X International Conference on Computational Methods for Coupled Problems in Science and Engineering
COUPLED PROBLEMS 2023
M. Papadrakakis, B. Schrefler and E. Oñate (Eds)

MODIFIED KIRSCH PROBLEM INCORPORATING SURFACE STRESSES UNDER PLANE STRESS

ALEKSANDRA B. VAKAEVA AND MIKHAIL A. GREKOV

St. Petersburg State University (SPbU)
7/9 Universitetskaya emb., 199034 St. Petersburg, Russia
e-mail: a.vakaeva@spbu.ru, m.grekov@spbu.ru, web page: https://spbu.ru/

Key words: Surface Stress, Gurtin–Murdoch Model, 2-D Boundary Conditions, Kirsch Problem, Circular Hole, Plane Stress

Abstract. We consider the Kirsch problem, taking into account the surface stresses at the boundary of the circular hole and on the front surfaces of the plate, in the framework of the original Gurtin–Murdoch model. The boundary conditions on the cylindrical surface of a circular hole in a nanoplate are derived in terms of a complex variable in the case of the plane stress state. The solution of the two-dimensional problem for an infinite plane with a circular hole under remote loading is explicitly obtained. Based on the analytical solution, we investigated the dependence of the elastic stress field on the nanosised plate thickness and dimension of the hole. Numerical examples are given in the paper to illustrate quantitatively the effect of the plate thickness at the nanoscale on the stress field at and near the cylindrical surface. The results are presented graphically as the dependence of the components of the stress tensor on the polar angle.

1 INTRODUCTION

Typical representatives of inhomogeneous materials are classes of fibrous composites with metal, polymer, and ceramic matrices. In these materials, fibers of different scale levels, ranging in diameter from hundreds of microns to several nanometers, are distributed in different ways. Strength and physicochemical properties of materials depend on the features of the stress-strain state of near-surface and near-boundary layers in inhomogeneous systems. The development of plastic deformation and fracture processes in these areas determines the mechanical behavior of the material as a whole and is therefore of great interest [1]. The traditional consideration of micro- and nanoscale heterogeneities in the framework of the classical theory of elasticity may lead to inaccuracies in determining the levels of real deformations and stresses [2]–[4].

Due to the rapid development of nanotechnology, it becomes necessary to study the mechanical behavior of nanoscale structures used in the manufacture of electromechanical devices, such as vibration shock sensors, biosensors, accelerometers, resonators, and others [5]–[7]. Continuum mechanics offers tools to model their behavior. One of such tools is two-dimensional bending models of thin-walled nanostructures that combine volumetric and surface elasticity and, thus, can simulate a dimensional effect that is not found in micromechanics. This effect is manifested in the fact that the mechanical properties of an elastic object, expressed in dimensionless quantities, depend on the absolute dimensions of the nanostructure (for example, on its thickness). Thus, classical methods of mechanics in combination with models of surface elasticity [8] make it possible to model the behavior of nanostructures of a very diverse nature.

In this work, we present the modified Kirsch problem of a plane stress [9] allowing for the surface elasticity and residual surface stress by the Gurtin–Murdoch model [10]. The problem of a plane stress of a plate in the presence of surface stresses differs essentially from the corresponding problem of a plane strain of a body, as the elastic parameters of the plate depend on the elastic parameters of the surface and plate thickness. The boundary conditions are derived according to the corresponding generalized Laplace–Young law. With the help of Goursat–Kolosov complex potentials and Muskhelishvili representations, the solution of the problem is reduced to the singular integro-differential equation. The algorithm for solving the integral equation is constructed in the form of a power series, as in [11, 12]. Based on the explicit forms of the analytical solution, we present numerical results for the stress field near the boundary of the nanohole. The effect of plate thickness on the stress field at the surface of the hole and the role of surface tension at this surface are shown.

