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A comparison between new adaptive remeshing strategies based on point wise stress error estimation and energy norm error estimation

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SUMMARY

Traditionally, the most commonly used mesh adaptive strategies for linear elastic problems are based on the use of an energy norm for the measurement of the error, and a mesh refinement strategy based on the equal distribution of the error between all the elements. However, little attention has been paid to the study of alternative error norms and alternative refinement strategies. This paper studies the feasibility of using alternative mesh refinement strategies based on

— the use of the classical error energy norm and an optimality criterion based on the equal distribution of the density of error,
— the use of alternative error norms based on measurements of the point wise error contained in the main magnitudes that control the equilibrium problem and/or the material constitutive equations such as the stresses (e.g. the Von Mises stress).

The feasibility of using all the described strategies is demonstrated through the solution of a benchmark example. This example is also used for comparison between the described refinement criteria. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: mesh adaptivity; adaptive remeshing; adaptive optimality criteria

1. INTRODUCTION

Traditionally, the most commonly used mesh adaptive strategies for linear elastic problems are based on the use of the energy norm for the measurement of the error, and a remeshing strategy based on the equal distribution of the error between all the elements [1–11]. Taking into account these two ideas different authors have derived alternative expressions to obtain the element size distribution for a new mesh based on the element size distribution of the old mesh and the corresponding error estimation [1–3, 10–12]. The main

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reasons for the use of the energy norm for the measurement of the error include the following:

- There is a clear mathematical theory that explains the different properties of the energy norm of the error.
- The energy norm has a clear mathematical meaning that is very much related to the finite element discretization process.
- There is a large family of different error estimators based on different approaches (residual-type, stress recovery, etc.) that are quite reliable and provide satisfactory estimations of the existing error in a given finite element solution.

However, the use of a mesh optimality criterion using the equal distribution of the energy error between all the elements is based on the fact that, for a given value of the global energy error norm, this criterion provides the cheapest mesh with the minimum number of degrees of freedom. A justification of this property can be seen in References [10–12]. Any other remeshing criteria will produce more expensive solutions for the same fixed amount of global energy error.

Nevertheless, very little attention has been paid to the use of alternative mesh refinement strategies [9–16], with alternative measurements of the error. In fact, the equal distribution of the energy error between all the elements produces a concentration of the energy error in the zones where the density of elements is largest. However, normally these are the zones where the highest stresses are found and where the attention of the structural engineer is focused. This apparent contradiction is due to the equal distribution of the energy error between the elements being based on the optimization of the necessary computational cost to obtain a given value of the global energy norm, but this does not necessarily agree with the principal interest of the engineer making the analysis.

Today, the computational power of the computers allows one to use alternative mesh refinement strategies that are closer to engineering intuition and not so much related to the computational cost. Bugeda proposes an alternative mesh refinement criterion based on the equal distribution of the density of error [13, 14, 16]. For the same amount of global energy error this criterion produces much more expensive meshes than the classical method, but these meshes result in more reasonable distribution of the error in stresses [14]. Díez proposes an alternative remeshing criterion that is, in some way, intermediate between the classical and the equal distribution of the error density [15]. These concepts will be revisited in the following sections and different refinement strategies will be compared.

The main objective of mesh adaptive strategies is to provide a finite element solution with a prescribed level of error in a specific magnitude with some physical meaning. However, the main objective of a linear structural finite element analysis is to provide accurate values for the displacements, strains and stresses that will allow a check on the safety of a given structure under a given set of loads. Taking into account that the majority of safety criteria are based on the stress values obtained everywhere in the structure, the concern of the structural engineer is focused more on the error contained in the stress solution of the structural problem than on the error contained in the energy norm of the displacements. Thus, it seems logical to base the adaptive remeshing strategies on the control of the error in stresses, instead of the control of the error in the energy norm of the solution. Bugeda proposes a set of alternative mesh adaptive strategies based on the control of the point wise error in stresses instead of strategies controlling the energy norm of the error [14]. These concepts will be revisited in the following sections and the corresponding strategies will be compared with the refinement strategies based on the energy error norm.

