A Mixed Finite Element Formulation for Incompressibility using Linear Displacement and Pressure Interpolations

D. Christ
M. Cervera
M. Chiumenti
C. Agelet de Saracibar

Monograph CIMNE Nº-77, September 2003

A Mixed Finite Element Formulation for Incompressibility using Linear Displacement and Pressure Interpolations

D. Christ
M. Cervera
M. Chiumenti
C. Agelet de Saracibar

Monograph CIMNE Nº-77, September 2003

International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING
Edificio C1, Campus Norte UPC
Gran Capitán s/n
08034 Barcelona, Spain
www.cimne.edu

First edition: September 2003

A MIXED FINITE ELEMENT FORMULATION FOR INCOMPRESSIBILITY USING LINEAR DISPLACEMENT AND PRESSURE INTERPOLATIONS

Monograph CIMNE M77

ISBN: 84-95999-35-8

Depósito legal: B-40323-2003

Abstract

In this work shall be presented a stabilized finite element method to deal with incompressibility in solid mechanics. A mixed formulation involving pressure and displacement fields is used and a continuous linear interpolation is considered for both fields. To overcome the Ladyzhenskaya-Babuška-Brezzi condition, a stabilization technique based on the orthogonal sub-grid scale method is introduced. The main advantage of the method is the possibility of using linear triangular finite elements, which are easy to generate for real industrial applications. Results are compared with several improved formulations, as the enhanced assumed strain method (EAS) and the Q1P0-formulation, in nearly incompressible problems and in the context of linear elasticity and J2-plasticity.

Contents

Chapter 1

Introduction

The finite element method is a mathematic tool whose usage for the solving of various problems in physics or in engineering is getting more and more extended and important. In engineering it will be actually integrated into the designing process of structures by using several commercial programs. Simultaneously extended codes will be developed which can describe divers physical phenomena and its applications. Within the development of finite element methods the element formulation is an intensive investigation field of its own, named *element technology*. The object of the element technology is to develop robust and efficient elements in view of the computational effort and the appliance to general problems. Series of desirable characteristics and requirements of finite elements can be considered which should be fulfilled to reflect in an adequate way local problems of solid mechanics:

- satisfying comportment in incompressible situations
- satisfying comportment in bending dominated problems
- little sensibility against distorted elements
- adequate precision in coarser meshes
- low computational cost
- applicable to triangular elements to simplify the mesh generation
- facility of the implementation of nonlinear constitutive models

Actually, in the practical applications it is tried to optimize low-order elements because of their simplicity, their efficiency and other advantageous aspects. The low-order elements are less sensible against the distortion and as well produce a smaller bandwidth in the global matrix of the linear equation system which leads for example to a lower computational cost and lesser need of memory.
Nevertheless in the field of solid mechanics, elements of a low order like triangles or quadrilaterals with linear displacement interpolations turn out to present dissatisfied results as

a matter of extreme conditions such as in case of nearly incompressibility. In these cases the numerical predictions that are obtained with these elements by using meshes which are naturally adequate, differ significant of the physical response. Hence it is necessary to know the causes which produce these insufficiencies and develop affording formulations that give an adequate result satisfying the physical response.

1.1 Motivation - locking of standard elements

The results that are obtained generally by using the standard displacement-based finite elements are satisfactory for a great variety of problems. By formulating a continuum problem via the finite element methods, a space of test functions is adopted. Essentially, what has to be done to find the solution of the given problem inside of the defined space is to introduce restrictions to the displacement solutions of the continuum problem whereby the finite element method allows to find the best approximation in this space. Nevertheless, in some situations exist proper restrictions of the medium which, established in the discrete form, lead to an over restricted system. The space of the test functions emerges as too poor for the problem and incapable to represent the deformations that have to occur. As a consequence the solution offered by this, in general under estimated function space could be nearly zero. This phenomenon is also known as *locking*. The locking of elements are presented mostly in two different situations: on one hand in bending problems and on the other hand in case of incompressibility or nearly incompressibility, respectively.

1.1.1 Locking in a bending dominated problem

The standard linear elements are attractive because of their simplicity and the low computational cost by solving a problem. They offer the minimum number of nodes as well as the minimum degrees of freedom. This restriction can effect an internal stiffness of the element which is unknown to the physical conditions of the problem. Particularly in cases of bending dominated problems the linear elements offer unsatisfied results.
In this type of problem the comparison between the theoretical displacement field and the description by the linear element can show the difficulty of the problem. The pure bending of an element as shown in Fig.**1.1** leads to the following considerations.
Let the displacement field in a bilinear element be described as

$$u_j = \sum_{i=1}^{4} N_i(\xi, \eta)\, u_{ij} \tag{1.1}$$

with the shape function

$$N_i(\xi, \eta) = \frac{1}{4}\,(1 + \xi\,\xi_i + \eta\,\eta_i + \xi\,\xi_i\,\eta\,\eta_i) \tag{1.2}$$

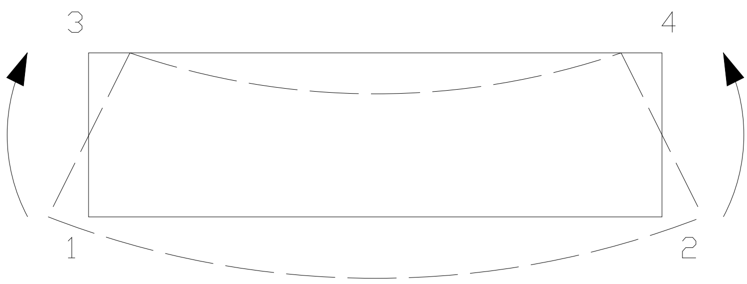


Figure 1.1: Bending dominated element

Additionally we can say in case of a constant bending moment that

$$u_{11} = -u_{21} = u_{31} = -u_{41} \tag{1.3}$$

Applying this for example on node 1 leads to

$$\begin{aligned} u_1 &= \sum_{i=1}^{4} N_i \, u_{i1} \\ &= \frac{1}{4}\,(1-\xi-\eta+\xi\eta)\,u_{11} + \frac{1}{4}\,(1+\xi-\eta-\xi\eta)\,u_{21} \\ &\quad + \frac{1}{4}\,(1+\xi+\eta+\xi\eta)\,u_{31} + \frac{1}{4}\,(1-\xi+\eta-\xi\eta)\,u_{41} \\ &= \xi\,\eta\,u_{11} = \xi\,\eta\,u_{31} \end{aligned} \tag{1.4}$$

The condition for pure bending says that

$$u_{1,2} = 0 \tag{1.5}$$

With $x_1 = \xi\, l$ and $x_2 = \eta\, l$ the derivations of the displacements are defined as

$$\frac{\partial(\cdot)}{\partial x_1} = \frac{1}{l}\,\frac{\partial(\cdot)}{\partial \xi}\,, \quad \frac{\partial(\cdot)}{\partial x_2} = \frac{1}{l}\,\frac{\partial(\cdot)}{\partial \eta} \tag{1.6}$$

And so we come to the conclusion that

$$(\eta - \xi)\,\frac{u_{31}}{l} = 0 \quad \rightarrow \quad u_{31} = 0 \tag{1.7}$$

which is the typical effect of locking. This means that the linear quadrilateral element does not have the capacity to represent the real comportment because of its lack of deformation modes which permit the longish fiber to adopt the curvature that is characteristic for the bending. This difficulty to represent the deformation can be apprehended as well as an

extreme stiffness of the element.
To delete or at least weaken the effect of locking, it could be refined the mesh size or improved the elements quality to a higher order of interpolations for the displacement field.

1.1.2 Locking in incompressibility

This kind of locking is beyond doubt one of the most important difficulties that are found in problems solved by the finite element methods. The practical applications which are presented in (nearly) incompressible solid mechanics are for example the computation of elastomer in case of elasticity or the computation of metals in case of J2-plasticity. This kind of constitutive model is used in most instances in the field of numerical simulations of metallic materials in case of elasto-plasticity. The J2-model considers as an assumption that the plastic deformations are isochore which means that they do not cause a volume change.
As a simple example, let us behold the problem in Fig.**1.2**. Given is an element which shall be clamped at the nodes 1, 2 and 4. Additionally we postulate at the node 3 a displacement which has the same absolute value in the x-direction as well as in the y-direction in the way that

$$u_1 = \sum_{i=1}^{4} N_i(\xi,\eta)\, u_{i1} = \frac{1}{4}\,(1+\xi+\eta+\xi\eta)\, u_{31} \tag{1.8}$$

$$u_2 = \sum_{i=1}^{4} N_i(\xi,\eta)\, u_{i2} = \frac{1}{4}\,(1+\xi+\eta+\xi\eta)\, u_{32} \tag{1.9}$$

with $u_{31} = -u_{32}$. The requirement of incompressibility leads to

$$u_{1,1} + u_{2,2} + u_{3,3} = 0 \tag{1.10}$$

If we define the derivations as in Eq.(1.6) we come to the conclusion that

$$(\eta - \xi)\,\frac{u_{31}}{l} = 0 \quad \rightarrow \quad u_{31} = 0 \tag{1.11}$$

which shows the volumetric locking of the linear element in case of incompressibility. This means in view of the impossibility to represent an isochore displacement field, that the standard element offers zero displacements to adjust to the condition of zero volumetric deformations.
Unfortunately this difficulty cannot be neglected by using finer meshes. As a difference to the bending locking, the volumetric locking can only be weakened by the appliance of higher order elements but neither deleted if we do not use elements of such a high order that the computational time is overpayed. In this case only the use of special element

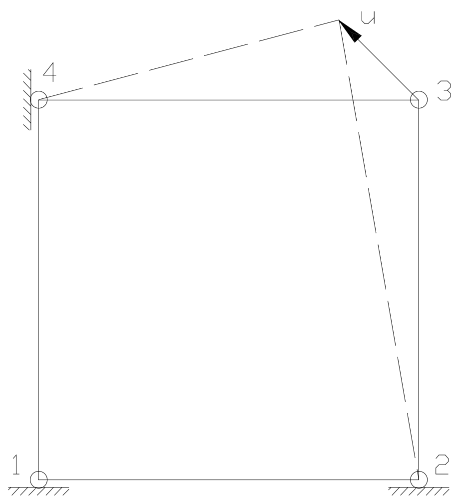


Figure 1.2: Incompressible element with postulated displacement

formulations leads to improved results. The investigation in this field is very intensive and there are several alternatives to affront this problem like the element formulation developed in this work.

1.2 Object of the thesis

The object of the thesis is the development and implementation of an explicit formulation which shall be robust and accurate in representing the problem of incompressibility in solid mechanics under the use of the constitutive models of elasticity and J2-plasticity. With the aim of being valuable in practical and industrial applications it is necessary that the developed elements pass certain requirements like low computational costs, good comportments in coarse meshes, offering a facility to implement the nonlinear constitutive model and an easy mesh generation in case of complex geometries. The specific object will be the development of a mixed triangular element with linear interpolations for the displacement and the pressure field.

1.3 Contents of the thesis

The next chapter shall introduce the problem of (nearly) incompressibility in solid mechanics and the difficulty of the standard displacement-based formulation to treat this constitutive border case. It will be presented the mixed finite element method with the approximation of the displacements $\mathbf{u}$ and the pressure p to deal successfully with the problem of incompressibility and afterward will be shown its own limits and the necessity of applying a stabilization technique on this mixed element formulation. At last will be presented the sub-grid scale approach as a kind of stabilization technique for the mixed finite element method with the particular interest in the orthogonal sub-grid scales.

In the third chapter will be developed a triangular mixed finite element formulation with linear interpolations for the displacement and the pressure field stabilized by the previous specified orthogonal sub-grid scale. It will be considered the case of elasticity as well as the J2-plasticity.
The fourth chapter treats of numerical results of this developed element in comparison to other standard and enhanced finite element formulations like the standard quadrilateral and triangular elements, the EAS-element, the Q1P0-element and a triangular mixed element without any stabilization. The comportment is shown by examples in cases of elasticity and J2-plasticity.
Finally conclusions of the thesis are drawn and possible future investigation aspects of this work will be emphasized.

Chapter 2

The Mixed Finite Element Formulation

2.1 Introduction

Many problems of physical importance involve motions that essentially preserve volumes locally. That means that after a deformation each small portion of the medium has the same volume as before the deformation. Media that behave in this fashion are termed *incompressible*. In this case when the Poisson's ratio ν becomes 0.5 the standard displacement formulation of elastic problems fails. Indeed, problems arise even when the material is nearly incompressible with $\nu > 0.4$ and the simple linear approximation with triangular elements gives highly oscillatory results in such cases.
The application of a mixed formulation for such problems can avoid the difficulties and is of great practical interest as *nearly* incompressible behavior is encountered in a variety of real engineering problems ranging from soil mechanics, as example the modeling of clammy clay, to aerospace engineering where a usage of rubber like materials is fundamental. Identical problems also arise when the flow of incompressible fluids is encountered as in the case of Stoke's flow.
In this chapter the mixed approaches to incompressible problems shall be discussed, generally using a two-field formulation where the displacements $\mathbf{u}$ and the pressure p are the free variables. Such formulation allows the dealing with full incompressibility as well as near incompressibility as it occurs. There are two ways to express this kind of two-field formulation with displacement and pressure variables which are very familiar. They differ in the way of defining the pressure variable in the elements.

The u/p-formulation: In this formulation the pressure variable owns only to the just contemplating element. These allows the static condensation of the pressure variable in every element before they are assembled to the global structure which saves a lot of evaluation time. We will observe later on that this static condensation does not work for exact incompressibility (see section 2.3.2). The continuity of the pressure over the elements is not forced but appears with the refining of the mesh.

The u/p-c formulation: This formulation assumes the continuity of the pressure over the elements what means that the pressure is defined in the nodes of the elements which are connected to neighboring elements. This excludes from the outset the static condensation of the pressure variables but leads to a continuous pressure field over the elements independent of the size of the mesh.

The question which element formulation is more efficient depends on the kind of problem that has to be solved. Where the u/p-formulation is very fast in solving problems of nearly incompressibility this property looses its function in exact incompressibility and it has to be said that the results of the pressure can be very inaccurate if large meshes are used or extreme problems occur as it is demonstrated in chapter 4. The u/p-c formulation needs particularly in case of nearly incompressibility more evaluation time but has the advantage that the results of the pressure are very accurate. In the presented work shall be developed a mixed finite element formulation based on a continuous u/p-formulation also with the aim to circumvent the inaccuracy of the pressure distributions.

2.2 The basic differential equations for mixed element formulations

In many cases of calculations in solid mechanics the properties of the material of a body is leading to a incompressible behavior. For example some rubber like materials or materials in an inelastic stadium can give an incompressible reply. In fact the compressibility can be so small that it may be disregarded. In that case the material could be idealized as completely incompressible. The difficulty in the computation of incompressible mediums is to predict the pressure. Whether it is possible to get adequate results for nearly incompressible materials using finite elements with only the displacements as free variables, the number of necessarily elements to get a default accuracy of the solution is much larger than in case of problems with compressible properties.
To display the difficulty of this problem we should consider an arbitrary body as in Fig.**2.1**. The material of the body is isotropic and will be described by the Young's modulus E and the Poisson's ratio ν. The boundary-value problem for this continuously differentiable body including the equilibrium conditions and the boundary conditions can be defined as

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \partial_t \Omega \tag{2.2}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u \Omega \tag{2.3}$$

where the surface of the body Ω is defined as $\partial\Omega = \partial_u\Omega \cup \partial_t\Omega$ and $\partial_u\Omega \cap \partial_t\Omega = \emptyset$.
Additionally to the equilibrium and the boundary conditions we need the constitutive equations which give the relation between the strains $\boldsymbol{\varepsilon}(\mathbf{u})$ as a function of the deformations $\mathbf{u}$ and the stresses $\boldsymbol{\sigma}$. In linear elasticity we use the Hookean law in the form of

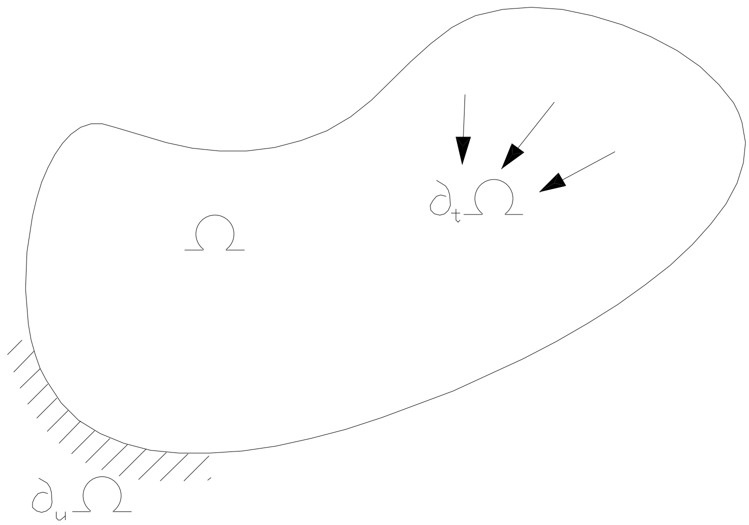


Figure 2.1: Arbitrary body Ω with boundary conditions $\partial_u\Omega$ and $\partial_t\Omega$

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \tag{2.4}$$

where the components of the constitutive 4th order tensor are defined as

$$C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{2.5}$$

λ and μ are the so called LAMÉ parameters and can also be described as parameters dependent on the Young's modulus E and the Poisson's ratio ν.

$$\lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)} \tag{2.6}$$

$$\mu = G = \frac{E}{2\,(1+\nu)} \tag{2.7}$$

μ or G are also known as the shear modulus of the material.
In fact, in linear elasticity the material properties can be expressed in terms of only two appropriate parameters. In case of incompressibility it is of use to write the constitutive equations against the Poisson's ratio ν and the bulk modulus K which is a measure for the behavior of volumetric deformation of the material. It can be defined as

$$K = \frac{E}{3\,(1-2\,\nu)} \tag{2.8}$$

Finally the stresses $\boldsymbol{\sigma}$ can be written as

$$\boldsymbol{\sigma} = K\varepsilon_{\mathrm{V}}\mathbf{1} + 2\mu\,\mathrm{dev}(\boldsymbol{\varepsilon}) \tag{2.9}$$

Thereby is

$$\varepsilon_{\mathrm{V}} = \nabla \cdot \mathbf{u} \tag{2.10}$$

the volumetric strain, the matrix **1** the identity matrix and

$$\mathrm{dev}(\boldsymbol{\varepsilon}) = \mathrm{dev}(\nabla^{s}\mathbf{u}) \tag{2.11}$$

the deviatoric strains.
As ν approaches 0.5, resistance to volume change is greatly increased assuming resistance to shearing remains constant. This may be seen by calculating the ratio of the bulk modulus to the shear modulus.

$$\frac{K}{\mu} = \frac{2\,(1+\nu)}{3\,(1-2\,\nu)} \tag{2.12}$$

Clearly, as $\nu \rightarrow 0.5$, the ratio approaches infinity. The limiting value $\nu = 0.5$ thus represents incompressibility. This limit creates problems in the equations of the constitutive law (2.9). A possible formulation to circumvent this problem can be written considering the hydrostatic pressure p as an independent unknown additionally to the displacement field $\mathbf{u}$. The pressure p in the body is defined as

$$p = -K\varepsilon_{\mathrm{V}} = -\frac{1}{3}\,\sigma_{\mathrm{V}} \tag{2.13}$$

with σ_{V} as the volumetric stresses. In case of $\nu = 0.5$, K is going to infinity and ε_{V} is getting zero. The pressure of course has nevertheless a finite value. With regard to this the stress tensor can be expressed such as

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + 2\mu\,\mathrm{dev}(\nabla^{s}\mathbf{u}) \tag{2.14}$$

where the displacements u and additionally the pressure p are the two unknown variables. With the definition of the pressure in (2.13) we now get a new condition which has to be fulfilled at all times.

$$\frac{p}{K} + \varepsilon_{\mathrm{V}} = \frac{p}{K} + \nabla \cdot \mathbf{u} = 0 \tag{2.15}$$

In case of incompressibility ($K \rightarrow \infty$) the condition changes into

$$\nabla \cdot \mathbf{u} = 0 \tag{2.16}$$

Now we can summarize the considerations to the following set of differential equations:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega \tag{2.17}$$

$$\frac{p}{K} + \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{2.18}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \partial_t \Omega \tag{2.19}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u \Omega \tag{2.20}$$

wherein the components of the stress tensor are now a function of the displacements $\mathbf{u}$ and the pressure p (2.14).

2.3 The variational formulation for mixed finite elements

The previous considerations point to significant difficulties which appear if we use finite elements applying only to displacements in case of incompressible media. The very small volumetric strain, so called dilatation, which approach to zero if exists complete incompressibility, will be dominated by the derivations of the displacements. This derivations can not be predicted as accurate as the displacements themselves. And every imprecision in the volumetric strains will appear also as an error in the stresses. Finally this error on his part will influence the forecasting of the displacements because the external forces are balanced by the stresses via the principal of the virtual work. Although solutions are possible it could be necessary to use a very fine discretization of finite elements to get an appropriate solution.

With regard to this aspect it would be useful to develop a finite element formulation that, independently of the Poisson's ratio, gives accurate solutions even if ν is nearly 0.5. This leads to a formulation in which the deformations and the stresses can be defined independently of the bulk modulus. Finite element formulations which show this special property could also be called as (volumetric) locking-free elements. The *locking* of elements has the effect that the value of the computed displacements are much smaller than the expected displacements which should appear. Whether the normal displacement-related formulation locks in case of nearly incompressibility, it is as well an important aspect that also the stresses and the pressure respectively are very imprecise. Effective elements which neglect the locking behavior in nearly incompressible problems can be deduced from the interpolation of the displacement and additionally the pressure as done in the previous section.

