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ABSTRACT

This study investigates the utility of the newly modified Kies–Rayleigh distribution for analyzing progressively first-failure censored samples. We develop both classical and Bayesian estimators for the distribution's parameters and derived reliability measures, including the reliability function and hazard rate. Interval estimation is addressed via approximate confidence intervals and Bayesian credible intervals. Classical estimates are obtained numerically by solving the likelihood equations, whereas Bayesian inference is conducted through Markov chain Monte Carlo sampling from the posterior distribution. To assess and compare the performance of the classical and Bayesian procedures, we carry out a comprehensive simulation study under multiple experimental scenarios. We also consider the design problem of choosing an optimal progressive sampling plan and evaluate competing plans using four standard optimality criteria. We analyze a dataset comprising time-to-failure observations for turbocharged engines from a single model of diesel engine. The results highlight the flexibility of the modified Kies–Rayleigh model and provide guidance on estimation and design choices for progressively censored reliability data.

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1 Introduction

In many life-testing trials, the researcher may not receive comprehensive knowledge of the units being tested. This indicates that only a subset of the units have exact lifetimes, while the rest have lifetimes that exceed specific values. The resulting data is referred to as censored data. There are numerous censorship schemes in the literature that try to reduce both the cost and duration of testing, see Prakash et al. [1]. According to Balasooriya [2], in a circumstance when a product's lifetime is rather long and test facilities are restricted but test equipment is relatively inexpensive one may assess $m \times n$ units by examining m groups, each consisting of n units. Wu and Kus [3] used this technique to design a

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progressive first-failure censoring (PFFC) scheme. The framework of this scheme can be described as follows: Assume there are n unique groups, each with k elements, allocated to a life-testing experiment at time zero. Consider r represent the previously set number of failures, and $\mathbf{S} = (S_1, \ldots, S_r)$ be a predetermined removal design. When the first failure occurs (its time denoted by $Y_{1:r}$), the S_1 groups are eliminated from the experiment, along with the group which generated the first failure. After observing the second failure at failure time $Y_{2:r}$, S_2 groups, including the group that experienced the failure, are removed from the remaining groups, and so on. The procedure continues till the r-th failure occurs, with failure time $Y_{r:r}$. At this point, the remaining groups $S_r = n - r - \sum_{i=1}^{r-1} S_i$ are removed from the test. Let $\underline{\mathrm{y}} = (y_1, \ldots, y_r)$ be the realization of $Y_{1:r}, \ldots, Y_{r:r}$.

The joint likelihood function of a PFFC sample $\underline{\mathrm{y}}$ from a continuous population with PDF $g(.)$ and CDF $G(.)$. can be written as follows

$$\mathfrak{L}(\underline{\mathrm{y}}) = Ck^r \prod_{i=1}^{r} g(y_i)[1 - G(y_i)]^{k(1+S_i)-1}, \tag{1}$$

where C is a constant. Wu and Kus [3] used the PFFC data to estimate the maximum likelihood estimates (MLEs) of Weibull distribution. The complete, Type-II censored and progressively Type-II censored samples can be obtained directly as special cases from the PFFC sample. Due this importance, many researchers considered this plan to investigate various estimation issues for many lifetime models using classical and Bayesian estimation approaches. Some recent studies include, Ashour et al. [4], Saini et al. [5], Shi and Shi [6], Kumar et al. [7], Abu-Moussa et al. [8], Eliwa and Ahmed [9], Shi and Shi [10], Lin et al. [11], and Saini and Garg [12], among others.

In order to analyze the collected PFFC data, it is essential to choose a probability distribution that accurately corresponds to the population from which the sample is selected. One of the most important distributions is the Rayleigh (R) distribution, which can be acquired as a special case from the well-known Weibull distribution. The R distribution was initially developed by Rayleigh [13] to study acoustics and optics. It provides an effective model for positively skewed data gathered from various fields such as physics, medicine, and engineering. Additionally, it has garnered the attention of engineers and physicists for modeling radiation and wave propagation. It also has applications in medical imaging, such as ultrasound, where it predicts the distribution of speckle noise, enhancing image quality and diagnostic accuracy, see detail Wagner et al. [14]. For further instances of using the R distribution, see the work of Hoffman and Karst [15], Kuruoglu and Zerubia [16], and Sarti et al. [17], among others.

The conventional R distribution has only one scale parameter, which limits its ability to simulate diverse forms of data, such as data with bathtub failure rates, which are significant in many fields. To address this limitation, several generalizations of the R distribution have been proposed in recent years. See for example, Gomes et al. [18], Iriarte et al. [19], Bashir and Rasul [20], Shen et al. [21], and Ieren et al. [22]. One of the simplest extensions is the modified Kies R (MKR) distribution proposed by Algarni and Almarashi [23]. The MKR distribution includes one scale parameter and one shape parameter. Its hazard rate function (HRF) is capable of modeling data with increasing and bathtub shapes. Through data analysis, it has been shown that the MKR distribution outperforms commonly used models such as gamma, Weibull, exponentiated R, and exponentiated exponential distributions. If the random variable Y is known to follow the MKR distribution, then its probability density function (PDF) can be expresses according to Algarni and Almarashi [23] as given below

$$g(y;\, \mu, \sigma) = 2\mu\sigma y \left(1 - e^{-\mu y^2}\right)^{\sigma-1} e^{\mu\sigma y^2 - \left(e^{\mu y^2} - 1\right)^{\sigma}},\ y > 0,\ \mu, \sigma > 0, \tag{2}$$

where μ and σ are the scale and shape parameters, respectively. The cumulative distribution function corresponds to the random variable Y is given by

$$G(y;\,\mu,\,\sigma) = 1 - e^{-\left(e^{\mu y^2}-1\right)^{\sigma}}. \tag{3}$$

In addition, two important reliability indices for the MKR distribution, namely reliability function (RF) and HRF at a distinct time t, can be obtained, respectively, as follow

$$R(t;\,\mu,\,\sigma) = e^{-\left(e^{\mu t^2}-1\right)^{\sigma}} \tag{4}$$

and

$$h(t;\,\mu,\,\sigma) = 2\mu\sigma t e^{\mu\sigma t^2}\left(1 - e^{-\mu t^2}\right)^{\sigma-1}. \tag{5}$$

Although the MKR distribution is known for its flexibility in modeling different types of data and its superior performance compared to other common distributions, no study has yet examined the estimation of its reliability metrics using censored data. Algarni and Almarashi [23] utilized progressive Type-II censoring to estimate the distribution's unknown parameters, but they did not discuss estimations for RF and HRF. Additionally, they solely relied on classical methods without considering Bayesian estimation or interval estimations. To address this issue, we utilize both maximum likelihood and Bayesian estimation methods to evaluate the estimations of the MKR distribution when PFFC data is present. The estimations include both the model parameters μ and σ, as well as the reliability measures RF and HRF as displayed in (4) and (5), respectively. In addition to the two-point estimation approaches, we also employ the use of approximate confidence intervals (ACIs) and Bayes credible intervals (BCIs). The Bayes estimates are computed based on the symmetric squared error (SE) loss function. A simulation research is carried out to evaluate the effectiveness of various estimates and compare their performance. It takes into account different removal designs and accuracy standards. From a practical standpoint, we consider one application by investigating a real dataset.

