A laminated structural finite element for the behavior of large non-linear reinforced concrete structures.

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Abstract

In order to correctly predict the kinematics of complex structures, analysis using three-dimensional finite elements (3DFEs) seems to be the best alternative. However, simulation of large multi-layered structures with many plies can be unaffordable with 3DFEs because of the excessive computational cost, especially for non-linear materials. In addition, the discretization of very thin layers can lead to highly distorted FEs carrying numerical issues, therefore, reduced models arise as an affordable solution.

This paper describes a new finite element formulation to perform numerical simulations of laminated reinforced concrete structures. The intention of this work is that the proposed scheme can be applied in the analysis of real-life structures where a high amount of computational resources are needed to fulfill the meshing requirements, hence the resulting formulation has to be a compromise between simplicity and efficiency.

So that, the condensation of a dimension (thickness), mandatory to model three-dimensional structures with two-dimensional finite elements (2DFEs), leads to refer all layers contained within such FEs to a plane, which is typically named middle plane or geometrical plane, since its sole function is to serve as a geometrical reference. This work is based on the assumption that the geometrical plane has to be distinguished from a mechanical plane, which is where the resultant stiffness of all layers is contained. It is also assumed in this work that the mechanical plane changes its position due to non-linear response of the component materials.

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1. Introduction

Current theories that allow the use of two-dimensional FEs to model composite materials, yet powerful, lack the necessary simplicity for their application in complex structures where a large amount of FE is required for a good approximation in the obtained results.

Thus, simpler and more efficient techniques are required for modelling laminated structures, where the three-dimensional description can be reduced to a two-dimensional model by introducing hypotheses on the

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displacements and/or on the stresses field, since laminate thickness is at least one order of magnitude lower than in-plane dimensions.

Reference [1] provides an overview of the available theories and finite elements that have been developed for multi-layered, anisotropic, composite plate and shell structures. Multi-scale approaches [2, 3] can also be used to model non-linear multi-layered materials. In this method a macroscopic model is used to obtain the global response of the structure whereas the material behaviour, modelled with a constitutive law, is solved with a microscopic model.

Many reduced approaches have been developed and improved since 19th century. In order to facilitate their classification, they could be distinguished according to [4]:

a) The type of unknown variable chosen, so they could be Displacement Based Theories (DB), Stress-Based Theories (SB), or if both stress and displacement are considered as unknowns, a Mixed Approach (MB) is obtained.

b) How the unknown variables are described. In this classification it may be an Equivalent Single Layer (ESL) description, where governing equations are written for the whole plate, or a Layer-Wise (LW) description, where each layer is treated independently assuming separate displacement/stress fields within each ply, which leads to write the governing equations for each layer.

From the previous classification, the most basic DB-ESL model is the Classical Theory (CT) [5], whereas an improvement to the CT theory is the First Order Shear Deformation Theory (FSDT) [6] which enhances the CT kinematics by adding shear effects. Although CT and FSDT are excellent alternatives to accurately model homogeneous thin and thick structures, they lead to poor prediction in the cases listed below

- Where component materials have a high level of transverse anisotropy.
- When applied to the analysis of composite laminated with embedded debounding.
- When it is necessary to provide regions with 3-D stresses fields, i.e. $\sigma_Z \neq 0$.
- When it is required to capture the so called zig-zag pattern of in-plane displacements (ZZ condition).
- When it is required to satisfy the condition of continuous transverse shear along the thickness direction (TC condition).

The cause is found in the linear thickness distribution of the axial displacement, which does not match the ZZ pattern depicted in figure 1 [4].

Fig. 1 Continuous zigzag in-plane displacement a), discontinuous in-plane stress b), and continuous transverse stress c).
In order to fulfill the previously listed condition it must be necessary to use either a theory based on 3-D kinematics, or a LW based theory. Although LW theories accurately fulfill both, the ZZ and the TC condition, the number of unknown variables is proportional to the number of analyzed layers. As a result, these models yield not only a high level of accuracy but also to an amount of unknown variables similar to the 3D analysis. For this reason, LW models may result unattractive for simulating large laminated structures with many plies. Therefore, these models should be employed to analyze complex problems where other less expensive approaches fail to give realistic predictions [4].

A special case of LW model where the number of unknowns is independent of the number of analyzed layer is the Zigzag theory (ZZT), which is a good compromise between the accuracy of MB-LW theories and the computational efficiency of DB-ESL models. One of the most important advantages of these theories is that the number of kinematics unknowns are independent of the number of analyzed layers.

Have to be remarked that, the study of the cases where it is mandatory to use a LW description are out of the scope of this work, and the only feasible solution from a computational point of view, which allows to achieve good results, is to adopt a ESL scheme.