2 FORMULATION OF THE PROBLEM

The problem of a plane stress on a plate in the presence of surface stresses differs essentially from the problem of a plane strain on a body because of the existence of the second unknown function in the boundary equation. The elastic parameters of the plate depend on the surface elastic parameters and plate thickness, which is another characteristic of the plane stress problem incorporating the surface effect. As a result, the formulation and resolution of the associated Kirsch problem for a circular hole in a thin plate should be expressed in terms of averaged stresses σ_{ij} and strains ε_{ij} that fulfill constitutive equations based on the thickness of the plate and the surface elasticity parameters.

Using the Gurtin–Murdoch model, we derive the boundary conditions in complex variables for the 2-D problems assuming that the surface of a bulk material is cylindrical with a generatrix parallel to the x_3-axis of the Cartesian coordinates x_j, $j = 1, 2, 3$. We assume that the bulk material occupies a three-dimensional cylindrical region $|x_3| \leq h/2$ for the plane stress state. The cross section of this region is an infinite domain of the complex variable $z = x_1 + ix_2$ with the boundary $\Gamma = \{z : z = \zeta, |\zeta| = R\}$.

The constitutive equations for the generalized plane stress [13] with surface tension τ_0^s at the face surfaces of the plate in terms of stress-strain components in the local coordinate system n, t, x_3 as follows:

$$\sigma_{nn} = \sigma_0^s + (\lambda^* + 2\mu^*)\varepsilon_{nn} + \lambda\varepsilon_{tt}, \quad \sigma_{tt} = \sigma_0^s + (\lambda^* + 2\mu^*)\varepsilon_{tt} + \lambda^*\varepsilon_{nn}, \quad \sigma_{nt} = 2\mu^*\varepsilon_{nt}, \tag{1}$$

where σ_0^{s*} and effective elastic modules λ^*, μ^* equal

$$\sigma_0^{s*} = \frac{2\tau_0^s}{h}, \quad \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu} + \frac{2\lambda^s}{h}, \quad \mu^* = \mu + \frac{2\mu^s}{h}. \tag{2}$$

In equations (1), (2) σ_0^s is the average surface tension on the surface of the hole, λ_s, μ_s are the surface elastic constants similar to the Lamé constants λ, μ of the bulk material.

3 BOUNDARY INTEGRAL EQUATION AND SOLUTION

Consider the infinite elastic plate $\{D = (x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, |x_3| \leq h/2$ with the circular hole of the radius R under the plane stress conditions, subjected to the remote loading uniformly distributed along the thickness of the plate and directed parallel to the plate faces:

$$\lim_{|z|\to\infty} \sigma_{11} = s_{11}^* = s_{11} + \frac{2\tau_0^s}{h}, \quad \lim_{|z|\to\infty} \sigma_{22} = s_{22}^* = s_{22} + \frac{2\tau_0^s}{h}, \quad \lim_{|z|\to\infty} \sigma_{12} = s_{12}^* = s_{12}, \tag{3}$$

where s_{ij} is the average stress in the bulk material of the plate at infinity. Moreover, the rotation angle at infinity satisfies the condition $\lim\limits_{|z|\to\infty} \omega = 0$. Taking into account the surface stresses on the cylindrical surface, the boundary condition is introduced according to [9]

$$\sigma_{nn} + i\sigma_{nt} = \left[\frac{\sigma_0^s}{R} + \frac{K_1}{R}\mathrm{Re}\frac{\partial u}{\partial \zeta} - \frac{K_2}{R}\varepsilon_{nn} + \sigma_0^s \mathrm{Im}\left(\frac{\partial^2 u}{\partial \zeta^2}\right)\right] - \tag{4}$$

$$-i\left[K_1 \mathrm{Re}\left(\frac{\partial^2 u}{\partial \zeta^2}\right) - K_2 \frac{\partial \varepsilon_{nn}}{\partial \zeta} e^{i\alpha_0} - \frac{\sigma_0^s}{R}\mathrm{Im}\frac{\partial u}{\partial \zeta}\right] + q(\zeta) = q^s(\zeta) + q(\zeta),$$

$$K_1 = M_s - \frac{\lambda L_s}{\lambda + 2\mu}, \quad K_2 = \frac{\lambda L_s}{\lambda + 2\mu}, \quad M_s = \lambda_s + 2\mu_s, \quad L_s = \lambda_s + \sigma_0^s,$$

where $\zeta = re^{i\theta}$, θ is the angle of the polar coordinates r, θ with the origin in the center of the circular hole; α_0 is the angle between the tangent to Γ and the x_1-axis at the point ζ; $\varepsilon_{nn} = \partial u_n/\partial n$, u is unknown complex displacement; q is the known complex traction at Γ.