2. ERROR ESTIMATION

When dealing with adaptive mesh refinement finite element analysis it is necessary to measure the quality of a given solution. Since the 'exact' solution is not known, a method to approximately evaluate the error of the finite element solution must be defined. The most popular measurement of the discretization error contained in the solution of a structural elliptic problem is based on the energy norm that can be expressed as

$$\|\mathbf{e}\| = \left[\int_{\Omega} [\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}]^{\mathrm{T}} \mathbf{D}^{-1} [\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}] \, \mathrm{d}\Omega\right]^{1/2} \tag{1}$$

where $\boldsymbol{\sigma}$ are the exact stresses, $\hat{\boldsymbol{\sigma}}$ are the stress values obtained from the finite element solution and $\mathbf{e}$ is the displacements error. Since the exact stresses are usually not known they can be approximated by

$$\boldsymbol{\sigma} \simeq \boldsymbol{\sigma}^* = \mathbf{N}_\sigma \bar{\boldsymbol{\sigma}}^* \tag{2}$$

where $\mathbf{N}_\sigma$ are stress interpolating functions and $\boldsymbol{\sigma}^*$ are nodal stress values obtained by either the simple nodal average of the finite element values, local or global least-square smoothing, or other adequate nodal stress recovery techniques [1–5].

Till now, the use of adaptive mesh refinement techniques based on point wise stress errors instead of the energy norm error has not received much attention due to the lack of reliable and simple point wise stress error estimators. Nevertheless, some very promising advances in this direction have arised recently. The most useful approaches available nowadays seem to be the Superconvergent Patch Recovery (SPR) technique proposed by Zienkiewicz and Zhu [6], the Recovery by Equilibrium in Patches (REP) proposed by Boroomand and Zienkiewicz [17, 18] and the Constitutive Relation Error (CRE) technique proposed by Ladevéze *et al.* [19]. Despite the fact that further advances are expected in this field, these three approaches seem to be good enough to be used in conjunction with adaptive mesh refinement techniques based on point wise stress errors.

The approximation in Equation (2) leads to the following error estimation:

$$\|\mathbf{e}\| = \left[\int_{\Omega} [\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}}]^{\mathrm{T}} \mathbf{D}^{-1} [\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}}] \, \mathrm{d}\Omega\right]^{1/2} \tag{3}$$

Despite the fact that most adaptive remeshing strategies are based on the control of the energy norm of the error other possibilities are still available. The fact that the SPR, the REP, the CRE and other recovery techniques provide very much improved stress fields $\boldsymbol{\sigma}^*$ compared with the finite element stress solution $\hat{\boldsymbol{\sigma}}$, makes them suitable for the development of adaptive remeshing strategies based on the control of only the point wise error in stresses. In this case, the error in stresses $\mathbf{e}_\sigma$ can be estimated as

$$\mathbf{e}_\sigma = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \simeq \boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}} \tag{4}$$

It must be emphasized that the estimation of the error in stresses defined in Equation (4) is just a point wise measurement of the error, and that it should be controlled at a series of *a priori* fixed points such as the integration and/or the nodal points. On the other hand, the energy norm of the error is a global norm, in the sense that the error in the energy is integrated over the elemental or the global domain.

3. STRATEGIES BASED ON ENERGY NORMS

As mentioned, such strategies are based on the use of an energy norm of the error for the measurement of the quality of a solution. Due to the additive contribution of the elemental norms, separate global and local parameters can be defined. It allows the control of the global error norm as well as its distribution over the domain depending on the optimality criteria chosen for the mesh. These concepts were detailed in References [14, 16], and are revisited below.

3.1. Definition of acceptable solution

It is usually agreed that a solution is 'acceptable' if the two following conditions are satisfied:

(a) The global error in energy norm is below a specified value of the total energy norm:

$$\|\mathbf{e}\| \leqslant \eta \|\mathbf{u}\| \tag{5}$$

where η is the user's specified value of the permissible relative global error. Equation (5) allows a definition of a *global error parameter* as

$$\xi_{\mathrm{g}} = \frac{\|\mathbf{e}\|}{\eta} \|\mathbf{u}\| \tag{6}$$

Clearly, the values $\xi_{\mathrm{g}} \leqslant 1$ denote satisfaction of the global error criterion, whereas $\xi_{\mathrm{g}} > 1$ indicates that further refinement is necessary. A mesh refinement criterion based only on ξ_{g} would lead to a uniform refinement $((\xi_{\mathrm{g}} > 1))$ or derefinement $((\xi_{\mathrm{g}} < 1))$ of all element sizes (with the usual corrections to preserve the dimensions of the analysis domain).