That is why it has been proposed a scheme capable of reproducing the bending damage of a laminated material without the need of additional degrees of freedom than the ones listed below.

\[
d = \begin{bmatrix}
    w_1 & \theta_{x1} & \theta_{y1} & w_2 & \theta_{x2} & \theta_{y2} & w_3 & \theta_{x3} & \theta_{y3}
\end{bmatrix}^T
\]  

(1)

This simplification is justified by the fact that stiffness of the simple materials used for reinforced concrete (RC) structures never exceeds an order of magnitude. In addition, in order to avoid shear locking situations [7], the proposed scheme has been implemented using a Discrete Kirchhoff Triangle [8] where the shear transverse strains are postulated to be neglected with respect to other strains.

The paper is organized as follows. We start by presenting the basic framework of the plate theory, and later the proposed modification to finally detail an implementation into the FE framework. Section 3 describes the application of the proposed scheme to non-linear materials, and consequently, to plates with bending degradation. Finally, in section 4 it is shown the performance of the proposed plate scheme, and are presented some numerical examples; in particular, a linear clamped beam, a linear clamped plate with a notch, a non-linear clamped beam, an unreinforced concrete frame and a reinforced concrete frame.

2.1. Geometry and load.

The plate term is referred to a flat slender body, occupying the domain

\[\Omega = \{(x,y,z) \in \mathbb{R} \mid z \in \left[-\frac{t}{2}, \frac{t}{2}\right], \ (x,y) \in \mathcal{A} \subset \mathbb{R}^2\}\]

(2)

where the plane \(Z = 0\) coincides with the mid-plane (also referred as the geometrical along this work) of the undeformed plate and the transverse dimension, of thickness \(t\), is small compared with the other two dimensions.

2.2. Kinematics

The infinitesimal kinematic for a plate always assume the following displacement fields:

\[u(x, y, z) = -z\theta_x(x, y), \quad v(x, y, z) = -z\theta_y(x, y), \quad w(x, y, z) = w(x, y)\]

(3)

where \(u\), \(v\) and \(w\) are the displacements along \(x\), \(y\) and \(z\) axes respectively, and \(\theta_x\) and \(\theta_y\) are the rotations of the transverse line elements about the \(x\) and \(y\) axes. Accordingly, a straight line element, normal to the plate mid-surface in the undeformed configuration could either remain straight and normal according to the classical Kirchhoff plate theory, or not necessarily normal according to Reissner [9] and Mindlin [10], being the difference among such theories the effect of the transverse shear deformation.
The use of classical theories for thin and thick plates applied to homogeneous plates is the best alternative. However, when it comes to laminated plates formed by layers made of composite materials, the use of such theories is not sufficient to assess the in-plane deformation fields of their components. That is why this paper proposes a scheme in which the in-plane displacements field $u$ and $v$ are evaluated according to the mechanical and constitutive properties of the constituent materials.

First, let us assume the existence of points $a$ and $b$. Point $a$ is defined as the intersection of the middle plane and the undeformed normal of the plate, whereas point $b$ is defined as the intersection of the theoretical deformation of the normal and the neutral (or mechanical) plane. The purpose of using points $a$ and $b$ is to refer the displacement field $w$ to point $a$, whereas point $b$ will be the reference for rotations $\theta_x$ and $\theta_y$, consequently, it will be used as the reference for in-plane displacements $u$ and $v$ of the layers within the laminate material. Figures 2.a and 3.a are used as an aid to express this idea. The case when $a = b$ is where the plate is formed by a homogeneous material, as depicted in figure 2.

Now, let us assume the particular laminate from figure 2.b in its layer 8 has a material stiffness than the rest, leading to the case illustrated in figure 3.b. This is $a \neq b$, and the consequences is that the strains distribution although remain linear, no longer remain symmetrical to the middle plane, also there is a variation in the stress distribution and in the generalized stresses $\hat{\sigma}$, which in figure 3.c are represented using bending moment $M$. From figure 3.b can also be noticed that at point $b$ (neutral plane) the in-plane displacements are equal to $u = 0$ and $v = 0$.

The same as for both thick and thin plate theories, the integration of the in-plane stresses $\sigma_x, \sigma_y$ and $\gamma_{xy}$, and the transverse shears stresses $\gamma_{xz}$ and $\gamma_{yz}$ define the stress resultant per unit length

$$\hat{\sigma}_x = \int_{-l/2}^{l/2} \sigma_x \, dz, \quad \hat{\sigma}_y = \int_{-l/2}^{l/2} \sigma_y \, dz, \quad \hat{\sigma}_{xy} = \int_{-l/2}^{l/2} \sigma_{xy} \, dz$$

$$\hat{\gamma}_x = \int_{-l/2}^{l/2} \gamma_{xz} \, dz, \quad \hat{\gamma}_y = \int_{-l/2}^{l/2} \gamma_{yz} \, dz$$

where $z$ has to be redefined as will be seen later.

