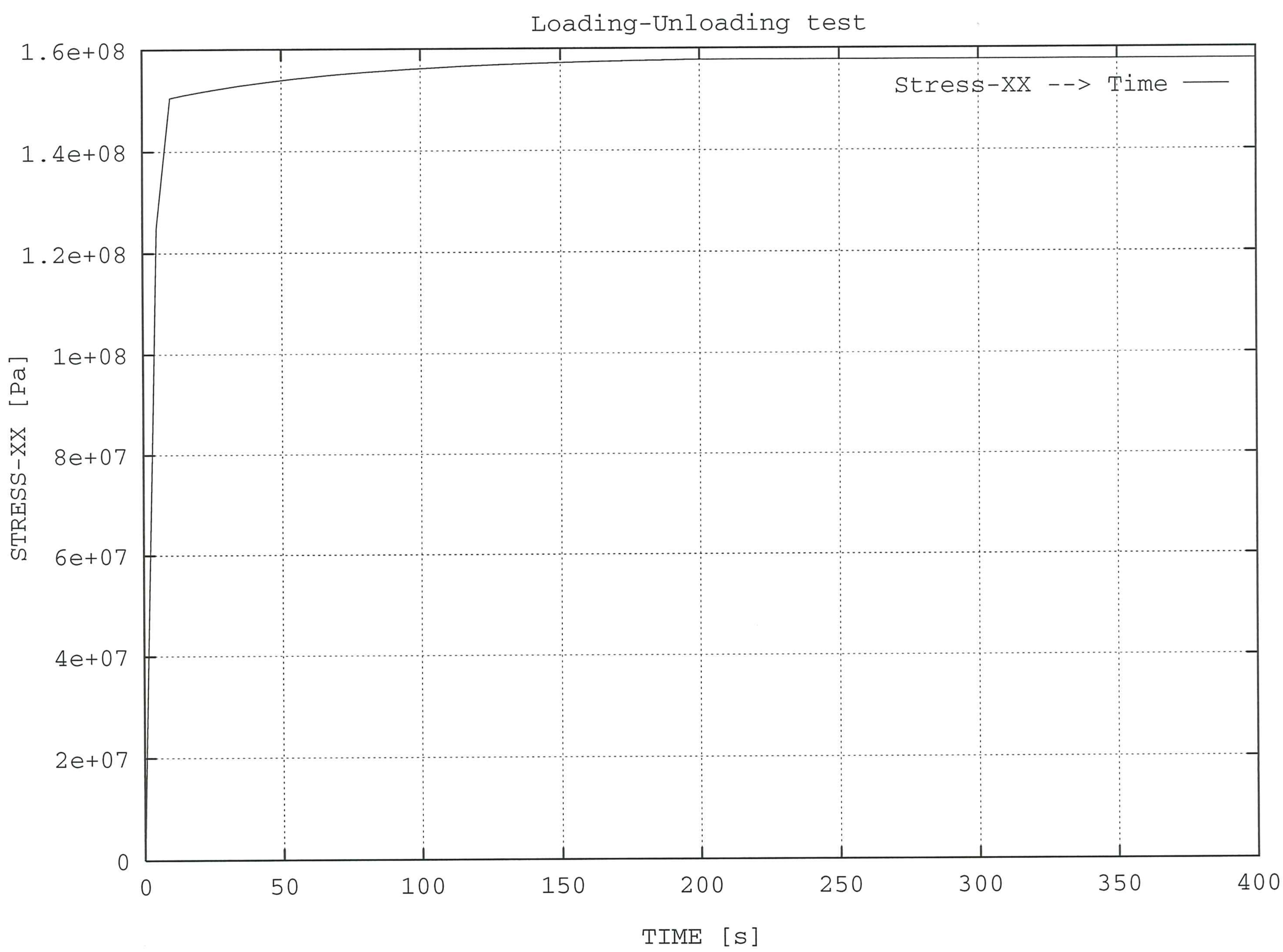
Material properties to be input:

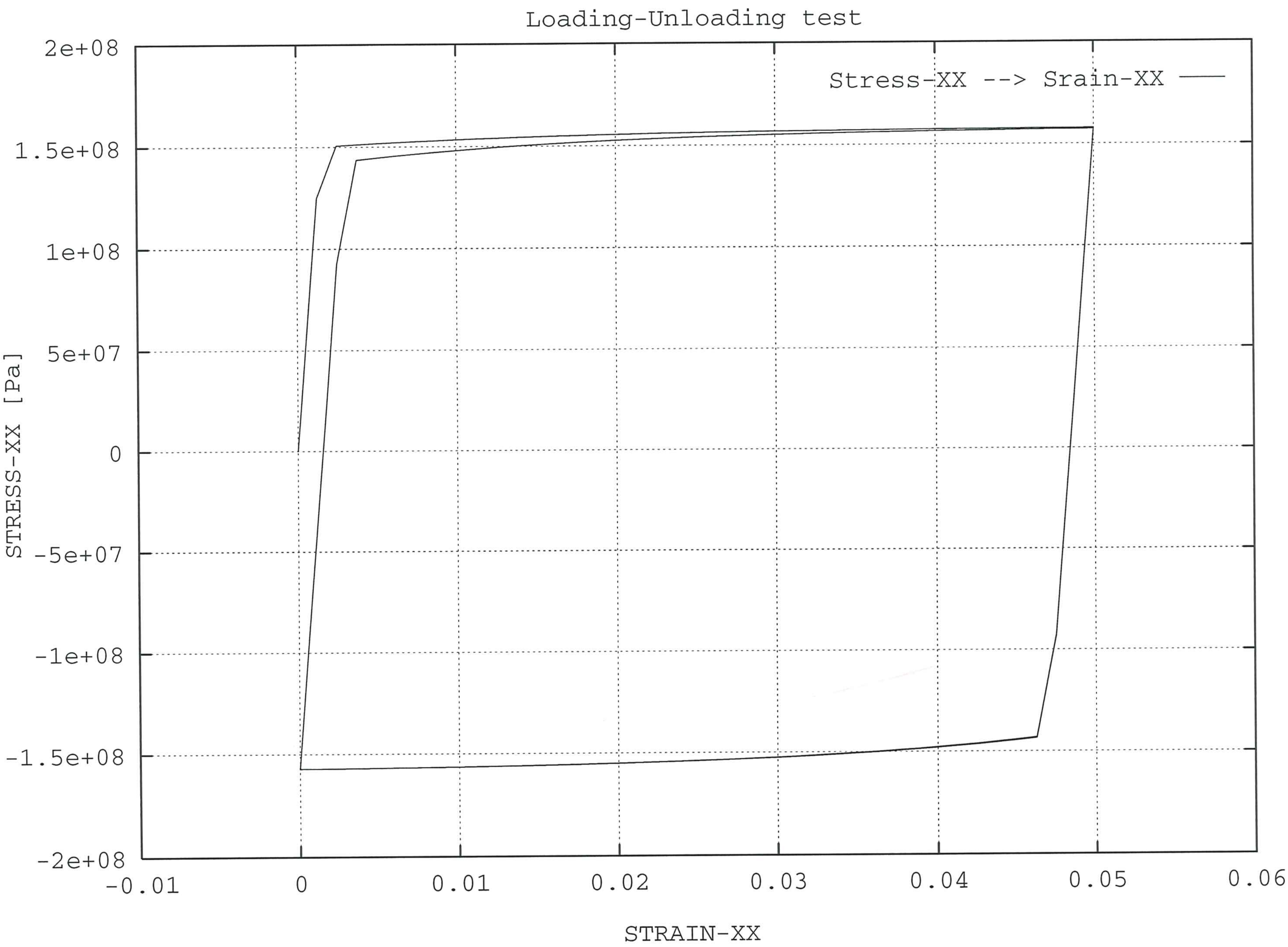
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
K_H	Linear kinematic hardening coefficient
A	Non-linear kinematic hardening parameter

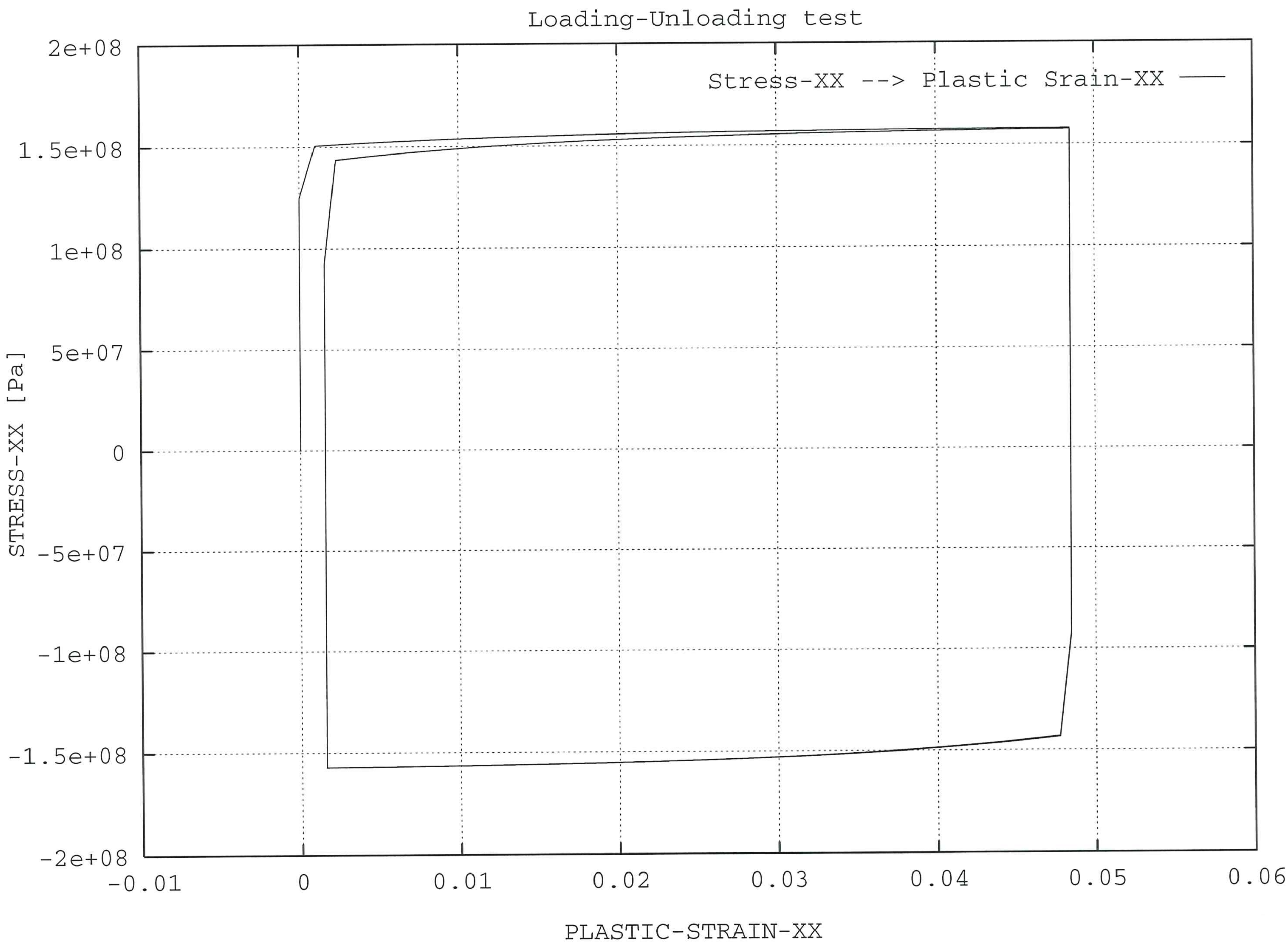
Constitutive model:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $\mathbf{q} = -\frac{2}{3} K_H\, \boldsymbol{\zeta}$
Plastic potential	$\Phi = \lVert\mathbf{s} - \mathbf{q}\rVert - \sqrt{\frac{2}{3}}\sigma_o$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\boldsymbol{\zeta}} = \dot{\gamma}\left(\frac{\partial \Phi}{\partial \mathbf{q}} - A\,\boldsymbol{\zeta}\right) = \begin{cases} -\dot{\gamma}\,(\mathbf{n}+A\,\boldsymbol{\zeta}) \\ or \\ -\dot{\boldsymbol{\varepsilon}}^p - \dot{\gamma}\,A\,\boldsymbol{\zeta} \end{cases}$ $\dot{\gamma} = \lVert\dot{\boldsymbol{\varepsilon}}^p\rVert$
Elastic domain	$J_2\,(\boldsymbol{\sigma} - \mathbf{q}) < \sigma_o$ $J_2\,(\boldsymbol{\sigma} - \mathbf{q}) = \sqrt{\frac{3}{2}}\,\lVert\mathbf{s} - \mathbf{q}\rVert$