We will use (4) to obtain the analytical solution to the problem of a circular hole in an infinite elastic matrix under any arbitrary remote load. The problem is reduced to the result of Riemann – Hilbert's boundary problem [9, 14]. The solution can be conditionally divided into two parts since the function q^s depends on the unidentified complex displacement u. First, the complex potentials are evaluated in terms of the function u, and then the integral equation is constructed in form

$$2\mu^* v'(\tau) + \varkappa I^-(\tau) + I^+(\tau) = (\varkappa + 1)T(\tau) - \widetilde{\sigma} - \varkappa J^-(\tau) - J^+(\tau), \quad |\tau| = 1, \tag{5}$$

where $v = u/R$, $\tau = \zeta/R$, $\varkappa = (\lambda^* + 3\mu^*)/(\lambda^* + \mu^*)$, $\widetilde{\sigma} = \mu^*\sigma_0^{s*}/(\lambda^* + \mu^*)$, function T depends on stresses at infinity (3), functions I, J are known and understood in the sense of the principal value of Cauchy type integrals dependent on tractions at Γ.

The solution of the system of the integral equation (5) can be found in the form of the power series for function $v'(\tau)$, coupled with the series

$$v'(\tau) = \sum_{k=-\infty}^{\infty} b_k \tau^k.$$

According to [9], the following are the final formulations for the stress tensor components in the polar coordinates r, θ:

$$\sigma_{rr} = -(2\mu^* a_0 + \widetilde{\sigma})r^{-2} + \frac{s_{11}^* + s_{22}^*}{4}\left(2 + (\varkappa - 1)r^{-2}\right) - \tag{6}$$

$$-\left[1 - 2(2 - K_1 d_2^+ + \sigma_0^s d_2^- + 2K_2 d_3)r^{-2} + 3(1 - K_1 d_2^+ + 2K_2 d_3)r^{-4}\right]\Xi_{cs},$$

$$\sigma_{\theta\theta} = (2\mu^* a_0 + \widetilde{\sigma})r^{-2} + \frac{s_{11}^* + s_{22}^*}{4}\left(2 - (\varkappa - 1)r^{-2}\right) + \left[1 + 3(1 - K_1 d_2^+ + 2K_2 d_3)r^{-4}\right]\Xi_{cs}, \tag{7}$$

$$\sigma_{r\theta} = \left[1 + (2 - K_1 d_2^+ + \sigma_0^s d_2^- + 2K_2 d_3) r^{-2} - 3(1 - K_1 d_2^+ + 2K_2 d_3) r^{-4}\right] \Xi_{sc}, \tag{8}$$

where

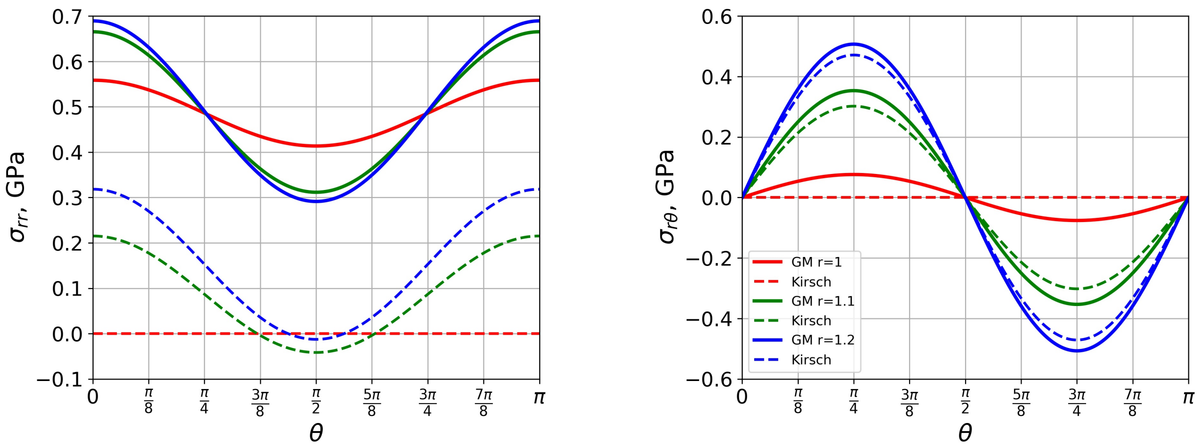
$$\Xi_{cs} = \frac{s_{22}^* - s_{11}^*}{2} \cos 2\theta - s_{12}^* \sin 2\theta, \quad \Xi_{sc} = \frac{s_{22}^* - s_{11}^*}{2} \sin 2\theta + s_{12}^* \cos 2\theta.$$