According to the classification of the existing theories, the proposed scheme is a modification of the DB-ESL, where it is taken into account the evolution of the eccentricity of geometric and mechanical planes of a
2.3. FE implementation

Although the proposed integration scheme is easily extensible to any bi-dimensional plate element, here the two-dimensional triangular FE developed by Batoz et al. [8], depicted in figure 4 and commonly referred to as the DKT element, has been taken as the starting point. The D.O.F. for element DKT have been presented in equation 1, and such element is based upon the assumption listed below.

1. Rotations vary quadratically over the element

\[
\theta_x = \sum_{i=1}^{6} N_i \theta_{x_i} ; \quad \theta_y = \sum_{i=1}^{6} N_i \theta_{y_i}
\]

Fig. 4 Bi-dimensional triangular plate element, commonly called DKT element.
where \( \theta_x \) and \( \theta_y \) are the nodal values at the corners and at the mid-nodes [fig.4], and \( N_i(\xi, \eta) \) are shape functions.

\[
\begin{align*}
N_1 &= 2(1 - \xi - \eta)(\frac{1}{2} - \xi - \eta) \\
N_2 &= \xi(2\xi - 1) \\
N_3 &= \eta(2\eta - 1) \\
N_4 &= 4\xi\eta \\
N_5 &= 4\eta(1 - \xi - \eta) \\
N_6 &= 4\xi(1 - \xi - \eta)
\end{align*}
\]

2. The Kirchhoff hypothesis is imposed at corners (nodes 1, 2, 3)

\[
\theta_x + \frac{\partial w}{\partial x} = 0 ; \quad \theta_y + \frac{\partial w}{\partial y} = 0
\]

and at the mid-nodes \((k = 4, 5, 6)\)

\[
(\theta_x)_k + \left( \frac{\partial w}{\partial s} \right)_k = 0
\]

3. The variation of \( w \) along the sides is cubic

\[
\left( \frac{\partial w}{\partial s} \right)_k = \frac{3}{2A} w_i - \frac{3}{4} \left( \frac{\partial w}{\partial \zeta} \right)_i + \frac{3}{4} \left( \frac{\partial w}{\partial \zeta} \right)_j - \frac{3}{4} \left( \frac{\partial w}{\partial \zeta} \right)_j
\]

with \( k = 4, 5, 6 \) denoting the mid-node of side \( ij \) = 23, 31, 12 respectively, and \( l_i \) equal to the length of the side \( ij \).

4. A linear variation of \( \theta_n \) is imposed along sides

\[
(\theta_n)_k = \frac{1}{2} [(\theta_n)_i + (\theta_n)_j]
\]

The evaluation of the stiffness matrix for the DKT element follow the standard procedures of the finite element method, as depicted in equation 6.

\[
K = \int_A B^T D_b B \ dA = 2A \int_0^1 \int_0^{1-\xi_0} B^T D_b B \ d\zeta_2 d\zeta_3
\]  \hspace{1cm} (6)

where \( K \) is the stiffness matrix, \( A \) is the area of the FE, the deformation matrix \( B \) is defined as

\[
B = \frac{1}{2A} \begin{bmatrix}
y_{31}H_1^T + y_{12}H_3^T \\
-x_{31}H_2^T - x_{12}H_4^T \\
-x_{31}H_3^T - x_{12}H_4^T + y_{31}H_1^T + y_{12}H_3^T
\end{bmatrix}
\]  \hspace{1cm} (7)

being vectors \( H_1 - H_4 \) functions of \( \zeta_2 \) and \( \zeta_3 \) defined in [8].

As will be seen later, the evaluation of the flexural stiffness for a plate element \( D_b \) is going to be the cornerstone of this work, let us first point out that the corresponding value for a continuum material, as the one depicted in figure 2 can be evaluated using equation 8.

\[
D_b = \int_{-\frac{1}{2}}^{+\frac{1}{2}} z^2 D d\zeta
\]  \hspace{1cm} (8)
The matrix $\mathbf{D}$ corresponds to the constitutive matrix of the material in plane stress. As can be seen in equation 9, $\mathbf{D}$ has not been integrated along the thickness of the plate.

$$
\mathbf{D} = \begin{bmatrix}
    d_{11} & d_{12} & 0 \\
    d_{21} & d_{22} & 0 \\
    0 & 0 & d_{33}
\end{bmatrix}
$$

However, for a laminated material formed by materials with different mechanical properties, the previous approach is limited, and it becomes necessary to express the flexural stiffness $\mathbf{D}_b$ in terms of the geometrical and mechanical properties of the $k_{th}$ layer [11], leading to equation 10, which basically is a more general form of equation 8, since $\mathbf{D}$ is no longer constant along the thickness, hence

$$
\mathbf{D}_b = \sum_{k=1}^{n} \frac{1}{3} (z_k^2 - z_{k-1}^2) \mathbf{D}^k
$$

where $\mathbf{D}_{ij}$ has been defined in equation 9, $k = 1, ..., n$ is the current layer, and $n$ the total number of layers within the laminated material.