The remainder of this study is organized as follows: The classical point and interval estimations of the MKR distribution using the PFFC data are discussed in Section 2. Section 3 employs the Markov Chain Monte Carlo (MCMC) technique to develop the Bayes estimates as well as the BCIs for various parameters. Section 4 presents the simulation design and the numerical outcomes. In Section 5, two groups of real data are examined. Some conclusions are highlighted in Section 6.

2 Likelihood Estimation

In this section, we investigate the MLEs and ACIs of μ and σ, as well as the RF and HRF of the MKR distribution with PFFC data. To obtain these estimates, we need to formulate the likelihood function of the observed data. Thus, based on a PFFC sample $\underline{y}$ drawn from the MKR population with removal design $\mathbf{S} = (S_1, \ldots, S_r)$, the likelihood function can be obtained using (1)–(3), after omitting constant terms for simplicity, as given below

$$L(\mu,\,\sigma;\,\underline{y}) = (\mu\sigma)^r \exp\left[\mu\sigma\sum_{i=1}^{r} y_i^2 + (\sigma-1)\sum_{i=1}^{r}\log\left(1 - e^{-\mu y_i^2}\right) - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\right], \tag{6}$$

where $D_i = k(1+S_i)$. To find the MLEs of μ and σ, it is more convenient to maximize the log-likelihood function provided below with respect to μ and σ

$$l(\mu,\sigma;\underline{y}) = r\log(\mu\sigma) + \mu\sigma\sum_{i=1}^{r} y_i^2 + (\sigma-1)\sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}. \tag{7}$$

The MLEs $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ, respectively, in this case can be found by solving the following system of non-linear equations

$$\frac{\partial l(\mu,\sigma;\underline{y})}{\partial\mu} = \frac{r}{\mu} + \sigma\sum_{i=1}^{r} y_i^2 + (\sigma-1)\sum_{i=1}^{r}\frac{y_i^2}{1-e^{-\mu y_i^2}} - \sigma\sum_{i=1}^{r} D_i y_i^2 e^{\mu y_i^2}\left(e^{\mu y_i^2}-1\right)^{\sigma-1} = 0 \tag{8}$$

and

$$\frac{\partial l(\mu,\sigma;\underline{y})}{\partial\sigma} = \frac{r}{\sigma} + \mu\sum_{i=1}^{r} y_i^2 + \sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\log\left(e^{\mu y_i^2}-1\right) = 0. \tag{9}$$

It is clear from the system of equations above that the MLEs cannot be obtained in explicit forms. To overcome this challenge, one can easily use any numerical technique to obtain these estimates from (8) and (9). It is important to mention here that the work of Algarni and Almarashi [23] using the progressively Type-II censored sample can be derived as a special case by setting $k = 1$ in (7). Once these estimates are reached, one can then utilize the invariance property of the MLEs to directly estimate the reliability metrics in (4) and (5), by substituting each true parameter with its MLE, as shown below

$$\hat{R}(t) = e^{-\left(e^{\hat{\mu}t^2}-1\right)^{\hat{\sigma}}} \text{ and } \hat{h}(t) = 2\hat{\mu}\hat{\sigma}te^{\hat{\mu}\hat{\sigma}t^2}\left(1-e^{-\hat{\mu}t^2}\right)^{\hat{\sigma}-1}.$$

On the other hand, to construct the ACIs of the parameters μ and σ we need the following second derivatives in order to obtain the variance-covariance matrix

$$\frac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\mu^2} = -\frac{r}{\mu^2} - (\sigma-1)\sum_{i=1}^{r}\frac{y_i^4 e^{-\mu y_i^2}}{(1-e^{-\mu y_i^2})^2} + \sigma\sum_{i=1}^{r} D_i y_i^4 e^{\mu y_i^2}\left(e^{\mu y_i^2}-1\right)^{\sigma-2}\left(1-\sigma e^{\mu y_i^2}\right),$$

$$\frac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\sigma^2} = -\frac{r}{\sigma^2} - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\log^2\left(e^{\mu y_i^2}-1\right)$$

and

$$\frac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\mu\partial\sigma} = \sum_{i=1}^{r} y_i^2 + \sum_{i=1}^{r}\frac{y_i^2}{1-e^{-\mu y_i^2}} - \sum_{i=1}^{r} D_i y_i^2 e^{\mu y_i^2}\left(e^{\mu y_i^2}-1\right)^{\sigma-1}\left[1+\sigma\log\left(e^{\mu y_i^2}-1\right)\right].$$

Obtaining the variance-covariance matrix involves finding the exact expressions of the Fisher information matrix, which includes the expectations of the second derivatives mentioned above. However, due to the complexity of these expressions, we estimate the variance-covariance matrix by calculating the inverse of the observed Fisher information matrix as follows

$$\boldsymbol{J}^{-1}(\hat{\mu},\hat{\sigma}) = \begin{pmatrix} -\dfrac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\mu^2} & -\dfrac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\mu\partial\sigma} \\ & -\dfrac{\partial^2 l(\mu,\sigma;\underline{y})}{\partial\sigma^2} \end{pmatrix}^{-1}_{(\mu,\sigma)=(\hat{\mu},\hat{\sigma})} = \begin{pmatrix} \widehat{Var}(\hat{\mu}) & \widehat{Cov}(\hat{\mu},\hat{\sigma}) \\ & \widehat{Var}(\hat{\sigma}) \end{pmatrix}. \tag{10}$$

After getting the required estimated variances, we can utilize the asymptotic normality of the MLEs to developed the ACIs with $100(1-\alpha)\%$ confidence level for the parameters μ and σ, respectively, as

$$\left[\hat{\mu} - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\mu})},\ \hat{\mu} + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\mu})}\right],\ \text{and}\ \left[\hat{\sigma} - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\sigma})},\ \hat{\sigma} + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\sigma})}\right].$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentile of the standard normal distribution.

To obtain such intervals for the RF and HRF, we employ the delta method to approximate the estimated variances $\widehat{Var}(\hat{R})$ and $\widehat{Var}(\hat{h})$. Using the estimated variance-covariance matrix in (10), the required variances can be obtained as follow

$$\widehat{Var}(\hat{R}) \approx \begin{pmatrix} \frac{\partial R(t)}{\partial \mu} & \frac{\partial R(t)}{\partial \mu} \end{pmatrix}_{(\mu,\sigma)=(\hat{\mu},\hat{\sigma})} \begin{pmatrix} \widehat{Var}(\hat{\mu}) & \widehat{Cov}(\hat{\mu},\hat{\sigma}) \\ & \widehat{Var}(\hat{\sigma}) \end{pmatrix} \begin{pmatrix} \frac{\partial R(t)}{\partial \mu} \\ \frac{\partial R(t)}{\partial \mu} \end{pmatrix}_{(\mu,\sigma)=(\hat{\mu},\hat{\sigma})}$$

and

$$\widehat{Var}(\hat{h}) \approx \begin{pmatrix} \frac{\partial h(t)}{\partial \mu} & \frac{\partial h(t)}{\partial \mu} \end{pmatrix}_{(\mu,\sigma)=(\hat{\mu},\hat{\sigma})} \begin{pmatrix} \widehat{Var}(\hat{\mu}) & \widehat{Cov}(\hat{\mu},\hat{\sigma}) \\ & \widehat{Var}(\hat{\sigma}) \end{pmatrix} \begin{pmatrix} \frac{\partial h(t)}{\partial \mu} \\ \frac{\partial h(t)}{\partial \mu} \end{pmatrix}_{(\mu,\sigma)=(\hat{\mu},\hat{\sigma})},$$