(b) In addition to the first condition, the present author considers that the distribution of the elements in the mesh must satisfy an additional local condition. This local condition can be expressed as

$$\|\mathbf{e}\|_i = \|\mathbf{e}\|_{r_i} \tag{7}$$

where $\|\mathbf{e}\|_i$ is the actual error norm in each element i and $\|\mathbf{e}\|_{r_i}$ is the 'required' error norm in the element. The *local error indicator* $\bar{\xi}_i$ is defined as

$$\bar{\xi}_i = \frac{\|\mathbf{e}\|_i}{\|\mathbf{e}\|_{r_i}} \tag{8}$$

Note that a value of $\bar{\xi}_i = 1$ defines an 'optimal' element size (in the sense of satisfaction of (7)), whereas $\bar{\xi}_i > 1$ and $\bar{\xi}_i < 1$ indicate that the size of element i needs further refinement or derefinement, respectively. The definition of the required error in each element $\|\mathbf{e}\|_{r_i}$ is a key issue and it strongly affects the distribution of element sizes in the mesh. This definition can be based on different mesh acceptance criteria and a selection of these are presented later.

3.2. Element refinement parameter

In practice, this paper will aim to satisfy both global and local conditions (a) and (b) defined in previous subsection. This allows the definition of an *element refinement parameter* using Equations (7) and (8) as

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|\mathbf{e}\| \|\mathbf{e}\|_i}{\eta \|\mathbf{u}\| \|\mathbf{e}\|_{r_i}} \tag{9}$$

The element refinement parameter ξ_i was first introduced in Reference [10] and since then many authors have used it as the basis for defining the new element size in a general adaptive remeshing strategy. The expression of ξ_i given in Equation (9) can be interpreted as the result of trying to satisfy the global and local error conditions in a successive manner. Clearly, $\xi_i \geqslant 1$ indicates that the element should be further refined, whereas $\xi_i \leqslant 1$ implies that the element satisfies both the local and global error conditions.

3.3. Equal distribution of the global error

The classical and most popular criterion for adaptive remeshing in elliptic problems is that a mesh is ‘optimal’ if the distribution of the energy norm of the error is equal between all the elements [1–6, 9–12]. On the basis of this assumption, the required error for each element can be defined as the ratio between the global error and the total number of elements in the mesh n. Thus, noting that only the square norm is additive, we have

$$\|\mathbf{e}\|_{r_i} = \frac{\|\mathbf{e}\|}{\sqrt{n}} \tag{10}$$

Combining Equations (8) and (10) yields the expression of the local error parameter as

$$\bar{\xi}_i = \frac{\|\mathbf{e}\|_i}{\|\mathbf{e}\| n^{-1/2}} \tag{11}$$

The element refinement parameter is obtained, viz. Equation (9), as

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|\mathbf{e}\|_i}{\eta \|\mathbf{u}\| n^{-1/2}} \tag{12}$$

The new element size $\bar{h}_i$ in terms of the existing size h_i is then obtained using the expression [14, 16]

$$\bar{h}_i = \frac{h_i}{\beta_i}, \quad \beta_i = \xi_g^{1/p} \bar{\xi}_i^{2/(2p+d)} \tag{13}$$

where p is the degree of the shape functions polynomials. The expression of the element size parameter β_i as given by Equation (13) takes into account the different convergence rates of the element and global error norms. Other authors propose different expressions for the obtainment of the element size distribution for the new mesh using different arguments

[1–6, 10–12]. The outcome of this paper is that Equation (13) has always converged to the required optimal mesh despite the fact that some of the alternative expressions such as those proposed in References [10–12] can also provide the same satisfactory results.

3.4. *Equal distribution of the density of error*

An alternative criterion is to assume that a mesh is optimal if the density of the square of the energy norm of the error is constant over the whole mesh. It is clear then that in the optimal mesh:

$$\frac{\|\mathbf{e}\|_i^2}{\Omega_i} = \frac{\|\mathbf{e}\|^2}{\Omega} \tag{14}$$

Obviously, in Equation (14) Ω_i and Ω denote the element and the total area (or volume) respectively. Comparing Equations (14) and (7) gives the expression of the required error norm for each element as

$$\|\mathbf{e}\|_{r_i} = \|\mathbf{e}\| \left(\frac{\Omega_i}{\Omega}\right)^{1/2} \tag{15}$$

The local error parameter $\bar{\xi}_i$ is now obtained using (8) and (15) as

$$\bar{\xi}_i = \frac{\|\mathbf{e}\|_i}{\Omega_i^{1/2}} \left[\frac{\|\mathbf{e}\|}{\Omega^{1/2}}\right]^{-1} = \frac{\|\mathbf{e}\|_i}{\|\mathbf{e}\|} \left(\frac{\Omega}{\Omega_i}\right)^{1/2} \tag{16}$$

The element refinement parameter ξ_i is obtained from Equations (6), (8) and (16) as

$$\xi_i = \bar{\xi}_i \xi_{\mathrm{g}} = \frac{\|\mathbf{e}\|_i}{\eta\|\mathbf{u}\|} \left(\frac{\Omega}{\Omega_i}\right)^{1/2} \tag{17}$$

The new element size is obtained from Equation (13) with the element size parameter β_i given now by [14, 16]

$$\beta_i = (\bar{\xi}_i \xi_{\mathrm{g}})^{1/p} = (\xi_i)^{1/p} \tag{18}$$

To the knowledge of the authors, the criterion of equal distribution of the density of error was first introduced in the form given here by Bugeda [13].