Figure 2.c depicts a typical distribution of stresses and strains within a laminated material, although properly speaking, it is not a laminated material at all, since it is composed by a one-single material. In any case, figure 2 is necessary as a reference to locate the layers, the middle plane and to state the position parameters $z_{k-1}$ and $z_k$ within a laminated material (fig. 2.b).

In order to perform the proposed integration scheme over the thickness, equation 10 has to be redefined, and position parameters $z_k$ and $z_{k-1}$ have to be redefined also. For convenience, they are named $\hat{z}_k$ and $\hat{z}_{k-1}$ and referred to the bottom of the plate now, as shown in figure 3.b. This leads us to the cornerstone of the proposed scheme, which is the assessment of the neutral axis $\hat{Z}$. To define $\hat{Z}$, let us start defining the auxiliary matrix $\hat{\mathbf{D}}_b$.

$$
\hat{\mathbf{D}}_b = \int_{0}^{\hat{z}_k} \hat{y}_k \mathbf{D}^k dz = \sum_{k=1}^n \hat{y}_k t_k \mathbf{D}^k
$$

where the membrane stiffness $\mathbf{D}_m$ also has a role to play in the scheme and is defined as

$$
\mathbf{D}_m = \sum_{k=1}^{n} (\hat{z}_k - \hat{z}_{k-1}) \mathbf{D}^k
$$

Using equation 12, the bending stiffness tensor $\mathbf{D}_b$ from equation 10 is rewritten into equation 14.

$$
\mathbf{D}_b = \sum_{k=1}^{n} \left[ \frac{\hat{z}_k^2}{12} + t_k (\hat{Z}^k)^2 \right] \mathbf{D}^k
$$
finally, $\hat{Z}^k$ from equation 14 is defined in equation 15.

$$\hat{Z}^k = \hat{y}_k - Z$$

(15)

With this new expression (eq. 14) the bending stiffness tensor is modified by non-linear effects, result of material failure, by two different means. One corresponds to the variation of the material stiffness matrix, which now has a lower stiffness as it is replaced by the secant tensor, and also by the position of the layer with respect to the new neutral axis of the laminate.

Shall be remarked that $\hat{Z}$ will be evaluated at the beginning of a quasistatic loading process, and then in every iteration once damage has occur within the given FE. Also, has to be pointed out that this integration scheme is intended to be used with a secant tensor $D_{sec}$ instead of $D$ in equations 11, and 14, this in order to reproduce the change in the position of the mechanical plane using non-elastic constitutive equation.

Using equation 16 is now possible to evaluate the in-plane strains for each of the layers.

$$\varepsilon^k = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = -\hat{Z}^k \begin{bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{bmatrix}$$

(16)

the corresponding stresses can be evaluated using equation 17.

$$\sigma^k = D^k \varepsilon^k$$

(17)

where $\sigma^k = [\sigma_x \sigma_y \sigma_{xy}]^T$. Finally, using $\hat{Z}$ the integration through the thickness can be carried out using equation 18.

$$\sigma = \sum_{k=1}^{n} \hat{Z}^k \sigma^k$$

(18)
3. Constitutive Formulation

In this section a brief description of the expected bending degradation is carried out, focusing on the evaluation of the secant constitutive tensor \( D^{sec} \) which is required to reproduce the bending degradation that arises when some layers within a laminated plate are beyond the elastic threshold and others are not.

3.1. Mechanical plane of a bending plate

Let us start considering the example shown in figure 5 where the layer distribution of a plate under a monotonic bending stress is depicted. Layers of the laminated plate are formed by a homogeneous and isotropic material, in such a way mechanical plane lays in the geometrical plane. Let us also consider that the simple material forming the laminated plate is a concrete-like material, where there is a pronounced difference among tension and compression damage threshold.

As bending moment monotonic load starts being applied, the expected distribution of strains along the thickness would be the shown in figure 5.a, whereas stress distribution would be like the ones shown in figure 5.b. Both stresses and strains corresponding to figures 5.a and 5.b would be in the elastic range.