2.3.1 The weak form

The variational or weak counterpart could be deduced directly from the strong formulation of the previous chapter (2.17)-(2.18). However, it is necessary to define at first the right spaces of the used functions which can solve the given problem.

$$\mathcal{L}^2(\Omega) = \left\{ v \mid \int_\Omega v^2 \, d\Omega < \infty \right\} \tag{2.21}$$

$$\mathcal{H}^1(\Omega) = \left\{ v \in \mathcal{L}^2(\Omega) \mid \frac{\partial v}{\partial x_j} \in \mathcal{L}^2(\Omega) \; , \quad j = 1, \ldots, n_{dim} \right\} \tag{2.22}$$

The space $\mathcal{L}^2$ is the space of the quadratic integrable functions in the considered volume and $\mathcal{H}^1$ the so called Hilbert-space, in which the functions $\mathbf{v}$ and its derivatives are elements of the space $\mathcal{L}^2$.
To write the variational formulation in dependency of the variables $\mathbf{u}$ and p the stress should be defined as in Eq.(2.14). After specified multiplication with the variations of the displacements and the pressure the following abstract formulation can be obtained

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = l(\mathbf{v}) \qquad \forall \, \mathbf{v} \in \mathcal{V} \tag{2.23}$$

$$b(q, \mathbf{u}) = 0 \qquad \forall \, q \in \mathcal{Q} \tag{2.24}$$

where

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \nabla^s \mathbf{v} : \nabla^s \mathbf{u} \, d\Omega \tag{2.25}$$

$$b(p, \mathbf{v}) = \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega \tag{2.26}$$

$$b(q, \mathbf{u}) = \int_\Omega q (\nabla \cdot \mathbf{u} - \frac{1}{K} p) \, d\Omega \tag{2.27}$$

$$l(\mathbf{v}) = (\mathbf{v}, \mathbf{f}) + (\mathbf{v}, \bar{\mathbf{t}})_{\partial_t \Omega} = \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega + \int_{\partial_t \Omega} \mathbf{v} \cdot \bar{\mathbf{t}} \, d\Gamma \tag{2.28}$$

and the spaces

$$\mathcal{V} = \left\{ \mathbf{v} \in \mathcal{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \; , \; in \; \partial_u \Omega \right\} \tag{2.29}$$

$$\mathcal{Q} = \mathcal{L}^2(\Omega) \tag{2.30}$$

Finally the discretized formulation of the weak form can be deduced by choosing the interpolation functions for the displacements, the pressure and their variations for every element in the way that the spaces comply with $\mathcal{V}_h \subset \mathcal{V}$ and $\mathcal{Q}_h \subset \mathcal{Q}$. Referring to this Eqs.(2.23) and (2.24) can be rewritten as

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = l(\mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h \tag{2.31}$$
$$b(q_h, \mathbf{u}_h) = 0 \qquad \forall\, q_h \in \mathcal{Q}_h \tag{2.32}$$

The approximation of the displacement field and its variations can be defined as

$$\mathbf{u}_h|_{\Omega^e} = \mathbf{N}\,\mathbf{u}^e\ , \qquad \mathbf{v}_h|_{\Omega^e} = \mathbf{N}\,\mathbf{v}^e \tag{2.33}$$

where

$$\mathbf{N} = [\mathbf{N}^A\ \mathbf{N}^B,\ \dots\ \mathbf{N}^{NN}]^T\ , \qquad \mathbf{N}^A = N^A\,\mathbf{1}^{ND}\ , \quad \text{with } A = 1, \dots, NN \tag{2.34}$$

are the interpolation functions and

$$\mathbf{u}^e = [\mathbf{u}^A\ \mathbf{u}^B\ ...\ \mathbf{u}^{NN}]^T\ , \qquad \mathbf{v}^e = [\mathbf{v}^A\ \mathbf{v}^B\ ...\ \mathbf{v}^{NN}]^T \tag{2.35}$$

the displacements at the respective nodes. Each vector is subdivided into the number of dimensions $\mathbf{u}^A = \{u_1^A\ ...\ u_{ND}^A\}$ and respectively $\mathbf{v}^A = \{v_1^A\ ...\ v_{ND}^A\}$. The same scheme can also be applied on the pressure field and its variation

$$p_h|_{\Omega^e} = \mathbf{N}_p\,\mathbf{p}^e\ , \qquad q_h|_{\Omega^e} = \mathbf{N}_p\,\mathbf{q}^e \tag{2.36}$$

with

$$\mathbf{N}_p = [N_p^A\ N_p^B\ \dots N_p^{NN}]^T \tag{2.37}$$

and

$$\mathbf{p}^e = [p^A\ p^B\ \dots p^{NN}]^T\ , \qquad \mathbf{q}^e = [q^A\ q^B\ \dots q^{NN}]^T \tag{2.38}$$

2.3.2 The matrix formulation

If we now insert the deduced relations into the weak form in Eqs. (2.23) and (2.24) and make use of the Lemma for variational calculations (that the formulations holds for any arbitrary variation v and q) we get the elementwise linear equation system

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix}^e \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}^e = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}^e \tag{2.39}$$

with

$$\mathbf{K}^e_{uu} = \int_{\Omega^e} (\mathbf{B}^A)^T \, \mathbf{D} \, \mathbf{B}^B \, d\Omega \tag{2.40}$$

$\mathbf{K}^e_{uu}$ is the element stiffness matrix which connects the displacements, in form of $\mathbf{B}^A$ and $\mathbf{B}^B$ as the differential operators of the shape functions, with the deviatoric part of the constitution matrix $\mathbf{D}$.

$$\mathbf{K}^e_{up} = (\mathbf{K}^T_{pu})^e = \int_{\Omega^e} \mathbf{b}^A \, \mathbf{N}^B_p \, d\Omega \,, \qquad \text{with } \mathbf{b}^A = \left[\; N^A_{,1} \dots N^A_{,ND} \; \right]^T \tag{2.41}$$

are mixed terms depending on the displacements as well as on the pressure and finally

$$\mathbf{K}^e_{pp} = -\int_{\Omega^e} (\mathbf{N}^A_p)^T \, \mathbf{N}^B_p \, d\Omega \tag{2.42}$$

is a term which depends only on the pressure. After assembling the element matrices we come to the global formulation

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \tag{2.43}$$

These are now the basic equations for the formulation of finite elements with the displacements and the pressure as unknown variables. If incompressibility exists the term depending only on the pressure $\mathbf{K}_{pp}$ will be dropped and the linear equation system changes into

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \tag{2.44}$$

where we can see that the second equation only contains the displacements as unknowns. This is generally important for particular formulations like the discontinuous $\mathbf{u}/p$-formulation, mentioned before (see chapter 2). After the derivation of the basic equations for displacement/pressure-based elements the main problem is now to find effective interpolation functions for the displacements and the pressure so that accurate solutions are obtained. If the ratio of the interpolation functions of the pressure to the displacements is too high, the element can behave like a displacement-based element that would be very inaccurate. On the other hand if the ratio is too low, the inaccuracy appears in the solution of the pressure which could be of a higher order.

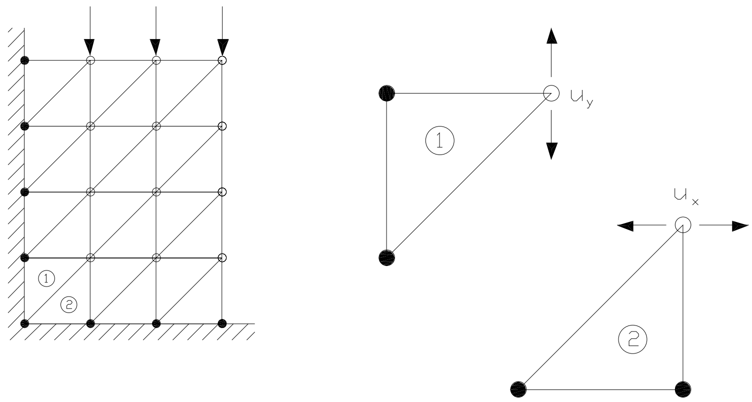


Figure 2.2: Locking (zero displacements) of a simple assembly of triangles with linear interpolations of the displacements and constant interpolations of the pressure if incompressibility is fully required

2.4 Illustration of a fundamental difficulty

Unfortunately, the arbitrary combination of displacement and pressure interpolations may prove ineffective in incompressible cases. An example of one of the difficulties is illustrated by the mesh in Fig.**2.2**.
It is supposed that linear displacement and constant pressure triangular elements are being employed and furthermore, the left-hand side and bottom edges of the mesh shall be fixed (i.e., the displacements are identically zero). On the top of the structure affects an arbitrary load distributed on the free nodes. If we assume incompressibility, the area of each triangle must necessarily remain constant and the equation

$$\int_{\Omega} q_h \nabla \cdot \mathbf{u}_h \, d\Omega = 0 \tag{2.45}$$

has to be fulfilled. The pressure field shall be formulated with constant functions distributed over every element and so we have for its variation also a constant value. Hence, q can be drawn out of the integral and it remains

$$\int_{\Omega} \nabla \cdot \mathbf{u}_h \, d\Omega = 0 \tag{2.46}$$

Due to the fact that $\mathbf{u}_h$ is a linear polynomial over each triangle and thus the divergence of the displacements is constant, Eq.(2.46) actually implies the stronger pointwise condition

$$\nabla \cdot \mathbf{u}_h = 0 \qquad \text{on each } \Omega_e \tag{2.47}$$

If we pick two elements out of the given mesh, here defined as element 1 and element 2, we can regard the elemental displacements of the common node A and see that in case of

incompressibility (the Area of every element may not increase) both elements could only achieve contrary displacements. This leads to the fact that the displacement has to be zero in this node if the elements shall remain compatible. Identical reasoning may be used to conclude that every node in the entire mesh must have zero displacement. Thus the only possible incompressible displacement is $\mathbf{u}_h = \mathbf{0}$. This result is preserved, no matter how many elements are present in each direction. Clearly, this type of mesh offers no approximation power whatsoever and is a simple device for the mesh locking. It is but one of the difficulties afflicting problems of incompressibility.
In the nearly incompressible case, the same phenomenon occurs, only this time $\mathbf{u}_h \cong \mathbf{0}$. Thus introducing slight compressibility does not solve the problem. For these kinds of problems a mathematical convergence theory for mixed finite element methods of the type under consideration has been established by some authors and is known as the Ladyzhenskaya-Babuška-Brezzi condition (LBB). To establish whether or not this condition is satisfied for elements of interest is not so simple. Hence, the next chapter gives attention to some previous properties of finite element solutions and then comes to a short explanation of the LBB-condition.

2.5 The Mixed Patch Test

2.5.1 Important properties of finite element solutions

To understand the necessity of specific conditions, mixed finite element methods has to fulfill, it is helpful to look primarily at some important properties of basic finite element methods. The problem of elasticity can be written as

$$\begin{aligned} &Find\ \mathbf{u} \in \mathcal{V},\ such\ that:\\ &\quad a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathcal{V} \end{aligned} \tag{2.48}$$

whereas the vector space $\mathcal{V}$ is defined as

$$\mathcal{V} = \left\{ \mathbf{v} \,|\, \mathbf{v} \in L^2(\Omega) \,|\, \frac{\partial v_i}{\partial x_j} \in L^2(\Omega)\ ,\ \text{with: } i,j = 1,2,3 \,|\, \mathbf{v} = \mathbf{0} \quad \text{on } \partial_u \Omega \right\} \tag{2.49}$$

Here is

$$L^2(\Omega) = \left\{ \mathbf{w} \,|\, \int_\Omega \left(\sum_{i=1}^{3} (w_i)^2 \right) d\Omega = \|\mathbf{w}\|_{L^2(\Omega)} < \infty \right\} \tag{2.50}$$

the room of the quadratic integrable functions in the considered volume. Thus (2.49) defines a general three dimensional function space whereas the functions disappear on the edge of the body and the squares of the functions as well as its first derivatives are integrable. Accordingly to $\mathcal{V}$ we can define a kind of energy norm

$$\|\mathbf{v}\|_E^2 = a(\mathbf{v}, \mathbf{v}) \tag{2.51}$$

This norm describes the double of the strain energy which contains a body if it is exposed to the displacement field $\mathbf{v}$. Assumed that the body is bedded in the right way, the norm $\|\mathbf{v}\|_{\mathrm{E}}^2$ is bigger than zero for all $\mathbf{v} \neq \mathbf{0}$. Additionally we have to define the so called Sobolev-norms of the order $s = 0$ and $s = 1$.

$$\begin{aligned} s &= 0: \\ & (\|\mathbf{v}\|_0)^2 = \int_\Omega \left(\sum_{i=1}^{3} (v_i)^2 \right) d\Omega \end{aligned} \tag{2.52}$$

$$\begin{aligned} s &= 0: \\ & (\|\mathbf{v}\|_1)^2 = (\|\mathbf{v}\|_0)^2 = \int_\Omega \left[\sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} \right)^2 \right] d\Omega \end{aligned} \tag{2.53}$$

Hence we get the following properties of the bilinear form $a(\cdot, \cdot)$

$$\exists\, M > 0 \text{ so, that } \forall\, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}, \qquad |a(\mathbf{v}_1, \mathbf{v}_2)| \leq M \|\mathbf{v}_1\|_1 \|\mathbf{v}_2\|_1 \tag{2.54}$$

$$\exists\, \alpha > 0 \text{ so, that } \forall\, \mathbf{v} \in \mathcal{V}, \qquad a(\mathbf{v}, \mathbf{v}) \geq \alpha (\|\mathbf{v}_1\|_1)^2 \tag{2.55}$$

whereas (2.54) is the property of the continuity of the bilinear form $a(\cdot, \cdot)$ and (2.55) the ellipticity or coercivity. Thereby the constants α and M depend on the actually considered elastic problem including the material parameters but they are independent of the displacement field $\mathbf{v}$. These two properties of continuity and ellipticity are fulfilled if feasible norms are used, like in this case the Sobolev-norm of the order one, and if we assume a stable bearing of the structure. The mathematic device of this assumptions are explained in [6]. On this two properties bases the inequation

$$c_1 \|\mathbf{v}\|_1 \leq [a(\mathbf{v}, \mathbf{v})]^{1/2} \leq c_2 \|\mathbf{v}\|_1 \tag{2.56}$$

with the two constants c_1 and c_2 independent of $\mathbf{v}$. From this inequality we can now rewrite the known convergence criteria

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \rightarrow 0 \qquad \text{as } h \rightarrow 0 \tag{2.57}$$

in the form

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \rightarrow 0 \qquad \text{as } h \rightarrow 0 \tag{2.58}$$

whereas h specifies the size of the used elements. This shows that the energy norm by solving of problems converges with the same order like the Sobolev-1-norm.
These conclusions in case of elasticity problems contains an important aspect that the definite solution to the problem has to come up to a finite strain energy (Eqs. (2.56) and (2.57)). A must for the definite solution is also its uniqueness. If for example $\mathbf{u}_1$ and $\mathbf{u}_2$ shall be two various solutions of a problem, then

$$a(\mathbf{u}_1, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathcal{V} \tag{2.59}$$

and as well

$$a(\mathbf{u}_2, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathcal{V} \tag{2.60}$$

The subtraction leads to

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in \mathcal{V} \tag{2.61}$$

Appointing in particular

$$\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2 \tag{2.62}$$

leads to

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = 0 \tag{2.63}$$

and with regard to (2.56) we get

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_1 = 0 \tag{2.64}$$

which is the same as $\mathbf{u}_1 \equiv \mathbf{u}_2$. For this reason there is no possibility for two various solutions.
Now we define $\mathcal{V}_h$ as a space of finite element displacement functions and $\mathbf{v}_h$ shall be an arbitrary element of this space which can be extracted of the displacement interpolations. Additionally $\mathbf{u}_h$ is the finite element solution and therewith an element of $\mathcal{V}_h$, too. Then the finite element solution of this problem can be written as

$$\begin{aligned}&Find \mathbf{u}_h \in \mathcal{V}_h, such\ that:\\ &\quad a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h\end{aligned} \tag{2.65}$$

The space is again defined as

$$\mathcal{V}_h = \left\{ \mathbf{v}_h \,|\, \mathbf{v}_h \in L^2(\Omega) \,|\, \frac{\partial (v_h)_i}{\partial x_j} \in L^2(\Omega)\,,\; i,j = 1,2,3 \,|\, \mathbf{v}_h = \mathbf{0} \quad on\ \partial_u \Omega \right\} \tag{2.66}$$

and it is obvious that $\mathcal{V}_h \subset \mathcal{V}$.
The relation (2.65) is the principle of virtual forces for the to $\mathcal{V}_h$ corresponding finite element discretization. According to this solution space the conditions of the continuity (2.54) and the ellipticity (2.55) are fulfilled by the use of $\mathbf{v}_h \in \mathcal{V}_h$ and we get for arbitrary $\mathcal{V}_h$ a positive stiffness matrix. The relation between the finite element solution $\mathbf{u}_h$ and the discrete solution $\mathbf{u}$ of the problem is characterized by three important properties.
The first property is

$$a(\mathbf{e}_h, \mathbf{v}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h \tag{2.67}$$

where

$$\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h \tag{2.68}$$

is the error between the discrete solution $\mathbf{u}$ and the finite element solution $\mathbf{u}_h$. This seems evident if we know that the principle of virtual work implies that

$$a(\mathbf{u}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h \tag{2.69}$$

and additionally

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h \tag{2.70}$$

Because if we subtract these two equations we come finally to Eq.(2.68). So it can be said that the error $\mathbf{e}_h$ is orthogonal in $a(\cdot,\cdot)$ to all $\mathbf{v}_h$ in $\mathcal{V}_h$ and by increasing the space $\mathcal{V}_h$, whereas $\mathcal{V}_h \subset \mathcal{V}$ still holds, the accuracy of the solution increases, too.
From this first property can be derivated two other properties. Thus the second property states that

$$a(\mathbf{u}_h, \mathbf{u}_h) \leq a(\mathbf{u}, \mathbf{u}) \tag{2.71}$$

This can be proved by

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}) &= a(\mathbf{u}_h + \mathbf{e}_h, \mathbf{u}_h + \mathbf{e}_h) \\ &= a(\mathbf{u}_h, \mathbf{u}_h) + 2a(\mathbf{u}_h, \mathbf{e}_h) + a(\mathbf{e}_h, \mathbf{e}_h) \\ &= a(\mathbf{u}_h, \mathbf{u}_h) + a(\mathbf{e}_h, \mathbf{e}_h) \end{aligned} \tag{2.72}$$

where $\mathbf{u}_h$ is used in case of $\mathbf{v}_h$. For every error $\mathbf{e}_h \neq 0$, we get $a(\mathbf{e}_h, \mathbf{e}_h) > 0$ if we assume that $\|\mathbf{v}\|_E \geq 0$. That means that strain energy obtained by the finite element solution is always smaller or equal the strain energy obtained by the discrete solution.
The third property states that

$$a(\mathbf{e}_h, \mathbf{e}_h) \leq a(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h \tag{2.73}$$

As proof it can be used the Eq.(2.72) to get for an arbitrary $\mathbf{w}_h \in \mathcal{V}_h$

$$a(\mathbf{e}_h + \mathbf{w}_h, \mathbf{e}_h + \mathbf{w}_h) = a(\mathbf{e}_h, \mathbf{e}_h) + a(\mathbf{w}_h, \mathbf{w}_h) \tag{2.74}$$

And from this

$$a(\mathbf{e}_h, \mathbf{e}_h) \leq a(\mathbf{e}_h + \mathbf{w}_h, \mathbf{e}_h + \mathbf{w}_h) \tag{2.75}$$

With defining $\mathbf{w}_h = \mathbf{u}_h - \mathbf{v}_h$ we come to the statement in (2.73).
The third property signifies that the finite element solution $\mathbf{u}_h$ is chosen from all possible displacement functions $\mathbf{v}_h \in \mathcal{V}_h$ in this way that the strain energy, corresponding to the difference between the discrete and the finite element displacement $\mathbf{u} - \mathbf{u}_h$, gets a minimum.
Moreover we get by using Eq.(2.73) as well as the ellipticity (2.55) and the continuity (2.54) of the bilinear form

$$\begin{aligned} \alpha \|\mathbf{u} - \mathbf{u}_h\|_1^2 &\leq a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \\ &= \inf_{\mathbf{v}_h \in \mathcal{V}_h} a(\mathbf{u} - \mathbf{v}_h, \mathbf{u} - \mathbf{v}_h) \\ &\leq M \inf_{\mathbf{v}_h \in \mathcal{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_1 \, \|\mathbf{u} - \mathbf{v}_h\|_1 \end{aligned} \tag{2.76}$$

with inf as the Infimum, i.e. the lower limit. Let

$$d(\mathbf{u}, \mathcal{V}_h) = \lim_{h \to 0} \inf_{\mathbf{v}_h \in \mathcal{V}_h} \|\mathbf{u} - \mathbf{v}_h\| \tag{2.77}$$

be the minimal distance between the discrete and the finite element solution if the element size goes to zero, then we come to the property

$$\|\mathbf{u} - \mathbf{u}_h\| \leq c d(\mathbf{u}, \mathcal{V}_h) \tag{2.78}$$

where $c = \sqrt{M/\alpha}$ is a constant which does not depend on the element size h but on the material properties. This inequality is also called the Lemma of Cea [6].
These three derived properties give a valuable insight in which way the finite element

solutions has to be chosen from the possible displacement functions in a given finite element mesh. The inequality (2.78) which is based on the property (2.73) means that it is a sufficient condition of convergence if $\lim_{h \to 0} \inf \|\mathbf{u} - \mathbf{v}_h\|_1 = 0$ holds for any arbitrary $\mathbf{u} \in \mathcal{V}$. Furthermore from the second and the third property it can be deduced that for the finite element solution the error of the strain energy within the possible pattern of displacements in a given mesh is minimized and that the finite element dependent strain energy approaches from below to the discrete strain energy if increasingly finer meshes are used.
After these considerations to the properties of the finite element solutions, it is now easier to follow the next section in which the mathematical requirements for mixed finite element formulations shall be introduced.

2.5.2 The basic requirements for mixed finite elements

As seen in the previous derivation of the mixed finite element method, to prevent the use of inefficient elements it is necessary to find adequate conditions for the solutions of the problem which should be independent of the material parameters in order to avoid the influence of incompressibility where $K \to \infty$. In case of mathematical aspect we search a condition for the space $\mathcal{V}_h$ so, that

$$\|\mathbf{u} - \mathbf{u}_h\| \leq c d(\mathbf{u}, \mathcal{V}_h) \tag{2.79}$$

whereas c now should be a constant independent of the element size h and the material parameters, particularly the bulk modulus K. The distance $d(\mathbf{u}, \mathcal{V}_h)$ shall be defined as

$$d(\mathbf{u}, \mathcal{V}_h) = \inf_{\mathbf{v}_h \in \mathcal{V}_h} \|\mathbf{u} - \mathcal{V}_h\| = \|\mathbf{u} - \tilde{\mathbf{u}}_h\| \tag{2.80}$$

where $\tilde{\mathbf{u}}_h$ is an element in $\mathcal{V}_h$ but not necessarily the finite element solution. For the better understanding what is the meaning of this distance the Fig.**2.3** shall give a possible idea. The inequation (2.79) means that the difference between the discrete solution $\mathbf{u}$ and the finite element solution $\mathbf{u}_h$ will be smaller than a reasonable measured constant c times $d(\mathbf{u}, \mathcal{V}_h)$. In condition (2.78) we came to the same conclusion except for the independence of the material parameters in terms of the bulk modulus K. If condition (2.79) is fulfilled with a reasonable measured constant c it is said that the accuracy of the finite element solution will increase independently of the bulk modulus K if the element size h will be decreased and the finite element discretization is getting reliable. So the condition (2.79) is the basic requirement for the general finite element discretization to get non-locking element formulations. The condition in (2.79) seems very general and it is difficult to show it for any element formulation. Therefore it shall be an aim to rewrite this condition in a much practical manner what will lead to the so called *inf-sup-condition*.