Loading-Unloading test
Stress-XX --> Time
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
STRESS-XX [Pa]
Stress-XX --> Time
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.7 Model - V1

Model characterization:

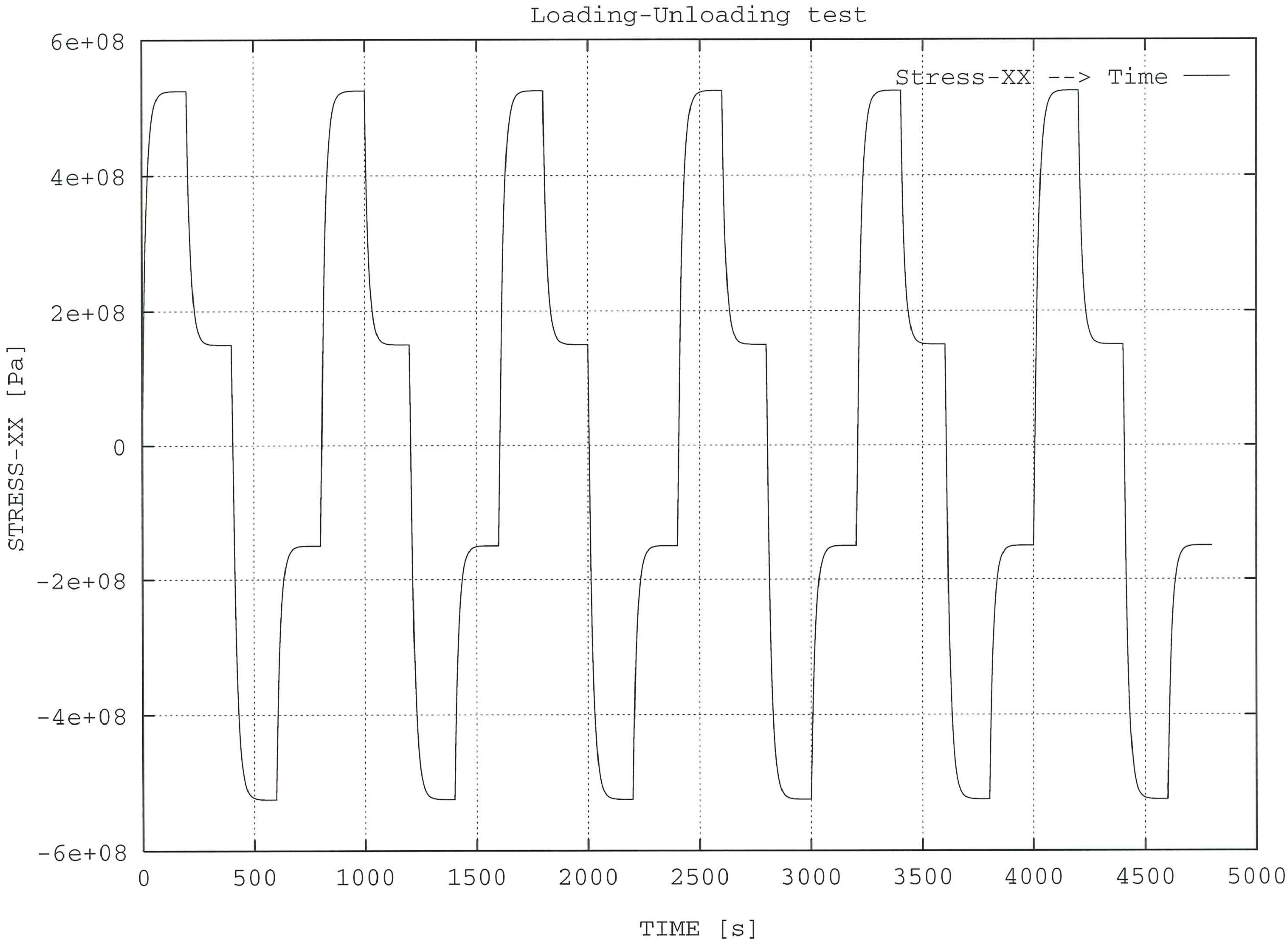
Linear isotropic hardening	OFF
Isotropic hardening saturation law	OFF
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	ON
Non-linear viscous law	OFF

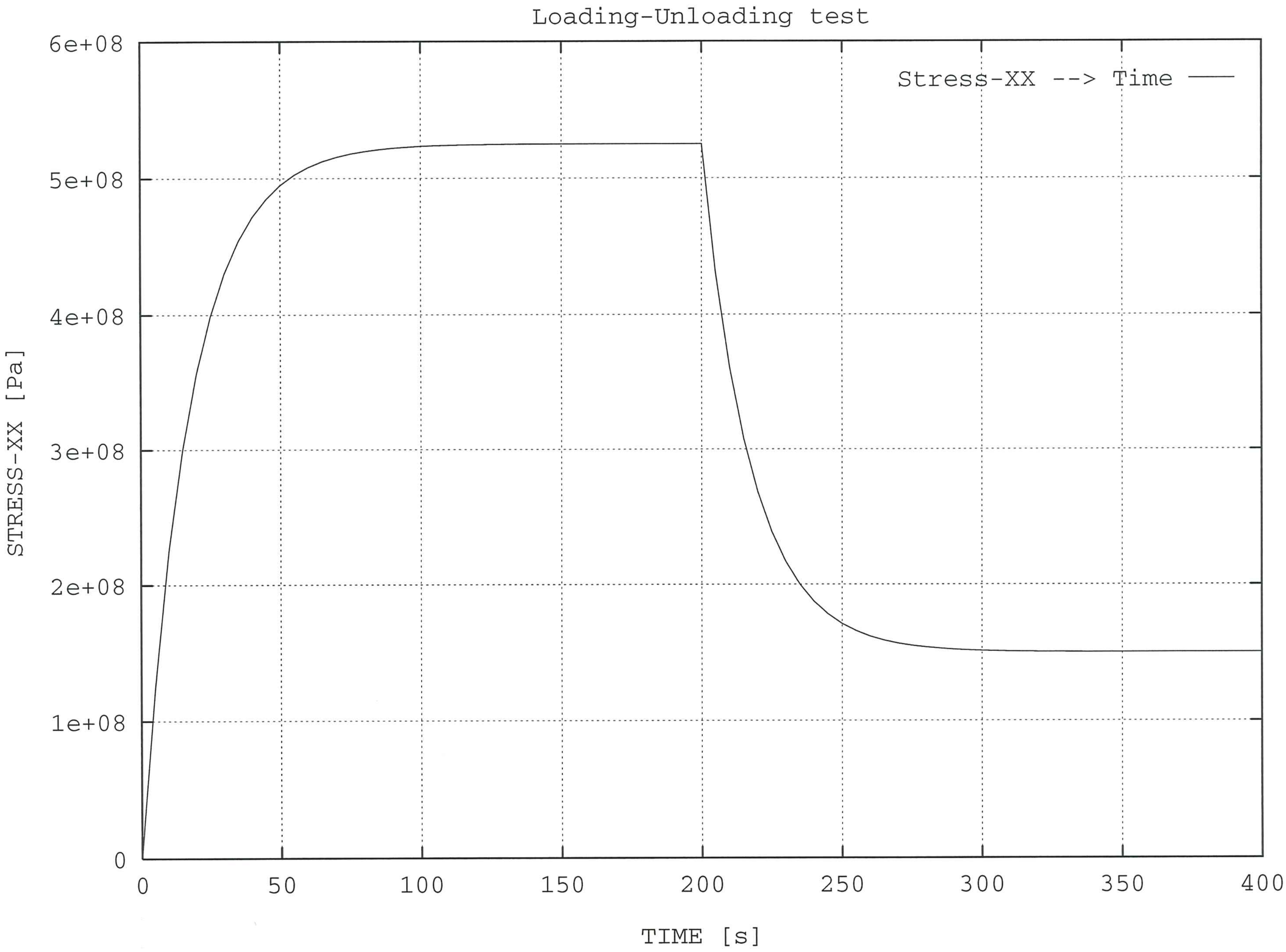
Material properties to be input:

K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
η	Viscosity

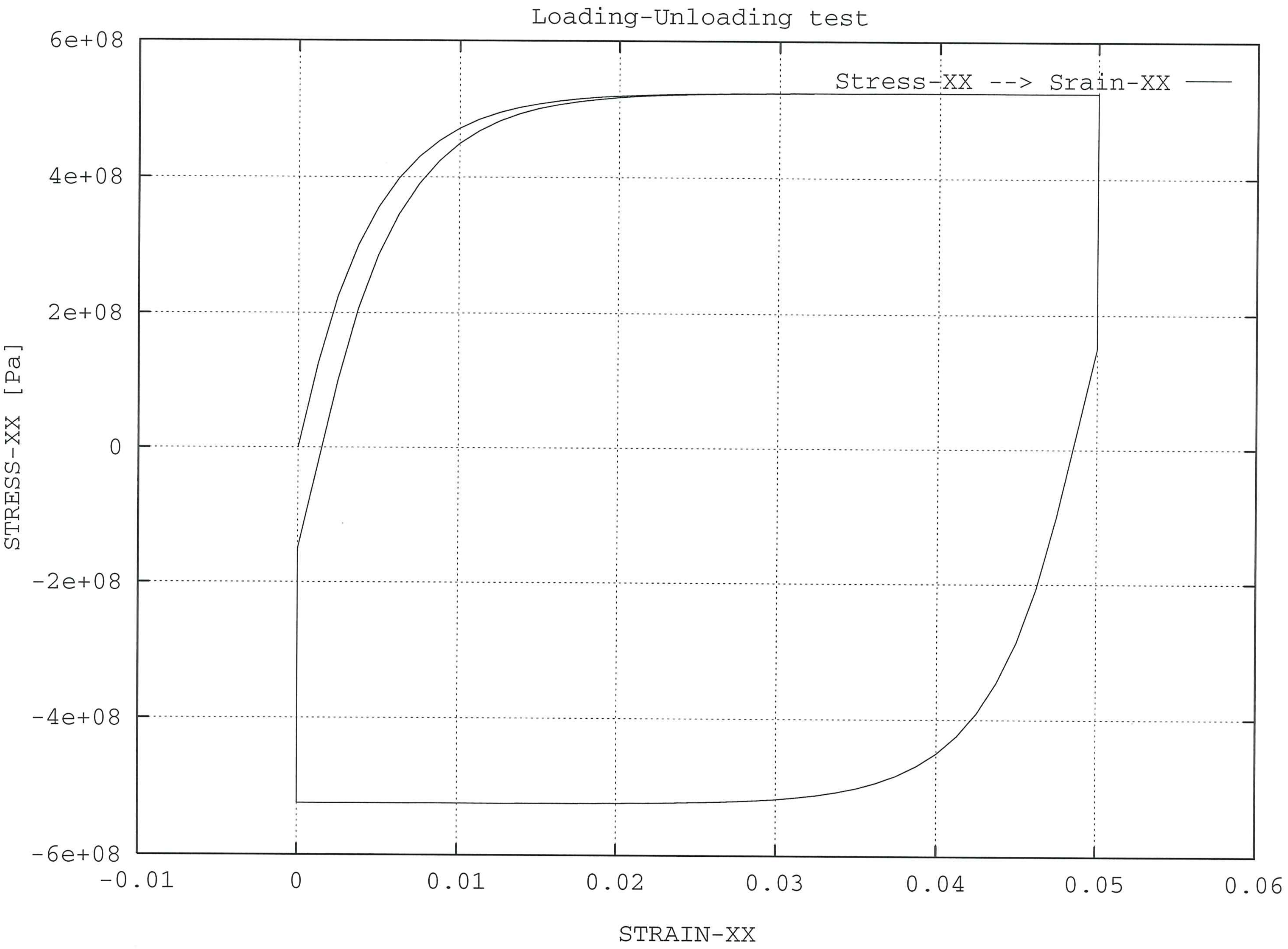
Constitutive model:

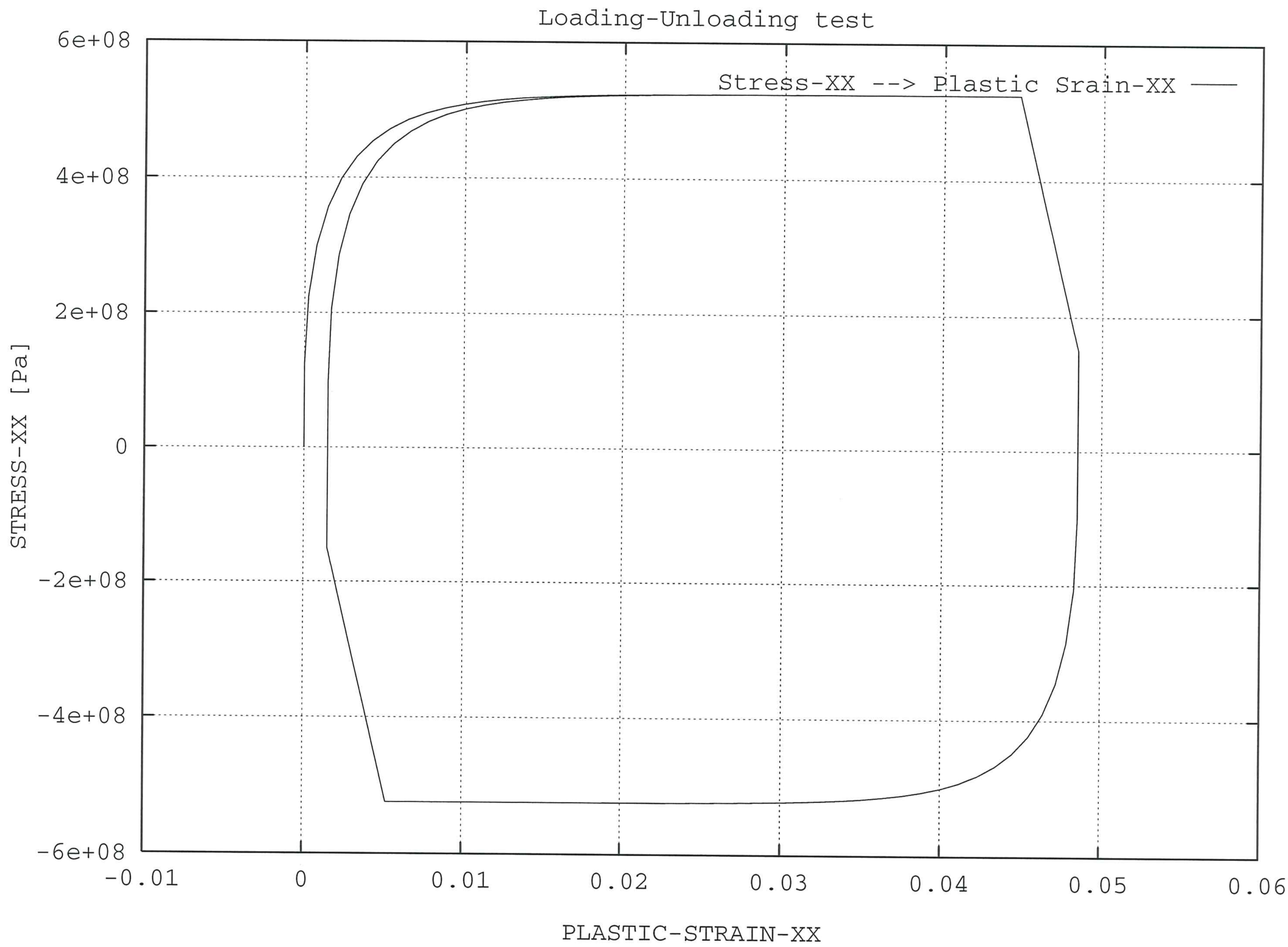
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp})$
Plastic potential	$\Omega = \frac{\eta}{2}\left\langle \frac{\Phi}{\eta} \right\rangle^2$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^{vp} = \frac{\partial \Omega}{\partial \mathbf{s}} = \left\langle \frac{\Phi}{\eta} \right\rangle \frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\, \mathbf{n}$ $\dot{\gamma} = \left\langle \frac{\Phi}{\eta} \right\rangle = \lVert \dot{\boldsymbol{\varepsilon}}^{vp} \rVert$
Visco-elastic domain	$J_2(\boldsymbol{\sigma}) < \sigma_o + \sigma_v$ $\begin{cases} J_2(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}}\,\lVert \mathbf{s} \rVert \\ \sigma_v = \eta \sqrt{\frac{3}{2}}\dot{\gamma} \end{cases}$


Loading-Unloading test
Stress-XX --> Time
6e+08
4e+08
2e+08
0
-2e+08
-4e+08
-6e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]



Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
6e+08
5e+08
4e+08
3e+08
2e+08
1e+08
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
6e+08
4e+08
2e+08
0
-2e+08
-4e+08
-6e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
6e+08
4e+08
2e+08
0
-2e+08
-4e+08
-6e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.8 Model - V2

Model characterization:

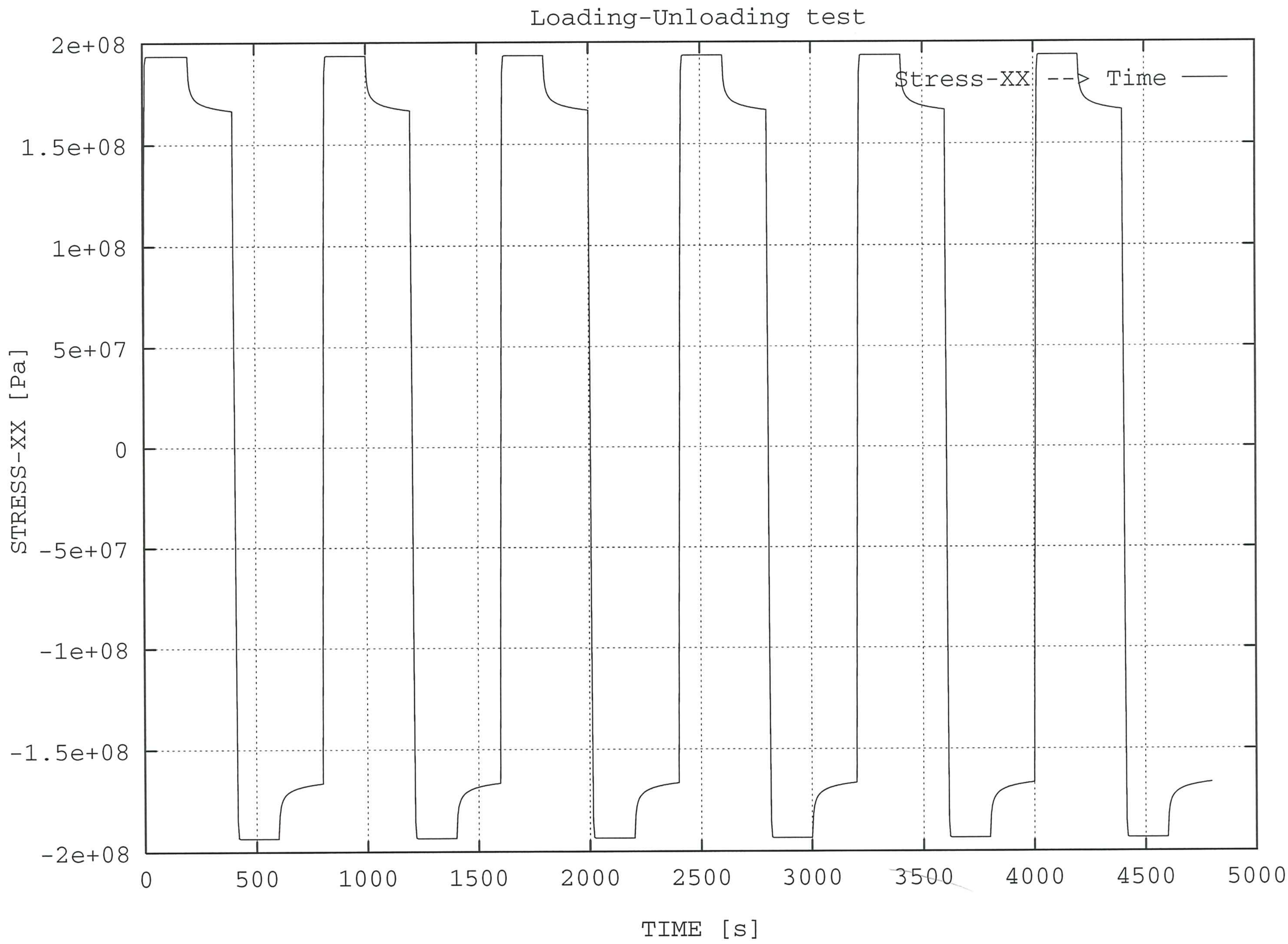
Linear isotropic hardening	OFF
Isotropic hardening saturation law	OFF
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	ON
Non-linear viscous law	ON

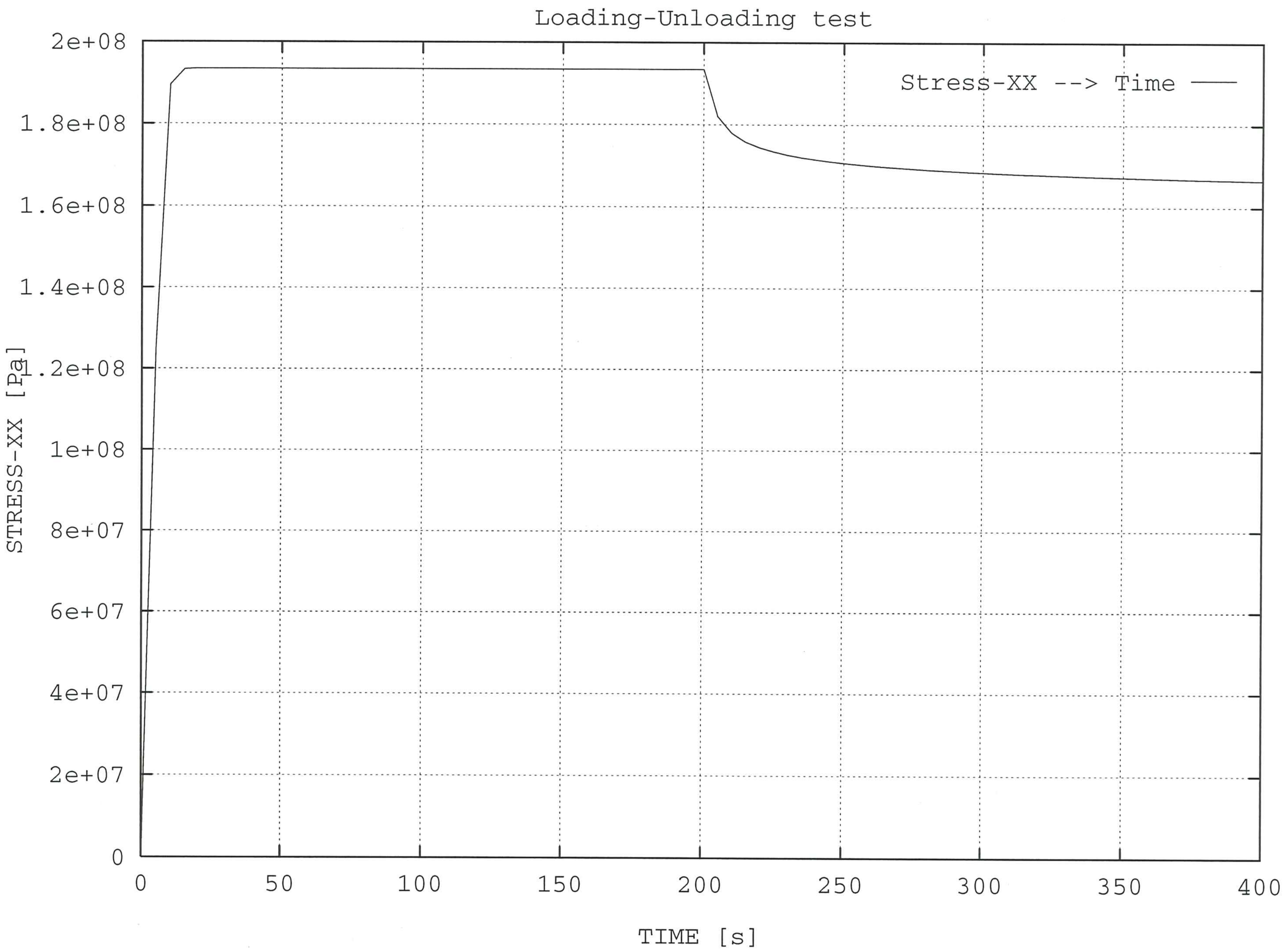
Material properties to be input:

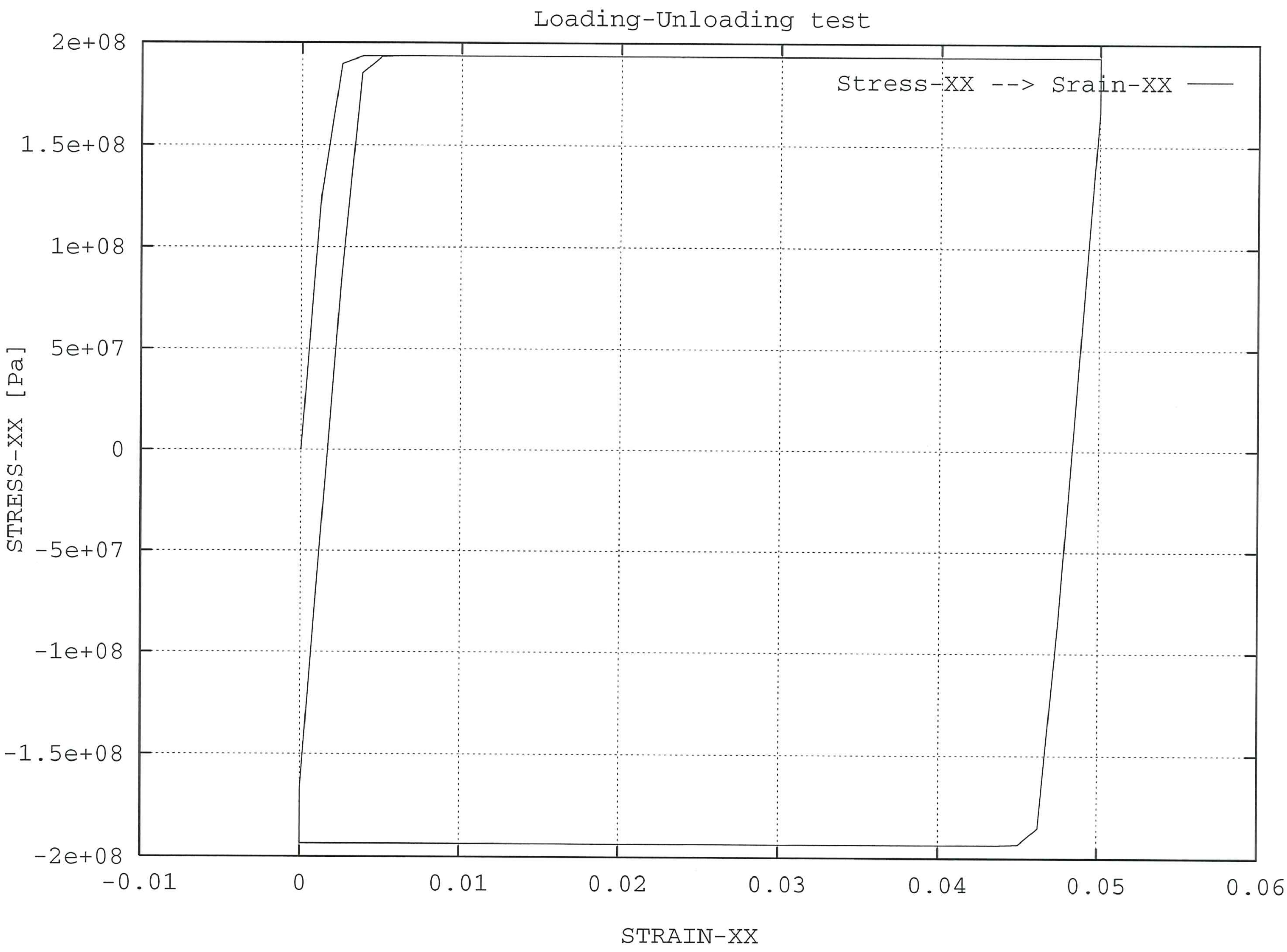
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
η	Viscosity
m	Exponent of the non-linear viscous law

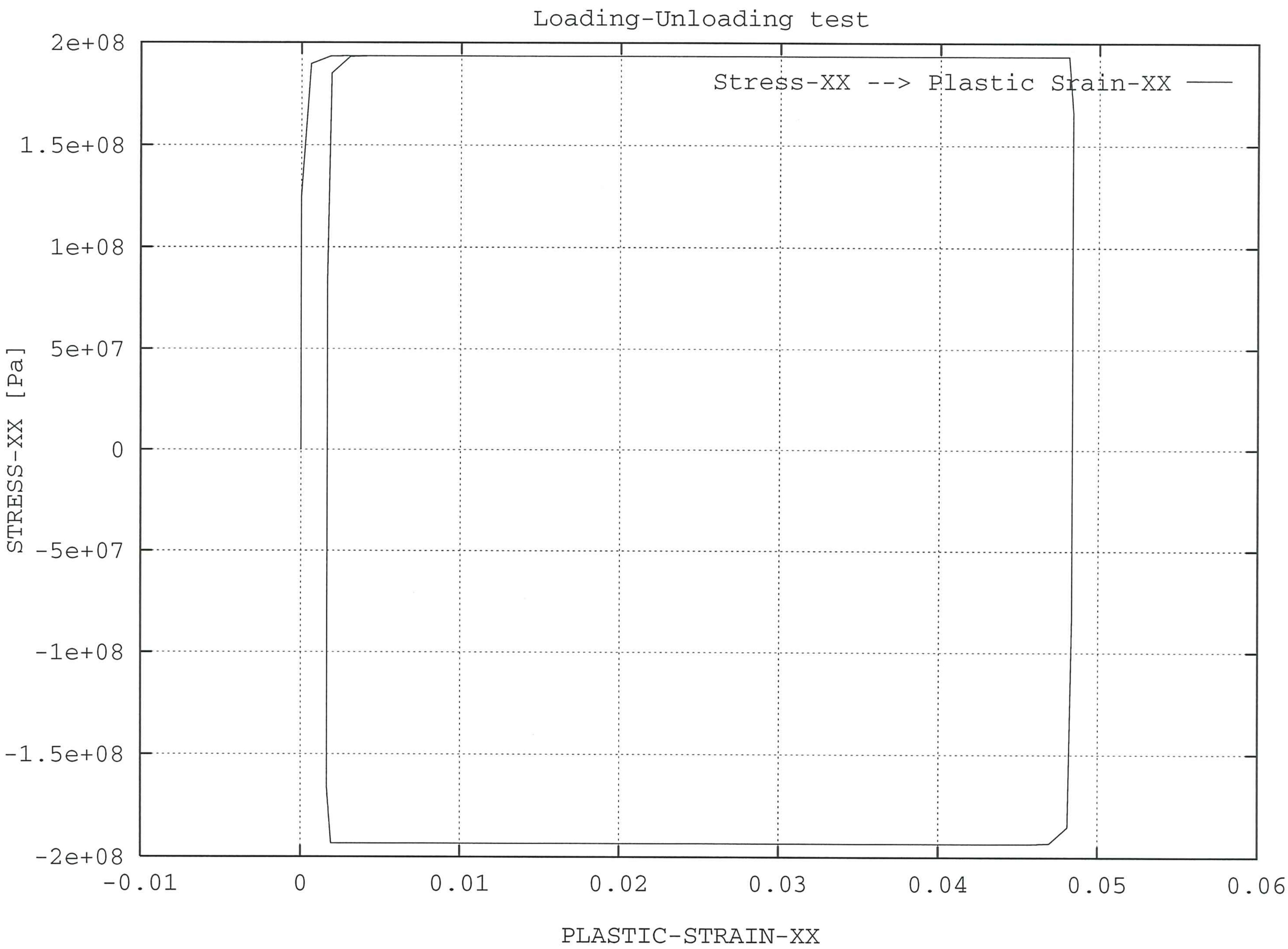
Constitutive model:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$
Constitutive laws	$\mathbf{s} = 2G\, dev\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp}\right)$
Plastic potential	$\Omega = \dfrac{\eta}{m+1}\left\langle\dfrac{\Phi}{\eta}\right\rangle^{m+1}$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^{vp} = \dfrac{\partial\Omega}{\partial\mathbf{s}} = \left\langle\dfrac{\Phi}{\eta}\right\rangle^{m}\dfrac{\partial\Phi}{\partial\mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\gamma} = \left\langle\dfrac{\Phi}{\eta}\right\rangle^{m} = \lVert\dot{\boldsymbol{\varepsilon}}^{vp}\rVert$
Visco-elastic domain	$J_2(\boldsymbol{\sigma}) < \sigma_o + \sigma_v$ $\begin{cases} J_2(\boldsymbol{\sigma}) = \sqrt{\dfrac{3}{2}}\,\lVert\mathbf{s}\rVert \\ \sigma_v = \eta\sqrt{\dfrac{3}{2}}\,\dot{\gamma}^{\frac{1}{m}} \end{cases}$


Loading-Unloading test
Stress-XX --> Time
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2e+08
1.8e+08
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.9 Model - HK

Model characterization:

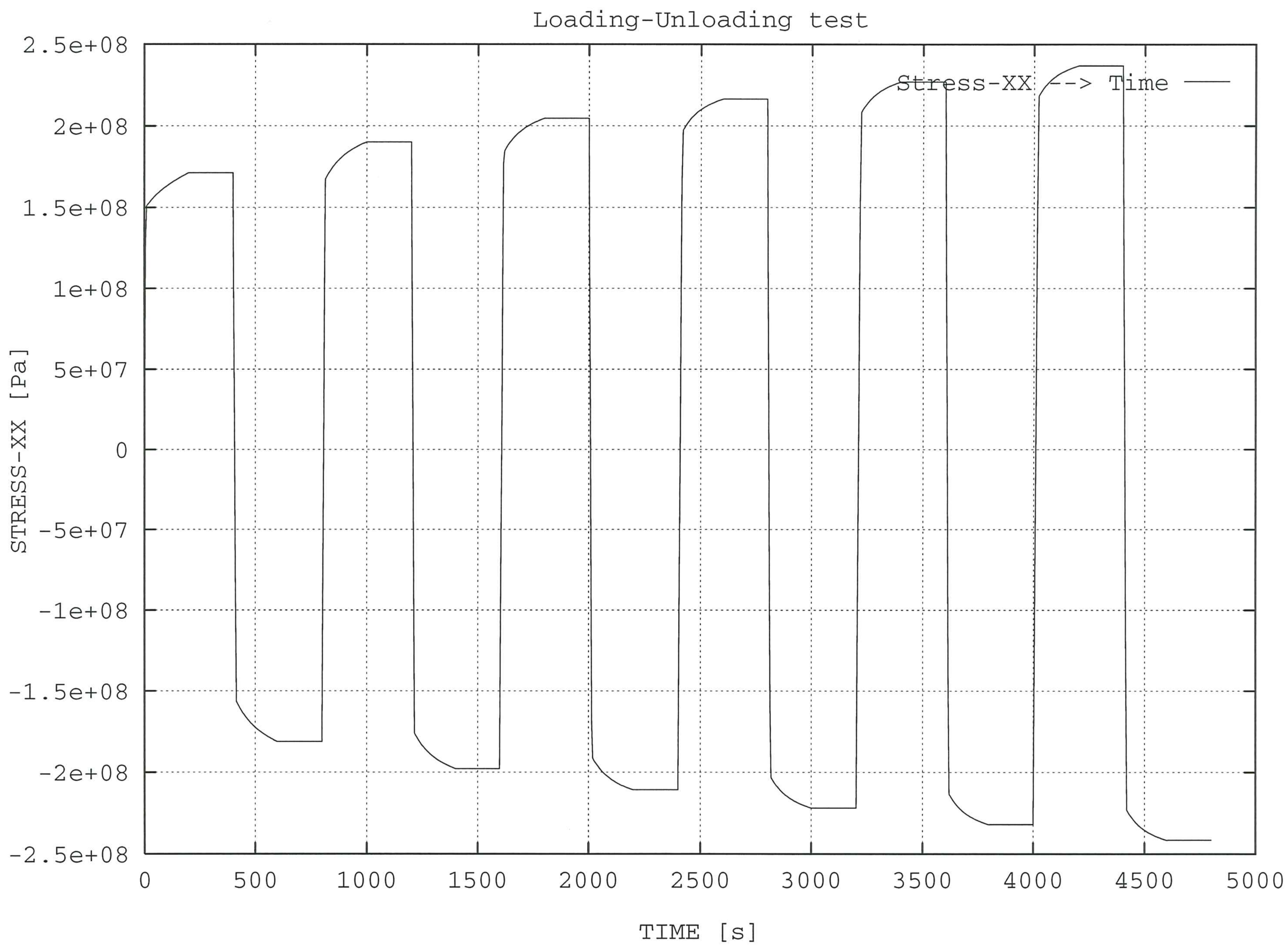
Linear isotropic hardening	ON
Isotropic hardening saturation law	ON
Kinematic hardening	ON
Non-linear kinematic law	ON
Viscosity	OFF
Non-linear viscous law	OFF