4. STRATEGIES BASED ON POINT WISE MEASUREMENTS

These types of strategies are based on the control of a specific magnitude whose point wise error is limited to a maximum value everywhere. In this case, the quality of the solution is measured using an evaluation of the point wise error. The advantage of this type of strategies is that they allow the control of a specific magnitude with a clear physical meaning such as the stresses at each point, strains, etc. From an engineering point of view, the interpretation of this type of criteria can be much easier than in previous cases.

The following adaptive strategies are based on the fact that, assuming the necessary regularity conditions, the error in stresses at each point behaves as h^p [20]. For any regular point P, and for each component of the stress tensor we can write

$$|\sigma_{ij}(P) - \hat{\sigma}_{ij}(P)| \leqslant Ch^p \tag{19}$$

The size h that appear in (19) is a global representation of the element size of the whole mesh. Expression (19) indicates how the size of all the elements of the mesh should be uniformly reduced in order to provide a specific value of the error in the corresponding component of stress at point P. A uniform reduction of the element size h over the whole mesh ensures the reduction of both the local and the pollution errors [7, 8]. On the other hand, if only the size of the element containing the point P is reduced only the local part of the error is reduced and the pollution error could remain constant.

Nevertheless, it is accepted that for globally adapted grids which satisfy a sufficient small tolerance for error in global energy norm, the influence of the pollution error is very small and the local accuracy of the recovered gradient is improved [7, 8]. In these conditions, expression (19) can be used for the definition of adaptive strategies based on the control of the stress error not only at a specific point but over a global set of points. In this way, the influence of the pollution error is minimized and the mentioned stress recovery methods become reliable.

Expression (19) does not hold for zones around singularities due to the lack of regularity. At this zones, the behaviour of the error in stresses is governed by the intensity of the singularity λ instead of the degree of the shape functions polynomials and the p parameter should be substituted by λ. Next subsections show different adaptive remeshing strategies based on (19) that do not contemplate any specific treatment of singularities. Nevertheless, the substitution of p by λ at these zones could provide the necessary alternatives for zones around singularities.

4.1. Maximum error in stresses

The first obvious possibility is to maintain the error in the stresses below to a certain limit everywhere. This can be done by estimating the error in stresses $\mathbf{e}_\sigma$ defined in Equation (4). Due to the tensorial nature of the stresses, the error in stresses defined in Equation (4) will also have a tensorial magnitude. For this reason, this error can also be written in terms of its components. Taking this into account, we have

$$\mathbf{e}_\sigma = \begin{bmatrix} e_{\sigma_x} & e_{\tau_{xy}} & e_{\tau_{xz}} \\ e_{\tau_{xy}} & e_{\sigma_y} & e_{\tau_{yz}} \\ e_{\tau_{xz}} & e_{\tau_{yz}} & e_{\sigma_z} \end{bmatrix} = \begin{bmatrix} \sigma_x - \hat{\sigma}_x & \tau_{xy} - \hat{\tau}_{xy} & \tau_{xz} - \hat{\tau}_{xz} \\ \tau_{xy} - \hat{\tau}_{xy} & \sigma_y - \hat{\sigma}_y & \tau_{yz} - \hat{\tau}_{yz} \\ \tau_{xz} - \hat{\tau}_{xz} & \tau_{yz} - \hat{\tau}_{yz} & \sigma_z - \hat{\sigma}_z \end{bmatrix}$$

$$\simeq \begin{bmatrix} \sigma_x^* - \hat{\sigma}_x & \tau_{xy}^* - \hat{\tau}_{xy} & \tau_{xz}^* - \hat{\tau}_{xz} \\ \tau_{xy}^* - \hat{\tau}_{xy} & \sigma_y^* - \hat{\sigma}_y & \tau_{yz}^* - \hat{\tau}_{yz} \\ \tau_{xz}^* - \hat{\tau}_{xz} & \tau_{yz}^* - \hat{\tau}_{yz} & \sigma_z^* - \hat{\sigma}_z \end{bmatrix} \tag{20}$$