where

$$\frac{\partial R(t)}{\partial \mu} = -\sigma t^2 e^{\mu t^2} e^{-\left(e^{\mu t^2}-1\right)^{\sigma}} \left(e^{\mu t^2} - 1\right)^{\sigma-1},$$

$$\frac{\partial R(t)}{\partial \sigma} = e^{-\left(e^{\mu t^2}-1\right)^{\sigma}} \left(e^{\mu t^2} - 1\right)^{\sigma} \log\left(e^{\mu t^2} - 1\right)^{\sigma},$$

$$\frac{\partial h(t)}{\partial \mu} = 2\sigma t e^{\mu\sigma t^2} \left(1 - e^{-\mu t^2}\right)^{\sigma-2} \left[1 + \mu t^2 + e^{-\mu\sigma t^2}(\mu\sigma t^2 - 2\mu t^2 - 1)\right]$$

and

$$\frac{\partial h(t)}{\partial \sigma} = 2\mu t e^{\mu\sigma t^2} \left(1 - e^{-\mu t^2}\right)^{\sigma-1} \left[1 + \sigma \log\left(1 - e^{-\mu\sigma t^2}\right)\right].$$

Thus, the $100(1-\alpha)\%$ ACIs for the RF and HRF can be expressed, respectively, as

$$\left[\hat{R}(t) - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{R})},\ \hat{R}(t) + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{R})}\right]$$

and

$$\left[\hat{h}(t) - z_{\alpha/2}\sqrt{\widehat{Var}(\hat{h})},\ \hat{h}(t) + z_{\alpha/2}\sqrt{\widehat{Var}(\hat{h})}\right].$$

3 Bayesian Estimation

In this part, we employ the SE loss function and MCMC technique to obtain the Bayesian estimations for various parameters of the MKR distribution using PFFC data. Bayesian estimation is important for parameter estimation because it includes prior knowledge and evidence, providing a structure that is stronger than classical likelihood estimation. Although likelihood estimation is based on observable data, Bayesian approach combines this data with prior information to provide an integrated view that incorporates both past and present information. This combination enables more flexible estimates, particularly in circumstances with limited data. Moreover, Bayesian estimation inherently facilitates the quantification of uncertainty by means of posterior distributions. This, in turn, enhances our understanding of the variability of the estimated parameters.

To begin with Bayesian estimation, we need to choose appropriate prior distributions for the unknown parameters that reflect our understanding of them. It is clear that there are no conjugate prior distributions for the unknown parameters μ and σ. To address this issue, we propose adopting the gamma distribution as a prior distribution for both parameters. This selection is based on the gamma distribution's straightforward form and lack of complicated computing concerns. Let $\mu \sim$ *Gamma* (v_1, w_1) and $\sigma \sim$ *Gamma* (v_2, w_2), where $v_j, w_j > 0$, $j = 1, 2$, are the hyper-parameters which are assumed to be known. Then, the joint prior distribution of μ and σ, can be written as

$$p(\mu, \sigma) \propto \mu^{v_1-1}\sigma^{v_1-1}e^{-w_1\mu-w_2\sigma}, \;\; \mu, \sigma > 0. \tag{11}$$

Let $Q(\mu,\sigma|\underline{y})$ denotes the posterior distribution of the unknown parameters μ and σ which can be derived by taking the product of likelihood function in (6) and the joint prior distribution given by (11) and then applying Bayes rule as follows

$$Q(\mu,\sigma|\underline{y}) = \frac{1}{A}\mu^{r+v_1-1}\sigma^{r+v_2-1}e^{-w_1\mu-w_2\sigma} \times \exp\left[\mu\sigma\sum_{i=1}^{r}y_i^2 + (\sigma-1)\sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - \sum_{i=1}^{r}D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\right], \tag{12}$$

where

$$A = \int_0^\infty\int_0^\infty \mu^{r+v_1-1}\sigma^{r+v_2-1}e^{-w_1\mu-w_2\sigma} \times \exp\left[\mu\sigma\sum_{i=1}^{r}y_i^2 + (\sigma-1)\sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - \sum_{i=1}^{r}D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\right]d\mu d\sigma.$$

In Bayesian statistics, the Bayes estimator for any function of unknown parameters, denoted as $\varpi(\mu,\sigma)$, is an estimator that minimizes the expected value of a loss function. Specifically, for the SE loss, the Bayes estimator of $\varpi(\mu,\sigma)$ is equal to the mean of the posterior distribution obtained from

$$\tilde{\varpi}(\mu, \sigma) = \frac{\int_0^\infty\int_0^\infty \varpi(\mu, \sigma)L(\mu, \sigma; \underline{y})p(\mu, \sigma)d\mu d\sigma}{\int_0^\infty\int_0^\infty L(\mu, \sigma; \underline{y})p(\mu, \sigma)d\mu d\sigma}. \tag{13}$$

The key challenge in computing the Bayes estimator in (13) is evaluating the posterior mean which necessitates the computation of integrals that can be difficult to resolve analytically. The complex nature of these integrals presents major challenges, frequently requiring the employment of computational techniques such as Monte Carlo integration and the MCMC methodology, among others. In this study, we consider using the MCMC technique which attempts to simulate samples from the posterior distribution in (12), and then use the acquired samples to compute the Bayes estimates and BCIs of different parameters. To sample from the posterior distribution given by (12), we first formulate the full conditional distributions for each unknown parameter as listed below

$$Q_1(\mu|\sigma,\underline{y}) = \mu^{r+\nu_1-1} \times \exp\left[\mu\left(\sigma\sum_{i=1}^{r} y_i^2 - w_1\right) + (\sigma-1)\sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\right] \tag{14}$$

and

$$Q_2(\sigma|\mu,\underline{y}) = \sigma^{r+\nu_2-1} \times \exp\left\{\sigma\left[\mu\sum_{i=1}^{r} y_i^2 + \sum_{i=1}^{r}\log\left(1-e^{-\mu y_i^2}\right) - w_2\right] - \sum_{i=1}^{r} D_i\left(e^{\mu y_i^2}-1\right)^{\sigma}\right\}, \tag{15}$$

The upcoming task is to check if the full conditional distributions in (14) and (15) match any known distribution. This serves as a crucial stage in Bayesian estimation since it specifies how the MCMC procedure will be run. The graph of the conditional distributions $Q_1(\mu|\sigma,\underline{y})$ and $Q_2(\sigma|\mu,\underline{y})$ in (14) and (15), respectively, suggests that they seem like a normal distribution, despite its unfamiliar form. As a result, we implement the Metropolis-Hastings (MH) algorithm to produce random samples from the distributions of interest. This is accomplished by taking advantage of the normal distribution for the proposal distribution with variances obtained based on the MLEs. The procedures required to generate the samples are outlined below

Step 1. Set $[\mu^{(0)},\sigma^{(0)}] = [\hat{\mu},\hat{\sigma}]$, we use the MLEs as starting points.

Step 2. Put $k = 1$.