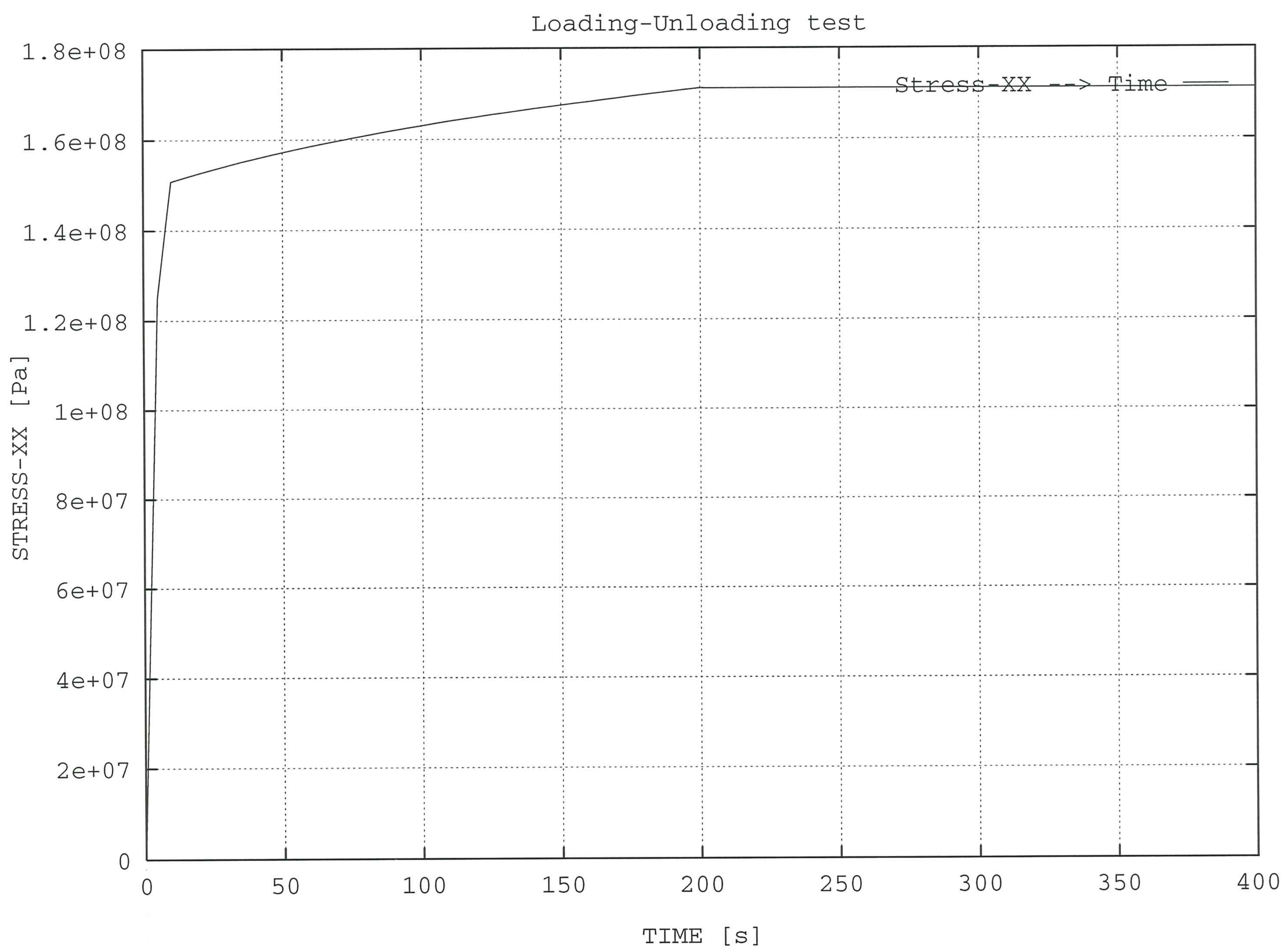
Material properties to be input:

K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law
H	Linear isotropic hardening coefficient
K_H	Linear kinematic hardening coefficient
A	Non-linear kinematic hardening parameter

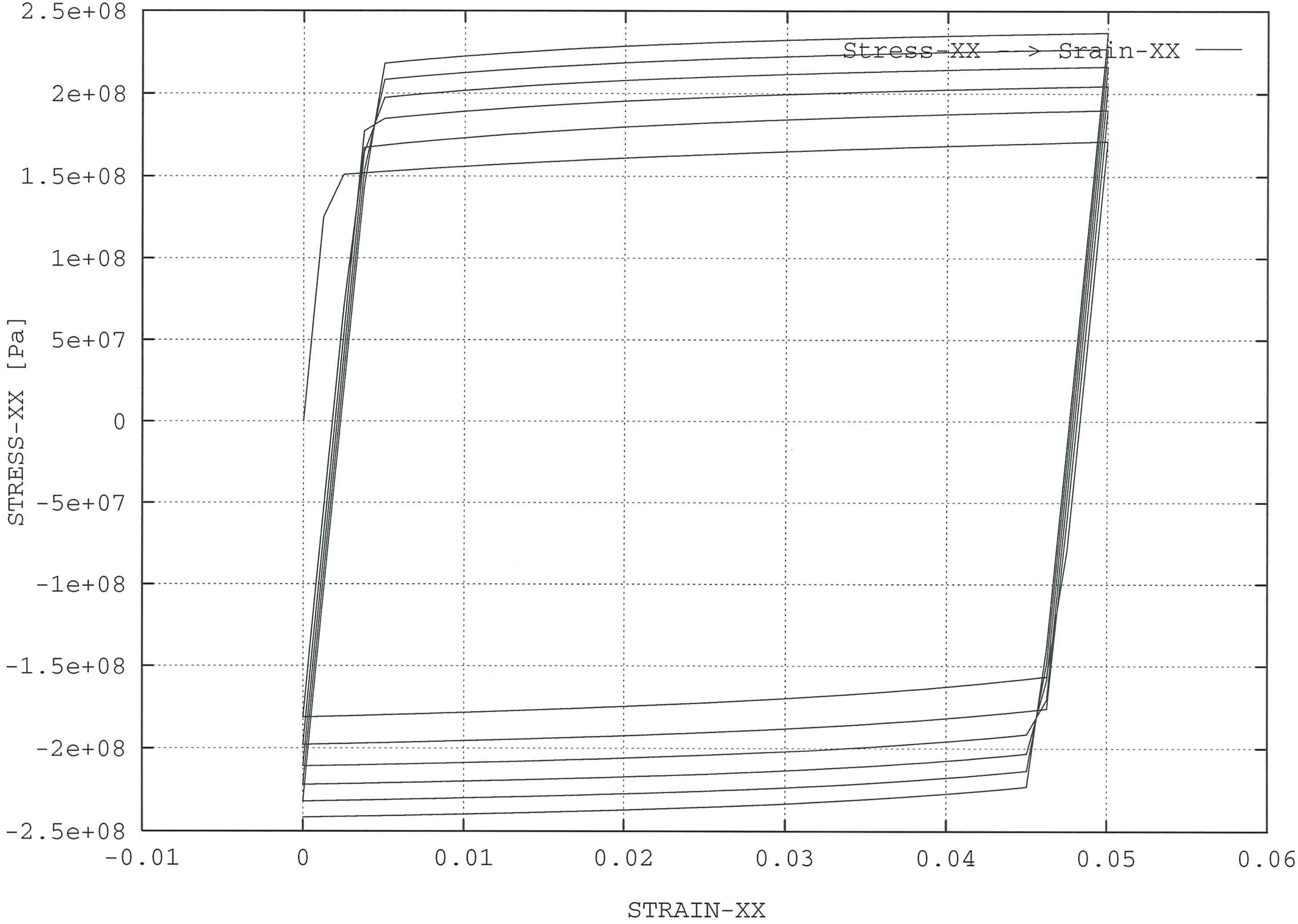
Constitutive model:

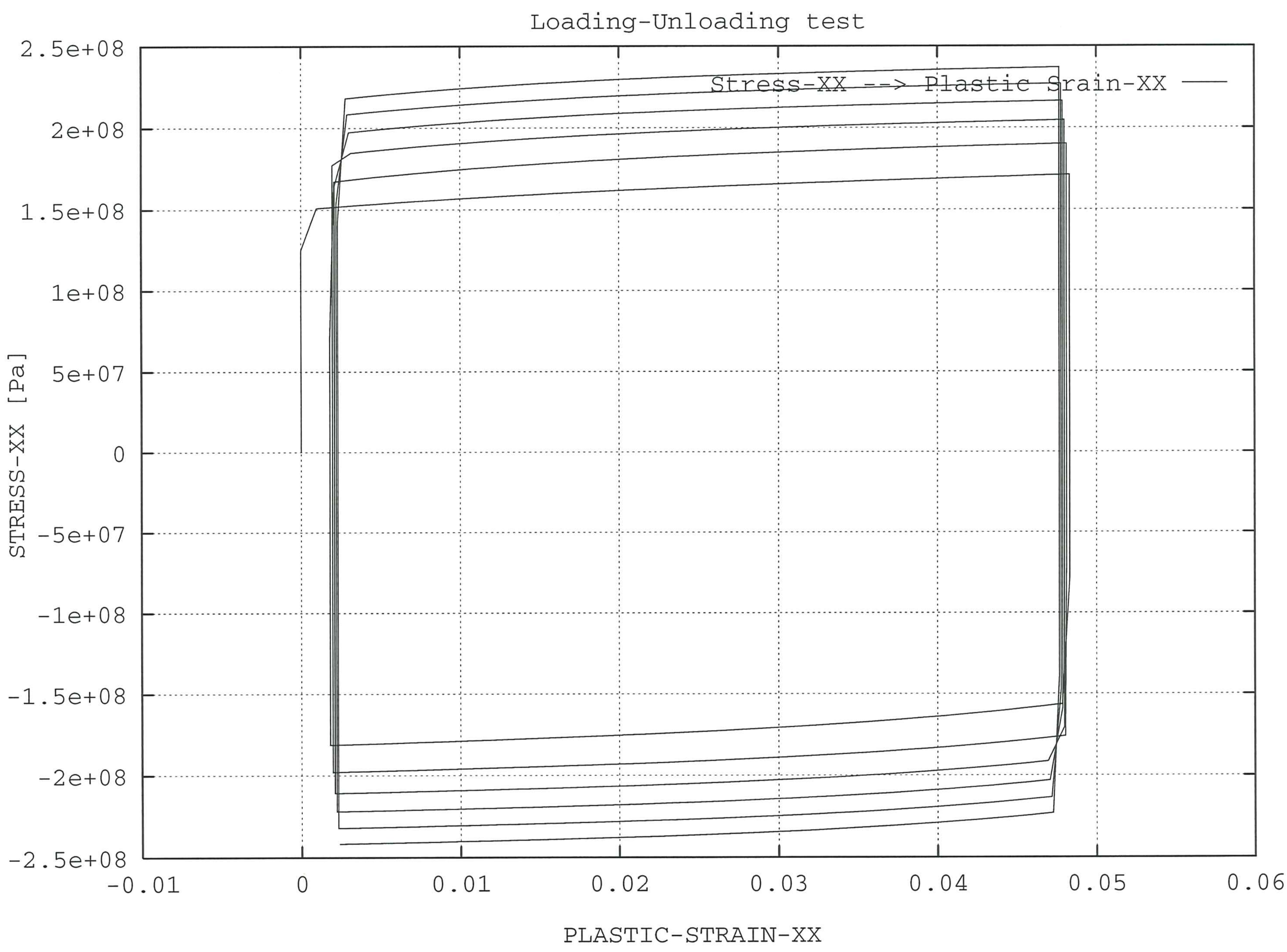
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$ $\mathbf{q} = -\frac{2}{3} K_H\, \boldsymbol{\zeta}$ $q = -(\sigma_\infty - \sigma_o)\,[1 - \exp\,(-\delta\,\xi)] - H\xi$
Plastic potential	$\Phi = \lVert\mathbf{s} - \mathbf{q}\rVert - \sqrt{\frac{2}{3}}\,(\sigma_o - q)$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\frac{\partial\Phi}{\partial\mathbf{s}} = \dot{\gamma}\,\mathbf{n}$ $\dot{\boldsymbol{\zeta}} = \dot{\gamma}\left(\frac{\partial\Phi}{\partial\mathbf{q}} - A\,\boldsymbol{\zeta}\right) = \begin{cases} -\dot{\gamma}\,(\mathbf{n}{+}A\,\boldsymbol{\zeta}) \\ or \\ -\dot{\boldsymbol{\varepsilon}}^p - \dot{\gamma}\,A\,\boldsymbol{\zeta} \end{cases}$ $\dot{\xi} = \dot{\gamma}\,\frac{\partial\Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}}$ $\dot{\gamma} = \lVert\dot{\boldsymbol{\varepsilon}}^p\rVert$
Elastic domain	$J_2\,(\boldsymbol{\sigma} - \mathbf{q}) < \sigma_o + R$ $\begin{cases} J_2\,(\boldsymbol{\sigma} - \mathbf{q}) = \sqrt{\frac{3}{2}}\,\lVert\mathbf{s} - \mathbf{q}\rVert \\ R = (\sigma_\infty - \sigma_o)\,[1 - \exp\,(-\delta\,\xi)] + H\xi \end{cases}$