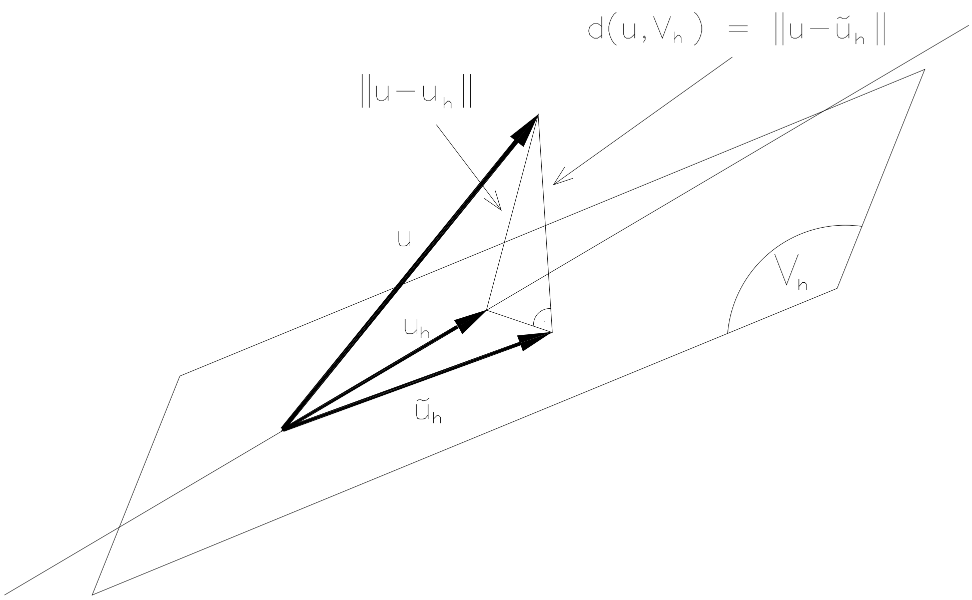


Figure 2.3: Illustration of distances between different solutions; for optimal convergence is $\|\mathbf{u}-\mathbf{u}_h\| \leq c d(\mathbf{u}, \mathcal{V}_h)$ where c is independent of h and K.

2.5.3 The derivation of the inf-sup-condition

At first we declare some necessary spaces like $\mathcal{K}$ and $\mathcal{D}$ in the way that

$$\begin{aligned} \mathcal{K}(q) &= \{\mathbf{v} \,|\, \mathbf{v} \in \mathcal{V}\,,\ \mathrm{div}\mathbf{v} = q\} && (2.81)\\ \mathcal{D} &= \{q \,|\, q = \mathrm{div}\mathbf{v} \text{ for some } \mathbf{v} \in \mathcal{V}\} && (2.82) \end{aligned}$$

and accordingly the spaces $\mathcal{K}_h$ and $\mathcal{D}_h$ for the discretization as

$$\begin{aligned} \mathcal{K}_h(q_h) &= \{\mathbf{v}_h \,|\, \mathbf{v}_h \in \mathcal{V}_h\,,\ \mathrm{div}\mathbf{v}_h = q_h\} && (2.83)\\ \mathcal{D}_h &= \{q_h \,|\, q_h = \mathrm{div}\mathbf{v}_h \text{ for some } \mathbf{v}_h \in \mathcal{V}_h\} && (2.84) \end{aligned}$$

That means for a given q_h the space $\mathcal{K}_h(q_h)$ corresponds to all elements $\mathbf{v}_h \in \mathcal{V}_h$ which fulfill $\mathrm{div}\mathbf{v}_h = q_h$. Furthermore the space $\mathcal{D}_h$ corresponds to all elements q_h with $q_h = \mathrm{div}\mathbf{v}_h$ which are achieved by the elements $\mathbf{v}_h \in \mathcal{V}_h$. So that for every $q_h \in \mathcal{D}_h$ exists at least one element $\mathbf{v}_h \in \mathcal{V}_h$ which fulfills $q_h = \mathrm{div}\mathbf{v}_h$. The same conclusions can be made for the spaces $\mathcal{K}$ and $\mathcal{D}$.

The elasticity problem with the case of incompressibility says that $q = 0$ and then the displacement $\mathbf{u}$ belongs to $\mathcal{K}(q)$ or better to $\mathcal{K}(0)$. For $\mathbf{u}_h \in \mathcal{K}(0)$ always holds $\|\mathbf{u}-\mathbf{u}_h\| \leq \tilde{c}d[\mathbf{u}, \mathcal{K}(0)]$, with $\tilde{c}$ as another constant independent of K, condition (2.79) can be rewritten as

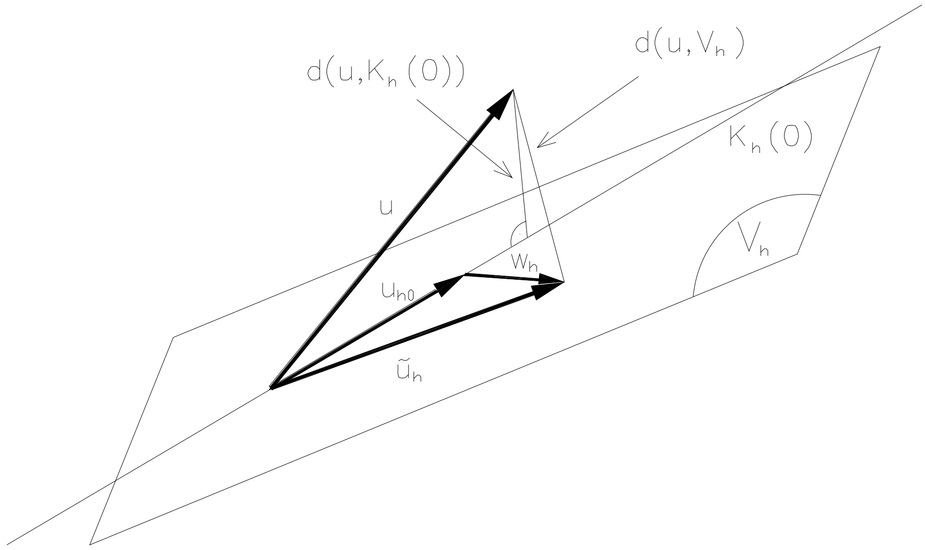


Figure 2.4: Illustration of the contemplated spaces and vectors

$$d[\mathbf{u}, \mathcal{K}(0)] \leq c d(\mathbf{u}, \mathcal{V}_h) \tag{2.85}$$

That means that if we let $h \to 0$ the distance between $\mathbf{u}$ and $\mathcal{K}(0)$ decreases with the same measure as the distance between the space $\mathcal{K}(0)$. To depict these relations see also Fig.**2.4**. Here $\mathbf{u}_{h0}$ shall be an chosen vector in $\mathcal{K}_h(0)$ and $\mathbf{w}_h$ the vector which produces

$$\tilde{\mathbf{u}}_h = \mathbf{u}_{h0} + \mathbf{w}_h \tag{2.86}$$

Then it can be shown that condition (2.85) is fulfilled if $\forall\, q_h \in \mathcal{D}_h$ exists a $\mathbf{w}_h \in \mathcal{K}_h(q_h)$ so that

$$\|\mathbf{w}_h\| \leq c' \|q_h\| \tag{2.87}$$

with c' independent of the element size h and the bulk modulus K.
Further on

$$\|\text{div}(\mathbf{u} - \tilde{\mathbf{u}}_h)\| \leq \alpha \|\mathbf{u} - \tilde{\mathbf{u}}_h\| \tag{2.88}$$

and accordingly with regard to incompressibility

$$\|\text{div}\tilde{\mathbf{u}}_h\| \leq \alpha\, d(\mathbf{u}, \mathcal{V}_h) \tag{2.89}$$

It is also obvious the

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h0}\| &= \|\mathbf{u} - \tilde{\mathbf{u}}_h + \mathbf{w}_h\| \\
&\leq \|\mathbf{u} - \tilde{\mathbf{u}}_h\| + \|\mathbf{w}_h\| \qquad (2.90)
\end{aligned}
$$

Now it shall be assumed that condition (2.87) holds for $q_h = \text{div}\tilde{\mathbf{u}}_h$. For the reason that $\text{div}\mathbf{u}_{h0} = 0$ and with respect to Eq.(2.80) we get

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h0}\| &\leq d(\mathbf{u}, \mathcal{V}_h) + c'\|q_h\| \\
&= d(\mathbf{u}, \mathcal{V}_h) + c'\|\text{div}\tilde{\mathbf{u}}_h\| \qquad (2.91) \\
&\leq d(\mathbf{u}, \mathcal{V}_h) + c'\alpha\, d(\mathbf{u}, \mathcal{V}_h) \qquad (2.92)
\end{aligned}
$$

Due to the fact that $\mathbf{u}_{h0} \in \mathcal{K}_h(0)$ we can finally write

$$
d[\mathbf{u}, \mathcal{K}_h(0)] \leq \|\mathbf{u} - \mathbf{u}_{h0}\| \leq (1 + \alpha c')\, d(\mathbf{u}, \mathcal{V}_h) \qquad (2.93)
$$

This condition corresponds to (2.85) for $c = 1 + \alpha c'$ and we see that c is independent of the element size h and the bulk modulus K. The result shows that condition (2.85) can be deviced by using only the inequation (2.87) and this inequation therefore is the basic requirement to get a finite element formulation with an ideal rate of convergence.
With the variables q_h and $\mathbf{w}_h \in \mathcal{K}_h(q_h)$ condition (2.87) can be deduced as

$$
\|\mathbf{w}_h\|\,\|q_h\| \leq c'\|q_h\|^2 = c' \int_\Omega q_h \,\text{div}\,\mathbf{w}_h \, d\Omega \qquad (2.94)
$$

or as the condition that for all $q_h \in \mathcal{D}_h$ exists such a $\mathbf{w}_h \in \mathcal{K}_h(q_h)$ that

$$
\frac{1}{c'}\|q_h\| \leq \frac{\int_\Omega q_h \,\text{div}\,\mathbf{w}_h \, d\Omega}{\|\mathbf{w}_h\|} \qquad (2.95)
$$

Thus it shall be

$$
\frac{1}{c'}\|q_h\| \leq \sup_{\mathbf{v}_h \in \mathcal{V}_h} \frac{\int_\Omega q_h \,\text{div}\,\mathbf{v}_h \, d\Omega}{\|\mathbf{v}_h\|} \qquad (2.96)
$$

and from this we come to the condition

$$
\inf_{q_h \in \mathcal{D}_h} \sup_{\mathbf{v}_h \in \mathcal{V}_h} \frac{\int_\Omega q_h \,\text{div}\,\mathbf{v}_h \, d\Omega}{\|\mathbf{v}_h\|\|q_h\|} \geq \beta > 0 \qquad (2.97)
$$

with $\beta = 1/c'$ as a constant independent of h and K. This so called inf-sup-condition or LBB-condition states that for an effective finite element discretization there has to be for any arbitrary $q_h \in \mathcal{D}_h$ such a $\mathbf{v}_h \in \mathcal{V}_h$ that the quotient in (2.97) is bigger or equal β and bigger than 0 and if this condition is passed the discretization possesses the approximation criteria that (2.85) will be fulfilled. After all the goal is now to develop an efficient mixed element formulation which even pass the LBB-condition to be sure that the formulation is adequate for quality of the finite element solutions. For elements that satisfy the LBB-condition, error estimates of the following form

$$\|\mathbf{u}_h - \mathbf{u}\|_1 + \|p_h - p\|_0 = \mathcal{O}(h^{\min[k,l+1]}) \tag{2.98}$$

may be established. Where k and l are the orders of the displacement and pressure interpolations, respectively. If $k = \min[k, l+1]$ the rate of convergence is said to be *optimal*. Clearly, elements that satisfy the LBB-condition will not lock. But to establish whether or not this condition is satisfied for elements of interest is not a trivial task because the formulation of the condition to be fulfilled has a very mathematical character and can not be applied easily nether on every finite element formulation nor on arbitrary mesh geometries. Thus it is desirable to have a simple procedure of constraint counting proves quite effective, see also [16].

2.5.4 Constraint counts

The constraint counts method is a heuristic approach for determining the ability of an element to perform well in incompressible and nearly incompressible applications. It should be emphasized that this is not a precise mathematical method for assessing elements but rather a quick and simple tool for obtaining an indication of element potential. However, it does seem to be able to predict a propensity for locking. There are, of course, other issues that need to be considered in an overall evaluation of element performance. To picture this method very intelligible it should be demonstrated with a simple example illustrated in Fig.**2.5**. Here n_{eq} shall represent the total number of displacement equations after boundary conditions have been imposed and n_c represents the total number of incompressibility constraints. As long as the pressure equations are linearly independent, n_c will be equal to the number of pressure equations $\tilde{n}_{eq}$. Additionally it shall be defined the so called constrained ration r by

$$r = \frac{n_{eq}}{n_c} \tag{2.99}$$

Now the interest lies in values of r as n_{es}, the number of elements per side approaches infinity. The conjecture is that r should mimic the behavior of the number of equilibrium equations divided by the number of incompressibility conditions for the governing system of partial differential equations. These are the number of space dimensions n_{sd} and 1, respectively. So in two dimensions, the ideal value of r would be 2. A value of r less than 2 would indicate a tendency to lock. If $r \leq 1$, there are more constraints of the pressure

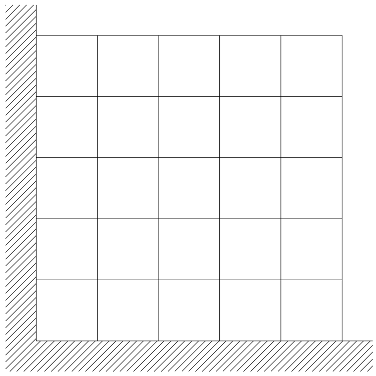


Figure 2.5: Discretized cotter

than there are displacement degrees of freedom available and thus severe locking would be anticipated, such as was seen for the linear displacements and constant pressure triangle in section 2.4. A value of r much greater than 2 indicates that not enough incompressibility conditions are present, so the incompressibility condition may be poorly approximated in some problems.
As a summary it could be said:

$r > 2$	too few incompressibility constraints
$r = 2$	optimal
$r < 2$	too many incompressibility constraints
$r \leq 2$	locking

Examples for elements with discontinuous pressure which come from the u/p-formulation are illustrated in Fig.**2.6**.
For elements with continuous pressure like in the u/p-c formulation occur a grave problem when using identical displacement and pressure interpolations as shown in Fig.**2.7**. Although the constraint ratio is 2 it is known that convergence of these elements is from below and they may exhibit spurious pressure modes. These pressure modes are very bad features which can downgrade the solution of a problem. In the following chapter a new kind of stabilization method shall be deduced which affects the formulation of a mixed finite element with linear displacement and linear pressure interpolations. In this case it will be applied to a triangular element which does not mean that it can not be applied on quadrilateral elements, too.

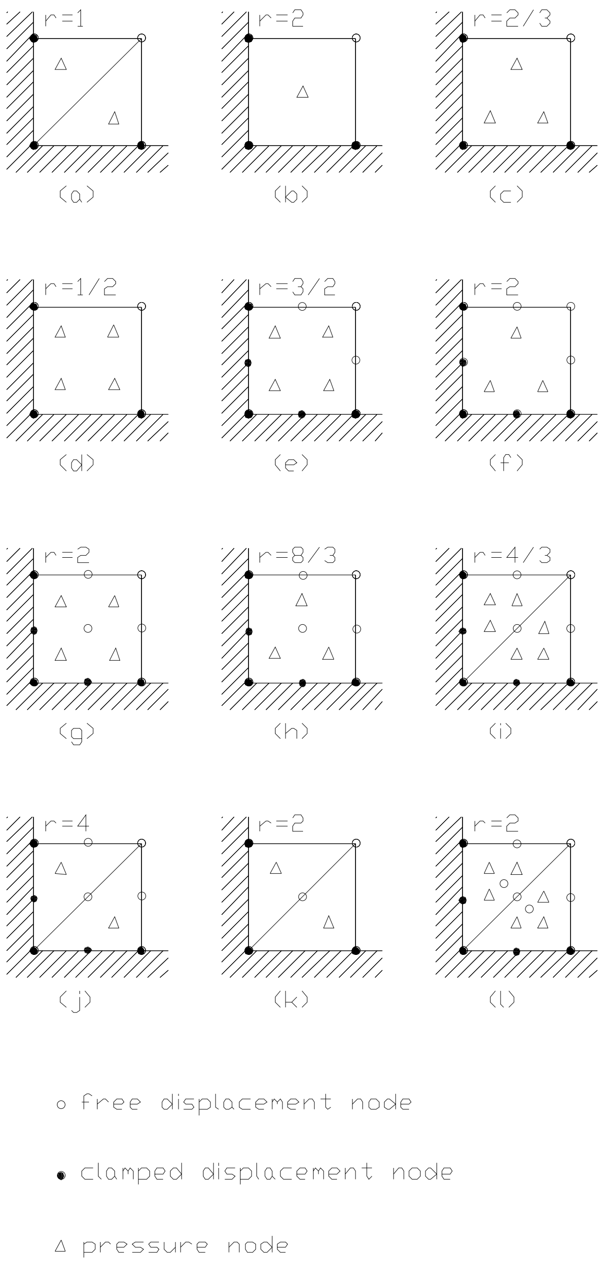


Figure 2.6: Discontinuous pressure field elements

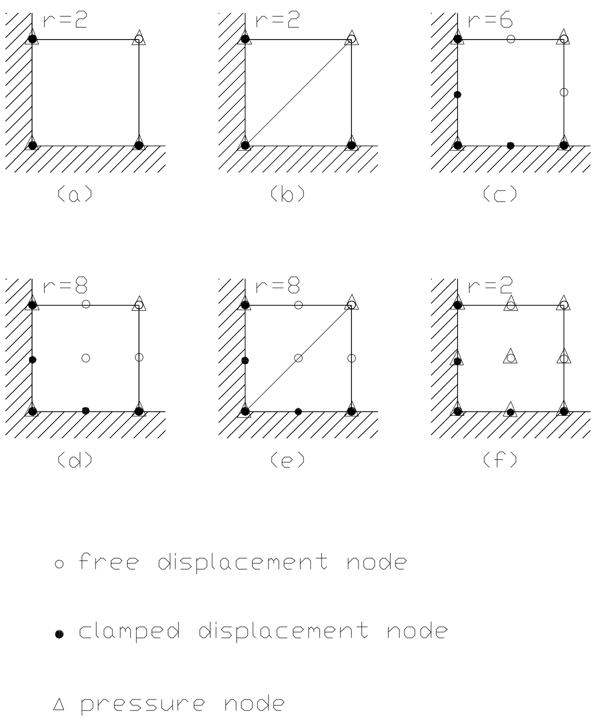


Figure 2.7: Continuous pressure field elements

Chapter 3

A Stabilized Mixed Element Formulation

3.1 Introduction

In this chapter a formulation of triangular and tetrahedral elements shall be developed with linear interpolation functions for the displacements and the pressure field with regard to small deformations. As we have seen in the previous chapter, the possibility to use a linear/linear formulation requires a kind of stabilization method for the element to circumvent the Ladyzhenskaya-Babuška-Brezzi condition. The stabilization technique which shall be used in this work will be deduced from the orthogonal sub-grid scales approach. It should be pronounced that this kind of stabilization method can also be applied to elements of higher order and different geometries.

3.2 The sub-grid-scale approach

The solution of a problem for finite element methods consists in searching in the space of functions with finite dimension $\mathcal{V}_h \subset \mathcal{V}$ an approximation to the solution which has a required accuracy. The basic idea of the sub-grid scale approach is to consider that the continuous displacement field can be split in two components, one coarse and a finer one, corresponding to different scales or levels of resolution. The solution of the continuous problem contains components from both scales. For the solution of the discrete problem to be stable, it is necessary to, somehow, include the effect of both scales in the approximation. The coarse scale can be appropriately solved by a standard finite element interpolation, which however can not solve the finer scale. Nevertheless, the effect of this finer scale can be included, at least locally, to enhance the stability of the pressure in the mixed formulation. But at first this method shall be applied only on the displacement formulation to present it rather clearly. Afterward it will be enhanced on the mixed formulation.

3.2.1 The basic idea

The problem which has been already defined in the previous chapter was:

Find the displacement field $\mathbf{u} : \bar{\Omega} \to \mathbf{R}^{ND}$ so that, by given external forces $\bar{\mathbf{t}} : \partial_t\Omega \to \mathbf{R}^{ND}$, body forces $\mathbf{b} : \Omega \to \mathbf{R}^{ND}$ and also the displacements on the boundary $\partial_u\Omega$, following equations are fulfilled:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega \tag{3.1}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \partial_t\Omega \tag{3.2}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial_u\Omega \tag{3.3}$$

where the stresses $\boldsymbol{\sigma}$ at first only depends via the constitutive equation on the displacements $\mathbf{u}$. To keep track of the derivation of the sub-scales, this problem shall be rewritten in a more abstract form as:
Find $\mathbf{U} \in \mathcal{V}$, with regard to the boundary conditions, so that

$$\mathcal{L}(\mathbf{U}) = \mathcal{F} \tag{3.4}$$

holds, where $\mathcal{L}(\cdot)$ is the differential operator of this problem and $\mathcal{F}$ represents the force vector

$$\mathcal{L}(\mathbf{U}) = -\nabla \cdot \boldsymbol{\sigma}(\mathbf{U}) \tag{3.5}$$

$$\mathcal{F} = \mathbf{f} \tag{3.6}$$

The variational formulation of this problem can be written as

$$\int_\Omega \mathbf{V} \cdot [\mathcal{F} - \mathcal{L}(\mathbf{U})] \, d\Omega = 0\,, \qquad \forall\, \mathbf{V} \in \mathcal{V} \tag{3.7}$$

where the term $\int_\Omega \mathbf{V} \cdot \mathcal{L}(\mathbf{U})$ includes derivations of second order of the displacements $\mathbf{U}$. With the aid of the partial integration the order can be compensated and we get

$$\int_\Omega \mathbf{V} \cdot \mathcal{L}(\mathbf{U}) \, d\Omega = B(\mathbf{U}, \mathbf{V}) - \int_{\partial_t\Omega} \mathbf{V} \cdot [\boldsymbol{\sigma}(\mathbf{U}) \cdot \mathbf{n}] \, d\Gamma \tag{3.8}$$

in which the Green's theorem is used in the first term of the right side $B(\mathbf{U}, \mathbf{V})$. The vector $\mathbf{n}$ is the unit normal exterior to the integration domain and the term $[\boldsymbol{\sigma}(\cdot) \cdot \mathbf{n}]$ stands for the tractions vector.
After inserting this equation in (3.5) we come to

$$B(\mathbf{U},\mathbf{V}) = L(\mathbf{V}) \, , \qquad \forall\, \mathbf{V} \in \mathcal{V} \tag{3.9}$$

with

$$L(\mathbf{V}) = \int\limits_{\Omega} \mathbf{V} \cdot \mathcal{F} \, d\Omega + \int_{\partial_t \Omega} \mathbf{V} \cdot [\boldsymbol{\sigma}(\mathbf{U}) \cdot \mathbf{n}] \, d\Gamma \tag{3.10}$$

To apply the method of sub-grid scales, the displacements have to be enhanced by an additionally displacement field in the way that

$$\mathbf{U} = \mathbf{U}_h + \tilde{\mathbf{U}} \tag{3.11}$$

As the solution of the finite elements is defined in the space referable to the finite elements $\mathbf{U}_h \in \mathcal{V}_h$, the enhanced displacements shall be defined in an equivalent space as $\tilde{\mathbf{U}} \in \tilde{\mathcal{V}}$, the sub-grid scale. Considering the same result for the variations leads to

$$\mathbf{V} = \mathbf{V}_h + \tilde{\mathbf{V}} \tag{3.12}$$

where $\mathbf{V}_h \in \mathcal{V}_h$ and $\tilde{\mathbf{U}} \in \tilde{\mathcal{V}}$. After this we come to the conclusion that the space of the displacement field $\mathbf{U}$ and its variation $\mathbf{V}$ can be composed by the two different spaces so we get

$$\mathcal{V} \simeq \mathcal{V}_h \oplus \tilde{\mathcal{V}} \tag{3.13}$$

It is reasonable to assume that the sub-grid displacements $\tilde{\mathbf{U}}$ will be sufficiently small compared to the finite element solution $\mathbf{U}_h$. They can be viewed as a *high-frequency* perturbation of the finite element field which cannot be resolved in the finite element space $\mathcal{V}_h$. It can also be assumed that $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ vanish on the boundary $\partial\Omega$.
With this conclusions we can rewrite Eq. (3.9) and come to

$$B(\mathbf{U}_h + \tilde{\mathbf{U}}, \mathbf{V}_h) = L(\mathbf{V}_h) \, , \qquad \forall\, \mathbf{V}_h \in \mathcal{V}_h \tag{3.14}$$

$$B(\mathbf{U}_h + \tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = L(\tilde{\mathbf{V}}) \, , \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.15}$$

If the terms $B(\mathbf{U}_h + \tilde{\mathbf{U}}, \mathbf{V}_h)$ and $B(\mathbf{U}_h + \tilde{\mathbf{U}}, \tilde{\mathbf{V}})$ are a linear function of the displacements, we can split these terms to get the following formulation

$$B(\mathbf{U}_h, \mathbf{V}_h) + B(\tilde{\mathbf{U}}, \mathbf{V}_h) = L(\mathbf{V}_h) \, , \qquad \forall\, \mathbf{V}_h \in \mathcal{V}_h \tag{3.16}$$

$$B(\mathbf{U}_h, \tilde{\mathbf{V}}) + B(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = L(\tilde{\mathbf{V}}) \, , \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.17}$$

As we see, the first equation (3.14) is defined in the finite element scale and it solves the balance of momentum including a term depending on the sub-grid scale which can be seen as a kind of stabilization term. The second equation (3.15) is formulated in the sub-grid scale and will be used to find an adequate definition for the enhanced displacement field $\tilde{\mathbf{U}}$. That means, after evaluating the second equation, the solution for $\tilde{\mathbf{U}}$ can be utilized to resolve the first equation. This is the essential idea of the proceeding, nevertheless some important aspects of the application of the method of sub-grid scales have to be emphasized.