Step 3. Obtain $\mu^{(k)}$ from (14) through the MH steps:

- Generate μ^* from $N[\mu^{(k-1)},\widehat{Var}(\hat{\mu})]$.
- Compute

$$\varphi(\mu^{(k-1)}|\mu^*) = \min\left[1,\frac{Q_1(\mu^*|\sigma^{(k-1)},\underline{y})}{Q_1(\mu^{(k-1)}|\sigma^{(k-1)},\underline{y})}\right].$$

- Generate u, from $U \sim U(0,1)$.
- Put $\mu^{(k)} = \mu^*$ if $u \le \varphi(\mu^{(k-1)}|\mu^*)$ and $\mu^{(k)} = \mu^{(k-1)}$, otherwise.

Step 4. Repeat step 3 to obtain $\sigma^{(k)}$ using $Q_2(\sigma|\mu,\underline{y})$ in (15).

Step 5. Compute RF and HRF using (4) and (5), respectively, as

$$R^{(k)} = e^{-\left(e^{\mu^{(k)}t^2}-1\right)^{\sigma^{(k)}}}$$

and

$$h^{(k)} = 2\mu^{(k)}\sigma^{(k)}te^{\mu^{(k)}\sigma^{(k)}t^2}\left(1-e^{-\mu^{(k)}t^2}\right)^{\sigma^{(k)}-1}.$$

Step 6. Change k with $k+1$.

Step 7. Rerun the steps 2–6 M repeats to get $\left[\mu^{(k)},\sigma^{(k)},R^{(k)}\right]$, $k = 1,\ldots,M$.

After removing the first B generated samples as burn-in period, the Bayes estimates of μ, σ, $R(t)$ and $h(t)$ using the SE loss function can be obtained as

$$\tilde{\mu} = \frac{1}{M^*}\sum_{k=B+1}^{M}\mu^{(k)},\ \tilde{\sigma} = \frac{1}{M^*}\sum_{k=B+1}^{M}\sigma^{(k)}$$

and

$$\tilde{R}(t) = \frac{1}{M^*}\sum_{k=B+1}^{M}R^{(k)},\ \tilde{h}(t) = \frac{1}{M^*}\sum_{k=B+1}^{M}h^{(k)},$$

where $M^* = M - B$.

In addition, to compute the BCIs, we first sort the obtained MCMC samples as $\mu^{[B+1]} < \ldots, \mu^{[M]}$, $\sigma^{[B+1]} < \ldots, \sigma^{[M]}$, $R^{[B+1]} < \ldots, R^{[M]}$ and $h^{[B+1]} < \ldots, h^{[M]}$. Thus, the $100(1-\alpha)\%$ are

$$\left\{\mu^{[\alpha M^*/2]},\ \mu^{[(1-\alpha/2)M^*]}\right\},\ \left\{\sigma^{[\alpha M^*/2]},\ \sigma^{[(1-\alpha/2)M^*]}\right\}$$

and

$$\left\{R^{[\alpha M^*/2]},\ R^{[(1-\alpha/2)M^*]}\right\},\ \left\{h^{[\alpha M^*/2]},\ h^{[(1-\alpha/2)M^*]}\right\}.$$

4 Numerical Evaluations

In this part, several Monte Carlo simulation experiments are used to compare the performance of the provided estimators σ, μ, $R(t)$, and $h(t)$, developed in the proceeding sections.

4.1 Monte Carlo Scenarios

We replicated PFFC 1000 times from the MKR lifetime model with $(\sigma, \mu) = (0.4, 0.8)$ with given selections of k (group length), n (number of independent groups), and r (effective sample), and $\mathbf{S}$ (removal design). Assuming $t = 0.2$, the genuine values of $R(t)$ and $h(t)$ are 0.775685 and 1.032374, respectively. By assigning k (= 2, 5), a number of failure percentages (FPs) (of each n(= 20, 40, 80)) are taken into consideration, such as $\frac{r}{n}$ (= 50, 80)%. Moreover, where $0^{[m]}$ indicates that 0 is repeated m times, four progressive plans $\mathbf{S}$ are also utilized for each group of (k, n, m); see Table 1.

Table 1: Various progressive plans used in Monte Carlo comparisons

n	FP%	S			
Plan →		1	2	3	4
20	50%	$(1^{[10]})$	$(10, 0^{[9]})$	$(0^{[4]}, 10, 0^{[5]})$	$(0^{[9]}, 10)$
	80%	$(1^{[4]}, 0^{[12]})$	$(4, 0^{[15]})$	$(0^{[7]}, 4, 0^{[8]})$	$(0^{[15]}, 4)$
40	50%	$(1^{[20]})$	$(20, 0^{[19]})$	$(0^{[9]}, 20, 0^{[10]})$	$(0^{[19]}, 20)$
	80%	$(1^{[8]}, 0^{[24]})$	$(8, 0^{[31]})$	$(0^{[15]}, 8, 0^{[16]})$	$(0^{[31]}, 8)$
80	50%	$(1^{[40]})$	$(40, 0^{[39]})$	$(0^{[19]}, 40, 0^{[20]})$	$(0^{[39]}, 40)$
	80%	$(1^{[16]}, 0^{[48]})$	$(16, 0^{[48]})$	$(0^{[31]}, 16, 0^{[32]})$	$(0^{[63]}, 16)$

After collecting 1000 PFFC samples using the 'maxLik' software (by Henningsen and Toomet [24]), the MLEs (along with their 95% ACI) of σ, μ, $R(t)$, and $h(t)$ are developed. In Bayes' evaluations, we have utilized several informative priors for σ and μ when those hyperparameter values are selected such that the prior mean becomes the mean value of each model parameter in order to illustrate the implications of the studied priors. Thus, two sets of v_j, $w_j > 0$, $j = 1, 2$, are used for each set of σ and μ, namely:

- Prior [A]: $(v_1, v_2) = (2, 4)$ and $w_j = 5$, $j = 1, 2$;
- Prior [B]: $(v_1, v_2) = (4, 8)$ and $w_j = 10$, $j = 1, 2$.

According to the MH technique, via the package 'coda' (by Plummer et al. [25]), the Bayes' MCMC and 95% BCI estimates of σ, μ, $R(t)$, or $h(t)$ are obtained by eliminating the first 2000 (out of 10,000) MCMC iterations. Moreover, considering the acceptance rates of the MH sampler under two different simulated samples with $k = 2$, n [FP%] = 20 [50%], Prior [A], and Scheme ($1^{[10]}$) (as an illustrative case), the obtained rates were 98.45% and 97.63%, respectively. These high acceptance rates indicate that the generated MCMC chains exhibit good mixing behavior and adequate convergence properties. Consequently, the collected posterior samples can be regarded as reliable approximations of the target posterior distributions, ensuring the validity and efficiency of the subsequent Bayesian inference.

Sensitivity analysis of posterior distributions with respect to prior choices aims to evaluate the extent to which Bayesian inference (posterior mean, standard deviation, and credible intervals) depends on the selected prior. Using the collected 1000 PFFC samples from the MKR model when $(\sigma, \mu) = (0.4, 0.8)$ with $k = 2$, n [FP%] = 20 [50%], Prior [A], and Scheme ($1^{[10]}$) (as an illustrative case), we investigate the sensitivity of posterior distributions under four distinct prior settings, namely: informative, improper, weakly informative, and overdispersed; see Fig. 1. The posterior densities illustrate that the informative gamma prior (as previously assigned) produces concentrated and stable posterior distributions centered near the prior mean. In contrast, the improper prior (e.g., Uniform$(0.1, 10)$), weak prior (e.g., Gamma$(1.5, 1)$), and overdispersed prior (e.g., Gamma$(3, 2)$) allow greater influence from the observed data, yielding broader credible intervals and higher posterior variances. To sum up, Fig. 1 reveals that the posterior estimates of μ show substantial sensitivity to prior choice, with the improper prior resulting in notably higher variability and wider distributions compared to the more concentrated and regularized posteriors under informative and overdispersed priors. In contrast, the posterior distributions of σ are less affected by the prior specification, though slight differences in spread suggest mild regularization effects, particularly with the overdispersed and informative priors.