Equations (4) and (20) define the error in stresses $\mathbf{e}_\sigma$ as the difference between the exact and the approximated stress tensors. Due to the tensorial properties of the stress tensor, $\mathbf{e}_\sigma$, it can also be written in terms of its eigenvalues:

$$\begin{bmatrix} e_{\sigma_{\rm I}} & 0 & 0 \\ 0 & e_{\sigma_{\rm II}} & 0 \\ 0 & 0 & e_{\sigma_{\rm III}} \end{bmatrix} \tag{21}$$

It seems logical to develop a strategy where the maximum error in the stresses is limited to a certain maximum value everywhere:

$$\max(\text{abs}(e_{\sigma_{\rm I}}), \text{abs}(e_{\sigma_{\rm II}}), \text{abs}(e_{\sigma_{\rm III}})) \leqslant e_{\sigma_{\max}} \tag{22}$$

A refinement parameter ξ can then be defined at each point as the ratio between the maximum error $\max(\text{abs}(e_{\sigma_{\rm I}}), \text{abs}(e_{\sigma_{\rm II}}), \text{abs}(e_{\sigma_{\rm III}}))$ and $e_{\sigma_{\max}}$:

$$\xi = \frac{\max(\text{abs}(e_{\sigma_{\rm I}}), \text{abs}(e_{\sigma_{\rm II}}), \text{abs}(e_{\sigma_{\rm III}}))}{e_{\sigma_{\max}}} \tag{23}$$

Using (19), the new element size $\bar{h}_i$ in terms of the existing size h_i can now be defined at each point using the expression

$$\bar{h}_i = \frac{h_i}{\xi^{1/p}} \tag{24}$$

Equation (24) provides the refinement strategy to improve the quality of the mesh. The refinement parameter ξ and new size $\bar{h}_i$ can be computed at any point of the domain. Nevertheless, taking into account that the biggest error in stresses are normally obtained at the nodal points the safest possibility is to compute ξ and $\bar{h}_i$ at those points. In addition, taking into account that different values of ξ and $\bar{h}_i$ will be obtained for each of the elements connected to a specific node, the safest possibility is to define the refinement strategy in terms of the minimum value of $\bar{h}_i$ obtained at each node.

4.2. Different maximum error in tensile and compression stresses

Equation (22) does not make any distinction between tensile and compression stresses. Nevertheless, in some structural problems where the mechanical properties of the material correspond with a frictional behaviour it can be convenient to have a stricter control in the tensile stress zones than in the rest of the domain. This type of control can provide more accurate information about the stresses in zones where cracking phenomena can arise, whereas in the rest of the domain a smaller computational effort will be required. In order to obtain this type of control two different values of the maximum stress error must be defined. At each point of the domain, one or other error will be selected depending on the presence of tensile stresses. Assuming the tensile stresses as positive, Equation (22) would then be complemented with:

$$e_{\sigma_{\max}} = \begin{cases} e_{\sigma_{\max}}^{\text{comp}} & \text{if } \max(\sigma_{\rm I}, \sigma_{\rm II}, \sigma_{\rm III}) \leqslant 0 \\ e_{\sigma_{\max}}^{\text{tens}} & \text{if } \max(\sigma_{\rm I}, \sigma_{\rm II}, \sigma_{\rm III}) > 0 \end{cases} \tag{25}$$

Clearly, for frictional materials it should be $e_{\sigma_{\max}}^{\text{tens}} < e_{\sigma_{\max}}^{\text{comp}}$. Equations (23) and (24) will also define the remeshing strategy for this error criteria.

4.3. Maximum error in Von Mises stress

Another possibility is to control the magnitude directly related with a failure criteria of a material. A typical case is the Von Mises stress $\bar{\sigma}$ defined as

$$\bar{\sigma} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} \tag{26}$$

where σ_1, σ_2 and σ_3 are the principal stresses. For these criteria, the error in the Von Mises stress can be defined as

$$\begin{aligned} e_{\bar{\sigma}} &= \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} - \sqrt{\frac{(\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2}{6}} \\ &\simeq \sqrt{\frac{(\sigma_1^* - \sigma_2^*)^2 + (\sigma_2^* - \sigma_3^*)^2 + (\sigma_3^* - \sigma_1^*)^2}{6}} - \sqrt{\frac{(\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2}{6}} \end{aligned} \tag{27}$$

Using Equation (27) a maximum error in the Von Mises stress $e_{\bar{\sigma}_{\max}}$ can be defined and the refinement parameter ξ can now be defined at each point as

$$\xi = \frac{e_{\bar{\sigma}}}{e_{\bar{\sigma}_{\max}}} \tag{28}$$

The new element size $\bar{h}_i$ in terms of the existing size h_i can now be defined at each point using again expression (24).