To obtain a solution for the enhanced displacements $\tilde{\mathbf{U}}$ from Eq.(3.17) and use it in Eq.(3.16), we have to accomplish some operations on element level which presupposes a notation for each element. As we define $\int_{\Omega'} := \sum_{e=1}^{NE} \int_{\partial\Omega^e}$ and $\int_{\partial\Omega'} := \sum_{e=1}^{NE} \int_{\Omega^e}$, respectively, where NE is the number of elements of the finite element partition. By using once again the differential operator, Eq.(3.17) can be rewritten as

$$\int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{L}(\mathbf{U}_h)\, d\Omega + \int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{L}(\tilde{\mathbf{U}})\, d\Omega + \int_{\partial\Omega'} \tilde{\mathbf{V}} \cdot [\ \boldsymbol{\sigma}(\mathbf{U}_h + \tilde{\mathbf{U}}) \cdot \mathbf{n}\ ]\, d\Gamma = \int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{F}\, d\Omega \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.18}$$

Observe that $\boldsymbol{\sigma}(\mathbf{U}_h + \tilde{\mathbf{U}}) \cdot \mathbf{n}$ represents the exact tractions, assumed to be continuous across interelement boundaries and thus the sum over these boundaries is zero. So we come to the conclusion that

$$\int_{\Omega'} \tilde{\mathbf{V}} \cdot \left[\mathcal{L}\left(\mathbf{U}_h + \tilde{\mathbf{U}}\right)\right] d\Omega = \int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{F}\, d\Omega\,, \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.19}$$

By means of the integration by parts, the second term of the left side in Eq.(3.16) can be defined by

$$B(\tilde{\mathbf{U}}, \mathbf{V}_h) = \int_{\Omega'} \tilde{\mathbf{U}} \cdot \mathcal{L}^*(\mathbf{V}_h)\, d\Omega + \int_{\partial\Omega'} \tilde{\mathbf{U}} \cdot (\boldsymbol{\sigma}(\mathbf{V}_h) \cdot \mathbf{n})\, d\Gamma \tag{3.20}$$

where $\mathcal{L}^*$ is the adjunct of the operator $\mathcal{L}$. Now it can be made a simplification in defining the sub-scales only in the interior of the element volume Ω^e. That means that the integral over the boundary in (3.20) disappears. This is comparable to the method of adding a bubble function to the element which values are zero at the boundary of the element like in the formulation of the MINI-element (see also [1], [17]).

The operator $\mathcal{L}^*$ can be calculated via the integration by parts with the following relation to $\mathcal{L}$ (without regard to the boundary)

$$\int_{\Omega} \mathbf{U} \cdot \mathcal{L}^*(\mathbf{V})\, d\Omega = \int_{\Omega} \mathbf{V} \cdot \mathcal{L}(\mathbf{U})\, d\Omega \tag{3.21}$$

Finally, Eq.(3.16) can be rewritten with the mentioned considerations as

$$B(\mathbf{U}_h, \mathbf{V}_h) + \int_{\Omega'} \tilde{\mathbf{U}} \cdot \mathcal{L}^*(\mathbf{V}_h)\, d\Omega = L(\mathbf{V}_h)\,, \qquad \forall\, \mathbf{V}_h \in \mathcal{V}_h \tag{3.22}$$

and Eq.(3.19) as

$$\int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{L}(\tilde{\mathbf{U}})\, d\Omega = \int_{\Omega'} \tilde{\mathbf{V}} \cdot [\mathcal{F} - \mathcal{L}(\mathbf{U}_h)]\, d\Omega\,, \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.23}$$

The expression (3.23) is more or less the key point to capture the effects of the component $\tilde{\mathbf{U}}$ used to enrich the standard finite element solution $\mathbf{U}_h$. It relates the enhanced displacements $\tilde{\mathbf{U}}$ to the residuum $\mathcal{R}_h$ of the differential approximation for the finite elements, represented by $\mathcal{R}_h = \mathcal{F} - \mathcal{L}(\mathbf{U}_h)$. Important is that it is not obliged to fulfill this equation pointwise, but only over the integral.
Up to now, the presentation of the method of the sub-grid scales is general and the current aim consists in obtaining a reliable approximation for $\tilde{\mathbf{U}} \in \tilde{\mathcal{V}}$ at a moderate computational effort. There already exist diverse possibilities to define the approximation of the sub-grid scales such as the definition of special functions for the enhanced displacement field $\tilde{\mathbf{U}}$ in a manner similar to the enhanced assumed strain method (EAS). In principle, the finer enhanced space could be any complementary space to the finite element space. The method which is used in this work shall be developed in the next section and based on considerations of R. Codina [8].

3.2.2 The orthogonal sub-scales

Codina proposed for the sub-grid scale as a reasonable possibility the orthogonal space to the finite element space. This definition gives the origin for precise and clever formulation named as the method of orthogonal sub-scales. With regard to this method the space of the enhanced elements can be approximated as

$$\tilde{\mathcal{V}} \approx \mathcal{V}_h^{\perp} \tag{3.24}$$

where $\mathcal{V}_h^{\perp}$ represents the orthogonal space of the finite element space $\mathbf{V}_h$. Hence, the enhanced elements $\tilde{\mathbf{U}}$ are now defined in this orthogonal space of $\mathbf{V}_h$

$$\tilde{\mathbf{U}} \in \mathcal{V}_h^{\perp} \tag{3.25}$$

Then Eq. (3.23) comes to

$$\int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{L}(\tilde{\mathbf{U}})\, d\Omega = \int_{\Omega'} \tilde{\mathbf{V}} \cdot [\mathcal{F} - \mathcal{L}(\mathbf{U}_h)]\, d\Omega\,, \qquad \forall\, \tilde{\mathbf{V}} \in \mathcal{V}_h^{\perp} \tag{3.26}$$

This equation represents the projection of the residuum of the differential equation $\mathcal{R}_h = \mathcal{F} - \mathcal{L}(\mathbf{U}_h)$ on the orthogonal space of the finite element space $\mathbf{V}_h^{\perp}$. Even though with this equation will not be found an exact value for the enhanced elements $\tilde{\mathbf{U}}$, it is at least possible to approximate the linear operator $\mathcal{L}$ by a matrix of parameters like

$$\mathcal{L}(\tilde{\mathbf{U}}) \approx \boldsymbol{\tau}_e^{-1}\tilde{\mathbf{U}} \tag{3.27}$$

If we additionally call $P_h\left(\cdot\right)$ the L_2 orthogonal projection on the finite element space and $P_h^{\perp}\left(\cdot\right) = \left(\cdot\right) - P_h\left(\cdot\right)$ the projection on the space $\mathcal{V}_h^{\perp}$, orthogonal to the finite element space, the enhanced elements $\tilde{\mathbf{U}}$ can be defined as

$$\tilde{\mathbf{U}} = \boldsymbol{\tau}_e P_h^{\perp}\left([\mathcal{F} - \mathcal{L}(\mathbf{U}_h)]\right) \qquad \text{in } \Omega_e \tag{3.28}$$

where $\boldsymbol{\tau}_e$ is the matrix of the stabilization parameters in each element. Codina developed in the context of the equations of Oseen (see [9]) a deduction for the values of the stabilization parameters by means of a Fourier series expansion, based for the first time on physical reasons. The values of the parameters of the matrix $\boldsymbol{\tau}_e$ depend on the coefficients of the differential equation, as material or geometry parameters, and establish via the comparison of the L_2-norm of the residuum and the $\tilde{\mathbf{U}}$ over each element.

3.2.3 Aspects to the stabilization of the mixed formulation in incompressibility

As we have seen in a previous chapter, the triangular and tetrahedral elements with the combination of linear interpolations for the displacements and the pressure do not pass the LBB-condition (see chapter 2.5). However, with the aid of stabilization techniques like the method of orthogonal sub-scales, it is possible to circumvent the LBB-condition and to present adequate results.
To develop the method of orthogonal sub-scales for mixed formulations we start once again from Eq.(3.4) where

$$\mathcal{L}(\mathbf{U}) = \begin{bmatrix} -\nabla p - 2\mu\nabla\cdot\operatorname{dev}(\nabla^s\mathbf{u}) \\ \nabla\cdot\mathbf{u} - \frac{1}{K}p \end{bmatrix} \tag{3.29}$$

is the differential operator deduced from 2.2 and

$$\mathcal{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \tag{3.30}$$

contains the volumetric forces. The vector $\mathbf{U} := [\mathbf{u}\ p]^T$ now contains the variables of the mixed formulation, as well the displacements $\mathbf{u} \in \mathcal{V}$ as the pressure $p \in \mathcal{Q}$. The space to which belongs $\mathbf{U}$ can be defined as $\mathcal{W} = \mathcal{V} \times \mathcal{Q}$. In the same manner, the terms of the weak form in (3.14) and (3.15) are

$$B(\mathbf{U}, \mathbf{V}) = a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u}) \tag{3.31}$$

$$L(\mathbf{V}) = (\mathbf{f}, \mathbf{w}) + (\bar{\mathbf{t}}, \mathbf{v})_{\Gamma_t} \tag{3.32}$$

with

$$a(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \nabla^s \mathbf{v} : \nabla^s \mathbf{u} \, d\Omega \tag{3.33}$$

$$b(p, \mathbf{v}) = \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega \tag{3.34}$$

$$b(q, \mathbf{u}) = \int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega \tag{3.35}$$

With the use of the sub-grid scale approach the vector $\mathbf{U} = \mathbf{U}_h + \tilde{\mathbf{U}}$ is generally defined as

$$\underbrace{\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix}}_{\mathbf{U}} = \underbrace{\begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix}}_{\mathbf{U}_h} + \underbrace{\begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{p} \end{bmatrix}}_{\tilde{\mathbf{U}}} \tag{3.36}$$

We see that there is also a possibility to enrich in addition to the displacements also the pressure by the sub-grid scales. But to circumvent the LBB-condition and as well to get adequate results for the solution, as we will see later on, it is not necessary to take additionally an enrichment of the pressure into account. What does not mean that it would not be reasonable. The effects depending on the enrichment of the pressure field are not yet investigated and there could be the possibility that it improves once more the results of the solution. However it should be obvious that the additional enrichment of the pressure field leads to a higher cost of computing time. Thus a more intensive investigation in this direction could be of a high value. The simplification of the method in this work leads finally to the conclusion that the enhanced pressure part becomes $\tilde{p} = 0$. To reconsider the way of defining the stabilization matrix $\boldsymbol{\tau}_e$ we should refer once again to Codina. In [9] considerations were established to define the stabilization matrix $\boldsymbol{\tau}_e$ in which the corresponding parameters were calculated in the scope of the problem of Oseen via the Fourier series expansion. In particular for the problem of Stoke's flow [8] the matrix of the stabilization parameters (3.28) is defined as a diagonal matrix in the way that

$$\boldsymbol{\tau}_e = \tau_e \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1_{ND} & \\ & & & 0 \end{pmatrix} \tag{3.37}$$

with

$$\tau_e = \frac{ch^2}{2\mu} \tag{3.38}$$

where h is the characteristic diameter of the mesh, μ the shear modulus and c a numerical constant that depends only on the element type. As a consequence of the previous considerations about the enrichment of the vector $\mathbf{U}$, the last term of the diagonal matrix in (3.37) which stands for the pressure is zero.
The definition of the stabilization parameter τ_e in (3.38) could also alter from the one of Codina in [9]. There are as well other considerations to be found in the literature how to define these stabilization parameters (see e.g. [11]). From the algorithmic point of view τ_e is a robust parameter which is not affected as much by the value of the constant c if the constant is defined in a reasonable order of magnitude. The numerical experiences have shown that for linear elements the value of c is located in the dimension of 0.5 to 0.75.
Having specified the manner of the stabilization, in the next section shall be deduced the formulation of stabilized mixed elements which is the main subject of this work.

3.3 Formulation in the elastic case

It can be proposed a mixed formulation of an elastic problem in which the incompressibility represents the limitation of the general problem of compressibility. In this formulation will be defined the pressure p as an independent variable additionally to the displacements $\mathbf{u}$.

3.3.1 Constitutive model

The constitutive comportment in infinite elasticity can be expressed by means of the following couple of equations

$$\boldsymbol{\sigma} = p\mathbf{1} + 2\mu \operatorname{dev}[\nabla^s \mathbf{u}] \tag{3.39}$$
$$p = K\varepsilon_v \qquad \text{with } \varepsilon_v = \nabla \cdot \mathbf{u} \tag{3.40}$$

where ε_v and $\operatorname{dev}[\nabla^s \mathbf{u}]$ are the volumetric and deviatoric components of the deformations, respectively. K is the bulk modulus already defined in (2.8) and now rewritten in subject to the LAMÉ parameters as

$$K = \lambda + \frac{2}{3}\mu \tag{3.41}$$

As it can be observed from the constitutive equations (3.39) and (3.40), the decomposition of the deformations in their volumetric and deviatoric parts leads to a decoupled

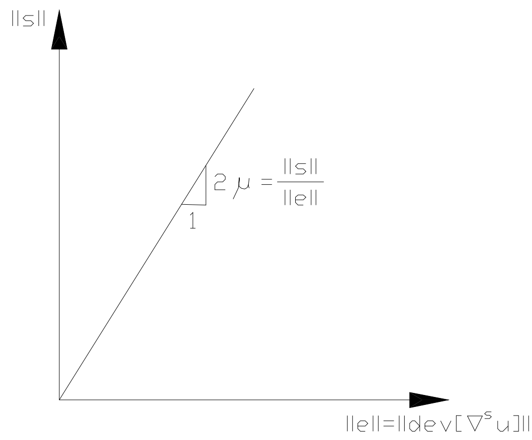


Figure 3.1: Graph of the deviatoric stresses against the deviatoric deformations in case of elasticity

expression of the stress tensor, too. The expression of the elastic constitutive tensor $\mathbf{C}$ associated to the displacements in volumetric and deviatoric parts is

$$C_{ijkl} = \underbrace{K\,\delta_{ij}\delta_{kl}}_{\mathbf{C}_{\text{vol}}} + \underbrace{\mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\,\delta_{ij}\delta_{kl}\right)}_{\mathbf{C}_{\text{dev}}} \tag{3.42}$$

where $\mathbf{C}_{\text{vol}}$ and $\mathbf{C}_{\text{dev}}$ are the volumetric and deviatoric parts of the constitutive tensor $\mathbf{C}$, respectively.
If we now observe that the deviatoric stresses are calculated by the deviatoric part of the constitutive tensor $\mathbf{C}_{\text{dev}}$ times the the total deformations and by understanding that via the contraction of both only the deviatoric part of the deformations remains, it can be written that

$$\mathbf{s} = \text{dev}[\boldsymbol{\sigma}] = \mathbf{C}_{\text{dev}} : \nabla^s\mathbf{u} = 2\mu\,\text{dev}[\nabla^s\mathbf{u}] \tag{3.43}$$

The consequence is that, as shown in Fig.**3.1**, the relation between the deviatoric stresses and the deviatoric strains is linear and depends only on the shear modulus:

$$2\mu = \frac{\|\mathbf{s}\|}{\|\text{dev}[\nabla^s\mathbf{u}]\|} \tag{3.44}$$

3.3.2 Formulation in the continuum

3.3.2.1 The strong form

The previous constitutive equations (3.39) and (3.40), the equilibrium condition (2.17), the additional equation for the pressure (2.18) and the boundary conditions (2.19) and

(2.20) define the problem of the strong form. Following the abstract nomenclature of the chapter 3.2, the linear elastic problem can be expressed as:
Given are the prescribed values of the external forces $\bar{\mathbf{t}} : \partial_t\Omega \rightarrow \mathbf{R}^N D$, the displacements on the boundary $\partial_u\Omega$ and the body forces $\mathbf{b} : \Omega \rightarrow \mathbf{R}^N D$; find then $\mathbf{U} := [\mathbf{u}, p]^T \in \mathcal{W} = \mathcal{V} \times \mathcal{Q}$, so that

$$\mathcal{L}(\mathbf{U}) = \mathcal{F} \tag{3.45}$$

where

$$\mathcal{L}(\mathbf{U}) = \begin{bmatrix} -\nabla p - 2\mu\nabla \cdot \mathrm{dev}(\nabla^s \mathbf{u}) \\ \nabla \cdot \mathbf{u} - \frac{1}{K} p \end{bmatrix} \tag{3.46}$$

$$\mathcal{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \tag{3.47}$$

together with the boundary conditions

$$\mathbf{u} = \mathbf{0} \qquad \partial_u\Omega \tag{3.48}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \qquad \partial_t\Omega \tag{3.49}$$

It is said that the equations of this formulation in the dominion are

$$\nabla p + 2\mu\, \nabla \cdot \mathrm{dev}[\nabla^s \mathbf{u}] + \mathbf{b} = \mathbf{0} \qquad \text{in } \Omega \tag{3.50}$$

$$\nabla \cdot \mathbf{u} - \frac{1}{K} p = 0 \qquad \text{in } \Omega \tag{3.51}$$

where in the equilibrium condition (3.50) has changed the expression of the stresses $\boldsymbol{\sigma}$ given as a function of $\mathbf{u}$ and p in case of the constitutive relation in (3.39).
As we already mentioned, this formulation holds as well in case of compressibility as in case of incompressibility. And it is also obvious that for $K \rightarrow \infty$ the equation (3.51) has to be changed into

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \tag{3.52}$$

The adjunct $\mathcal{L}^*(\cdot)$ of the differential operator $\mathcal{L}(\cdot)$ in Eq.(3.46) can be calculated via the Eq.(3.21). With the integration by parts this can be expressed as

$$\mathcal{L}^*(\mathbf{V}) = \begin{bmatrix} -\nabla p - 2\mu\, \nabla \cdot \mathrm{dev}(\nabla^s \mathbf{v}) \\ \nabla \cdot \mathbf{v} - \frac{1}{K} p \end{bmatrix} \tag{3.53}$$

3.3.2.2 The weak form

The variational or weak form of the problem is defined in that case as

Given are the prescribed values of the external forces $\bar{\mathbf{t}} : \partial_t\Omega \rightarrow \mathbf{R}^N D$, the displacements on the boundary $\partial_u\Omega$ and the body forces $\mathbf{b} : \Omega \rightarrow \mathbf{R}^N D$; find then $\mathbf{U} := [\mathbf{u}, p]^T \in \mathcal{W} = \mathcal{V} \times \mathcal{Q}$, so that

$$B(\mathbf{U}, \mathbf{V}) = L(\mathbf{V}) \;, \qquad \forall\, \mathbf{V} \in \mathcal{W} \tag{3.54}$$

where $\mathbf{V} \in \mathcal{W}$ is the vector of the variations of $\mathbf{U}$ and

$$\begin{aligned} B(\mathbf{U}, \mathbf{V}) &:= \langle \nabla^s \mathbf{v}, 2\mu\, \mathrm{dev}(\nabla^s \mathbf{u}) \rangle + \langle \nabla \cdot \mathbf{v}, p \rangle + \langle q, \nabla \cdot \mathbf{u} \rangle - \langle q, \frac{1}{K} p \rangle & (3.55) \\ L(\mathbf{V}) &:= \langle \mathbf{v}, \mathbf{b} \rangle + \langle \mathbf{v}, \bar{\mathbf{t}} \rangle_{\partial_t} & (3.56) \end{aligned}$$

This weak formulation can be separated into a couple of two equations associated to the variations of the displacements and the pressure, respectively

$$\begin{aligned} \langle \nabla^s \mathbf{v}, 2\mu\, \mathrm{dev}(\nabla^s \mathbf{u}) \rangle + \langle \nabla \cdot \mathbf{v}, p \rangle &= l(\mathbf{v}) \;, \qquad \forall\, \mathbf{v} \in \mathcal{V}_0 & (3.57) \\ \langle q, (\nabla \cdot \mathbf{u} - \frac{1}{K} p) \rangle &= 0 \;, \qquad \forall\, p \in \mathcal{Q} & (3.58) \end{aligned}$$

where, as before, $l(\mathbf{v}) = \langle \mathbf{v}, \mathbf{b} \rangle + \langle \mathbf{v}, \bar{\mathbf{t}} \rangle_{\partial_t\Omega}$ is the term of the external forces. This pair of equations can be expressed as well in a vector notation like

$$\mathbf{R}(\mathbf{U}, \mathbf{V}) = \mathbf{0} \qquad \forall\, \mathbf{V} \in \mathcal{V}_0 \tag{3.59}$$

where

$$\mathbf{R}(\mathbf{U}, \mathbf{V}) = \begin{bmatrix} \langle \nabla^s \mathbf{v}, 2\mu\, \mathrm{dev}(\nabla^s \mathbf{u}) \rangle + \langle \nabla \cdot \mathbf{v}, p \rangle - l(\mathbf{v}) \\ \langle q, (\nabla \cdot \mathbf{u} - \frac{1}{K} p) \rangle \end{bmatrix} \tag{3.60}$$

This notation is chosen because it permits more clearness for the upcoming deductions. The space in which exists $\mathbf{U}$ is defined as $\mathcal{W} = \mathcal{V} \times \mathcal{Q}$ where $\mathcal{V}$ and $\mathcal{Q}$ are the spaces for the displacements and the pressure, respectively

$$\begin{aligned} \mathcal{V} &= \left\{ \mathbf{v} \,|\, \mathbf{v} \in \mathcal{H}^1(\Omega) \;,\; \mathbf{v} = \mathbf{0} \quad in\ \partial_u\Omega \right\} & (3.61) \\ \mathcal{Q} &= \left\{ q \,|\, q \in \mathcal{L}^2(\Omega) \;,\; \int_\Omega q \, d\Omega = 0 \right\} & (3.62) \end{aligned}$$

with

$$\mathcal{L}^2(\Omega) = \left\{ \phi \,|\, \int_\Omega \phi^2 d\Omega < \infty \right\} \tag{3.63}$$

$$\mathcal{H}^1(\Omega) = \left\{ \phi \,|\, \phi \in \mathcal{L}^2(\Omega)\ ,\ \phi' \in \mathcal{L}^2(\Omega) \right\} \tag{3.64}$$

The stabilization of this mixed formulation depends on the compliance of the LBB-condition. This condition leads to the necessity of using different interpolations for the displacement field $\mathbf{u}$ and the pressure field p in the finite element formulation. However, the already appointed stabilization technique of the sub-grid scale approach copes the circumventing of the LBB-condition, although the same order of interpolations are used for the displacements and the pressure which shall be developed in the following section.