As bending moment increases, material non-linearity would be reached. The expected response of the laminate, in case of considering simple concrete, is such that only the layers under tension stress (since tension is by far the less resistant stress) reach the non-linear range, and due to this, layers subjected to compression forces undergo a gradual increase of stress. Consequently, strain level of the composite material slightly beyond the damage threshold would be as shown in figure 5.c, also, from such figure notice that mechanical and geometrical axes no longer correspond. On the other hand, expected stress distribution would be presented in figure 5.d.

Finally, if it stills imposing a bending moment up to a fully damaged state in the plate, strain distribution would be very much alike at the ones presented in figure 5.e and the corresponding stress distribution shall be the one shown in figure 5.f. The movement of the mechanical plane, using the same definition of equation 2, is

\[
\hat{Z}^{i,j} \in \left[ \frac{t_1^2}{t_1^2 + t_2^2} \right]
\]

and in order to get better accuracy determine to it, it becomes mandatory to perform a finer layer distribution in the following cases:

- At the farthest zones away from the geometric axis of the shell, since such layers will be subjected to the higher stresses while acting a bending stress.

- At zones where exists an abrupt change of stiffness, as is the case of the presence of steel reinforcement within the concrete, no matter their position within the overall thickness.

The main idea of such layer distribution is to endow the proposed integration scheme, with the capability of capturing the mechanical axis once the shell element has undergone a non-linear effect. The assumption is that, when the element damages, the mechanical axis moves away from its original position toward one end (according to the bending moment’s direction), and the nearest to one end, the more damage state is able to represent.

3.2. Isotropic Damage Model

Continuum damage models (CDM) have been widely accepted as an alternative to deal with complex constitutive behaviour [12, 13, 14, 15]. Among the different possibilities such a framework offers, the simplest is the one referring to isotropic damage models where the non-linear behavior is monitored through a single internal scalar variable called damage or degradation, \( d \). The meaning of internal variable \( d \) is the measurement of the loss of secant stiffness of the material, and it ranges from 0 for the undamaged material to 1 for the fully degraded one (0 ≤ \( d \) ≤ 1) [16, 17]. The constitutive equation for an isotropic damage model has the form:
Fig. 5 Schematic representation of range to move of mechanical axis.
\[ \sigma = (1 - d)D : \varepsilon \]

and the secant stiffness \( D^{\text{sec}} \) is defined as

\[ D^{\text{sec}} = (1 - d)D \]

It is important to remark that in the proposed scheme, during the degradation process, the gradual loss of bending stiffness will be evaluated using the secant constitutive tensor of each layer. An overview of the components conforming the model proposed by Oliver et al [16] is summarized in next paragraph.

a. a suitable norm, \( \tau \), used to compare different states of the deformation making possible to define concepts such as loading, unloading and reloading.

b. a damage criterion, \( F(\tau, r) \leq 0 \), formulated in the strain or the undamaged stress space. The simplest form of this criterion is:

\[ F(\tau, r) = G(\tau) - G(r) \leq 0 \]  

where \( \tau \) is the norm described in (a), \( r \) is the damage threshold, and \( G(\bullet) \) is a suitable monotonic scalar function. Using equation 22, now damage is numerically defined to occur when \( G(\tau) > G(r) \).

c. evolution laws for the damage threshold and the damage variable. Such laws are given by the rate expressions:

\[ \dot{r} = \dot{\mu} \]

\[ d = \dot{\mu} \frac{dG(\tau)}{d\tau} = \dot{\mu} \frac{dG(\tau)}{d\tau} \]  

where \( \dot{\mu} \) is the damage consistency parameter used to define loading/unloading conditions according to the Kuhn-Tucker relations.

\[ \dot{\mu} \geq 0 \ ; \ F(\tau, r) \leq 0 \ ; \ \dot{\mu}F(\tau, r) = 0 \]

The norm \( G(\tau) \) together with the damage criterion play the important role of defining the yield surface, this is when \( G(\tau) - G(r) = 0 \), this is particularly important, since most of the common yield surface can be reproduced using an isotropic damage model, like the ones proposed by are Tresca and Mohr-Coulomb, or the modification to the Mohr-Coulomb yield surface proposed by Oller [18] which are suitable to model concrete-like materials. Ductile materials such as metals, on the other hand, can also be modeled using a von Mises yield surface.

3.3. Secant tensor for elasto-plastic materials

Including the evolution of permanent deformations or plastic deformations in a CDM allows to represent the behavior of ductile materials with an elasto-plastic behavior.

Although in this work the proposed scheme only focuses on an isotropic CDM, the concepts presented here can also be extended to plastic damage models, like the one proposed by are Tresca and Mohr-Coulomb, or the modification to the Mohr-Coulomb yield surface proposed by Oller [18] where it is used a normalized internal variable to represent the plastic damage \( \kappa^p \). Range of values for \( \kappa^p \) are \( 0 \leq \kappa^p \leq 1 \), such that if \( \kappa^p = 0 \) there is no plastic damage, and \( \kappa^p = 1 \) defines the total damage of a solid.