Coefficients a_0, d_2^-, d_2^+, d_3 in the equations (6)–(8) aren't given here because of bulky expressions.

4 NUMERICAL RESULTS

To obtain the graphical dependencies, it is assumed that the surface properties are determined by the parameters $\lambda_s = 6.851$ N/m, $\mu_s = -0.376$ N/m and $\sigma_0^s = \tau_0^s = 0.9108$ N/m for aluminum [15]. Elastic volume constants are equal to $\lambda = 58.17$ GPa, $\mu = 26.13$ GPa [16]. Based on the relations (6)–(8), some numerical results for the stress field at the boundary of the cylindrical surface are shown in Fig. 1 and 2 by red curves for the radius of the hole $R = 2$ nm of the plate thickness $h = 0.3$ nm under the uniaxial load $s_{22} = 1$ GPa. Additionally, the green and blue curves represent the stress field at a distance from the hole's boundary at $r = 1.1$ and 1.2, respectively, in units of the hole's radius R. Dashed curves provide the solution to the traditional Kirsch problem for comparison.

It is important to think about the stress distribution along the lines that are the symmetry axes of the holes in the case of biaxial tension because of the potential future use of the aforementioned analytical solution to theoretical modeling of crack development at the hole or dislocation emission from it. Fig. 3 shows the distribution of the stress components $\sigma_{\theta\theta}$ and σ_{rr} along the lines $\theta = 0$ (red curves) and $\theta = \pi/2$ (green curves). In Fig. 4 we consider the dependence of the maximum hoop and normal stresses on the radius R for two values of thickness $h = 0.3, 1$ nm which are represented by green and blue curves, respectively; red curves correspond to the classical Kirsch solution and presented for comparison.



a b

Figure 1: Distribution of normal (a) and tangential (b) stresses at the boundary of the circular hole of the radius $R = 2$ nm and plate thickness $h = 0.3$.