3.3.3 Extension on multiscales

As preassigned, the spaces of the finite element functions for $\mathbf{u}$ and p shall be chosen as linear and continuous. So the spaces of the finite element approximations are defined as $\mathbf{U}_h \in \mathcal{W}_h = \mathcal{V}_h \times \mathcal{Q} \subset [H^1(\Omega)]^{ND+1}$.
By using the method of orthogonal sub-grid scales (OSGS), proposed by R. Codina [8], whereupon the complementary space $\tilde{\mathcal{W}}$ is specified as the orthogonal space of the finite element space. Agreeing with this, we come to $\tilde{\mathbf{U}} \in \tilde{\mathcal{W}}$ where we can approximate $\tilde{\mathcal{W}} \approx \mathcal{W}_h^{\perp}$. In the same way as in section 3.2.3, the refinement of the solution $\mathbf{U} = \mathbf{U}_h \tilde{\mathbf{U}}$ is used to enrich the displacement field with the expectation to improve the stabilization properties of the mixed finite element formulation. Therefore we get for the vector of unknowns

$$\begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{u}_h \\ p_h \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{u}} \\ 0 \end{bmatrix} \tag{3.65}$$

$$\begin{bmatrix} \mathbf{v} \\ q \end{bmatrix} = \begin{bmatrix} \mathbf{v}_h \\ q_h \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{v}} \\ 0 \end{bmatrix} \tag{3.66}$$

where we disregard once again the enrichment of the pressure field. The continuum problem transforms into:

Find $\mathbf{U}_h \in \mathcal{W}_h$ and $\tilde{\mathbf{U}} \in \tilde{W}$ so that

$$B(\mathbf{U}_h + \tilde{\mathbf{U}}, \mathbf{V}_h) = L(\mathbf{V}_h)\ , \qquad \forall\, \mathbf{V}_h \in \mathcal{W}_h \tag{3.67}$$

$$\int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{L}(\mathbf{U}_h + \tilde{\mathbf{U}})\, d\Omega = \int_{\Omega'} \tilde{\mathbf{V}} \cdot \mathcal{F}\, d\Omega\ , \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{W} \tag{3.68}$$

which correspond respectively to the expressions (3.16) and (3.17) with regard to the idea formulated in Eq.3.2.1. The equation (3.67), defined in the finite element space, is equivalent to the couple of two equations with dependency to the variations of the displacements and the pressure, respectively

$$\langle \nabla^s \mathbf{v}_h, 2\mu\, \mathrm{dev}[\nabla^s(\mathbf{u}_h + \tilde{\mathbf{u}})]\rangle_{\Omega'} + \langle \nabla \cdot \mathbf{v}_h, p_h\rangle_{\Omega'} = l(\mathbf{v}_h) \quad (3.69)$$

$$\langle q_h, (\nabla \cdot (\mathbf{u}_h + \tilde{\mathbf{u}}) - \frac{1}{K} p)\rangle_{\Omega'} = 0 \quad (3.70)$$

with $l(\mathbf{v}_h) = \langle \mathbf{v}_h, \mathbf{f}\rangle + \langle \mathbf{v}_h, \bar{\mathbf{t}}\rangle_{\partial_t \Omega}$. In the compact notation it can be written as

$$\mathbf{R}(\mathbf{U}, \mathbf{V}_h) = \mathbf{0}\,, \qquad \forall\, \mathbf{V}_h \in \mathcal{W}_h \quad (3.71)$$

In (3.69) and (3.70) the gradient and the divergence are applied on $(\mathbf{u}_h + \tilde{\mathbf{u}})$. Both operations are linear so that the term on the left side can be divided in two terms and the equation can be rewritten as

$$\mathbf{R}(\mathbf{U}, \mathbf{V}_h) = \mathbf{R}(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \mathbf{0}\,, \qquad \forall\, \mathbf{V}_h \in \mathcal{W}_h \quad (3.72)$$

where $\mathbf{R}(\mathbf{U}_h, \mathbf{V}_h)$ and $\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h)$ are respectively

$$\mathbf{R}(\mathbf{U}_h, \mathbf{V}_h) = \begin{bmatrix} \langle \nabla^s \mathbf{v}_h, 2\mu\, \mathrm{dev}(\nabla^s \mathbf{u}_h)\rangle_{\Omega'} + \langle \nabla \cdot \mathbf{v}_h, p_h\rangle_{\Omega'} - l(\mathbf{v}_h) \\ \langle q_h, (\nabla \cdot \mathbf{u}_h - \frac{1}{K} p_h)\rangle_{\Omega'} \end{bmatrix} \quad (3.73)$$

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} \langle \nabla^s \mathbf{v}_h, 2\mu\, \mathrm{dev}(\nabla^s \tilde{\mathbf{u}})\rangle_{\Omega'} \\ \langle q_h, \nabla \cdot \tilde{\mathbf{u}}\rangle_{\Omega'} \end{bmatrix} \quad (3.74)$$

The stabilization effect of the sub-scales appears in the terms contained in Eq.(3.74). In the same manner as in Eq.(3.22) the terms of (3.74) are evaluated via integration by parts. So we obtain an equivalent definition as in (3.74) but now expressed by the derivatives of the variations instead of the derivatives of $\tilde{\mathbf{u}}$

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} -\langle \tilde{\mathbf{u}}, \nabla \cdot [2\mu\, \mathrm{dev}(\nabla^s \mathbf{v}_h)]\rangle_{\Omega'} \\ -\langle \tilde{\mathbf{u}}, \nabla q_h\rangle_{\Omega'} \end{bmatrix} \quad (3.75)$$

In the same way the weak form in Eq.(3.68) can be put into a compact form so that we get

$$\mathbf{R}(\mathbf{U}_h + \tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = \mathbf{0}\,, \qquad \forall\, \tilde{\mathbf{V}} \in \mathcal{V}_0 \quad (3.76)$$

with

$$\mathbf{R}(\mathbf{U}_h + \tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = \begin{bmatrix} \langle \tilde{\mathbf{v}}, \nabla \cdot 2\mu\, \mathrm{dev}[\nabla^s(\mathbf{u}_h + \tilde{\mathbf{u}})]\rangle_{\Omega'} + \langle \tilde{\mathbf{v}}, \nabla p_h\rangle_{\Omega'} + \langle \tilde{\mathbf{v}}, \mathbf{b}\rangle_{\Omega'} \\ 0 \end{bmatrix} \quad (3.77)$$

It can be observed that there is no associated equation to the variations of the pressure in the sub-grid scale space because we only have taken the displacements into account. After these expressions and by considering the linearity of the divergence in (3.77) we get

$$-\langle \tilde{\mathbf{v}}, \nabla \cdot [2\mu \operatorname{dev}(\nabla^s \tilde{\mathbf{u}})] \rangle_{\Omega'} = \langle \tilde{v}, \nabla p_h + \nabla \cdot 2\mu \operatorname{dev}(\nabla^s \mathbf{u}_h) + \mathbf{b} \rangle_{\Omega'} \tag{3.78}$$

which is corresponding to Eq.(3.26). This expression connect, via the projection on the orthogonal space of the finite element space, the sub-scale $\tilde{\mathbf{u}}$ with the residuum of the equilibrium condition $\mathbf{R}_h = \nabla p_h + \nabla \cdot (2\mu \operatorname{dev}[\nabla^s \mathbf{u}_h]) + \mathbf{f}$. The sub-grid scale approach developed in 3.2, expressed in Eq.(3.28), consists by taking into account that the sub-grid scale is proportional to the projection on the orthogonal space of the finite element solutions of the residuum of the equilibrium conditions. This proportionality is expressed in every element via the matrix of the stabilization parameters (3.37). In this case the approximation results in

$$\tilde{\mathbf{u}} = \tau_e P_h^{\perp} \left(\nabla p_h + \nabla \cdot [2\mu \operatorname{dev}(\nabla^s \mathbf{u}_h)] + \mathbf{b} \right) \qquad in\ \Omega_e \tag{3.79}$$

where the stabilization parameter is defined as in Eq.(3.38). By using only elements with linear interpolations, the second derivations of the finite element functions like $\nabla \cdot (\nabla^s \mathbf{u})$ vanish. Furthermore it considers that the body forces $\mathbf{b}$ are approximated through elements of the finite element space so that $P_h^{\perp}(\mathbf{b}) = 0$. Hence the sub-grid scale approach can finally written as

$$\tilde{\mathbf{u}} = \tau_e [\nabla p_h - P_h(\nabla p_h)] \tag{3.80}$$

where the orthogonal projection of a variable is calculated as $P_h^{\perp}(\cdot) = (\cdot) - P_h(\cdot)$, and $(\cdot)$ is the identity of the variable.
It is only missing to insert the obtained approximation for $\tilde{\mathbf{u}}$ into the equations corresponding to the finite element spaces (3.69) and (3.70). By taking into account that linear functions are used and by inserting the definition for the sub-grid scales $\tilde{\mathbf{u}}$ of (3.80) into Eq.(3.75) we get at last the stabilization term

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} 0 \\ -\sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h \cdot (\nabla p_h - \mathbf{\Pi}_h) \rangle_{\Omega_e} \end{bmatrix} \tag{3.81}$$

with $\mathbf{\Pi}_h = P_h(\nabla p_h)$, the projection of the pressure gradient on the finite element space $\mathcal{W}_h$, which is defined as an additional nodal variable. This variable is calculated by the use of the relation between the pressure gradient and its projection $\mathbf{\Pi}_h$ in the way that

$$\langle \nabla p_h, \boldsymbol{\eta}_h \rangle = \langle \mathbf{\Pi}_h, \boldsymbol{\eta}_h \rangle \ , \qquad \forall\, \boldsymbol{\eta}_h \in \mathcal{V}_h \tag{3.82}$$

Finally, as a result of the developed proceeding, the proposed stabilized formulation to treat the elastic problem via linear triangular or tetrahedral elements is:

$$\langle \nabla^s \mathbf{v}_h, 2\mu \, \mathrm{dev}(\nabla^s \mathbf{u}_h) \rangle + \langle \nabla \cdot \mathbf{v}_h, p_h \rangle - l(\mathbf{v}_h) = 0 , \qquad \forall \, \mathbf{v}_h \in \mathcal{V}_h \tag{3.83}$$

$$\langle q_h, \nabla \cdot \mathbf{u}_h - \frac{1}{K} p_h \rangle - \sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla p_h - \mathbf{\Pi}_h \rangle_{\Omega_e} = 0 , \qquad \forall \, q_h \in \mathcal{Q}_h \tag{3.84}$$

$$\langle \nabla p_h, \boldsymbol{\eta}_h \rangle - \langle \mathbf{\Pi}_h, \boldsymbol{\eta}_h \rangle = 0 , \qquad \forall \, \boldsymbol{\eta}_h \in \mathcal{V}_h \tag{3.85}$$

It can be observed that the approach of the problem via the method of sub-grid scales leads to a system which includes a stabilization term in the equation of the volumetric deformation. This term involves an additional variable, the projection of the pressure gradient. The term is a function of the difference between the pressure gradient, which is discontinuous on the element level, and its correspondent continuous or smoothed variable, its projection as $\sum_{e=1}^{NE} \tau_e \langle \nabla q_h, \nabla p_h - \mathbf{\Pi}_h \rangle_{\Omega^e}$. This implies the finer is the finite element mesh the lesser is the effect of the stabilization term on the equations. The orthogonal sub-grid scale approach (OSGS) can be compared to the stabilization method GLS (Galerkin Least Squares), applied for example by O. Klaas [15] in problems of solid mechanics.
In this formulation the stabilization term for linear elements is $\sum_{e=1}^{NE} \tau_e \langle \nabla q_h, \nabla p_h \rangle_{\Omega^e}$. The linear stabilized elements for the GLS-method could also be obtained in the scope of the sub-grid scale method. This alternative solution differs from the orthogonal sub-grid scales approach because it does not consider the concept of projecting the residuum of the finite element equation on the orthogonal space of the finite element space but it states that the value of the sub-grid scale is defined by the proper residuum. This approach, also investigated by Codina [8], is named as the algebraic sub-grid scale approach (ASGS).
Outside of the scope of the orthogonal sub-grid scale method (OSGS), neither it's possible to deduce nor to justify the term corresponding to the projection of the pressure gradient in the stabilization term, but inside of the OSGS-formulation this term appears as in a natural manner. The physical meaning of the stabilization term is expained in the article of Codina [10] and refers to the context of fluid mechanics but can be deduced as well for the domain of solid mechanics.
This fact shows a great advantage of the OSGS-method over the GLS-method. Because the exact solution of the differential equations of the problem satisfies the equations of the stabilized formulation in case of the OSGS-method for linear elements by virtue of the projection term of the pressure gradient. The linear elements of the GLS-method does not fulfill this condition which is stricter than the usual definition of consistency, the stabilization term has no physical meaning. This involves that the OSGS-method is preciser and robuster than the GLS-method. This means that the proposed stabilized elements corresponding to the OSGS-method exhibit lesser sensibility with respect to the variation of the physical parameters.
By dint of some simplifications of the algorithm, one gains from the equations (3.83)-(3.85) a method with low costs in computation and good properties of the stabilization likewise in non-linear problems as seen later on.

3.3.4 Matrix formulation

The matrix expression of the associated algebraic problem can be written as:
Find $[\mathbf{U}\ \mathbf{P}\ \mathbf{\Pi}]^T$, so that

$$\begin{bmatrix} \mathbf{K}_{\text{dev}} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^T & (-\frac{1}{K}\mathbf{M}_p - \tau\mathbf{L}) & \tau\mathbf{G}^T \\ \mathbf{0} & \tau\mathbf{G} & -\tau\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \\ \mathbf{\Pi} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{3.86}$$

where $[\mathbf{U}\ \mathbf{P}\ \mathbf{\Pi}]^T$ is the vector of nodal unknowns, the displacements $\mathbf{U}$, the pressure $\mathbf{P}$ and the projection of the pressure gradient $\mathbf{\Pi}$, respectively.
The elemental submatrices corresponding to the nodes A and B are:

$$\left[\ \mathbf{K}^{AB}_{\text{dev}}\ \right]_e = \left[\ \int_{\Omega_e} \mathbf{B}^T_A\, \mathbf{D}\, \mathbf{B}_B\, d\Omega\ \right]_e \tag{3.87}$$

$$\left[\ \mathbf{G}^{AB}\ \right]_e = \left[\ \int_{\Omega_e} [\nabla N^A]\, N^B_p\, d\Omega\ \right]_e \tag{3.88}$$

$$\left[\ \mathbf{L}^{AB}\ \right]_e = \left[\ \int_{\Omega_e} [\nabla N^A]^T\, [\nabla N^B]\, d\Omega\ \right]_e \tag{3.89}$$

$$\left[\ M^{AB}_p\ \right]_e = \left[\ \int_{\Omega_e} N^A\, N^B\, d\Omega\ \right]_e \tag{3.90}$$

$$\left[\ \mathbf{M}^{AB}\ \right]_e = \left[\ \left(\int_{\Omega_e} N^A\, N^B\, d\Omega\right) \delta_{ij}\ \right]_e, \qquad i,j = 1,2,3 \tag{3.91}$$

$$[\,\mathbf{f}\,]_e = \left[\ \int_{\Omega_e} [\mathbf{N}^A]^T\, \mathbf{b}\, d\Omega + \int_{\partial_t\Omega_e} [\mathbf{N}^A]^T\, \bar{\mathbf{t}}\, d\Gamma\ \right]_e \tag{3.92}$$

where $\mathbf{D}$ is the correspondent matrix to the deviatoric component $\mathbf{C}_{\text{dev}}$ of the constitutive tensor. The matrix $\mathbf{G}$ represents the gradient operator and $\mathbf{G}^T$ the divergence operator. The matrix $\mathbf{L}$ stands for the Laplacian operator which comes from the term $\langle \nabla q_h, \nabla p_h \rangle$ and as the result of the product of the divergence operator and the gradient operator. The matrices $\mathbf{M}$ and $\mathbf{M}_p$ are the mass matrices associated to the displacements and the pressure, respectively.
The third equation of (3.86) corresponds to (3.85) and can be interpreted as a smoothing of the pressure gradient which is discontinuous between elements if linear interpolations are used. By converting this equation, we can express the projection of the pressure gradient $\mathbf{\Pi}$ as a function of the pressure $\mathbf{P}$ itself in the way that

$$\mathbf{\Pi} = \mathbf{M}^{-1}\mathbf{G}\mathbf{P} \tag{3.93}$$

If we now insert this expression into the second equation of (3.86) the projection in a formal manner so that we get only a formulation which depends only on the displacements $\mathbf{U}$ and the pressure $\mathbf{P}$

$$\begin{bmatrix} \mathbf{K}_{\text{dev}} & \mathbf{G} \\ \mathbf{G}^T & -\frac{1}{K}\mathbf{M}_p\mathbf{P} - \tau(\mathbf{L} - \mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}) \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{3.94}$$

In the proposed stabilized formulation the equation of the volumetric deformation is

$$\mathbf{G}^T\mathbf{U} - \frac{1}{K}\mathbf{M}_p\mathbf{P} - \tau\left(\mathbf{L} - \mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}\right)\mathbf{P} = \mathbf{0} \tag{3.95}$$

In case of incompressibility, if $K \to 0$, the term $-\frac{1}{K}\mathbf{M}_p$ goes as well to zero and the resulting equation represents the modified incompressibility condition

$$\mathbf{G}^T\mathbf{U} - \tau\left(\mathbf{L} - \mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}\right)\mathbf{P} = \mathbf{0} \tag{3.96}$$

It can be observed that this expression would be of the same structure as in the previous section 2.3.2 in Eq. (2.43) and (2.44) if the term $\tau\left(\mathbf{L} - \mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}\right)$ would not exist. This term represents formally the stabilization of the formulation of the sub-grid scale approach. The term $\mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}$ comes analogous to the matrix $\mathbf{L}$, from the term $\langle\nabla q_h, \mathbf{\Pi}_h\rangle$. Up to now, the stabilization term $\tau\left(\mathbf{L} - \mathbf{G}^T\mathbf{M}^{-1}\mathbf{G}\right)$ in (3.95) can be considered as a perturbation of the equation of the volumetric deformation which prevents the stabilization problems which presents the standard formulation.

3.3.5 Implementation aspects

Several considerations have to be made to resolve the equations (3.83)-(3.85) represented by (3.86). The additional variable $\mathbf{\Pi}_h$ in the formulation is continuous between the elements and consequently it is not possible to condensate this variable on the element level. The composed solution of the system would imply a huge additional computational cost. The development presented on the previous pages which leads to the formulation in (3.94) suggests a procedure of a staggered solution. In this case it can be applied an iterative method in which the displacements $\mathbf{U}^{(i)}$ and the pressure $\mathbf{P}^{(i)}$ are calculated by using the value $\mathbf{\Pi}^{(i-1)}$ of the previous step. Afterward the projection of the pressure gradient $\mathbf{\Pi}^{(i)}$ will be resolved in an explicit way via Eq.(3.93) in dependence of the pressure $\mathbf{P}^{(i)}$ calculated in the previous step. The iterative algorithm is presented Tab.**3.1**.
The calculation of the projections $\mathbf{\Pi}$ transforms in a trivial equation system if the lumped mass matrix is used for $\mathbf{M}$. The additional computational cost of this algorithm to calculate the projection is in proportion to the obtained improved results very low.

3.4 Formulation in the plastic case

The following section treats with the problem of elasto-plasticity. The development is based specifically on the case of J2-plasticity. This model considers that the plastic deformations are isochore. The comportment of the elasto-plastic model leads to incompressibility in such an extent how the plastic deformations predominate over the elastic ones. On the other hand, if the Poisson's ratio μ is nearly 0.5 the comportment is almost incompressible although the plastic deformations are not evolved in a certain manner.

i)	Solve at the global level $\mathbf{U}^{(i)}$and $\mathbf{P}^{(i)}$:
	$\begin{bmatrix} \mathbf{K}_{\text{dev}} & \mathbf{G} \\ \mathbf{G}^T & -\frac{1}{K}\mathbf{M}_p - \tau\,\mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{(i)} \\ \mathbf{P}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \tau\,\mathbf{G}^T\mathbf{\Pi}^{(i-1)} \end{bmatrix}$
ii)	Compute and store: $\mathbf{\Pi}^{(i)} = \mathbf{M}^{-1}(\mathbf{G}\mathbf{P}^{(i)}$
iii)	Go to next iteration: $i \leftarrow i+1$

Table 3.1: Algorithm to resolve the stabilized formulation (elastic case)

Most notably in these cases elements formulated via the GLS-method have difficulties to represent the incompressibility. The OSGS-formulation which already was developed in the previous section for the case of elasticity presents very good results also in case of plasticity (see chapter 4) and shall be formulated as next.

3.4.1 Constitutive model

The stress tensor $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma} = p\mathbf{1} + \mathbf{s}(\mathbf{u}) \tag{3.97}$$

where $\mathbf{s}(\mathbf{u}) = \text{dev}(\boldsymbol{\sigma})$ is the deviatoric component of the stress tensor and p the pressure. They can be expressed as

$$p = K\varepsilon_v \tag{3.98}$$

$$\mathbf{s}(\mathbf{u}) = 2\mu\,\text{dev}(\nabla^s\mathbf{u} - \boldsymbol{\varepsilon}^p) = 2\mu\text{dev}(\boldsymbol{\varepsilon}^e) \tag{3.99}$$

where ε_V is the volumetric part of the deformations, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$ the elastic and plastic strain tensors, respectively, μ the shear modulus and K the bulk modulus, as defined already before. With regard to (3.99) and the considerations of (3.44) it can be said that the relation between the deviatoric stresses and the elastic deviatoric deformations is linear in a function of μ, so that

$$2\mu = \frac{\|\mathbf{s}\|}{\|\text{dev}(\boldsymbol{\varepsilon}^e)\|} \tag{3.100}$$

Because of the isochore plastic deformations in the J2-plasticity the volumetric part of the plastic deformations has to be zero. That leads to

$$\text{tr}(\boldsymbol{\varepsilon}^e) = \varepsilon_{\text{V}} = \text{tr}(\boldsymbol{\varepsilon}) = \nabla\cdot\mathbf{u} \tag{3.101}$$

and it can be as well observed that the expression of the stress tensor is decoupled in volumetric and deviatoric components.