5. NUMERICAL EXAMPLE

The behaviours of the different adaptive strategies described in this paper are illustrated here through the solution of a typical structural example. It consists in the analysis of a 2D section of a gravity dam assuming an elastic plane strain model. Figure 1 shows the geometry of the analysed section. All the possible corner points of the geometry have been conveniently rounded in order to eliminate singularities that could mask the behaviour of each strategy. The main data used for the analysis are the following:

— Total height of the dam $h = 33.5$ m.
— Total height of water 32.5 m.
— Young modulus $E = 31.0$ GPa.
— Poisson's ratio $\gamma = 0.25$.
— Density of concrete $\rho = 2.3$ Mg/m^3.

The applied loads are the water pressure and the self-weight. The displacements of all the nodal points placed at the bottom line have been prescribed to be zero in both the vertical and horizontal directions, whereas the displacements of all the nodal points placed at the left and right sides have been prescribed to be zero only in the horizontal direction. Figure 1 also

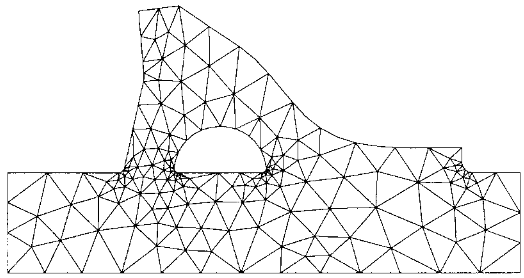

Figure 1. Geometry of the section of a gravity dam and initial mesh.

shows the initial mesh used for all the adaptive strategies. Quadratic triangular elements have been used for all the presented analysis.

The dam section described in Figure 1 has been analysed using each of the adaptive remeshing strategies described in this paper. For each strategy, the remeshing procedure has converged to the final mesh in a few remeshing steps. The strategies used for comparison are the following:

Strategy A: equal distribution of the global error energy norm between all the elements.
Strategy B: equal distribution of the density of the error energy norm.
Strategy C: control of the point wise error in stresses everywhere.
Strategy D: control of the point wise error in the Von Mises stress everywhere.
Strategy E: control of the point wise error in tensile stresses and in compression stresses everywhere.

In order to compare the results obtained with strategies A, B, C, and D two different sets of meshes have been obtained.

- The first set of meshes has been obtained by prescribing a 1.50% error of the global error energy norm ($\gamma = 0.015$) in strategy A, and prescribing the corresponding values in strategies B, C, and D in order to obtain a final mesh with a similar number of degrees of freedom to strategy A.
- The second set has been obtained by prescribing a 1.50% error of the global error energy norm ($\gamma = 0.015$) in strategy B, and prescribing the corresponding values in strategies A, C, and D in order to obtain a final mesh with a similar number of degrees of freedom to strategy B.

In addition, strategy E has also been tested by prescribing the maximum error in the stresses for the tensile elements to be $e_{\sigma_{\max}}^{\text{tens}} = 1.4$ kPa and for compression elements to be $e_{\sigma_{\max}}^{\text{comp}} = 25$ kPa. These values have been selected in order to obtain a final mesh with a similar number of degrees of freedom than those obtained with the second set of meshes, but with a much stricter control of the error in the stresses for the tensile elements than for the rest.

Figures 2 and 3 show, respectively, the final meshes obtained for each strategy and for each set. Figure 4 shows the final mesh obtained for strategy E. On the other hand, Table I shows a selection of the quality parameters obtained in the final mesh for each strategy.