There are also other CDM which include the presence of permanent deformations on the mechanical behavior of geomaterials [19, 20, 21, 22] especially in concrete subjected to compression. Such schemes make use of internal variables to represented damage and permanent plastic deformation.
The idea of using one single damage variable in a general form has been explored by Paredes [23] to be applied in the assess of the natural frequency of vibration of structures. Paredes considers the different existing possibilities to define the damage index \( d \), and proposes the index \( d^\text{eqv} \) to be a function of the yield surface \( f(\sigma) \) defined as:

\[
d^\text{eqv} = 1 - \frac{f(\sigma_c)}{f(\sigma)}
\]

hence, it is now possible to use the damage index \( d^\text{eqv} \) in equation 21 to obtain an equivalent secant tensor. In equation 25 \( f(\sigma) \) is evaluated with the current tensor of stress, whereas \( f(\sigma_c) \) is evaluated with the tensor of effective stresses that define the damage threshold of the material which is defined as

\[
\sigma^e = \begin{bmatrix} f^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

where \( f^* \) is nominal threshold of the material’s resistance. For the case of concrete, such threshold is defined by the resistance to compression, whereas for ductile materials such as steel, it is defined by the yield stress.

Finally, the use of equation 25 takes to the modification of equation 21 resulting in equation 27.

\[
D^\text{sec} = (1 - d^\text{eqv})D
\]

3.4. Two-scalar damage models

Other damage model highly extended to model the opening and closing of cracks due cyclic loads in geomaterials is the one proposed by Faria et al in [24], here two scalar damage variables \( d^+ \) and \( d^- \) are introduced as internal variables, so it is possible to distinguish among the damage produced by tensile stresses (cracking) and the damage produced due compressive stresses (crushing). The nature of the scheme proposed by Faria et al makes it more feasible for seismic analysis purposes.

Using the definition of the elastic secant constitutive tensor, the secant constitutive tensor can be evaluated for this scheme as [24]:

\[
D^\text{sec} = \frac{\partial \sigma}{\partial \varepsilon} = (1 - d^+) \frac{\partial^2 \Psi^+_0}{\partial \varepsilon^e \otimes \partial \varepsilon^e} + (1 - d^-) \frac{\partial^2 \Psi^-_0}{\partial \varepsilon^e \otimes \partial \varepsilon^e}
\]

The use of two internal variables to reproduce the damage is an advantage in terms of the constitutive analysis. On the other hand, it becomes cumbersome while trying to establish one single parameter that define the degradation of the material. This can only be achieved with an equivalent damage index [25] (eq. 27).

3.5. Secant tensor for composite materials

The use of mixing theories allows the numerical simulation of composite materials formed by \( n \) simple materials, being also possible to represent each of them with their own constitutive model. This is because mixing theories function as a manager of constitutive models whose main function is to preserve the compatibility of deformations, the equilibrium of internal forces, and to take into account the corresponding stiffness of each component.

Classical mixing theory (CMT), whose simpler expression is the rule of mixtures (ROM), was first studied in 1960 [26] establishing the basis for subsequent developments. CMT takes into account the volume fraction of the components but not its morphological distribution, since it assumes all component materials experiment a pure parallel behavior (same strain in all directions). To overcome this strong limitation Rastellini et al [27] developed a serial/parallel mixing theory (SP ROM) assuming that components behave as a parallel material in the fiber alignment direction and as serial material in the orthogonal direction. SP ROM will only be mentioned here, and reader may consult [27, 28] for a full reference for the basic
notations, definition and computational implementation. Also reader may consult references [29, 30] where SP ROM formulation has been used to characterize reinforced concrete structures.

During a loading process, if any of the component materials reach the non-linear range, the constitutive tensor of such material will change, and consequently, so the constitutive tensor of the composite material. Hence, to take into account such change, the use of equations 29, 30 and 31 depicts the way the secant tensor has to be evaluated using a SP ROM scheme.