3.4.2 Formulation in the continuum

3.4.2.1 The strong form

The problem of the strong form is defined by the previous constitutive equations (3.98) and (3.99), the equilibrium condition and the boundary conditions. Additionally, for the mixed formulation has to be fulfilled the condition for the volumetric deformation in (2.15). Altogether can be expressed once again as:

Given are the prescribed values of the external forces $\bar{\mathbf{t}} : \partial_t\Omega \rightarrow \mathbf{R}^N D$, the displacements on the boundary $\partial_u\Omega$ and the body forces $\mathbf{b} : \Omega \rightarrow \mathbf{R}^N D$; find then $\mathbf{U} := [\mathbf{u}, p]^T \in \mathcal{W} = \mathcal{V} \times \mathcal{Q}$, so that

$$\mathcal{L}(\mathbf{U}) = \mathcal{F} \tag{3.102}$$

where

$$\mathcal{L}(\mathbf{U}) = \begin{bmatrix} -\nabla p - \nabla \cdot \mathbf{s}(\mathbf{u}) \\ -\frac{1}{K}p + \nabla \cdot \mathbf{u} \end{bmatrix} \tag{3.103}$$

$$\mathcal{F} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{3.104}$$

and additionally the boundary conditions:

$$\mathbf{u} = \mathbf{0} \qquad in\ \partial\Omega_u \tag{3.105}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \qquad in\ \partial\Omega_t \tag{3.106}$$

where the expression of the stress tensor $\boldsymbol{\sigma}$ has been replaced by the dependency of $\mathbf{u}$ and p.

3.4.2.2 The weak form

The associated weak form can be written in the compact notation as:

Find $\mathbf{U} := [\mathbf{u}\ p]^T \in \mathcal{W} = \mathcal{V} \times \tilde{\mathcal{Q}}$, so that

$$\mathbf{R}(\mathbf{U}, \mathbf{V}) = \mathbf{0}\ , \qquad \forall\, \mathbf{W} \in \mathcal{W} \tag{3.107}$$

with

$$\mathbf{R}(\mathbf{U}, \mathbf{V}) = \begin{bmatrix} \langle \nabla^s \mathbf{v}, \mathbf{s} \rangle + \langle \nabla \cdot \mathbf{v}, p \rangle - \langle \mathbf{v}, \mathbf{f} \rangle - \langle \mathbf{v}, \bar{\mathbf{t}} \rangle_{\partial_t\Omega} \\ \langle q, \nabla \cdot \mathbf{u} \rangle - \langle q, \frac{1}{K}p \rangle \end{bmatrix} \tag{3.108}$$

The vectors $\mathbf{U} := [\mathbf{u}\ p]^T \in \mathcal{W}$ and $\mathbf{V} := [\mathbf{v}\ q]^T \in \mathcal{W}$ are the vector of the unknowns and its respective variation and $\mathcal{W}$ is the space of the admissible functions.
The spaces $\mathcal{V}$ and $\mathcal{Q}$ are the space of the displacements and the space of the pressure, respectively. In the continuum they are defined as

$$\mathcal{V} = \left\{\mathbf{v} \,|\, \mathbf{v} \in \mathcal{H}^1(\Omega)\ ,\ \mathbf{v} = \mathbf{0} \quad in\ \partial\Omega_u\right\} \tag{3.109}$$

$$\mathcal{Q} = \left\{q \,|\, q \in \mathcal{L}^2(\Omega)\ ,\ \int_\Omega q\, d\Omega = 0\right\} \tag{3.110}$$

3.4.3 Extension to multiscales

The solution of the unknowns are approximated by

$$\mathbf{U} = \mathbf{U}_h + \tilde{\mathbf{U}} \tag{3.111}$$

$$\mathbf{V} = \mathbf{V}_h + \tilde{\mathbf{V}} \tag{3.112}$$

The space in which the solution $\mathbf{U}$ is defined will be split as well in two parts as $\mathcal{W} = \mathcal{W}_h \oplus \tilde{\mathcal{V}}$ so that $\mathbf{U}_h \in \mathcal{W}_h$ and $\tilde{\mathbf{U}} \in \tilde{\mathcal{W}}$. Furthermore, by following a similar proceeding as in the elastic case, demonstrated in details in the previous chapter 3.3, the problem in the continuum (3.107) transforms into:
Find $\mathbf{U}_h \in \mathcal{W}_h$ and $\tilde{\mathbf{U}} \in \tilde{W}$, so that

$$\mathbf{R}(\mathbf{U}, \mathbf{V}_h) = \mathbf{R}(\mathbf{U}_h + \tilde{\mathbf{U}}, \mathbf{V}_h) = \mathbf{0}\ , \qquad \forall\, \mathbf{V}_h \in \mathcal{V}_{h,0} \tag{3.113}$$

$$\mathbf{R}(\mathbf{U}, \tilde{\mathbf{V}}) = \mathbf{R}(\mathbf{U}_h + \tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = \mathbf{0}\ , \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{V}} \tag{3.114}$$

It can be observed that equation (3.113) contains the equilibrium condition and the condition for the volumetric deformations defined in the finite element space $\mathcal{W}_h$. The equation (3.114) is defined in the space $\tilde{\mathcal{W}}$ which will be used to obtain an approximation of the sub-grid scales.
Preceding to the development of the method, it shall be introduced a consideration on the deviatoric stresses, taking into account that the effect of the sub-grid scales is of the order $\tilde{\mathbf{u}} \sim \mathcal{O}(h^2)$; see also [9]. Agreeing to this, the deviatoric components of the stresses $\mathbf{s}(\mathbf{u})$ can be approximated via a Taylor series of the first order outside of $\mathbf{u}_h$ as $f(x_0 + \Delta x) = f(x_0) + f'(x)|_{x=x_0}\Delta x$, without attention to the terms of quadratic or higher order and coming after all to

$$\mathbf{s}(\mathbf{u}_h + \tilde{\mathbf{u}}) = \mathbf{s}_h(\mathbf{u}_h) + \bar{\mathbf{C}}_s|_{u=u_h} : \nabla^s\tilde{\mathbf{u}} + \mathcal{O}(\tilde{\mathbf{u}})^2 \tag{3.115}$$

where $\bar{\mathbf{C}}_s|_{u=u_h}$ is the deviatoric part of the tangential constitutive tensor evaluated in $\mathbf{u} = \mathbf{u}_h$. This tensor relates the deviatoric stresses with the total stains. The tensorial product $\bar{\mathbf{C}}_s|_{u=u_h} : \nabla^s\tilde{\mathbf{u}}$ represents an increment of deviatoric stresses for the effect of the sub-grid scales $\tilde{\mathbf{s}}(\tilde{\mathbf{u}})$ which can be considered as a superposed perturbation to $\mathbf{u}_h$. This product involves only the deviatoric component of the deformations, so that as well the tensor $\bar{\mathbf{C}}_s$ is deviatoric. That means that

$$\tilde{\mathbf{s}}(\tilde{\mathbf{u}}) = \bar{\mathbf{C}}_s|_{u=u_h} : \nabla^s \tilde{\mathbf{u}} = \bar{\mathbf{C}}_s|_{u=u_h} : \mathrm{dev}[\nabla^s \tilde{\mathbf{u}}] \tag{3.116}$$

When plastic flow occurs, the total deviatoric deformations, particularly the plastic ones, are growing with respect to the deviatoric stresses more than in the elastic case. The increment of the deviatoric stresses corresponding to the total deviatoric deformations associated to the sub-grid scales can be calculated in a simple way if an approximation of $\bar{\mathbf{C}}_s|_{u=u_h}$ as a function of 2μ, as in Eq.(3.99), is made. It represents the relation between the deviatoric stresses and the elastic component of the deviatoric deformations. Now, with regard to the value 2μ, given by Eq.(3.100), the relation between the deviatoric stresses and the total deviatoric deformations can be estimated via the coefficient $2\mu^*$, calculated by

$$2\mu^* = 2\mu \left. \frac{\|\mathrm{dev}(\boldsymbol{\varepsilon}^e)\|}{\|\mathrm{dev}(\nabla^s \mathbf{u})\|} \right|_{u=u_h} = \left. \frac{\|\mathbf{s}\|}{\|\mathrm{dev}(\nabla^s \mathbf{u})\|} \right|_{u=u_h} \tag{3.117}$$

The coefficient $2\mu^*$ which could be named as the effective shear modulus, can be interpreted as the secant modulus in $\mathbf{u} = \mathbf{u}_h$ as shown in Fig.**3.2**. This coefficient represents the effect of the plastic flow in the relation between the deviatoric stresses and the total deviatoric deformations. If the plastic deformations are relatively small in comparison to the elastic deformations, it can be obtained adequate results and a stable comportment of the pressure by using the elastic shear modulus μ.
In case of elastic loading and unloading, the shear modulus shall have the value of μ and in case of plastic flow the value μ^*. This can be concluded to

$$\mu' = \begin{cases} \mu & \text{in elastic case} \\ \mu^* & \text{in plastic case} \end{cases} \tag{3.118}$$

The interpretation of the approximation of the shear modulus $2\mu^*$ as the secant to the curve of the deformations in $\mathbf{u} = \mathbf{u}_h$ against the stresses, allows to get an analogy of the case of plasticity to a model of fracture in which appears as well this kind of approximation. With these results, the tensorial product $\bar{\mathbf{C}}_s|_{u=u_h} : \mathrm{dev}[\nabla^s \tilde{\mathbf{u}}]$ can be finally approximated as

$$\bar{\mathbf{C}}_s|_{u=u_h} : \mathrm{dev}(\nabla^s \tilde{\mathbf{u}}) = \tilde{\mathbf{s}}(\tilde{\mathbf{u}}) \cong 2\mu' \, \mathrm{dev}(\nabla^s \tilde{\mathbf{u}}) \tag{3.119}$$

Although $\mathbf{s}(\cdot)$ is nonlinear, it is possible to decouple the deviatoric stress tensor into the sum of two contributions. On the one hand $\mathbf{s}_h(\mathbf{u}_h)$, coming from the finite element solution and on the other hand $\tilde{\mathbf{s}}(\tilde{\mathbf{u}})$, corresponding to the sub-grid scales. It is said that

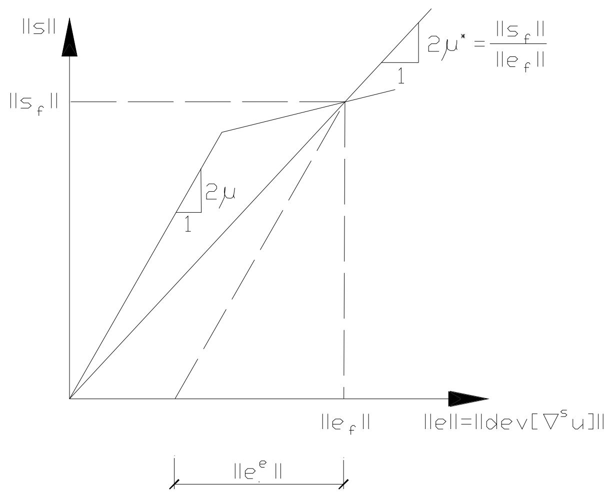


Figure 3.2: Graph of the deviatoric stresses against the deviatoric deformations in case of J2-plasticity

$$\mathbf{s}(\mathbf{u}_h + \tilde{\mathbf{u}}) \cong \mathbf{s}_h(\mathbf{u}_h) + \tilde{\mathbf{s}}(\tilde{\mathbf{u}}) \tag{3.120}$$

According to this and taking into account that the divergence operator is linear, the expression (3.113) can be separated into

$$\mathbf{R}(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \mathbf{0}\ , \qquad \forall\, \mathbf{V}_h \in \mathcal{W}_h \tag{3.121}$$

where $\mathbf{R}(\mathbf{U}_h, \mathbf{V}_h)$ and $\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h)$ are respectively

$$\mathbf{R}(\mathbf{U}_h, \mathbf{V}_h) = \begin{bmatrix} \langle \nabla^s \mathbf{v}_h, \mathbf{s}_h \rangle_{\Omega'} + \langle \nabla \cdot \mathbf{v}_h, p_h \rangle_{\Omega'} - l(\mathbf{v}_h) \\ \langle q_h, \nabla \cdot \mathbf{u}_h - \frac{1}{K} p_h \rangle_{\Omega'} \end{bmatrix} \tag{3.122}$$

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} \langle \nabla^s \mathbf{v}_h, 2\mu' \operatorname{dev}(\nabla^s \tilde{\mathbf{u}}) \rangle_{\Omega'} \\ \langle q_h, \nabla \cdot \tilde{\mathbf{u}} \rangle_{\Omega'} \end{bmatrix} \tag{3.123}$$

The second term represents the effect of the sub-grid scales in the finite element solution. The integration by parts of this term, with the aim to reduce the order of derivation of $\tilde{\mathbf{u}}$, leads to

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} -\langle \tilde{\mathbf{u}}, \nabla \cdot [2\mu' \operatorname{dev}(\nabla^s \mathbf{v}_h)) \rangle_{\Omega'} \\ -\langle \tilde{\mathbf{u}}, \nabla q_h \rangle_{\Omega'} \end{bmatrix} \tag{3.124}$$

If the additive decomposition of the stress tensor (3.120) is used in Eq.(3.114) this equation can be rewritten as

$$\mathbf{R}(\mathbf{U}, \tilde{\mathbf{V}}) = \begin{bmatrix} \langle \tilde{\mathbf{v}}, \nabla \cdot \mathbf{s}_h \rangle_{\Omega'} + \langle \tilde{\mathbf{v}}, \nabla \cdot \tilde{\mathbf{s}} \rangle_{\Omega'} + \langle \tilde{\mathbf{v}}, \nabla p_h \rangle_{\Omega'} + \langle \tilde{\mathbf{v}}, \mathbf{b} \rangle_{\Omega'} \\ 0 \end{bmatrix} = \mathbf{0} \tag{3.125}$$

It can be observed that there is no equation associated to the variations of the pressure because of considering only the sub-grid scales for the displacements. With the decoupling of the vector $\mathbf{U}$ in its parts $\mathbf{U}_h$ and $\tilde{\mathbf{U}}$ we get

$$\mathbf{R}(\mathbf{U}_h, \tilde{\mathbf{V}}) + \mathbf{R}(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = \mathbf{0}\ , \qquad \forall\, \tilde{\mathbf{V}} \in \tilde{\mathcal{W}} \tag{3.126}$$

which leads to

$$-\langle \tilde{\mathbf{v}}, \nabla \cdot [2\mu' \operatorname{dev}(\nabla^s \tilde{\mathbf{u}})] \rangle_{\Omega'} = \langle \tilde{\mathbf{v}}, \nabla \cdot \mathbf{s}_h + \nabla p_h + \mathbf{b} \rangle_{\Omega'} \tag{3.127}$$

This is the relation between the sub-grid scale and the residuum of the momentum equation $\mathbf{R}_h = \nabla \cdot \mathbf{s}_h + \nabla p_h + \mathbf{f}$ which is similar to the previous section in Eq.(3.78). According to the adopted considerations in section 3.2.2, it will be chosen the space of orthogonal sub-grid scales $\tilde{\mathcal{W}} = \mathcal{W}_h^{\perp}$ and with (3.127) the sub-grid scale $\tilde{\mathbf{u}}$ can be approximated in every element as

$$\tilde{\mathbf{u}} = \tau_e\, P_h^{\perp}\,(\mathbf{R}_h) \in \mathcal{W}_h^{\perp} \tag{3.128}$$

where τ_e is the stabilization parameter and $P_h^{\perp}(\cdot)$ represents the projection on the space $\mathcal{W}_h^{\perp}$. The stabilization parameter in linear elements will be calculated by

$$\tau_e = \frac{ch^2}{2\mu'} \tag{3.129}$$

where μ' accounts for the effect of plastic flow in the deviatoric deformations which in general is calculated in every integration point. In the case of linear triangular elements with only one integration point per element this value is the only one in each element. The main robustness of the proposed elements, stabilized by the OSGS-method, make them less sensible to the simplifications considered to the approximation of the stabilization parameter in comparison to the stabilized elements via others techniques, like the GLS-method used in [15].

As seen before, $\tilde{\mathbf{u}}$ is the projection of the residuum $\mathbf{R}_h$ on the orthogonal space of the finite element space multiplied by the stabilization parameter. Taking into account that the second derivatives of the linear functions of the finite elements are zero and that $P_h^{\perp}(\mathbf{f}) = \mathbf{0}$, similar to the considerations in the section 3.3.3, we get

$$\tilde{\mathbf{u}} = \tau_e\,(\nabla p_h - P_h(\nabla p_h)) \tag{3.130}$$

If this approximation is inserted into Eq.(3.125)and it is considered that $\langle \tilde{\mathbf{u}}, \nabla \cdot (2\mu' \text{dev}[\nabla^s \mathbf{v}_h])\rangle_{\Omega^e} \equiv 0$ for linear elements, Eq.(3.124) can be rewritten as

$$\mathbf{R}(\tilde{\mathbf{U}}, \mathbf{V}_h) = \begin{bmatrix} 0 \\ -\sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla p_h - \mathbf{\Pi}_h \rangle_{\Omega_e} \end{bmatrix} \tag{3.131}$$

where $\mathbf{\Pi}_h = P_h(\nabla p_h)$ is the projection of the pressure gradient on the finite element space $\mathcal{W}_h$.

Finally, the proposed stabilized version to resolve the incompressible elasto-plastic problem via linear triangular or tetrahedral elements is defined as:

Find $[\mathbf{u}_h,\ p_h,\ \mathbf{\Pi}_h]^T \in \mathcal{V}_h \times \mathcal{Q}_h \times \mathcal{V}_h$ so that

$$\begin{aligned} \langle \nabla^s \mathbf{v}_h, \mathbf{s}_h \rangle + \langle \nabla \cdot \mathbf{v}_h, p_h \rangle - l(\mathbf{v}_h) &= 0\,, \qquad \forall\, \mathbf{v}_h \in \mathcal{V}_h && (3.132) \\ \langle q_h, \nabla \cdot \mathbf{u}_h - \frac{1}{K} p_h \rangle - \sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla p_h - \mathbf{\Pi}_h \rangle_{\Omega_e} &= 0\,, \qquad \forall\, q_h \in \mathcal{Q} && (3.133) \\ \langle \nabla p_h, \boldsymbol{\eta}_h \rangle - \langle \mathbf{\Pi}_h, \boldsymbol{\eta}_h \rangle &= 0\,, \qquad \forall\, \boldsymbol{\eta}_h \in \mathcal{V}_h && (3.134) \end{aligned}$$

It can be observed that in the proposed formulation the stabilization term effects only on the equation of the volumetric deformation.

According to the adopted considerations in all developed equations appears the total value of the sub-grid scale instead of the incremental one. For example in the equation (3.127) from which is obtained the approximation of $\tilde{\mathbf{u}}$.

3.4.4 Implementation aspects

The plastic problem implies the nonlinearity of the equation system and consequently the solution of the equations (3.132)-(3.134) are executed via incremental-iterative procedures. In this scope diverse alternatives to calculate the projection Π are possible. According to the presented considerations in the section 3.3, the basic strategy consists in decoupling the projection Π and resolving the mixed system as from now several possibilities exist to build up the calculation algorithm. The adopted strategy in this work consists in resolving simultaneous the equations of the momentum balance and the volumetric deformations in the time step $t = t^{n+1}$. To find the equilibrium in the iterations, the value of the projection of the pressure gradient $\mathbf{\Pi}_h$ is used in the time step $t = t^n$. The value of the projection of the pressure gradient $\mathbf{\Pi}_h^n$ is evaluated on the basis of the converged value of the pressure gradient of the previous time step ∇p_h^n as

$$(\nabla p_h^n, \boldsymbol{\eta}_h) - (\mathbf{\Pi}_h^n, \boldsymbol{\eta}_h) = 0 \tag{3.135}$$

This equation transforms in a trivial system if the lumped mass matrix is used. With this value of $\mathbf{\Pi}_h^n$, the equations of the system which are resolved in the time step $n+1$ are

$$
\begin{aligned}
\langle \nabla^s \mathbf{v}_h, \mathbf{s}_h(\mathbf{u}_h^{n+1,i}) \rangle + \langle \nabla \cdot \mathbf{v}_h, p_h^{n+1,i} \rangle - l^{n+1}(\mathbf{w}_h) &= 0 && (3.136)\\
\langle q_h, \nabla \cdot \mathbf{u}_h^{n+1,i} - \frac{1}{K} p_h^{n+1,i} \rangle - \sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla p_h^{n+1,i} - \mathbf{\Pi}_h^n \rangle_{\Omega_e} &= 0 && (3.137)
\end{aligned}
$$

In each time step nonlinear equations has to be calculated. This system can be resolved, for example, by applying the Newton-Raphson method. The residuals corresponding to these equations in the iteration i can be expressed as

$$
\begin{aligned}
r_1^{n+1,i} &= \langle \nabla^s \mathbf{v}_h, \mathbf{s}_h(\mathbf{u}_h^{n+1,i} \rangle + \langle \nabla \cdot \mathbf{v}_h, p_h^{n+1,i} \rangle - l^{n+1}(\mathbf{v}_h) && (3.138)\\
r_2^{n+1,i} &= \langle q_h, \nabla \cdot \mathbf{u}_h^{n+1,i} \rangle - \sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla p_h^{n+1,i} - \mathbf{\Pi}_h^n \rangle_{\Omega_e} && (3.139)
\end{aligned}
$$

In each iteration shall be resolved the linear system

$$
\begin{aligned}
a\left(\Delta \mathbf{u}_h^{n+1,i+1}, \mathbf{v}_h\right) + \langle \nabla \cdot \mathbf{v}_h, \Delta p_h^{n+1,i+1} \rangle &= -r_1^{n+1,i} && (3.140)\\
\langle q_h, \nabla \cdot \Delta \mathbf{u}_h^{n+1,i+1} - \frac{1}{K} \Delta p_h^{n+1,i+1} \rangle - \sum_{e=1}^{n_{elm}} \tau_e \langle \nabla q_h, \nabla \Delta p_h^{n+1,i+1} \rangle_{\Omega_e} &= -r_2^{n+1,i} && (3.141)
\end{aligned}
$$