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
TIME [s]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
1.8e+08
1.6e+08
1.4e+08
1.2e+08
1e+08
8e+07
6e+07
4e+07
2e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]

Loading-Unloading test





Loading-Unloading test
Stress-XX --> Plastic Strain-XX
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
-5e+07
-1e+08
-1.5e+08
-2e+08
-2.5e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.10 Model - VH

Model characterization:

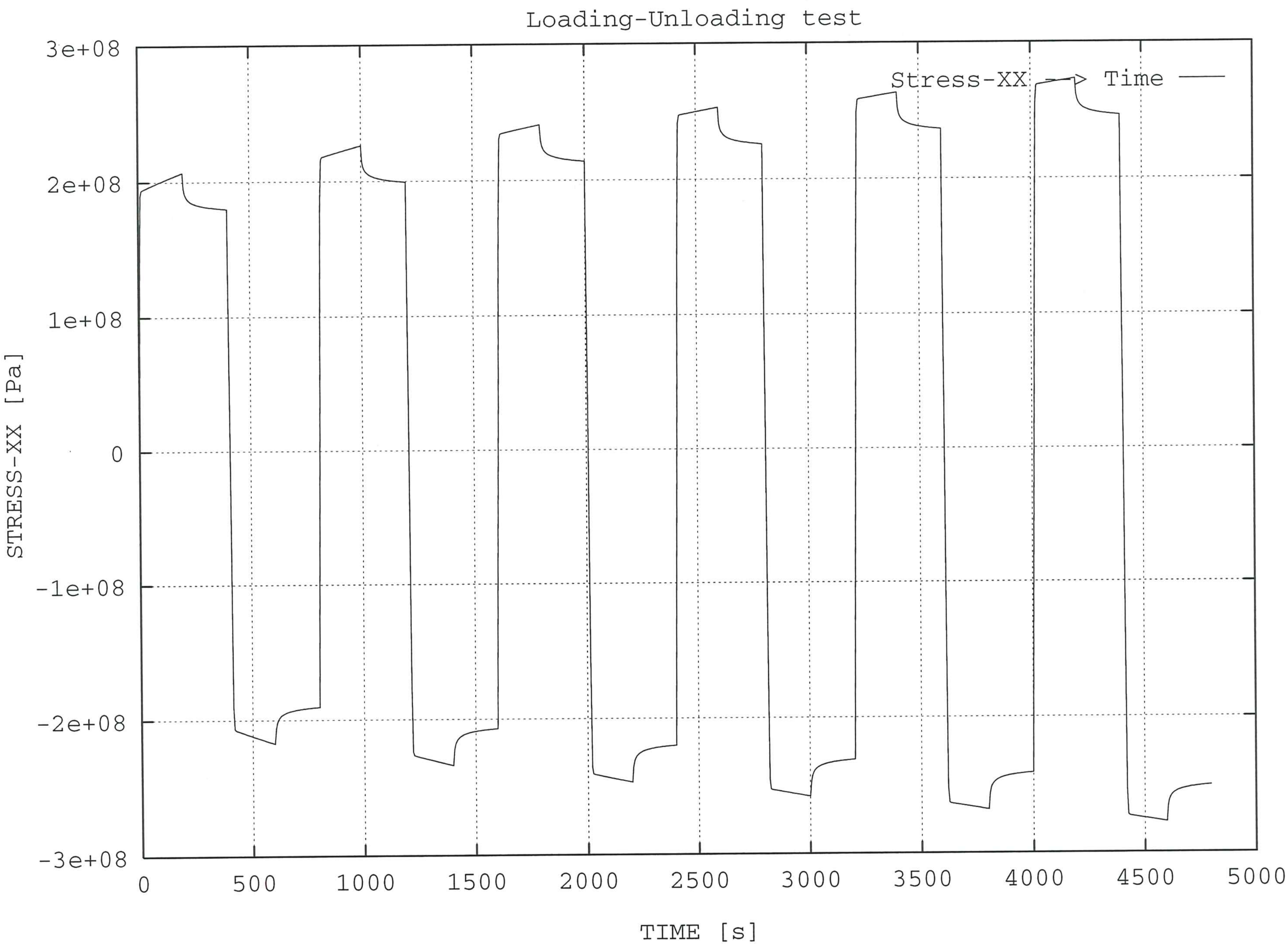
Linear isotropic hardening	ON
Isotropic hardening saturation law	ON
Kinematic hardening	OFF
Non-linear kinematic law	OFF
Viscosity	ON
Non-linear viscous law	ON

Material properties to be input:

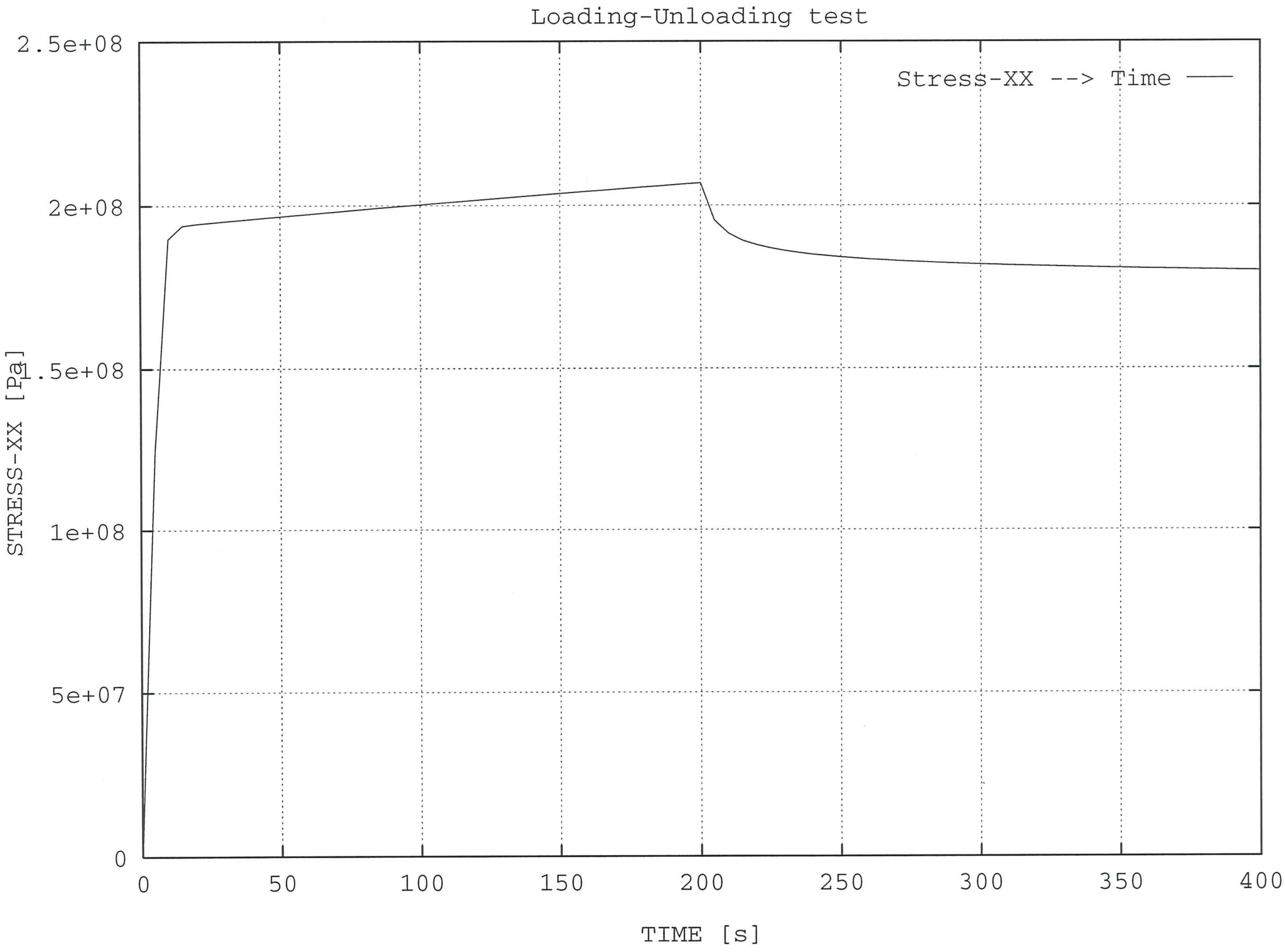
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law
H	Linear isotropic hardening coefficient
η	Viscosity
m	Exponent of the non-linear viscous law

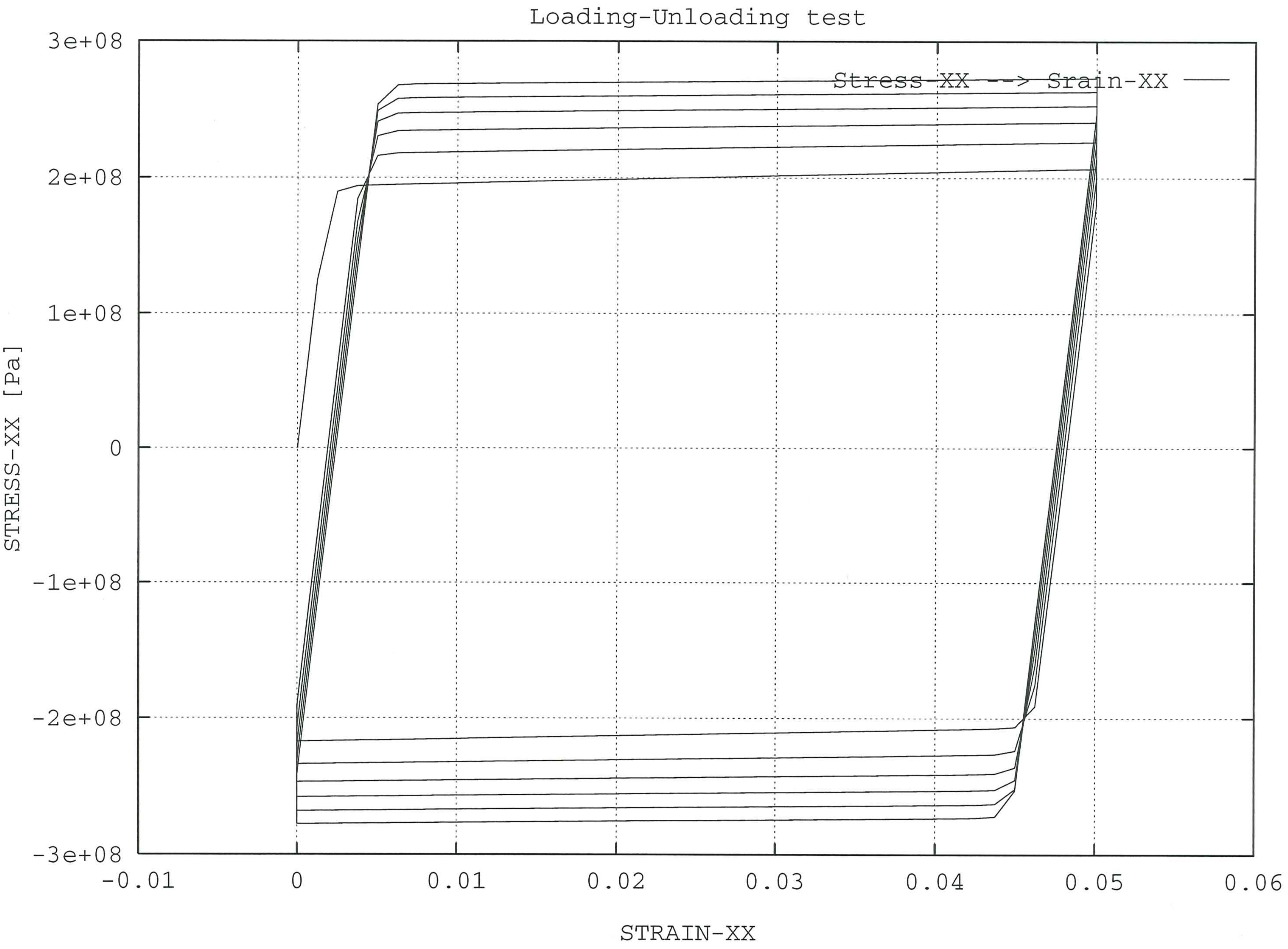
Constitutive model:

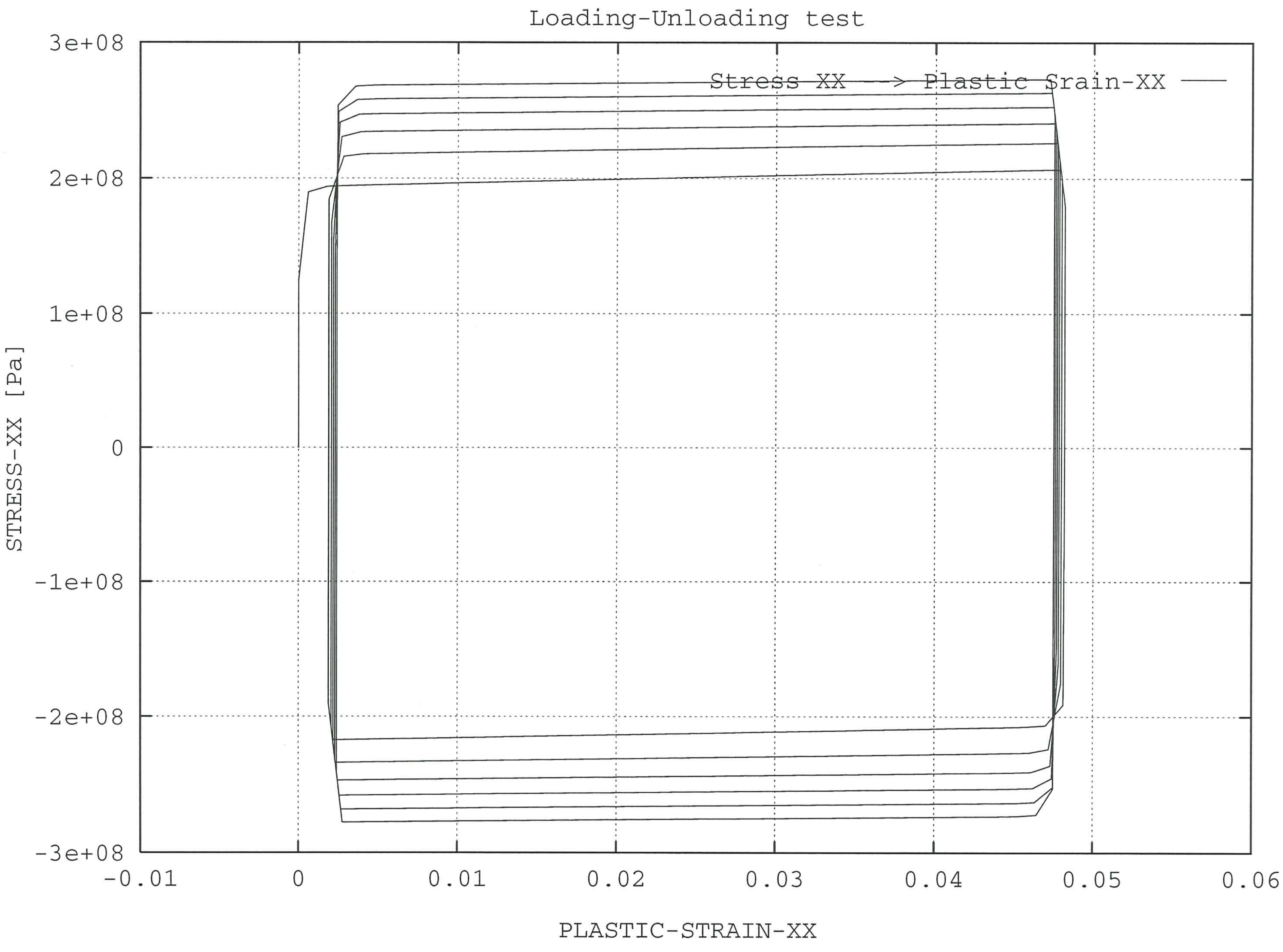
Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$
Constitutive laws	$\begin{aligned} \mathrm{s} &= 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp}) \\ q &= -\,(\sigma_\infty - \sigma_o)\,[1 - \exp\,(-\delta\,\xi)] - H\xi \end{aligned}$
Plastic potential	$\Omega = \dfrac{\eta}{m+1} \left\langle \dfrac{\Phi}{\eta} \right\rangle^{m+1}$
Evolution laws	$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^{vp} &= \frac{\partial \Omega}{\partial \mathrm{s}} = \left\langle \frac{\Phi}{\eta} \right\rangle^{m} \frac{\partial \Phi}{\partial \mathrm{s}} = \dot{\gamma}\,\mathbf{n} \\ \dot{\xi} &= \frac{\partial \Omega}{\partial q} = \left\langle \frac{\Phi}{\eta} \right\rangle^{m} \frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}} \\ \dot{\gamma} &= \left\langle \frac{\Phi}{\eta} \right\rangle^{m} = \lVert \dot{\boldsymbol{\varepsilon}}^{vp} \rVert \end{aligned}$
Visco-elastic domain	$J_2\,(\boldsymbol{\sigma}) < \sigma_o + R + \sigma_v$ $\left\{ \begin{aligned} J_2\,(\boldsymbol{\sigma}) &= \sqrt{\frac{3}{2}}\,\lVert \mathrm{s} \rVert \\ R &= (\sigma_\infty - \sigma_o)\,[1 - \exp\,(-\delta\,\xi)] + H\xi \\ \sigma_v &= \eta\sqrt{\frac{3}{2}}\,\dot{\gamma}^{\frac{1}{m}} \end{aligned} \right.$



Loading-Unloading test
Stress-XX ---> Time
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
STRESS-XX [Pa]
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
TIME [s]


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress XX --> Plastic Srain-XX
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
STRESS-XX [Pa]
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

3.11 Model - VHK

Model characterization:

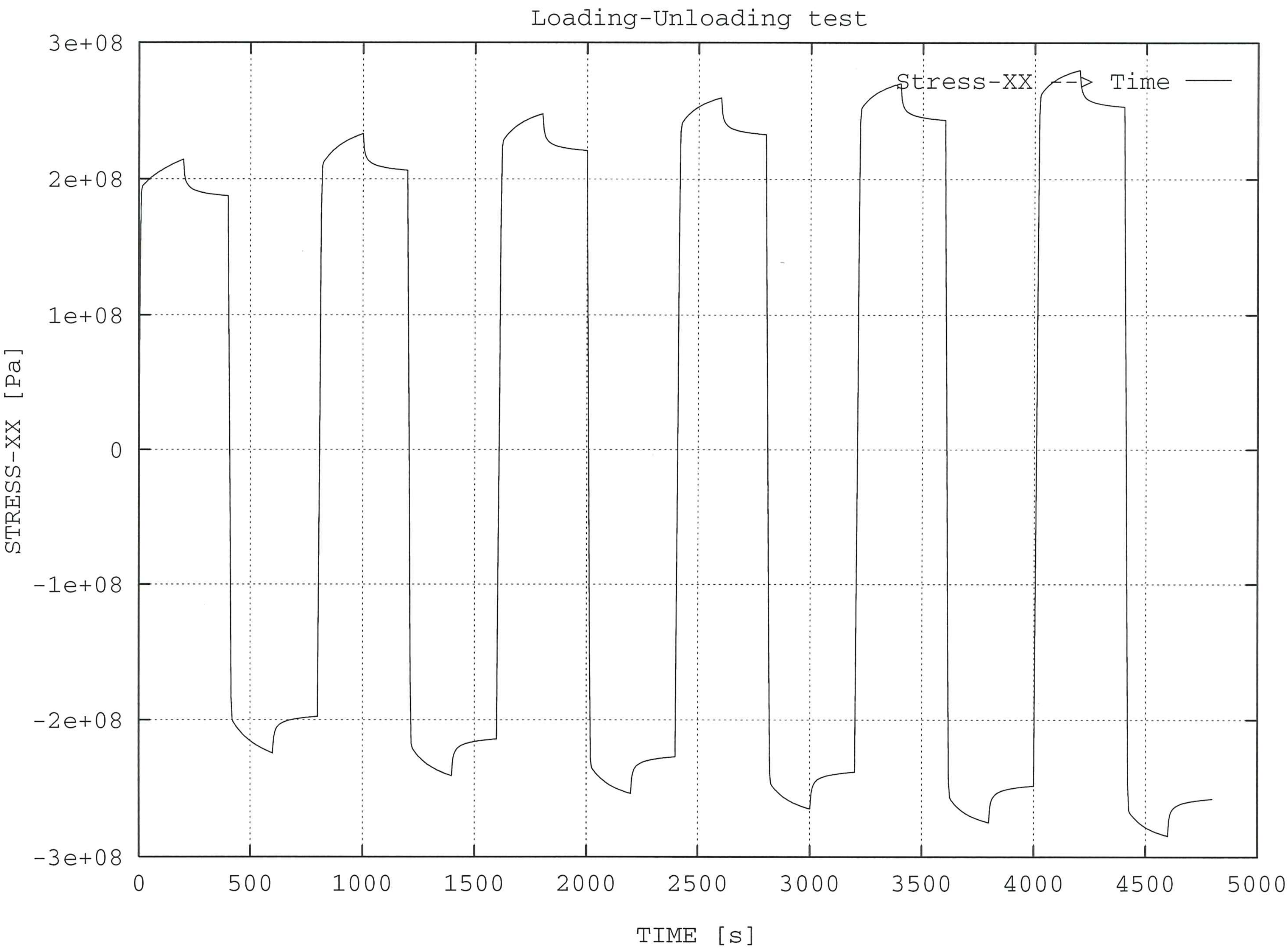
Linear isotropic hardening	ON
Isotropic hardening saturation law	ON
Kinematic hardening	ON
Non-linear kinematic law	ON
Viscosity	ON
Non-linear viscous law	ON

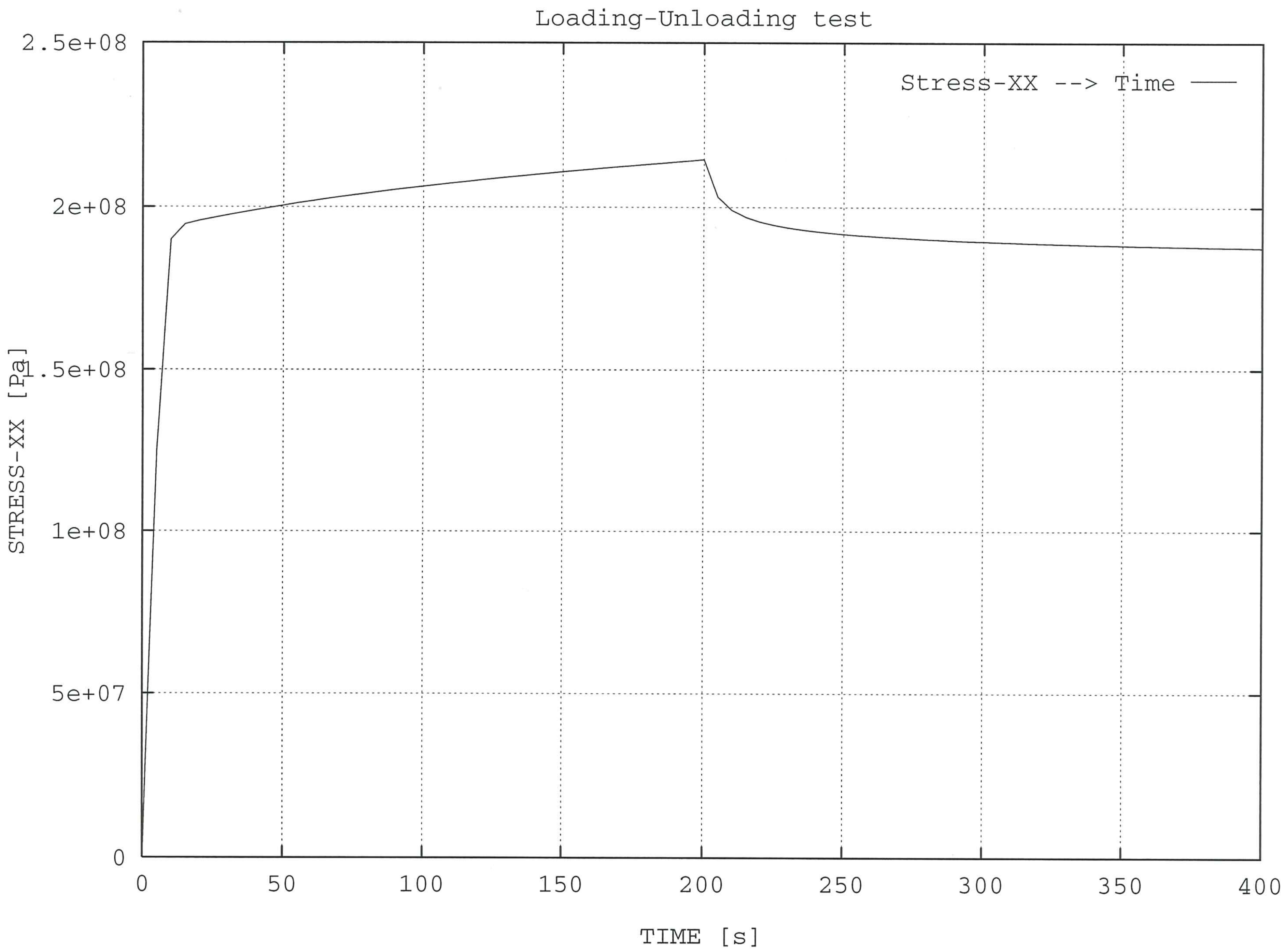
Material properties to be input:

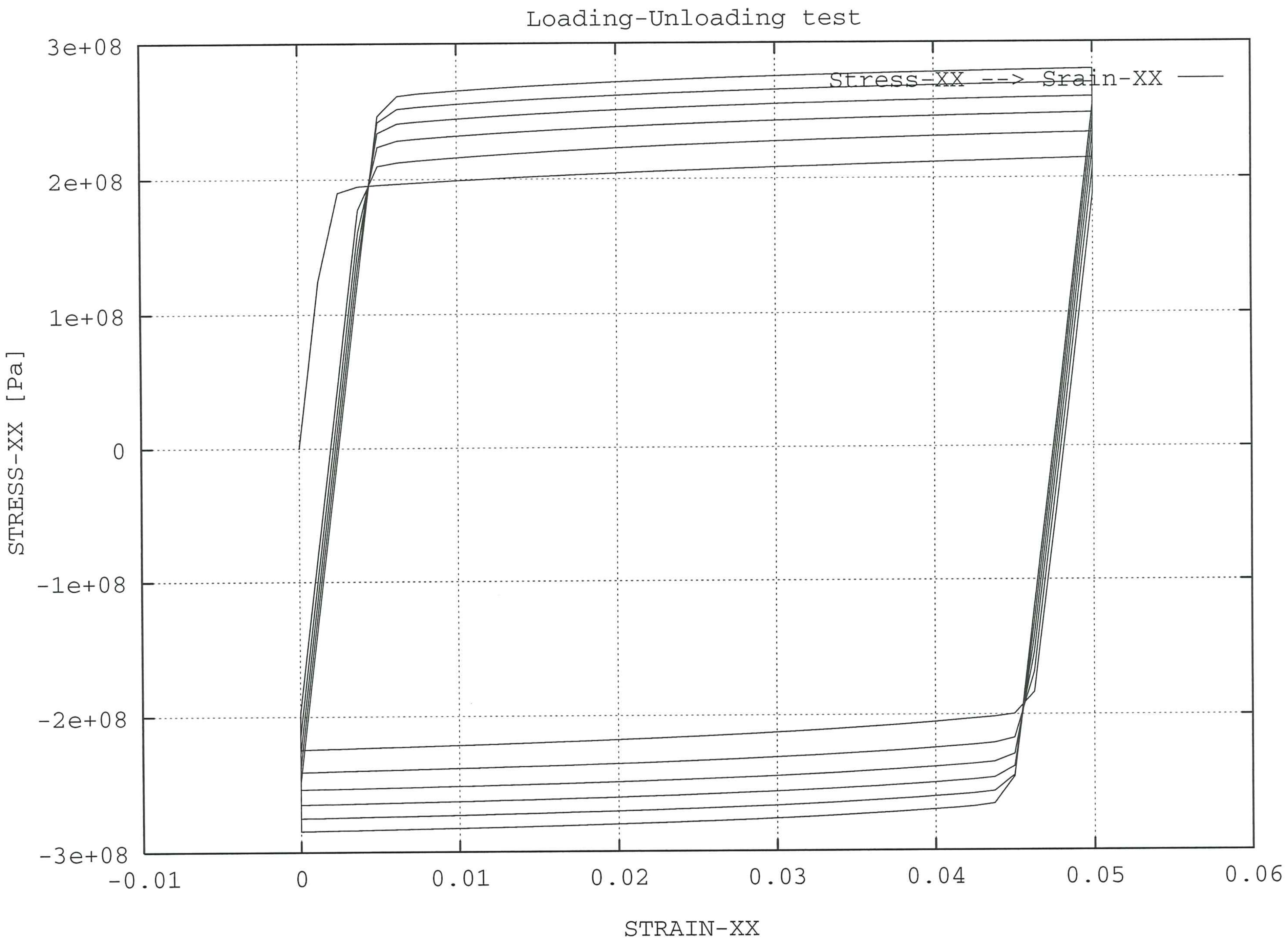
K	Bulk modulus
G	Shear modulus
σ_o	Initial flow stress
σ_∞	Isotropic hardening saturation flow stress
δ	Exponent of the isotropic hardening saturation law
H	Linear isotropic hardening coefficient
K_H	Linear kinematic hardening coefficient
A	Non-linear kinematic hardening parameter
η	Viscosity
m	Exponent of the non-linear viscous law

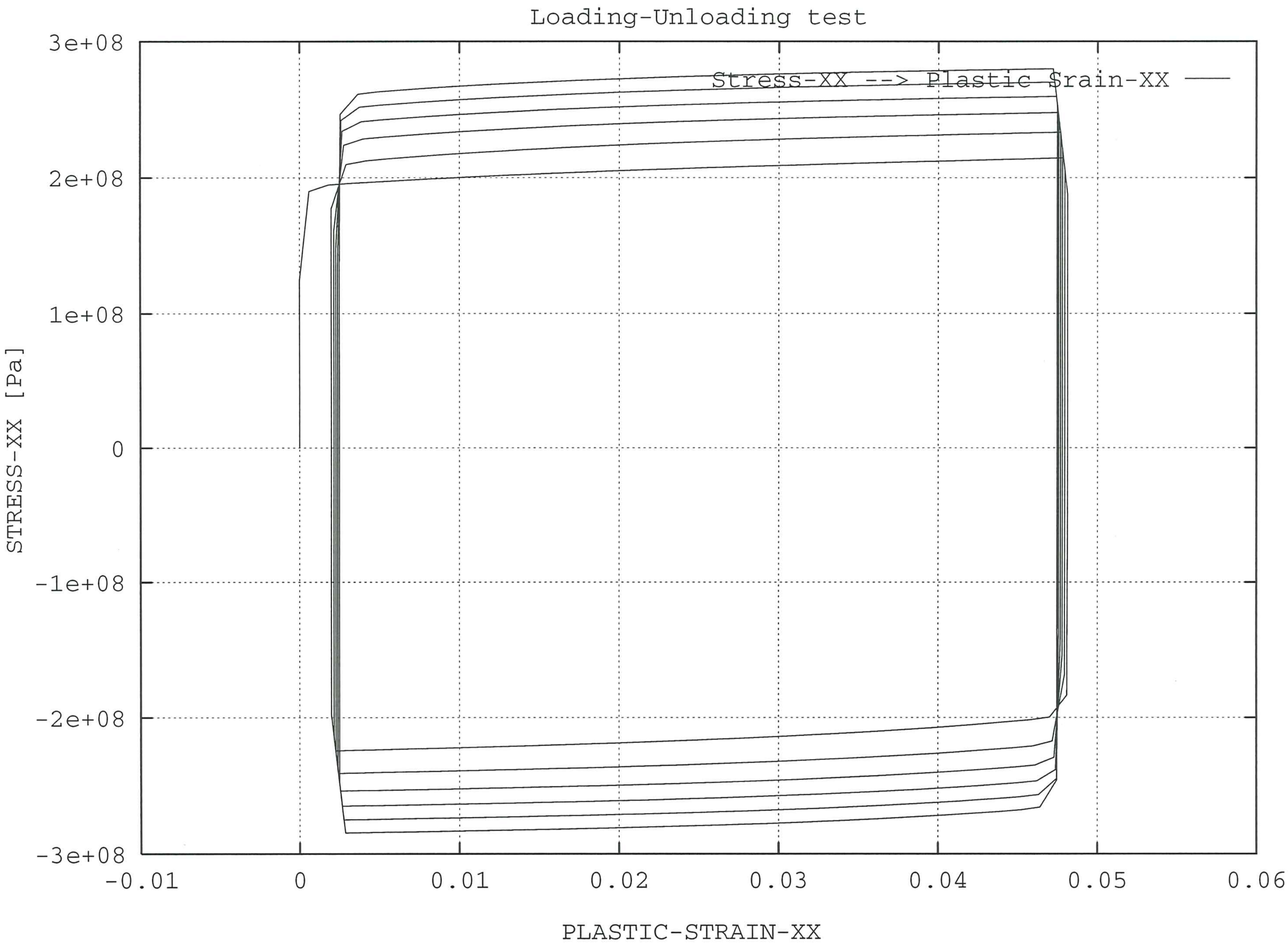
Constitutive model:

Additive decomposition	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{vp}$
Constitutive laws	$\mathbf{s} = 2G\, dev\,(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{vp})$ $\mathbf{q} = -\frac{2}{3} K_H\, \boldsymbol{\zeta}$ $q = -(\sigma_\infty - \sigma_o)\,[1 - \exp(-\delta\,\xi)] - H\xi$
Plastic potential	$\Omega = \frac{\eta}{m+1} \left\langle \frac{\Phi}{\eta} \right\rangle^{m+1}$
Evolution laws	$\dot{\boldsymbol{\varepsilon}}^{vp} = \frac{\partial \Omega}{\partial \mathbf{s}} = \left\langle \frac{\Phi}{\eta} \right\rangle^{m} \frac{\partial \Phi}{\partial \mathbf{s}} = \dot{\gamma}\, \mathbf{n}$ $\dot{\boldsymbol{\zeta}} = \begin{cases} -\dot{\gamma}\,(\mathbf{n} + A\,\boldsymbol{\zeta}) \\ or \\ -\dot{\boldsymbol{\varepsilon}}^p - \dot{\gamma}\, A\, \boldsymbol{\zeta} \end{cases}$ $\dot{\xi} = \frac{\partial \Omega}{\partial q} = \left\langle \frac{\Phi}{\eta} \right\rangle^{m} \frac{\partial \Phi}{\partial q} = \dot{\gamma}\sqrt{\frac{2}{3}}$ $\dot{\gamma} = \left\langle \frac{\Phi}{\eta} \right\rangle^{m} = \lVert \dot{\boldsymbol{\varepsilon}}^{vp} \rVert$
Visco-elastic domain	$J_2(\boldsymbol{\sigma}) < \sigma_o + R + \sigma_v$ $\begin{cases} J_2(\boldsymbol{\sigma} - \mathbf{q}) = \sqrt{\frac{3}{2}}\, \lVert \mathbf{s} - \mathbf{q} \rVert \\ R = (\sigma_\infty - \sigma_o)\,[1 - \exp(-\delta\,\xi)] + H\xi \\ \sigma_v = \eta \sqrt{\frac{3}{2}}\, \dot{\gamma}^{\frac{1}{m}} \end{cases}$


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
TIME [s]
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000


Loading-Unloading test
Stress-XX --> Time
STRESS-XX [Pa]
2.5e+08
2e+08
1.5e+08
1e+08
5e+07
0
0
50
100
150
200
250
300
350
400
TIME [s]


Loading-Unloading test
Stress-XX --> Srain-XX
STRESS-XX [Pa]
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
STRAIN-XX


Loading-Unloading test
Stress-XX --> Plastic Srain-XX
STRESS-XX [Pa]
3e+08
2e+08
1e+08
0
-1e+08
-2e+08
-3e+08
-0.01
0
0.01
0.02
0.03
0.04
0.05
0.06
PLASTIC-STRAIN-XX

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