In our calculations, the average estimate (Av.E), root mean squared-error (RMSE), mean absolute bias (MAB), average interval length (AIL), and coverage percentage (CP) of σ (as an instance) are given by

$$\text{Av.E}(\sigma) = \frac{1}{1000}\sum_{j=1}^{1000} \check{\sigma}^{[j]},$$

$$\text{RMSE}(\check{\sigma}) = \sqrt{\frac{1}{1000}\sum_{j=1}^{1000}\left(\check{\sigma}^{[j]} - \sigma\right)^2},$$

$$\text{MAB}(\check{\sigma}) = \frac{1}{1000}\sum_{j=1}^{1000}\left|\check{\sigma}^{[j]} - \sigma\right|,$$

$$\mathrm{AIL}_{95\%}(\sigma) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\mathcal{U}_{\check{\sigma}^{[j]}} - \mathcal{L}_{\check{\sigma}^{[j]}}\right),$$

and

$$\mathrm{CP}_{95\%}(\sigma) = \frac{1}{1000}\sum_{j=1}^{1000}\Im_{\left(\mathcal{L}_{\check{\sigma}^{[j]}};\mathcal{U}_{\check{\sigma}^{[j]}}\right)}(\sigma),$$

respectively, where $\check{\sigma}^{[j]}$ is the estimate of σ at jth sample, $\Im(\cdot)$ is the indicator, and $(\mathcal{L}(\cdot),\mathcal{U}(\cdot))$ denotes (lower-limit,upper-limit) of 95% ACI/BCI of σ.

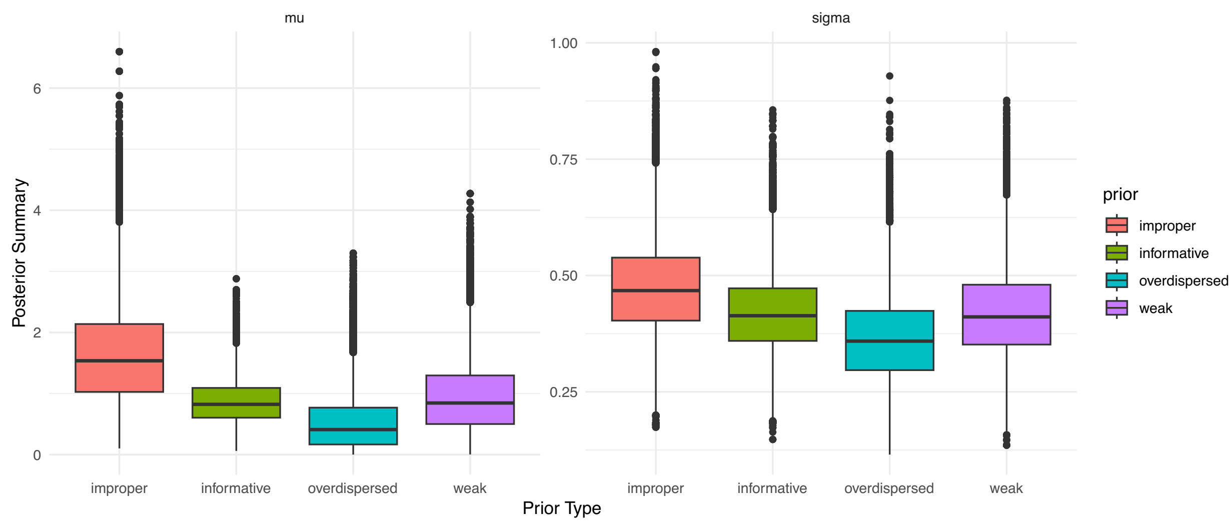


Figure 1: Sensitivity analysis of σ and μ based on several prior distributions

4.2 Monte Carlo Results

Tables 2–9 include a list of all the simulation results of σ, μ, $R(t)$, and $h(t)$. To differentiate, the AvEs, RMSEs, and MABs are given in the first, second, and third columns of Tables 2–5, respectively. Additionally, the AILs and CPs are listed in the first and second columns, respectively, in Tables 6–9.

Table 2: The point evaluations of σ

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
					$k = 2$					
20 [50%]	1	0.556	0.321	0.206	0.461	0.176	0.168	0.379	0.128	0.126
	2	0.453	0.366	0.311	0.451	0.296	0.262	0.363	0.139	0.137
	3	0.711	0.395	0.364	0.661	0.308	0.293	0.374	0.151	0.148
	4	0.488	0.418	0.401	0.492	0.384	0.375	0.367	0.168	0.154

(Continued)

Table 2 (continued)

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
20 [80%]	1	0.472	0.177	0.155	0.419	0.143	0.128	0.369	0.091	0.067
	2	0.455	0.194	0.163	0.439	0.151	0.135	0.365	0.097	0.067
	3	0.580	0.201	0.167	0.528	0.154	0.136	0.383	0.107	0.071
	4	0.546	0.238	0.186	0.519	0.159	0.142	0.385	0.118	0.077
40 [50%]	1	0.472	0.145	0.113	0.435	0.100	0.084	0.387	0.075	0.049
	2	0.429	0.152	0.114	0.401	0.105	0.087	0.324	0.077	0.050
	3	0.764	0.160	0.115	0.693	0.110	0.088	0.356	0.083	0.054
	4	0.467	0.167	0.128	0.454	0.135	0.112	0.337	0.088	0.060
40 [80%]	1	0.438	0.103	0.080	0.392	0.075	0.063	0.362	0.054	0.042
	2	0.431	0.127	0.094	0.398	0.078	0.065	0.330	0.059	0.043
	3	0.566	0.128	0.095	0.541	0.085	0.070	0.375	0.060	0.044
	4	0.525	0.133	0.112	0.485	0.097	0.083	0.359	0.063	0.045
80 [50%]	1	0.432	0.078	0.060	0.450	0.054	0.044	0.436	0.045	0.037
	2	0.416	0.081	0.062	0.456	0.067	0.054	0.448	0.046	0.039
	3	4.801	0.089	0.068	0.919	0.071	0.057	0.554	0.048	0.040
	4	0.456	0.096	0.071	0.483	0.073	0.059	0.467	0.049	0.041
80 [80%]	1	0.419	0.051	0.039	0.394	0.038	0.030	0.416	0.024	0.019
	2	0.418	0.057	0.040	0.439	0.047	0.038	0.439	0.041	0.024
	3	0.563	0.060	0.046	0.568	0.048	0.039	0.511	0.044	0.028
	4	0.514	0.076	0.059	0.535	0.049	0.040	0.495	0.045	0.034
					$k = 5$					
20 [50%]	1	0.554	0.309	0.205	0.404	0.168	0.147	0.405	0.120	0.077
	2	0.432	0.317	0.270	0.471	0.182	0.173	0.447	0.125	0.095
	3	0.669	0.345	0.313	0.541	0.201	0.219	0.467	0.139	0.111
	4	0.467	0.404	0.331	0.470	0.238	0.236	0.452	0.162	0.144
20 [80%]	1	0.473	0.163	0.130	0.397	0.126	0.114	0.398	0.092	0.058
	2	0.447	0.168	0.139	0.418	0.127	0.115	0.420	0.095	0.068
	3	0.559	0.180	0.143	0.450	0.154	0.137	0.436	0.105	0.071
	4	0.530	0.207	0.164	0.447	0.157	0.146	0.434	0.112	0.074
40 [50%]	1	0.468	0.133	0.100	0.408	0.082	0.065	0.362	0.056	0.047
	2	0.406	0.137	0.110	0.432	0.102	0.081	0.386	0.063	0.048
	3	0.690	0.143	0.113	0.473	0.105	0.082	0.455	0.079	0.052
	4	0.437	0.159	0.114	0.492	0.110	0.093	0.427	0.089	0.055
40 [80%]	1	0.435	0.096	0.072	0.419	0.064	0.053	0.373	0.044	0.035
	2	0.422	0.106	0.086	0.425	0.070	0.059	0.381	0.047	0.035
	3	0.543	0.115	0.086	0.547	0.074	0.060	0.424	0.048	0.036
	4	0.508	0.116	0.093	0.515	0.079	0.062	0.419	0.050	0.038
80 [50%]	1	0.430	0.070	0.051	0.448	0.056	0.044	0.448	0.030	0.024
	2	0.392	0.071	0.055	0.457	0.058	0.046	0.477	0.035	0.029