\[
D_{pp}^{\text{sec}} = \left( \frac{1}{k} D_{pp}^{\text{sec}} + \frac{m}{k} D_{pp}^{\text{sec}} \right) + m \frac{1}{k} \left( \frac{1}{D_{pp}^{\text{sec}}} - \frac{1}{D_{pp}^{\text{sec}}} \right) : A \cdot \left( \frac{D_{pp}^{\text{sec}}}{k} - \frac{D_{pp}^{\text{sec}}}{k} \right)
\]

\[
D_{ps}^{\text{sec}} = \left( \frac{1}{k} D_{ps}^{\text{sec}} : A \cdot \frac{m}{k} D_{ps}^{\text{sec}} : A \cdot \frac{1}{D_{ps}^{\text{sec}}} \right) + m \frac{1}{k} \left( \frac{1}{D_{ps}^{\text{sec}}} - \frac{m}{k} D_{ps}^{\text{sec}} : A \cdot \frac{1}{D_{ps}^{\text{sec}}} \right)
\]

\[
D_{sp}^{\text{sec}} = \left( \frac{1}{k} D_{sp}^{\text{sec}} : A \cdot \frac{m}{k} D_{sp}^{\text{sec}} : A \cdot \frac{1}{D_{sp}^{\text{sec}}} \right) + m \frac{1}{k} \left( \frac{1}{D_{sp}^{\text{sec}}} - \frac{m}{k} D_{sp}^{\text{sec}} : A \cdot \frac{1}{D_{sp}^{\text{sec}}} \right)
\]

\[
D_{ss}^{\text{sec}} = \frac{1}{2} \left( \frac{m}{k} D_{ss}^{\text{sec}} : A \cdot \frac{m}{k} D_{ss}^{\text{sec}} : A \cdot \frac{1}{D_{ss}^{\text{sec}}} \right)
\] *(29)*

where

\[
A = \left( \frac{m}{k} D_{ss}^{\text{sec}} + f D_{ss}^{\text{sec}} \right)^{-1}
\] *(30)*

and, \(D_{pp}^{\text{sec}}\) is the component in parallel direction of the secant tensor, \(D_{ps}^{\text{sec}}\) is the component in serial direction of the secant tensor, and \(D_{sp}^{\text{sec}}\) and \(D_{ss}^{\text{sec}}\) are the components in serial and parallel direction. Finally, the secant tensor for a composite material using a SP ROM is:

\[
D^{\text{sec}} = \begin{bmatrix}
D_{pp}^{\text{sec}} & D_{ps}^{\text{sec}} \\
D_{sp}^{\text{sec}} & D_{ss}^{\text{sec}}
\end{bmatrix}
\] *(31)*

Similarly, using the CMT, the secant constitutive tensor of a composite material is equal to

\[
D^{\text{sec}} = \sum_{i=1}^{n} i k D^{\text{sec}}_i
\] *(32)*

where \(i k \) and \(D^{\text{sec}}_i\) are respectively the volumetric participation and the secant constitutive tensor of material \(i\). In both cases, using either CMT or SP ROM \(D^{\text{sec}}\) of the constituent materials have to be obtained using either equation 21 or equation 27.

### 3.6. Mesh objectivity of FE formulation response with strain softening

In continuum mechanics, it is known that the inclusion of strain-softening leads to the increment of strains in narrow strips. This phenomenon is known as strain localization. Two major approaches may be distinguished for analyzing crack propagation, namely: Discontinuous (or Discrete) Crack Approach (DCA) and the Smeared Crack Approach (SCA).

The purpose of this section is only to point out in a briefly manner the strain-softening phenomena and to describe the SCA, since is the approach used in this work. Reader may abound in the subject by consulting reference [31] where a comprehensive review of the formulations used over the last 40 years for the solution of problems involving tensile cracking, with both the DCA and the SCA has been carried out. Some brief remarks on the SCA are listed below.

- **The simplicity of this approach.** SCA can be readily implemented in any non-linear FE code, by simply writing a routine for a new material constitutive model.

- **Re-meshing is unnecessary.** Since the cracking material is assumed to remain continuum and material properties (stiffness and strength) are modified to account for the effect of cracking.

- **Mesh dependency.** A drawback is that the total energy dissipated in the cracking process is proportional to the size of the element.
The requirement that the result from the numerical modelling should be independent of the mesh choice is named mesh objectivity. Several studies have been done to provide objectivity on the results independently of the size of the FE mesh (Oliver [32], Oller [33], Oliver et al [34], Cervera and Chiumenti [35] Lubliner et al [20], Martinez et al [36, 37], Paredes et al [25], among others). A discussion on this topic is beyond the scope of this work, and has to be clarified that the scheme adopted has been specified in the considered model, as described later.