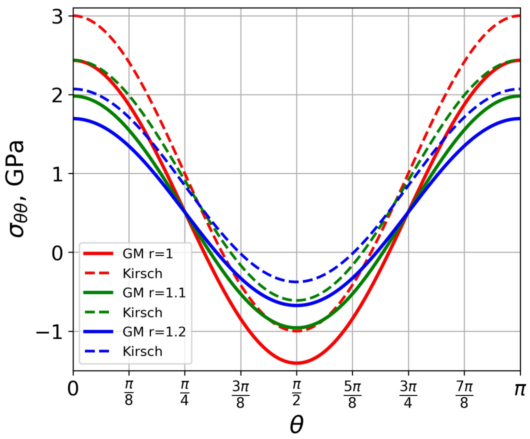
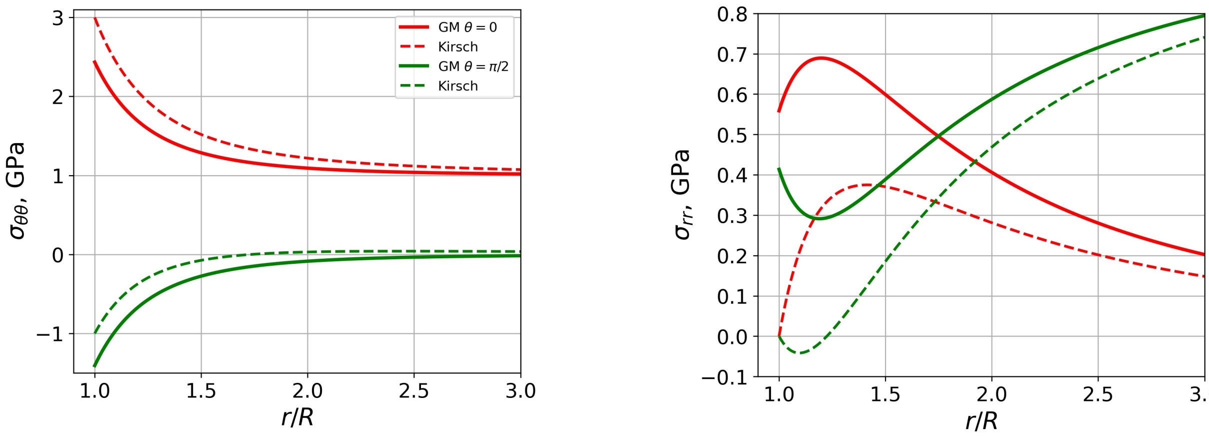


Figure 2: Distribution of hoop stresses at the boundary of the circular hole of the radius $R = 2$ nm and plate thickness $h = 0.3$.



a b

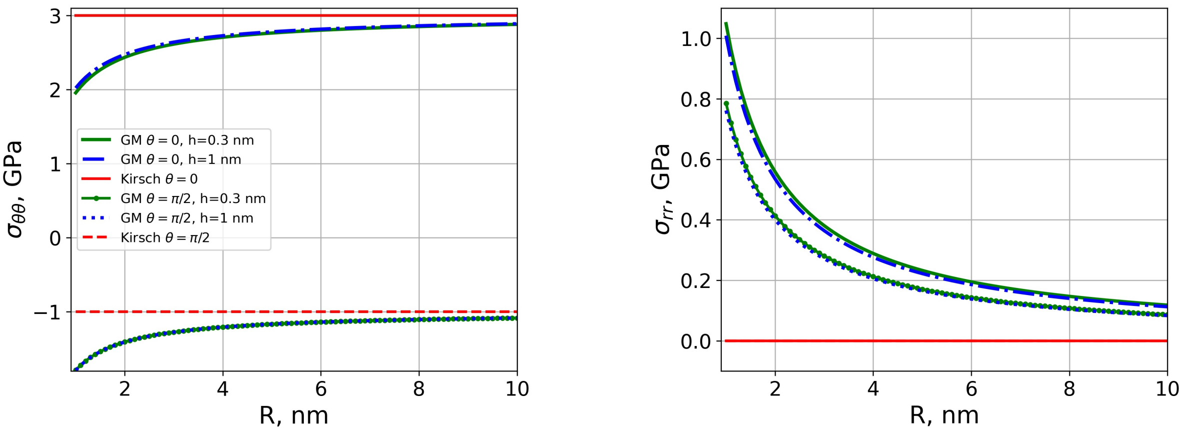
Figure 3: Distribution of the stress components $\sigma_{\theta\theta}$ (a) and σ_{rr} (b) along the lines $\theta = 0$ and $\theta = \pi/2$ when $R = 2$ nm and plate thickness $h = 0.3$.

5 CONCLUSIONS

From Figs. 1–4, it is clear that incorporating the surface tension in accordance with the full Gurtin–Murdoch model has a considerable impact on the value of the stress field at the hole boundary as well as farther away from the boundary deep within the plate.

- The highest difference in the values of the maximum hoop stresses at the hole boundary between the modified and the traditional Kirsch problems is 20%.

- The highest divergence of the stress field at the hole boundary is less than 3%, according to numerical analysis of the stress field calculated for various values of plate thickness $h = 0.3$, 1 nm. The qualitative behavior of the curves is maintained as we go away from the hole boundary, although the difference between the modified and classical problems gets smaller but is still noticeable.



a b

Figure 4: Dependence of the maximum hoop (a) and normal (b) stresses on the radius R at the boundary.