(Continued)

Table 2 (continued)

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
	3	0.694	0.082	0.063	0.793	0.061	0.048	0.461	0.037	0.030
	4	0.422	0.091	0.069	0.513	0.064	0.052	0.548	0.040	0.033
80 [80%]	1	0.417	0.046	0.035	0.448	0.040	0.032	0.455	0.018	0.014
	2	0.408	0.048	0.038	0.457	0.046	0.036	0.471	0.019	0.015
	3	0.530	0.053	0.042	0.573	0.050	0.039	0.537	0.025	0.020
	4	0.492	0.062	0.047	0.546	0.053	0.042	0.526	0.028	0.023

Table 3: The point evaluations of μ

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
					$k = 2$					
20 [50%]	1	0.658	0.830	0.725	0.468	0.522	0.512	0.669	0.141	0.131
	2	0.603	0.810	0.693	0.680	0.518	0.510	0.688	0.121	0.129
	3	0.453	0.894	0.864	0.490	0.588	0.584	0.659	0.152	0.142
	4	0.647	0.857	0.824	0.524	0.534	0.531	0.669	0.140	0.132
20 [80%]	1	0.879	0.527	0.468	0.428	0.451	0.438	0.672	0.138	0.124
	2	0.635	0.526	0.454	0.825	0.443	0.432	0.677	0.133	0.119
	3	0.864	0.747	0.507	0.411	0.490	0.481	0.671	0.139	0.129
	4	0.871	0.531	0.469	0.413	0.485	0.446	0.673	0.137	0.128
40 [50%]	1	0.550	0.488	0.390	0.319	0.406	0.335	0.706	0.106	0.112
	2	0.600	0.428	0.387	0.711	0.403	0.335	0.762	0.050	0.106
	3	0.341	0.521	0.428	0.627	0.435	0.340	0.686	0.128	0.116
	4	0.556	0.514	0.398	0.469	0.410	0.336	0.696	0.118	0.113
40 [80%]	1	0.756	0.366	0.353	0.402	0.327	0.296	0.711	0.101	0.097
	2	0.595	0.359	0.324	0.482	0.323	0.293	0.741	0.071	0.092
	3	0.743	0.422	0.372	0.572	0.393	0.330	0.698	0.115	0.104
	4	0.749	0.395	0.362	0.362	0.368	0.303	0.692	0.124	0.102
80 [50%]	1	0.499	0.300	0.251	0.447	0.286	0.235	0.722	0.088	0.065
	2	0.482	0.285	0.244	0.568	0.279	0.225	0.776	0.031	0.064
	3	0.293	0.347	0.319	0.288	0.322	0.288	0.681	0.136	0.078
	4	0.507	0.342	0.277	0.438	0.315	0.242	0.698	0.119	0.070
80 [80%]	1	0.695	0.234	0.211	0.597	0.193	0.176	0.739	0.068	0.045
	2	0.593	0.229	0.209	0.533	0.176	0.162	0.741	0.067	0.025
	3	0.682	0.272	0.224	0.576	0.247	0.214	0.731	0.079	0.061
	4	0.686	0.239	0.219	0.581	0.219	0.197	0.737	0.074	0.060
					$k = 5$					
20 [50%]	1	0.664	0.985	1.159	0.678	0.414	0.345	0.729	0.073	0.066
	2	0.406	0.924	1.081	0.880	0.372	0.315	0.740	0.071	0.062

(Continued)

Table 3 (continued)

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
	3	0.675	1.191	1.347	0.669	0.481	0.406	0.730	0.078	0.071
	4	0.423	1.044	1.289	0.747	0.416	0.386	0.734	0.077	0.070
20 [80%]	1	0.623	0.819	0.872	0.760	0.331	0.266	0.739	0.063	0.051
	2	0.691	0.790	0.852	0.932	0.329	0.258	0.744	0.062	0.049
	3	0.530	0.887	1.049	0.795	0.348	0.286	0.738	0.068	0.060
	4	0.699	0.821	0.924	0.802	0.337	0.271	0.739	0.065	0.056
40 [50%]	1	0.402	0.731	0.597	0.783	0.290	0.241	0.761	0.052	0.040
	2	0.954	0.651	0.571	1.083	0.287	0.240	0.774	0.051	0.036
	3	0.899	0.767	0.705	0.592	0.311	0.253	0.750	0.062	0.046
	4	0.935	0.758	0.675	1.065	0.303	0.252	0.763	0.058	0.043
40 [80%]	1	0.754	0.556	0.512	0.906	0.273	0.234	0.768	0.043	0.032
	2	0.980	0.533	0.500	1.013	0.266	0.231	0.771	0.042	0.030
	3	0.945	0.577	0.563	1.056	0.286	0.237	0.771	0.046	0.034
	4	0.661	0.560	0.520	1.144	0.284	0.236	0.774	0.044	0.033
80 [50%]	1	0.631	0.513	0.470	0.414	0.260	0.212	0.778	0.033	0.025
	2	0.823	0.511	0.368	0.725	0.257	0.209	0.808	0.031	0.023
	3	0.750	0.521	0.491	0.339	0.265	0.226	0.760	0.040	0.028
	4	0.994	0.518	0.471	0.745	0.262	0.220	0.784	0.037	0.026
80 [80%]	1	0.630	0.386	0.279	0.574	0.186	0.154	0.788	0.026	0.020
	2	0.740	0.313	0.266	0.672	0.169	0.142	0.803	0.023	0.018
	3	0.942	0.487	0.368	0.701	0.245	0.197	0.791	0.029	0.022
	4	0.898	0.450	0.321	0.755	0.227	0.175	0.796	0.027	0.021