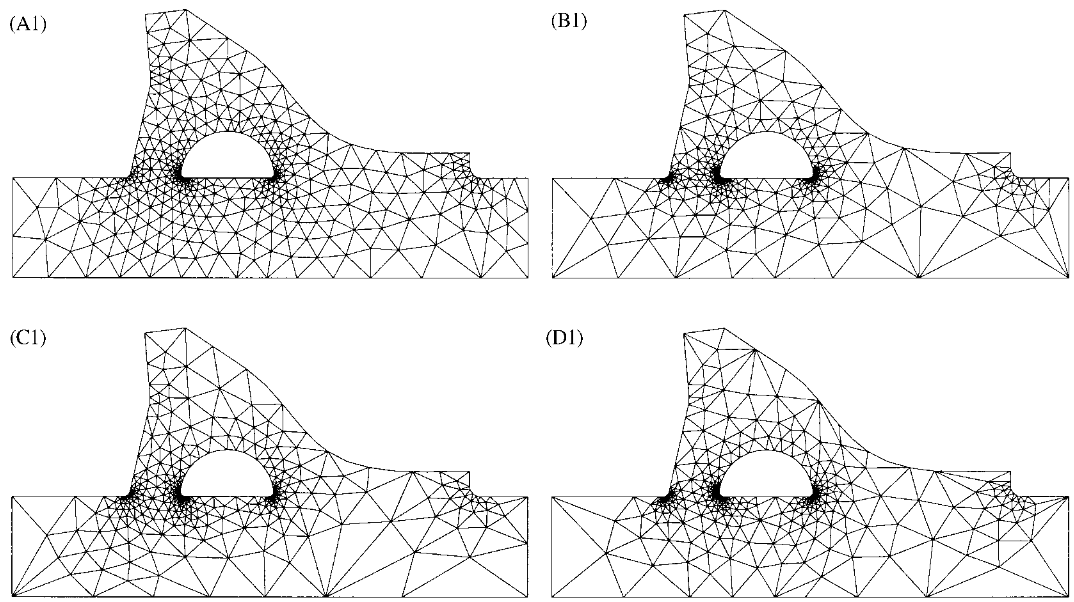

Figure 2. Final meshes obtained in the first set of applications of strategies A, B, C, and D.

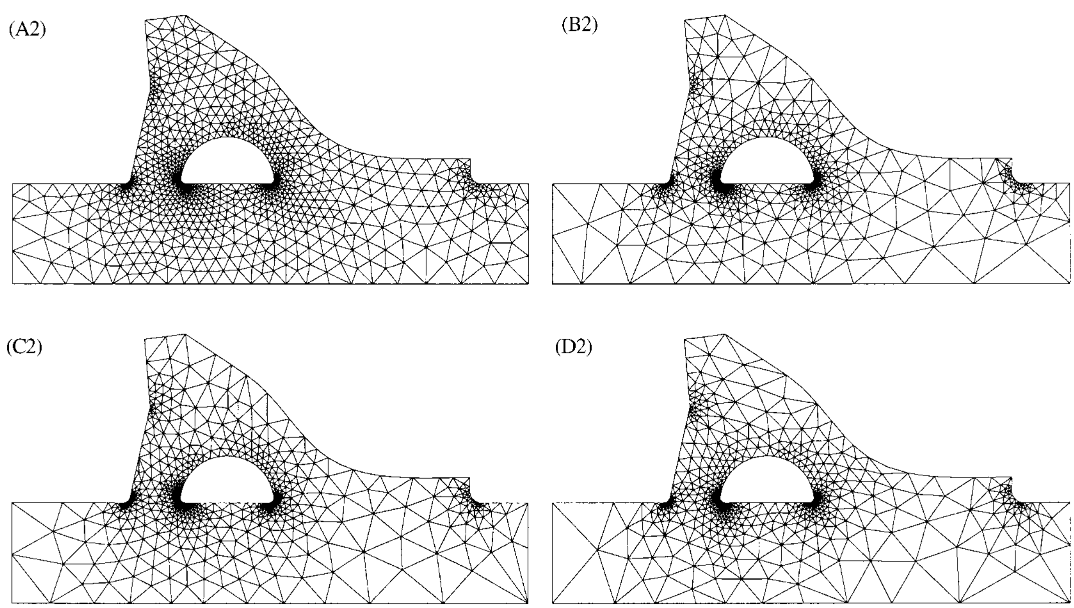

Figure 3. Final meshes obtained in the second set of applications of strategies A, B, C, and D.

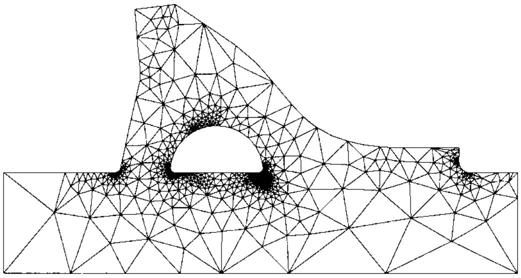

Figure 4. Final mesh obtained with strategy E.

In addition, the meaning of each of the parameters of Table I is as follows:

$(e_\sigma)_{\rm Max}$: Maximum value of the error in stresses
$(e_\sigma)_{\rm m}$: Mean value of the maximum error in stresses
$(e_\sigma)_\sigma$: Mean deviation of the maximum error in stresses
$(e_{\bar\sigma})_{\rm Max}$: Maximum value of the error in the Von Mises stress
$(e_{\bar\sigma})_{\rm m}$: Mean value of the error in the Von Mises stress
$(e_{\bar\sigma})_\sigma$: Mean deviation of the error in the Von Mises stress

The exact value for the maximum stress over the whole domain is 4122.7 kPa, and for the Von Mises stress is 3715.2 kPa. These values have been obtained by using a set of meshes with a extremely large number of degrees of freedom and checking for their convergence. In the case of strategy 4, the mean value of the error in stresses for the elements with tensile stresses is 1.3 kPa, whereas for the elements without tensile stresses it is 24.4 kPa.