The algorithm to resolve the elasto-plastic problem of the infinitesimals deformations is shown in Tab.**3.2**. $\mathbf{K}_{\text{dev}}$ is the matrix corresponding to the term $a(\Delta \mathbf{u}_h^{n+1,i+1}, \mathbf{v}_h)$, whose sub-matrix on the element level is

$$
\left[\ \mathbf{K}_{\text{dev}}^{AB}\ \right]^e = \int_{\Omega^e} (\mathbf{B}_A)^T \, \mathbf{D} \, \mathbf{B}_B \, d\Omega \qquad (3.142)
$$

where $\mathbf{D}$ is the matrix of the deviatoric component of the tangential constitutive tensor. Furthermore, the matrices $\mathbf{G}$, $\mathbf{L}$, $\mathbf{M}_p$ and $\mathbf{M}$ are defined in the same way as in section 3.3 like the vector of the nodal unknowns $[\mathbf{U},\ \mathbf{P}]^T$, too. The residuals $\mathbf{R}_1$ and $\mathbf{R}_2$ are the vectors associated to the residuals $r_1^{n+1,i}$ and $r_2^{n+1,i}$. These are respectively

$$
\begin{aligned}
\mathbf{R}_1 &= \mathbf{f}_1^{int} - \mathbf{f}_1^{ext} && (3.143)\\
\mathbf{R}_2 &= \mathbf{f}_2^{int} && (3.144)
\end{aligned}
$$

with the internal and external forces defined as:

$$\mathbf{f}_1^{int} = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} \left[\mathbf{f}_1^{int} \right]_e = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} \left[\int_{\Omega_e} \mathbf{B}_A^T \boldsymbol{\sigma} \, d\Omega \right]_e + \left[\mathbf{G}^{AB} p^B \right]_e \tag{3.145}$$

$$\mathbf{f}_1^{ext} = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} \left[\mathbf{f}_1^{ext} \right]_e = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} \left[\int_{\Omega_e} \left[\mathbf{N}^A \right]^T \mathbf{b} \, d\Omega + \int_{\partial_t \Omega_e} \left[\mathbf{N}^A \right]^T \bar{\mathbf{t}} \, d\Gamma \right]_e \tag{3.146}$$

$$\begin{aligned} \mathbf{f}_2^{int} = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} \left[\mathbf{f}_2^{int} \right]_e = \mathop{\mathbf{A}}_{e=1}^{n_{elm}} & \left[(\mathbf{G}^{AB})^T \mathbf{u}^B \right]_e - \left[\mathbf{M}_p^{AB} p^B \right]_e \\ & -\tau_e \left(\left[\mathbf{L}^{AB} p^B \right]_e - \left[(\mathbf{G}^{AB})^T \boldsymbol{\Pi}^B \right]_e \right) \end{aligned} \tag{3.147}$$

i) Load step $n+1$: Initialize $i = 0 \rightarrow [\mathbf{U}\ \mathbf{P}]^{T(n+1,0)} = [\mathbf{U}\ \mathbf{P}]^{T(n)}$

ii) Iteration: Solve at global level: $\Delta\mathbf{U}^{(n+1,i+1)}$ and $\Delta\mathbf{P}^{(n+1,i+1)}$:

$$\begin{bmatrix} \mathbf{K}_{\text{dev}} & \mathbf{G} \\ \mathbf{G}^T & -\frac{1}{K}\mathbf{M}_p - \tau\mathbf{L} \end{bmatrix}^{(n+1,i)} \begin{bmatrix} \Delta\mathbf{U} \\ \Delta\mathbf{P} \end{bmatrix}^{(n+1,i+1)} = - \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix}^{(n+1,i)}$$

iii) Update: $[\mathbf{U}\ \mathbf{P}]^{T(n+1,i+1)} = [\mathbf{U}\ \mathbf{P}]^{T(n+1,i)} + [\Delta\mathbf{U}\ \Delta\mathbf{P}]^{T(n+1,i+1)}$

iv) Check the convergence:

No: New iteration step: $i \leftarrow i+1$; Go to *ii)*

Yes: Set: $[\mathbf{U}\ \mathbf{P}]^{T(n+1)} = [\mathbf{U}\ \mathbf{P}]^{T(n+1,+1)}$

Compute: $\boldsymbol{\Pi}^{(n+1)} = \mathbf{M}^{-1}\mathbf{G}\mathbf{P}^{(n+1)}$

v) New load step: $n \leftarrow n+1$; Go to *i)*

Table 3.2: Algorithm to resolve the stabilized formulation (plastic case)

Chapter 4

Numerical Results

The efficiency of the proposed formulation can be shown in a series of numerical tests. The triangular elements, named as T1P1 for their linear displacement and pressure interpolation, shall be compared in diverse examples to other finite element formulations like the simple linear triangular element without pressure interpolation P1, the simple quadrilateral element Q1, the enhanced assumed strain element Q1E4, the improved quadrilateral with continuous pressure interpolations Q1P0 and a tetrahedral mixed element without any stabilization formulation here signified by (P1P1). In the examples are considered as well the condition of incompressibility as the elasto-plastic comportment simulated by the J2-plasticity. Most of these examples were calculated by the finite element programme "COupled MEchanical and Thermal analysis (COMET)" and the pre- and postprocessing has been done by the programme "GiD".
The examples which will be treated are the patch test, a forced cotter, the Cook's membrane and finally a 2D-block under a displacement load.
The patch test is a basic test for the validation of the elements. The problem of the cotter leads to a comparison between the elements mentioned above. The cavity flow problem is a typical problem for mixed finite element methods and gives an example for the very good comportment of the developed element against the improved elements (EAS and Q1P0). In the problems of elasto-plasticity, like in the problem of the Cook's membrane and the 2D-block, will be demonstrated the very good results of the proposed element in spite of its simplifying considerations in case of plasticity against the other mentioned elements.

4.1 Problems in case of elasticity

4.1.1 The patch test

The patch test is the most important prove the finite element methods particularly for introducing new formulations. The test, first introduced in [13], bases on rational physics and proves if a patch of elements, subjected to constant deformations, has the capacity to reproduce the exact comportment of the material and returns the according stresses.

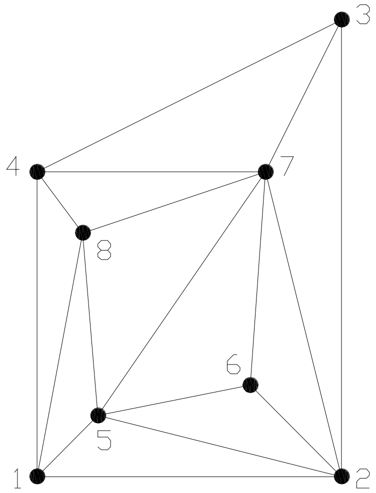


Figure 4.1: Triangular mesh for the verification of the patch test

	coord.		displacements		fuerzas	
node	x_i	y_i	u_x	u_y	F_x	F_y
1	0.0	0.0	0.0	0.0	-2.0	0.0
2	2.0	0.0	0.0040	0.0	3.0	0.0
3	2.0	3.0	0.0040	-0.00180	2.0	3.0
4	0.0	2.0	0.0	-0.00120	-3.0	0.0
5	0.4	0.4	0.0008	-0.00024	0.0	0.0
6	1.4	0.6	0.0028	-0.00036	0.0	0.0
7	1.5	2.0	0.0030	-0.00120	0.0	0.0
8	0.3	1.6	0.0006	-0.00096	0.0	0.0

Table 4.1: Used values for the patch test

The satisfaction of the patch test gives a sufficient condition of the convergence of an element and verifies that the implementation has succeeded. The patch which was used for the T1P1-element is shown in the figure **4.1**. It is the same patch test as shown in [22]. The material is linear elastic with a Young's modulus $E = 1000$ and the Poisson's ratio $\nu = 0.3$. The solution for the considered displacements is

$$u_x = 0.002x \tag{4.1}$$
$$u_y = -0.0006y \tag{4.2}$$

which produces in the plain strain formulation constant stresses over the whole patch

$$\sigma_x = 2.3460 \tag{4.3}$$
$$\sigma_y = 0.3462 \tag{4.4}$$
$$\sigma_z = 0.8077 \tag{4.5}$$

In Tab. **4.1** are given the nodal displacements corresponding to the displacement functions in (4.1). In a first test all the nodal displacements are prescribed to the specific values of Tab. **4.1** wherewith the theoretical values of the stresses given in (4.3) will be verified. Additionally the forces defined in the table are associated to the constant stresses in the patch

$$\sigma_x = 2.000 \tag{4.6}$$
$$\sigma_y = 0.000 \tag{4.7}$$
$$\sigma_z = 0.600 \tag{4.8}$$

This leads to a second test in which the node 1 is clamped and the node 4 is only blocked in x-direction. The forces given in the table are applied on the on the other two edge nodes 2 and 3. This test produces in the whole patch the theoretical stresses given in (4.6)-(4.8). With these results can be reasoned that the coding of the element has succeeded. There exist a lot of other kinds of patch tests with which the same results can be obtained.

4.1.2 Forced cotter in plane strain

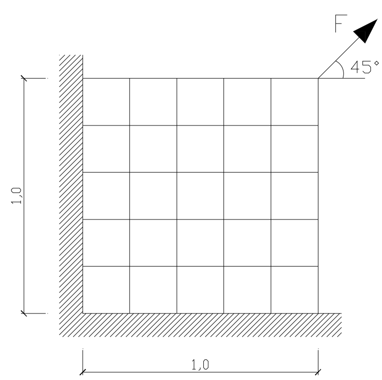


Figure 4.2: Forced cotter in plane strain

This test realizes the comportment of the element in an extreme situation with regard to the restrictions to the displacements, the singularity of the load and the natural incompressibility of the material.
Considered is a cotter in plane stain state, sectioned into five squares of $20 \times 20cm$ in each direction as shown in Fig.**4.2**. The vertical and horizontal displacements are totally restricted as well on the left side as on the bottom. The load is given by a concentrated

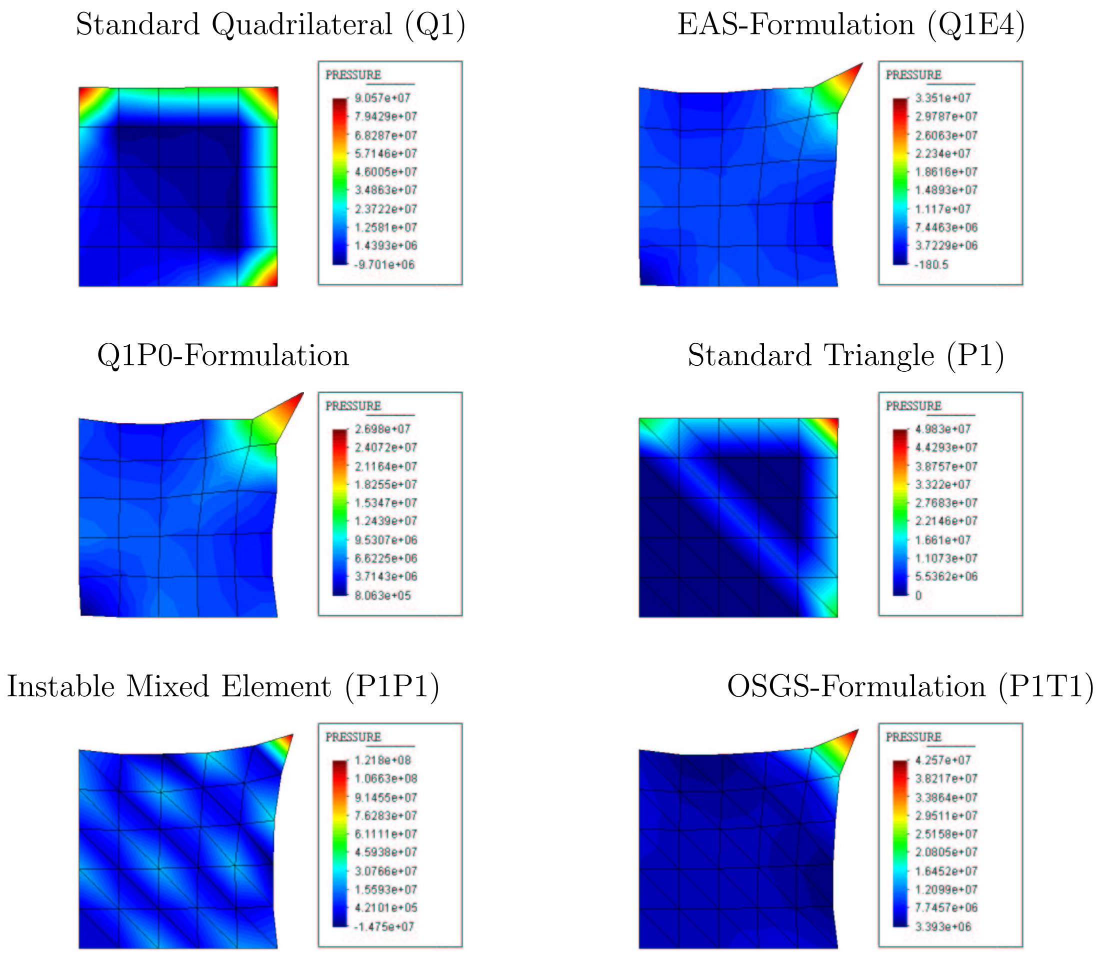


Figure 4.3: Forced cutter in plane strain: distribution to the pressure

load on the right upper corner of the cotter which has tension component as well to the x-direction as to the y-directions with the value of $F_x = F_y = 5 \cdot 10^6 N/m$. The material exhibits a Young's modulus of $E = 2.0 \cdot 10^5 MPa$ and a Poisson's ratio of $\nu = 0.4999$, which is nearly almost incompressible. Compared elements formulations are the P1, Q1, P1P1, Q1P0, Q1E4 and the T1P1 as defined above.
The resulted distributions of the pressure and the Von Mises stresses are shown in the deformed geometry of the cotter whereby the increase factor is for each element type the same. The locking effect of the two standard elements can be noticed in the results in Fig.**4.3**. The extreme stiffness of the elements is confirmed by the very small deformation of the mesh and also by the illustration of the pressure which is completely unphysically. The instable mixed element shows already better displacements but the very inhomogeneous pressure image reasons to a inaccurate pressure calculation. The pressure values of the Q1P0-element and the EAS-element are very low but their deformation figure shows a slight irregularity in the shape, in case of the Q1P0-element more than in case of the EAS-element. This is caused by the hourglass modes which can occur if the stabilization of the element formulation is accomplished by the static condensation of the additional variable, so in case of the Q1P0-element the volumetric deformations are constant in the

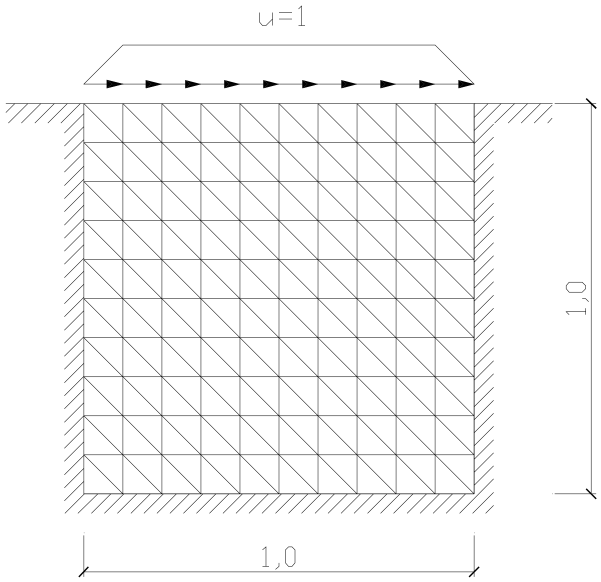


Figure 4.4: Problem of driven cavity flow

element and discontinuous over the boundaries. This effect does not exhibit in the OSGS-formulation because the elimination of the additional variable works globally. As result of this we detect the correct deformation of the cotter. It has to be added that the mesh that is used is very coarse and would normally be refined. So the bad effects which occur are somewhat enforced.

4.1.3 Driven cavity flow

Normally this problem is a standard test in the literature, particularly in the field of fluid mechanics. It is used to show the comportment of the mixed formulations in problems of incompressible media to represent the Navier-Stokes flow as shown for example in [12] or [7].
Fig.**4.4** shows the geometric data of the test including the mesh size of 10×10 elements which leads in cases of triangular elements not to 100 but to 200 elements. Additionally the test was executed as well with a mesh size of 50×50 elements. The enclosed medium in the the cavity is numerically incompressible with a value of $\nu = 0.49995$. The walls of the cavity to the right, to the left and the bottom are clamped, that means prescribed with zero displacements in x-direction as well as in y-direction. On the free surface the displacement in x-direction is prescribed with the value $u = 1$, except the nodes in the left and right corner, respectively which are prescribed by zero displacements. The Young's modulus is chosen by $E = 3$.
It shall be compared the total values of the displacement inside of the medium by using the standard quadrilateral and tetrahedral elements, the Q1P0-element and the EAS-element, an instable tetrahedral mixed element and the proposed OSGS-element. The results are presented in Fig.**4.5**. On the left are shown the total displacements obtained with the 10×10 mesh and on the right side with the 50×50 mesh.

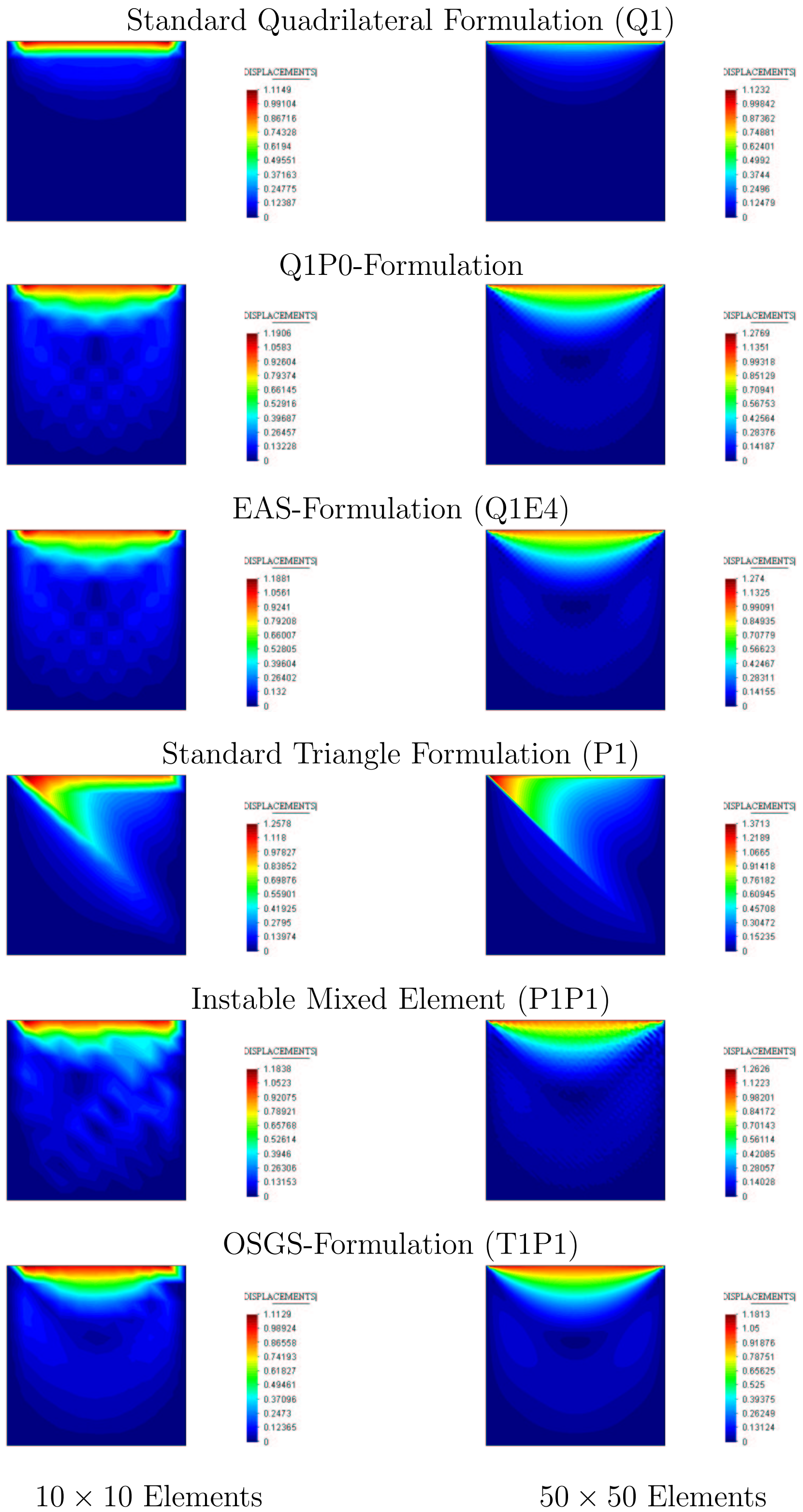


Figure 4.5: Driven cavity flow: Comparison of the total displacement distribution, calculated by using a mesh of 10×10 element, on the left and a mesh of 50×50 mesh, on the right

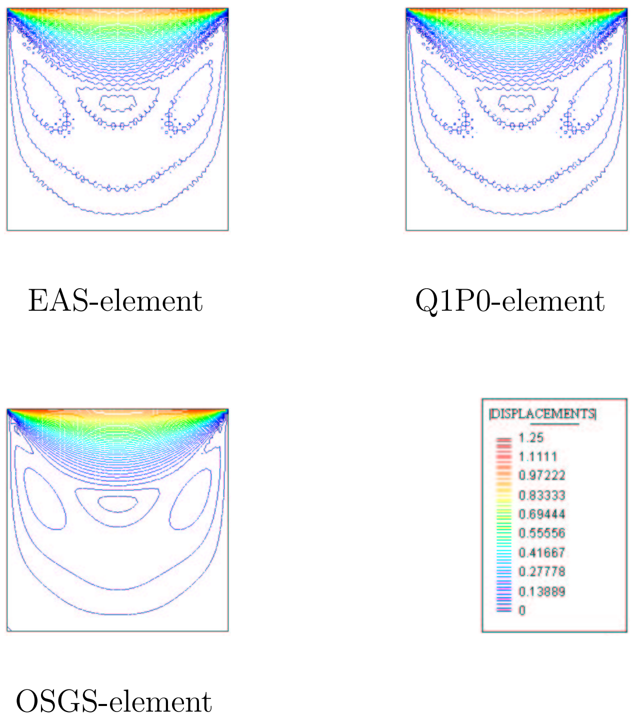


Figure 4.6: Driven cavity flow: Contour lines of the total displacements in case of a 50×50 mesh

At once we can see the big difference between the standard elements and the improved elements. The standard elements show again a extreme locking behavior and cannot represent the real displacement distribution. In the case of the standard triangular element is it such worse that the result reflects the geometry of the used mesh. Just there where the diagonal leads from the upper left corner to the lower right corner can be recognized a high jump of the displacement values. This dependency of the mesh geometry can appear quite often by using triangular elements but can be avoided if irregular meshes are used. The device of this alternative shall not be explicated in this work but is already known. Nevertheless, the triangular mixed elements are not effected by this geometrical dependency and give both much better results. Admittedly, the instable mixed element shows once more an oscillating result because of its lack of stabilization. But the OSGS-element shows a very homogeneous displacement distribution which represents clearly the so called cat-eye structure. This name has its seeds in the sickle-shaped distribution of the displacement and the appearing vortices at each side of the medium. This very homogeneous distribution presented by the OSGS-element cannot reached neither by the Q1P0-element nor by the EAS-element. As well this two improved elements offer slight oscillations in the displacement distribution.

Better observed in Fig.**4.6** where the contour lines of the total displacements, obtained with the 50×50 mesh, are compared between the Q1P0-element, the EAS-element and the OSGS-element. It shows the very unsteady run of the displacement level curves in case of the Q1P0-element and the EAS-element. In contrast to this the OSGS-element offers a very smooth run of the level curves. This discrepancy could be explained by the different ways of computing the additional variables. Where the the Q1P0- and the EAS-formulation uses the static condensation over each element to delete the additional variable, in case of the Q1P0-element the pressure and in case of the EAS-element the enhanced strains, so that the additional effects are introduced elementwise and discontinuous over the element boundaries, the OSGS-formulation compute the pressure at each element node which leads to a continuity over the element boundaries and stabilizes the results in a more efficient way. This shows one of the big advantages of the proposed element formulation against discontinuous formulations which should not be ignored as the results show.