Table 4: The point evaluations of $R(t)$

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
					$k = 2$					
20 [50%]	1	0.734	0.150	0.125	0.823	0.135	0.122	0.772	0.074	0.073
	2	0.819	0.167	0.160	0.856	0.165	0.142	0.762	0.078	0.077
	3	0.917	0.179	0.178	0.935	0.175	0.164	0.770	0.087	0.086
	4	0.866	0.199	0.198	0.888	0.178	0.177	0.766	0.110	0.107
20 [80%]	1	0.776	0.119	0.111	0.823	0.113	0.094	0.766	0.058	0.038
	2	0.799	0.123	0.113	0.841	0.117	0.097	0.764	0.061	0.043
	3	0.858	0.126	0.118	0.886	0.118	0.105	0.776	0.064	0.048
	4	0.845	0.129	0.121	0.883	0.122	0.110	0.778	0.067	0.052
40 [50%]	1	0.761	0.100	0.098	0.807	0.093	0.080	0.768	0.042	0.031
	2	0.821	0.105	0.100	0.839	0.098	0.084	0.727	0.045	0.033
	3	0.940	0.113	0.106	0.954	0.103	0.092	0.753	0.051	0.034

(Continued)

Table 4 (continued)

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
	4	0.872	0.115	0.108	0.889	0.109	0.093	0.740	0.057	0.035
40 [80%]	1	0.777	0.083	0.069	0.807	0.074	0.065	0.753	0.035	0.027
	2	0.799	0.085	0.070	0.822	0.078	0.070	0.731	0.039	0.028
	3	0.867	0.085	0.076	0.901	0.082	0.071	0.768	0.041	0.029
	4	0.850	0.095	0.085	0.875	0.088	0.077	0.756	0.042	0.030
80 [50%]	1	0.770	0.057	0.049	0.792	0.053	0.045	0.801	0.030	0.021
	2	0.820	0.060	0.052	0.851	0.058	0.051	0.814	0.031	0.023
	3	0.952	0.067	0.056	0.974	0.066	0.055	0.867	0.032	0.025
	4	0.876	0.079	0.063	0.890	0.069	0.058	0.829	0.034	0.026
80 [80%]	1	0.777	0.039	0.031	0.802	0.029	0.024	0.792	0.020	0.014
	2	0.798	0.042	0.034	0.822	0.037	0.031	0.807	0.025	0.015
	3	0.873	0.051	0.041	0.887	0.042	0.038	0.845	0.028	0.017
	4	0.852	0.056	0.046	0.874	0.049	0.041	0.838	0.029	0.018
					$k = 5$					
20 [50%]	1	0.643	0.119	0.102	0.772	0.099	0.097	0.785	0.074	0.053
	2	0.828	0.124	0.114	0.832	0.122	0.109	0.812	0.077	0.063
	3	0.867	0.157	0.122	0.858	0.139	0.117	0.823	0.084	0.070
	4	0.853	0.180	0.137	0.833	0.153	0.124	0.815	0.099	0.091
20 [80%]	1	0.742	0.096	0.086	0.762	0.084	0.079	0.780	0.046	0.041
	2	0.792	0.098	0.091	0.793	0.085	0.083	0.795	0.050	0.045
	3	0.808	0.111	0.093	0.807	0.091	0.083	0.805	0.054	0.047
	4	0.805	0.115	0.097	0.806	0.093	0.091	0.804	0.061	0.049
40 [50%]	1	0.719	0.076	0.063	0.756	0.059	0.053	0.751	0.033	0.030
	2	0.837	0.079	0.064	0.800	0.064	0.058	0.770	0.036	0.034
	3	0.888	0.083	0.065	0.902	0.067	0.059	0.812	0.041	0.037
	4	0.868	0.093	0.073	0.854	0.067	0.061	0.798	0.043	0.039
40 [80%]	1	0.764	0.066	0.051	0.769	0.046	0.038	0.760	0.031	0.026
	2	0.801	0.068	0.054	0.782	0.047	0.041	0.766	0.032	0.027
	3	0.822	0.071	0.062	0.828	0.049	0.046	0.793	0.032	0.028
	4	0.817	0.075	0.062	0.820	0.052	0.046	0.790	0.032	0.029
80 [50%]	1	0.750	0.056	0.046	0.816	0.037	0.029	0.805	0.023	0.020
	2	0.839	0.058	0.050	0.859	0.038	0.031	0.825	0.025	0.021
	3	0.897	0.059	0.050	0.938	0.040	0.033	0.883	0.027	0.022
	4	0.872	0.064	0.050	0.897	0.045	0.037	0.861	0.029	0.024
80 [80%]	1	0.768	0.035	0.027	0.821	0.028	0.022	0.809	0.014	0.011
	2	0.802	0.036	0.030	0.837	0.031	0.023	0.820	0.016	0.013
	3	0.826	0.045	0.037	0.873	0.034	0.027	0.853	0.021	0.017
	4	0.820	0.052	0.040	0.867	0.036	0.029	0.848	0.022	0.018

Table 5: The point evaluations of $h(t)$

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
					$k=2$					
20 [50%]	1	0.662	0.882	0.797	0.546	0.801	0.606	0.978	0.164	0.174
	2	0.553	0.838	0.720	0.393	0.728	0.562	0.978	0.143	0.160
	3	0.846	0.672	0.640	0.667	0.662	0.518	0.988	0.137	0.147
	4	0.742	0.620	0.510	0.872	0.519	0.475	0.985	0.128	0.137
20 [80%]	1	0.862	0.555	0.487	0.618	0.506	0.433	0.968	0.119	0.099
	2	0.827	0.496	0.478	0.609	0.491	0.413	0.971	0.117	0.078
	3	0.967	0.479	0.473	0.733	0.441	0.401	0.988	0.111	0.070
	4	1.195	0.446	0.423	0.795	0.438	0.377	0.987	0.107	0.067
40 [50%]	1	0.621	0.437	0.415	0.523	0.432	0.369	1.012	0.105	0.064
	2	0.441	0.418	0.398	0.312	0.414	0.310	1.006	0.093	0.061
	3	0.828	0.392	0.366	0.694	0.382	0.295	1.029	0.082	0.057
	4	1.429	0.362	0.351	0.929	0.354	0.284	1.017	0.081	0.053
40 [80%]	1	0.835	0.357	0.338	0.634	0.325	0.280	1.005	0.077	0.052
	2	0.786	0.325	0.310	0.554	0.312	0.277	0.988	0.066	0.052
	3	0.952	0.322	0.301	0.769	0.310	0.259	1.030	0.065	0.052
	4	1.104	0.314	0.290	0.832	0.306	0.236	1.025	0.064	0.051
80 [50%]	1	0.600	0.287	0.264	0.559	0.284	0.222	0.883	0.062	0.051
	2	0.372	0.280	0.239	0.235	0.267	0.221	0.777	0.057	0.046
	3	0.820	0.258	0.219	0.732	0.230	0.202	0.933	0.056	0.043
	4	1.176	0.251	0.214	1.061	0.211	0.175	0.981	0.055	0.035
80 [80%]	1	0.818	0.236	0.210	0.723	0.190	0.173	0.878	0.051	0.030
	2	0.756	0.193	0.167	0.682	0.184	0.154	0.860	0.046	0.029
	3	0.943	0.178	0.140	0.862	0.173	0.135	0.955	0.037	0.028
	4	1.063	0.144	0.117	0.870	0.132	0.103	0.982	0.036	0.027
					$k=5$					
20 [50%]	1	0.790	1.394	0.462	0.855	0.536	0.529	0.934	0.154	0.146
	2	1.079	1.330	0.535	0.815	0.480	0.473	0.917	0.139	0.132
	3	0.838	1.278	0.370	0.860	0.348	0.335	0.938	0.133	0.125
	4	0.785	1.250	0.468	1.053	0.306	0.282	0.992	0.126	0.115
20 [80%]	1	1.247	1.145	0.442	0.964	0.295	0.266	0.959	0.109	0.125
	2	1.314	1.020	0.494	0.964	0.269	0.248	0.958	0.086	0.116
	3	1.091	0.954	0.342	0.974	0.264	0.245	0.976	0.085	0.110
	4	1.290	0.844	1.433	1.089	0.262	0.225	1.002	0.083	0.094
40 [50%]	1	0.620	0.795	0.422	0.777	0.234	0.215	0.972	0.077	0.088
	2	0.825	0.663	0.319	0.752	0.231	0.187	0.948	0.076	0.087
	3	0.726	0.649	0.329	0.974	0.229	0.187	1.021	0.075	0.074
	4	1.254	0.464	0.880	1.160	0.180	0.155	1.048	0.065	0.073
40 [80%]	1	1.046	0.446	0.214	1.035	0.178	0.147	0.994	0.064	0.065
	2	1.089	0.388	0.234	1.043	0.166	0.137	0.990	0.063	0.058