4. Practical Examples

4.1. Linear clamped beam

The first example selected to demonstrate the efficiency of the proposed scheme is shown in figure 6. It is a beam fixed at one end with the other end free where a vertical load \( P = 10 \text{ N} \) is applied. The total length of the beam is \( L = 0.72 \text{ m} \), whereas the dimensions of the cross section are \( b = 0.04 \text{ m} \) and \( h = 0.10 \text{ m} \). The layer distribution of the laminated material is also depicted in figure 6, the thickness of all layers are \( t_k = 0.01 \text{ m} \), all except layer 08 are formed only by a concrete matrix. Layer 8 on the other hand, is a composite material formed by a volumetric participation of 90% of concrete which is reinforced with 10% of steel fibre. Composite material for layer 8 will be modeled using the SP ROM [28]. Both concrete and steel have been modeled using a elastic constitutive equation whose mechanical properties are \( E_c = 3.5 \times 10^7 \text{ N/m}^2 \), \( \nu_c = 0.20 \), \( E_s = 2.1 \times 10^7 \text{ N/m}^2 \) and \( \nu_s = 0.0 \), where subindexes \( c \) and \( s \) stands for concrete and steel respectively.

Considering only \( x \) direction in equations 11, 12, 14, and 15, then

\[
\begin{align*}
\mathbf{D}^{b}_{1,1} &= 1.592 \times 10^7 \text{N} \\
\mathbf{D}^m_{1,1} &= 2.13 \times 10^5 \text{N/m} \\
\mathbf{Z}_{1,1} &= 7.46 \times 10^{-2} \text{m} \\
\mathbf{D}^b_{1,1} &= 5.75 \times 10^{-6} \text{m}^4
\end{align*}
\]

also, using the analogy with the beam of Euler-Bernoulli with \( EI = \mathbf{D}^{b}_{1,1} \), leads to state that

\[
\delta_a = \frac{PL^3}{3\mathbf{D}^b_{1,1}} = -4.56 \times 10^{-3} \text{m} \quad ; \quad \theta_a = \frac{PL^2}{2\mathbf{D}^b_{1,1}} = 9.50 \times 10^{-3} \text{rad}
\]

where \( \delta_a \) and \( \theta_a \) are the displacement and the rotation at point \( a \) respectively (fig. 7).

The results shown in equation 34 will be used as the basis for the upcoming comparisons with two modelling strategies. The first strategy consist in a mesh with 365 nodes and 576 two-dimensional triangular FE combined with the proposed scheme, that for referencing purposes it will be named \textbf{Sh01} from now on. The second modelling strategy consists in 2880 elements and 14,837 nodes formed by 20-node Lagrangian FE, named \textbf{Hex01} from now on.
The layer distribution for the laminated composite material depicted in figure 6 has been selected in order to make comparison in terms of vertical displacement, but especially, in terms of horizontal displacements among both models.

The first point of comparison of the two models are the displacements shown in figure 7. The maximum value obtained for the \( z \) displacement for model \( \text{Hex01} \) is \( \delta_{\text{Hex01}} = -4.79 \times 10^{-3} \) m (figure 7.a), while the value obtained with the proposed scheme is \( \delta_{\text{Sh01}} = -4.55 \times 10^{-3} \) m (figure 7.b), the latter having better approximation compared with the value shown in equation 34.

On the other hand, figure 7.c is used in order to show a comparison among the \( x \) displacements of both models. Also in figure 7.c is depicted the value of \( Z_{1,1} \) obtained in eq. 33, which according to the proposed scheme, will be the reference plane to evaluate the in-plane displacements, and consequently the in-plane strains and stresses for a plate element. As can be seen from such figure, the relationship among the in-plane displacements for both models is exact. Values of the in-plane \( x \) displacements for model \( \text{Sh01} \) have been evaluated using the obtained value for the rotation \( \theta_{\text{Sh01}} = 9.48 \times 10^{-3} \) rad, whereas values for model \( \text{Hex01} \) are the global displacements in \( x \) direction.

Also, a comparison in terms of strains and stresses can be performed more accurately if layer 08 from model \( \text{Sh01} \) is split into 8 layers, which leads to a total of 17 layers for model \( \text{Sh01} \).

Comparison of the stresses undergone by steel fibres are presented in figure 8. Figure 8.a corresponds to obtained results for model \( \text{Hex01} \). In such figure it is also possible to appreciate the change in the stress (compressive and tensile) within the same FE. This effect has been possible to be captured due to the type of FE used, since 4-node and 8-node three-dimensional elements are not able to cast reliable results under bending conditions, such FE have not been considered in this work. Hence, for bending cases, it is highly recommended a quadratic approximation or higher [7], which makes computationally expensive the used strategy for model \( \text{Hex01} \), and naturally arises the reason of using the proposed scheme.

Figure 8.b, on the other hand, shows the stresses at the top of the reinforced layer for model \( \text{Sh01} \), as can be seen, the maximum tensile stress undergone by such layer is \( \sigma_{xx} = 2.710 \times 10^7 \) N/m\(^2\), which matches the obtained for model \( \text{Hex01} \) being \( \sigma_{xx} = 2.757 \times 10^7 \) N/m\(^2\). In both cases, such values are obtained in the fixed...