- It should be emphasized that, in contrast to the classical problem, normal and tangential stresses appear when the surface tension at the hole boundary is taken into account.
- It is important to keep in mind that, as a result of numerical research, the stress field tends toward the Kirsch solution as the hole's radius rises (size effect), that means the surface energy tends to minimum.

6 ACKNOWLEDGEMENTS

The research was supported by the Russian Science Foundation (project no. 22-11-00087).

REFERENCES

[1] Podstrigach, Ya.S. and Povstenko, Yu.Z. *An Introduction to the Mechanics of Surface Phenomena in Deformable Solids*, (in Russian). Kiev, USSR: Naukova Dumka, (1963).

[2] Grekov, M.A., Kostyrko, S.A. and Vakaeva, A.B. The model of surface nanorelief within continuum mechanics. *AIP Conference Proceedings.* (2017) **1909**:020062.

[3] Saeb, S., Steinmann, P. and Javili, A. On effective behavior of microstructures embedding general interfaces with damage. *Computational Mechanics.* (2019) **64**:1473–494.

[4] Nazarenko, L., Stolarski, H. and Altenbach, H. Effective properties of particulate nanocomposites including Steigmann–Ogden model of material surface. *Computational Mechanics.* (2021) **68**:651–665.

[5] Kachanov, M., Mishakin, V. and Pronina, Y. On low cycle fatigue of austenitic steel. Part II: Extraction of information on microcrack density from a combination of the acoustic and eddy current data. *International Journal of Engineering Science.* (2021) **169**:103569.

[6] Sedova, O. and Pronina, Y. The thermoelasticity problem for pressure vessels with protective coating, operating under conditions of mechanochemical corrosion. *International Journal of Engineering Science.* (2022) **170**:103589.

[7] Almazova, L.A., Sedova, O.S. and Kabritz, S.A. Interaction of Two Growing Corrosion Pits on the Surface of a Spherical Pressure Vessel. *Procedia Structural Integrity.* (2023) **47**:417–425.

[8] Javili, A., McBride, A. and Steinmann, P. Thermomechanics of solids with lower-dimensional energetics: on the importance of surface, interface, and curve structures at the nanoscale. A unifying review. *Applied Mechanics Reviews.* (2013) **65**:010802.

[9] Grekov, M.A. General approach to the modified Kirsch problem incorporating surface energy effect. *Continuum Mechanics and Thermodynamics.* (2021) **33(4)**:1675–1689.

[10] Gurtin, M.E. and Murdoch, A.I. A continuum theory of elastic material surfaces. *Archive for Rational Mechanics and Analysis.* (1975) **57**:291–323.

[11] Shuvalov, G.M. and Kostyrko, S.A. Stability analysis of nanoscale surface patterns in ultrathin film coating. *Materials Physics and Mechanics.* (2022) **48(2)**:232–241.

[12] Grekov, M.A., Vakaeva, A.B. and Müller, W.H. Stress field around cylindrical nanopore by various models of surface elasticity. *Continuum Mechanics and Thermodynamicsthis.* (2023) **35(1)**:231–243.

[13] Grekov, M.A. Fundamental solution for the generalized plane stress of a nanoplate. *Advanced Structured Materials.* (2019) **108**:157–164.

[14] Grekov, M.A. and Vakaeva, A.B. The perturbation method in the problem on a nearly circular inclusion in an elastic body. *Proceedings of the 7th International Conference on Coupled Problems in Science and Engineering, COUPLED PROBLEMS 2017.* (2017) 963–971.

[15] Miller, R.E. and Shenoy, V.B. Size-dependent elastic properties of nanosized structural elements. *Nanotechnology.* (2000) **11(3)**:139–147.

[16] Tian, L. and Rajapakse, R.K.N.D. Analytical solution for size-dependent elastic field of a nanoscale circular inhomogeneity. *Transaction ASME. Journal of Applied Mechanics.* (2007) **74**:568–574.