Looking at Figures 2–4 and at Table I we can see the following aspects:

— As expected, for a similar percentage of error in the global energy norm ($\simeq 1.5\%$) strategy A (A1) provides a mesh that is much more economical than the rest of strategies
(B2, C2 and D2). In fact, with a similar number of degrees of freedom strategy A provides meshes with less than one-half of the global energy norm than the rest of strategies. Nevertheless, meshes B2, C2 and D2 have much smaller errors in the stresses than mesh A1.
— However, with a similar number of degrees of freedom strategies B and C provide a much smaller maximum error in the stresses $(e_\sigma)_{\rm Max}$ than strategy A. The mean value of this error in stresses $(e_\sigma)_{\rm m}$ is similar for this three strategies, but the mean deviation $(e_\sigma)_\sigma$ of this error is much smaller for strategies B and C than for strategy A. This indicates a much more uniform distribution of the error in stresses for strategies B and C than for strategy A. Strategy D provides a mean deviation $(e_\sigma)_{\rm m}$ slightly higher than strategies B and C.
— Something similar happens with the error in the Von Mises stress, but in this case strategies B, C and D behaves in a more similar way.
— The meshes obtained with strategies B, C and D look quite similar and they tend to use more and smaller elements than strategy A in the zones where stress concentrations

Table I. Quality parameters obtained in the final mesh of each strategy.

Strategy	No. of elements	No. of points	% error	$(e_\sigma)_{\mathrm{Max}}$ (kPa)	$(e_\sigma)_{\mathrm{m}}$ (kPa)	$(e_\sigma)_\sigma$ (kPa)	$(e_{\bar\sigma})_{\mathrm{Max}}$ (kPa)	$(e_{\bar\sigma})_{\mathrm{m}}$ (kPa)	$(e_{\bar\sigma})_\sigma$ (kPa)
A1	803	1715	1.40	311.3	16.2	26.6	303.8	13.4	24.1
B1	806	1708	3.43	75.7	16.8	10.7	85.9	13.8	10.4
C1	809	1693	2.75	74.6	16.1	11.0	90.6	13.5	10.5
D1	809	1713	3.36	113.8	16.2	14.2	92.0	13.4	11.8
A2	1923	4023	0.55	78.6	6.4	9.3	81.0	5.3	7.9
B2	1898	3974	1.35	23.8	6.8	3.1	18.9	5.7	2.8
C2	1896	3968	1.32	24.0	6.6	3.1	33.2	5.5	2.9
D2	1906	3982	1.34	32.4	6.7	3.5	28.0	5.6	3.1
E	1854	3888	2.50	65.9	See text	See text	70.2	5.8	8.0

are present, whereas in the rest of the domain strategy A uses smaller elements than the other strategies.

— Strategy E allows a final mesh to be obtained with a similar number of elements than the other strategies, but with a much higher accuracy in the zones where tensile stresses are present. In this case, the mean value and the mean deviation of the maximum error in the stresses has not been included in Table I because the values of the error in the stresses at each point are concentrated around two different values depending on the existence or otherwise of tensile stresses. Figure 4 also shows that strategy E concentrates elements only in the zones where tensile stresses are present.

6. CONCLUDING REMARKS

The use of alternative adaptive remeshing strategies based on concepts that are different to the equal distribution of the energy error norm have been revisited and tested. It has been shown that the classical error estimation based on the energy norm can be used in conjunction with other mesh refinement criteria such as the equal distribution of the density of error. In addition, the use of alternative point wise error measurements of different magnitudes allows the derivation of new adaptive remeshing strategies based on magnitudes directly related with the equilibrium equations or the material constitutive equations.

The results of the examples presented here show that some of the described remeshing strategies are clearly related. In particular, the results produced by the equal distribution of the density of the error energy norm are very similar to those produced by the point wise control of the errors in stresses. In fact, for 1D problems it can be demonstrated that both strategies are completely equivalent. This equivalence was also shown in Reference [14] for a different 2D problem. In addition, despite the fact that the equal distribution of the energy error between all the elements provides the most economical mesh for a given degree of accuracy in the global error norm, the rest of strategies have demonstrated that they provide a much more reliable distribution of the error in stresses while still keeping a good control on the total error energy norm.

The feasibility of a different degree of demanded accuracy for tensile and compression stresses has been shown and tested. This produces an alternative strategy that allows a very strict control of the stresses in zones where cracking or damage phenomena can arise whereas in the rest of the domain the computational cost is not increased.

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