4.2 Problems in case of plasticity

4.2.1 Cook's membrane

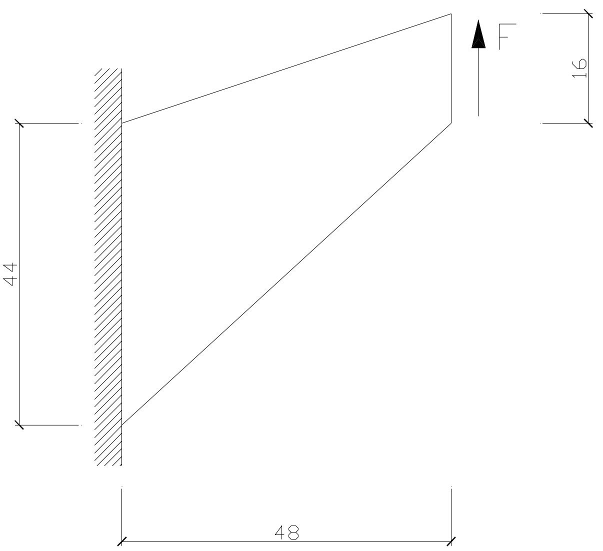


Figure 4.7: Cook's membrane

The problem of the Cook's membrane can be chosen to show the convergence comportment of an element in case of bending dominated problems. By minimizing the element size continuously, the value of displacement should converge to a given analytical value. This kind of problem was first introduced by Simo & Rifai in [20]. It is about a clamped wing which is forced on its free end by a force in the y-direction. This force has a value of $F = 1.8MN/m$. It is used a constitutive model base on the J2-plasticity whereas the Young's modulus $E = 70MPa$, the Poisson's ratio $\nu = 0.4999$, the yield stress has the value $\sigma_y = 0.243MPa$ and the coefficient of isotropic hardening is $H = 0.135$. The geometry parameters are given in Fig. **4.7**.
Discretized are seven different meshes which are strung in respect to the element number at each side in meshes of 2×2, 5×5, 10×10, 16×16, 20×20, 32×32 and finally 64×64. It can be observed that the element number in case of triangles is the double of

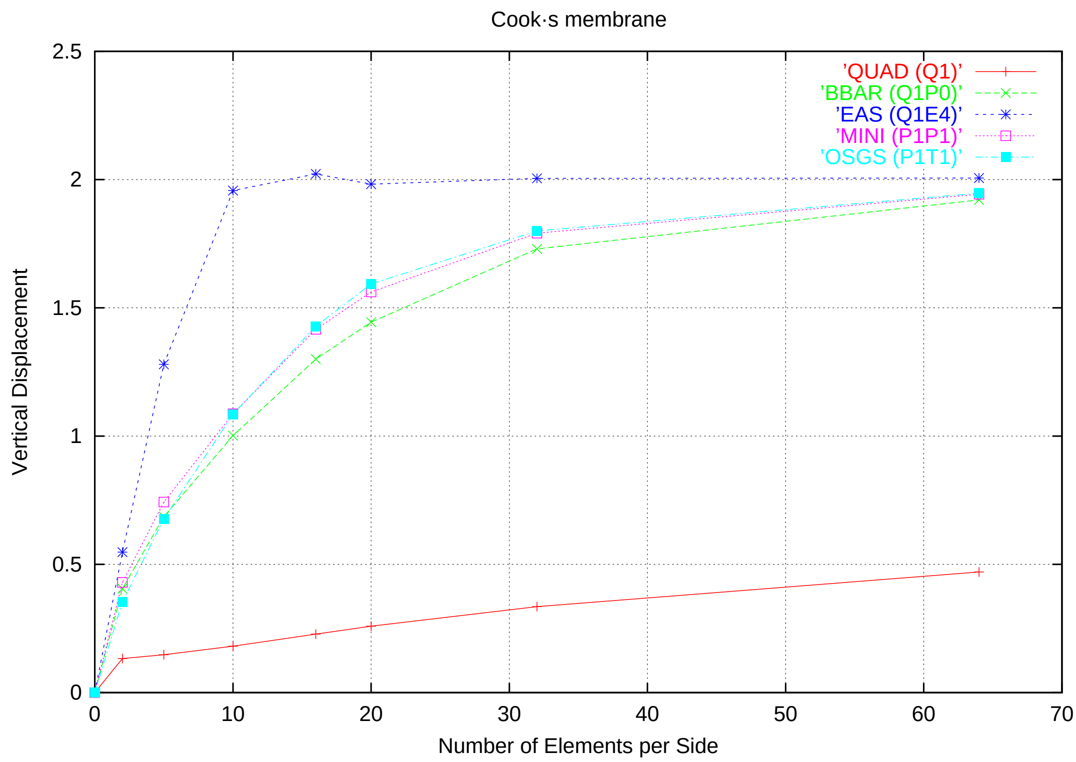
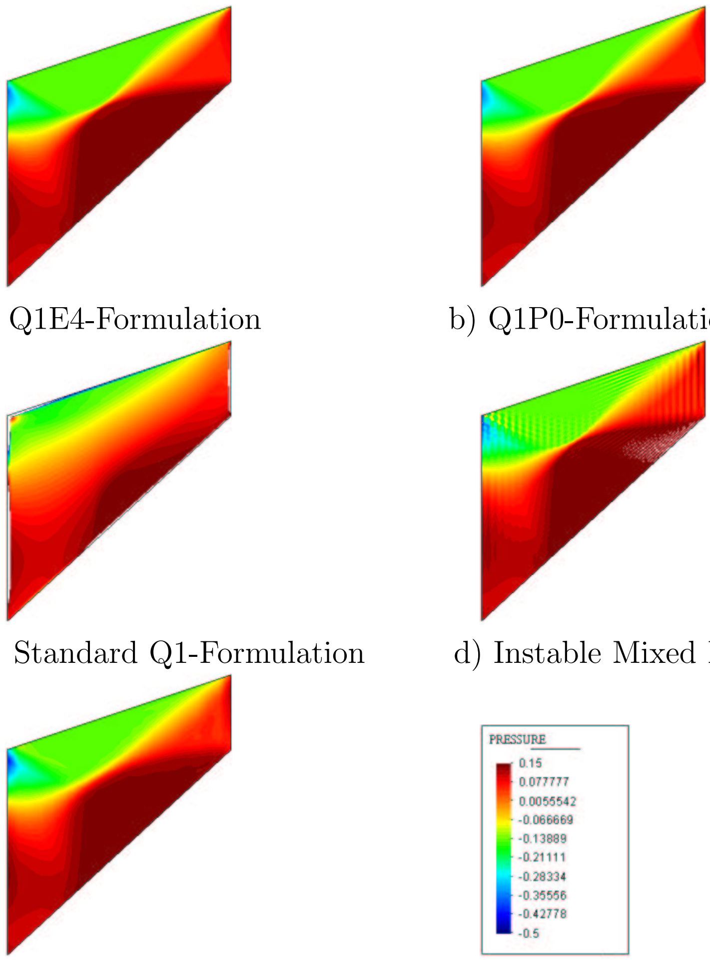


Figure 4.8: Cook's membrane problem: convergence of the displacement at the right upper corner of the clamped wing by using several mesh configurations

the quadrilateral but the number of nodes remains the same.
At first shall be analyzed the displacement of the edge node in the upper right corner by refining the mesh up to 64×64 elements. The refinement shall be lead to a convergence of the numerical solution to the analytical solution which has the value of 2.0. As we can see in the graph **4.8** only the EAS-element reaches to this analytical value already with coarse meshes. This has the reason in the fact that the EAS-formulation has a very good comportment in bending dominated problems much better than other improved element formulations. Nevertheless also the both triangular mixed elements, the instable and the OSGS formulated, reach by the mesh size 64×64 nearly to the analytical solution and show a better convergence as the Q1P0-element. The standard quadrilateral element locks at once and could never reach the analytical solution. The effect of the stabilization can shown once more at the results of the pressure. Where the deformation of the instable mixed element and the OSGS-element were nearly the same, the pressure values of the instable mixed element show an oscillating distribution. Again because of the lack of a stabilization of the element. Whereas the OSGS-element shows the same consistent image as the EAS-element and the Q1P0-element. The standard quadrilateral element cannot display the real pressure characteristics because it does not offer the required deformation modes.



a) Q1E4-Formulation

b) Q1P0-Formulation

c) Standard Q1-Formulation

d) Instable Mixed Element

e) OSGS-Formulation

Figure 4.9: Pressure values of the Cook's membrane, discretized by a mesh of 64×64 elements

4.2.2 Compression of a 2D-block

Finally shall be presented an example of inhomogeneous compression in plane strain state to show the effect of locking in plasticity of some element formulations.

The geometry of the 2D-block is given by the length of $0.6m$ and a height of $0.2m$. The compression will be accomplished by a given displacement in the middle of the upper edge of the block over a length of $0.2m$. The value of the prescribed displacement is defined as $\delta = 0.012m$. The boundary conditions are prescribed at the bottom of the block as zero in the y-direction and free in the x-direction. The node in the middle of the bottom is clamped so that the block is determinately supported. Once again the J2-plasticity is applied and it is supposed perfect plasticity with the yield stress $\sigma_y = 150MPa$. The Young's modulus is $E = 1.96 \cdot 10^5 MPa$ and the Poisson's ratio has the value $\nu = 0.3$, thus compressible. The dominion is discretized in case of quadrilaterals by a mesh of 300 elements, respectively 341 nodes, and in case of triangles by a unstructured mesh of 617 elements and 346 nodes so that the degrees of freedom of the whole structure are nearly the same. The total prescribed displacement is applied in 100 steps.

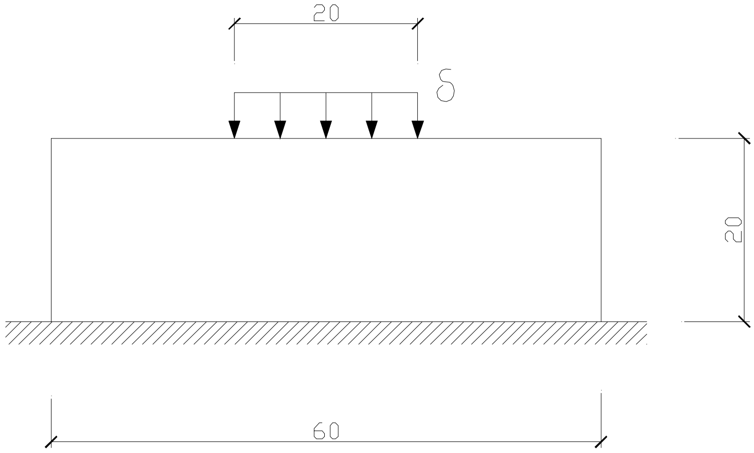


Figure 4.10: Inhomogeneous compression on a 2D-block

In the figure **4.11** are shown the distributions of the pressure on the left side and the Von Mises stresses on the right side. As in the previous examples once again the standard triangular and quadrilateral element cannot capture the physical correct stress distribution. This is attributed to the locking of the elements although no incompressibility exists which means that in this case the effect depends only on the isochore properties of the applied J2-plasticity. The three other formulations give quite the same distributions in the pressure as well as in the Von Mises stresses. Nevertheless it has to be emphasized that the pressure limit by using the OSGS-formulation is higher than in case of the EAS-formulation and the Q1P0-formulation. This could be referred to the approximation of the shear modulus in case of J2-plasticity. It shows that this simplification can offer good results as demonstrated in the Cook's membrane problem and also the results for this problem are much better than the results of the standard elements but could still be improved by a different and maybe better alternative as the proposed one which leads to further investigations.

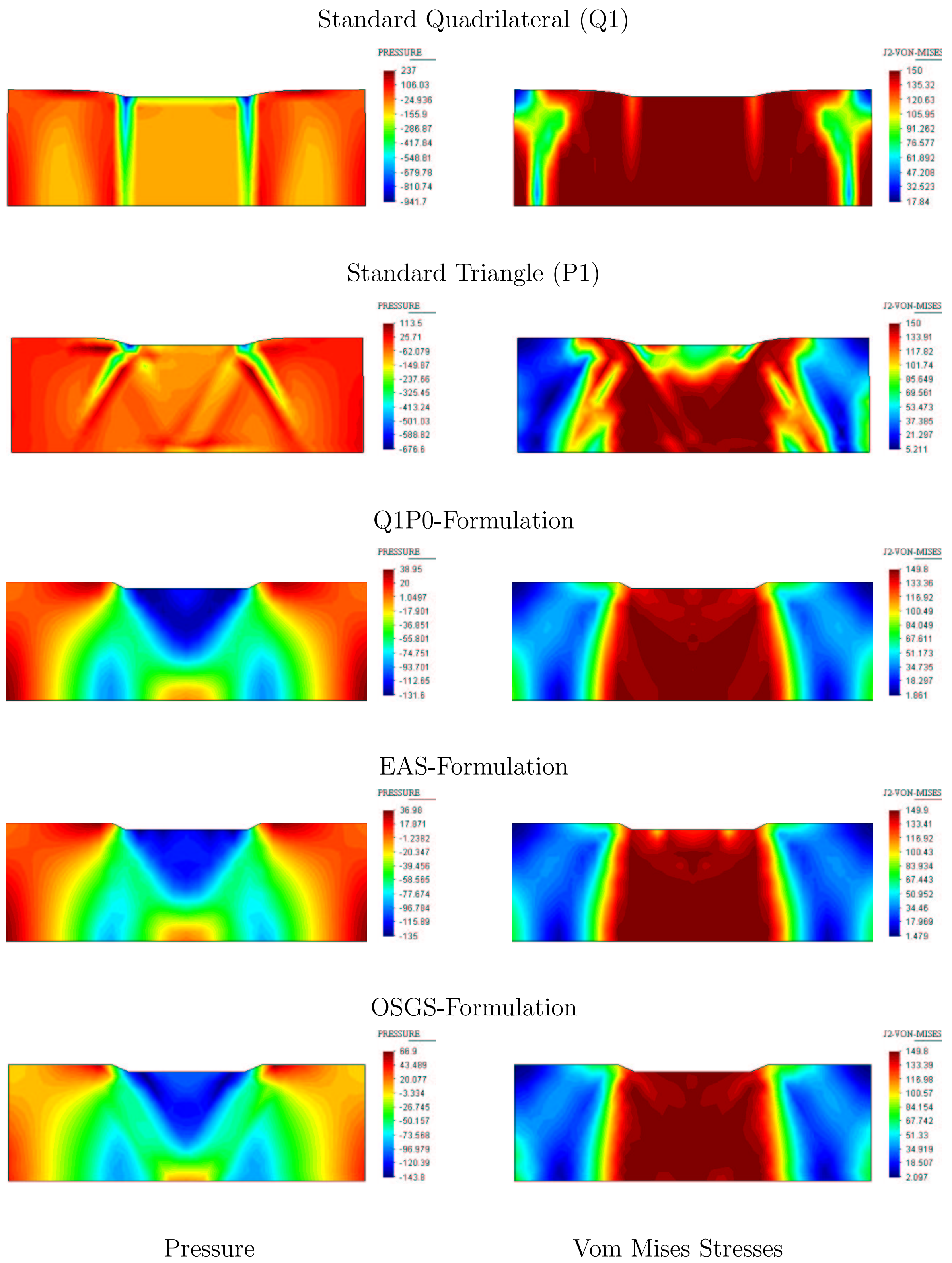


Figure 4.11: Inhomogeneous compression on a 2D-block

Chapter 5

Conclusion

5.1 Summary

The demonstrated object in this work was the development of an efficient low-order element formulation with the capacity to represent the problem of incompressibility in solid mechanics. The standard elements of low order unite various requirements for the utility in practical applications but present locking effects in the case of incompressibility. The alternative in view of this difficulty can be the mixed finite element formulation. The stability of the mixed elements depends on the compatibility between the involved interpolation fields. The mathematic condition that expresses this restriction is known as the stability condition of *Ladyzhenskaya-Babuška-Brezzi* or LBB-condition. The mixed elements with linear displacement and pressure interpolations do not satisfy this condition and therefore present stabilization problems which are manifested in the lack of the control over the pressure. This means that the feasibility of these mixed elements require the application of a stabilization method. The stabilization method used in this work bases on the concept of the sub-grid scale approach, particularly the method of orthogonal sub-grid scales (OSGS). From the point of view of the method, the numerical instabilities of certain formulations have the reason in the fact that the effects of the unresolved components of the solution, named as sub-grid scales, are not taken into account in the balance equations. The concept of the orthogonal sub-grid scales proposes that the orthogonal space to the finite element space is the natural space to search for the sub-grid scales of the solution. The stabilization method circumvents the LBB-condition and allows to obtain elements with interpolations of the same order for the displacement and the pressure fields.

5.2 Statements

The main conclusions of this work shall be resumed in the following statements:

- An adequate finite element formulation to represent the problem of incompressibility in solid mechanics has been developed, implemented and valuated; in case of elasticity as well as in case of J2-plasticity.

- The developed elements are mixed elements stabilized by the method of orthogonal sub-grid scales (OSGS) and exhibit linear and continuous interpolations of the displacement and the pressure fields. These elements offer low computational cost and show good comportments against other known element formulations like the EAS- and the Q1P0-element.

- The developed formulation can be applied to triangular as well as to quadrilateral elements. However the linear triangular element is more interesting for the practical application because of its possibility to generate meshes of complexer geometries as used in many industrial problems. This is a great advantage against the quadrilateral element formulations as for example the EAS-formulation or the Q1P0-element.

- The proposed element is able to represent the situation of incompressibility in a satisfying way. The condition of incompressibility is obliged in the global form. In contrast to this in elements like the Q1P0-element this condition is obliged element wise which can lead in extreme situations to the so called *hourglassing*, a kind of eigenmodes that degrade the result of the solution.

5.3 Aspects of future investigation

Following considerations could be made to improve or to expand the application of the developed element formulation:

- The introduction of the sub-grid scale to the pressure in the formulation. The capability of the orthogonal sub-grid scales has been utilized only in the formulation of the displacement field and it could be observed that the formulation gives satisfying results. Nevertheless, the integration of the sub-grid scales of the pressure could improve once more the stabilizing characteristics.

- The study of the influence in case of compressibility to the stabilization parameter.

- Enhancing the developed formulation to the case of finite deformations (already in progress and partially concluded by the work of M. Cervera and M. Chiumenti but not mentioned in this work).

- Finding alternatives or improvements for the approximation of the stabilization parameter τ and the shear modulus μ^* in case of J2-plasticity.

- Looking for further alternatives for the sub-grid scale approach and comparing them to the presented orthogonal sub-grid scales.

- The study of dynamic problems and their effect on the stabilization method.

- Developing other applications of the stabilization method in solid mechanics like coupled thermo-mechanical problems, contact problems, shell formulations, fracture models and so on.

()

References

[1]ARNOLD, D. N.; BREZZI, F.; FORTIN, M.: A stable finite element for the Stokes equations, Calculo, Vol. 21, pp. 337-344 (1984)

[2]BAIOCCHI, C.; BREZZI, F.; FRANCA, P. F.: Virtual bubbles and Galerkin-least-squares type methods, Computer Methods in Applied Mechanics and Engineering Vol. 105, pp. 125-141 (1993)

[3]BATHE, K. J.: Finite Elemente Methoden, Springer Verlag, 2. Auflage (2002)

[4]BREZZI, F.; FORTIN, M.: Mixed and Hybrid Finite Element Methods, Springer Verlag (1991)

[5]CHIUMENTI, M.; VALVERDE, Q.; AGELET DE SARACIBAR, C.; CERVERA, M.: A stabilized formulation for incompressible elasticity using linear displacement and pressure interpolations, Computational Methods in Applied Mechanics and Engineering, Vol. 191, pp. 5253-5264 (2002)

[6]CIARLET, P.G.: The Finite Element Method for Elliptic Problems, North Holland, New York (1978)

[7]CODINA, R.: A finite element model for incompressible flow problems, PhD. thesis, Universitat Polytecnica de Catalunya (1992)

[8]CODINA, R.: Stabilization of incompressibility and convection through orthogonal sub-scales in finite element methods, Computer Methods in Applied Mechanics, Vol. 190, pp. 1579-1599 (2000)

[9]CODINA, R.: Stabilized finite element approximation of transient incompressible flows using orthogonal sub-scales, Computer Methods in Applied Mechanics, Vol. 191, pp. 4295-4321 (2002)

[10]CODINA, R.; VASQUEZ, M.; ZIENKIEWICZ, O.: A general algorithm for compressible and incompressible flow - part III: The semi-implicit form, International Journal for Numerical Methods in Fluids, Vol. 27, pp. 13-32 (1998)

[11]HUGHES, T. J. R.; FRANCA, L.; BALESTRA, M.: A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, Computer Methods in Applied Mechanics, Vol. 59, pp. 85-99 (1986)

[12]HUGHES, T. J. R.: The Finite Element Method - Linear Static and Dynamic Finite Element Analysis, Dover Publications Inc. (2000)

[13]IRONS, B.; RAZZAQUE A.: Experiences with the patch test for convergence of finite elements, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, pp. 557587, Academic Press, New York (1972)

[14]KASPER, E. P.; TAYLOR, T. L.: A Mixed-Enhanced Strain Method: Linear Problems, UCB/SEMM-97/02 (1997)

[15]KLAAS, O.; MANIATTY, A.; SHEPHARD, M. S.: A stabilized mixed finite element method for finite elasticity formulation for linear displacement and pressure interpolations, Computer Methods in Applied Mechanics and Engineering, Vol. 180, pp. 65-79 (1999)

[16]MALKUS, D. S.; HUGHES, T. J. R.: Mixed Finite Element Methods - Reduced and Selective Integration Techniques: A Unification of Concepts, Computer Methods in Applied Mechanics and Engineering Vol. 15, pp. 68-81 (1978)

[17]PERCHAT, E.: MINI-Élément et factorisations incomplétes pour la parallélisation d'un solveur de Stokes 2D - Application au forgeage, Thése Docteur, L'École Nationale Supérieure des Mines de Paris (2000)

[18]QUECEDO, M.; PASTOR, M.; ZIENKIEWICZ, O. C.: Enhanced linear triangle for plasticity problems in J2-Solids, Computer Methods in Applied Mechanics and Engineering, Vol. 188, pp. 145-163 (2000)

[19]REESE, S.: Advanced Finite Element Technology, Numerische Mechanik und Simulationstechnik, Ruhr University Bochum (2001)

[20]SIMO, J. C.; RIFAI, M. S.: A class of mixed assumed strain methods and the method of incompatible modes, International Journal for Numerical Methods in Engineering, Vol. 29, pp. 1595-1638 (1990)

[21]VALVERDE GUZMÁN, Q. M.: Elementos estabilizados de bajo orden en mecnica de slidos, Tesis Doctoral, Universitat Politcnica de Catalunya (2002)

[22]ZIENKIEWICZ, O. C.; TAYLOR, R. L.: The Finite Element Method - Volume 1, Butterworth Heinemann, 5th edition (1999)