(Continued)

Table 5 (continued)

n FP%	Plan	MLE			MCMC					
					Prior [A]			Prior [B]		
	3	0.948	0.357	0.198	1.055	0.156	0.131	1.030	0.062	0.057
	4	1.241	0.355	0.359	1.116	0.154	0.125	1.038	0.060	0.050
80 [50%]	1	0.577	0.345	0.456	0.559	0.151	0.123	0.826	0.059	0.049
	2	0.756	0.327	0.293	0.503	0.150	0.123	0.758	0.056	0.042
	3	0.690	0.312	0.343	0.698	0.147	0.121	0.932	0.049	0.042
	4	1.141	0.303	0.551	0.922	0.142	0.117	0.984	0.040	0.040
80 [80%]	1	0.994	0.250	0.140	0.787	0.141	0.105	0.876	0.039	0.038
	2	1.032	0.188	0.149	0.784	0.125	0.096	0.865	0.034	0.034
	3	0.909	0.181	0.155	0.819	0.120	0.094	0.945	0.032	0.027
	4	1.136	0.172	0.217	0.892	0.119	0.093	0.974	0.030	0.026

Table 6: The interval evaluations of σ

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
			$k = 2$				
20 [50%]	1	0.456	0.937	0.365	0.942	0.178	0.950
	2	0.541	0.935	0.381	0.940	0.182	0.949
	3	0.611	0.932	0.398	0.939	0.195	0.948
	1	0.686	0.929	0.424	0.937	0.220	0.946
20 [80%]	1	0.370	0.940	0.322	0.947	0.164	0.952
	2	0.375	0.940	0.345	0.945	0.168	0.952
	3	0.396	0.939	0.350	0.945	0.171	0.951
	4	0.429	0.938	0.354	0.944	0.174	0.951
40 [50%]	1	0.307	0.942	0.239	0.952	0.140	0.955
	2	0.326	0.942	0.249	0.951	0.149	0.954
	3	0.346	0.941	0.250	0.951	0.155	0.953
	4	0.357	0.941	0.285	0.950	0.162	0.952
40 [80%]	1	0.241	0.945	0.205	0.954	0.125	0.957
	2	0.247	0.945	0.215	0.954	0.126	0.957
	3	0.266	0.944	0.227	0.953	0.127	0.957
	4	0.293	0.943	0.232	0.952	0.135	0.956
80 [50%]	1	0.205	0.946	0.189	0.957	0.109	0.960
	2	0.211	0.946	0.191	0.956	0.115	0.959
	3	0.218	0.946	0.194	0.955	0.119	0.958

(Continued)

Table 6 (continued)

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
	4	0.240	0.945	0.199	0.955	0.122	0.958
80 [80%]	1	0.149	0.951	0.146	0.961	0.078	0.964
	2	0.164	0.949	0.150	0.960	0.084	0.963
	3	0.177	0.948	0.164	0.959	0.089	0.963
	4	0.183	0.947	0.178	0.958	0.098	0.962
			$k = 5$				
20 [50%]	1	0.427	0.941	0.285	0.946	0.162	0.954
	2	0.454	0.939	0.348	0.944	0.167	0.953
	3	0.608	0.936	0.355	0.943	0.176	0.952
	1	0.620	0.933	0.366	0.941	0.184	0.950
20 [80%]	1	0.367	0.944	0.240	0.951	0.122	0.956
	2	0.375	0.944	0.250	0.949	0.134	0.956
	3	0.387	0.943	0.259	0.949	0.140	0.955
	4	0.387	0.942	0.285	0.948	0.152	0.955
40 [50%]	1	0.292	0.946	0.223	0.956	0.091	0.959
	2	0.299	0.946	0.224	0.955	0.092	0.958
	3	0.318	0.945	0.225	0.955	0.102	0.957
	4	0.364	0.945	0.234	0.954	0.110	0.956
40 [80%]	1	0.245	0.949	0.201	0.958	0.081	0.961
	2	0.245	0.949	0.214	0.958	0.082	0.961
	3	0.253	0.948	0.215	0.957	0.085	0.961
	4	0.260	0.947	0.220	0.956	0.088	0.960
80 [50%]	1	0.176	0.950	0.169	0.961	0.067	0.964
	2	0.203	0.950	0.174	0.960	0.070	0.963
	3	0.214	0.950	0.186	0.959	0.073	0.962
	4	0.244	0.949	0.195	0.959	0.079	0.962
80 [80%]	1	0.145	0.955	0.126	0.965	0.053	0.968
	2	0.165	0.953	0.146	0.964	0.056	0.967
	3	0.168	0.952	0.156	0.963	0.061	0.967
	4	0.172	0.951	0.163	0.962	0.065	0.966

Table 7: The interval evaluations of μ

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
				$k = 2$			
20 [50%]	1	1.760	0.913	0.892	0.925	0.183	0.949
	2	1.690	0.915	0.886	0.927	0.177	0.950
	3	2.032	0.907	1.066	0.922	0.221	0.947
	4	1.871	0.911	0.933	0.924	0.201	0.948
20 [80%]	1	1.385	0.920	0.579	0.935	0.154	0.952
	2	1.218	0.921	0.541	0.936	0.149	0.952
	3	1.590	0.916	0.741	0.930	0.166	0.951
	4	1.461	0.918	0.655	0.933	0.163	0.951
40 [50%]	1	1.003	0.925	0.446	0.941	0.124	0.954
	2	0.887	0.927	0.432	0.942	0.118	0.955
	3	1.117	0.923	0.523	0.937	0.134	0.953
	4	1.048	0.924	0.487	0.940	0.127	0.954
40 [80%]	1	0.679	0.932	0.375	0.943	0.106	0.956
	2	0.673	0.933	0.352	0.944	0.104	0.956
	3	0.863	0.928	0.427	0.942	0.116	0.955
	4	0.698	0.931	0.389	0.943	0.110	0.956
80 [50%]	1	0.563	0.937	0.309	0.946	0.097	0.958
	2	0.468	0.939	0.298	0.946	0.096	0.958
	3	0.627	0.934	0.333	0.945	0.102	0.957
	4	0.582	0.936	0.326	0.945	0.100	0.957
80 [80%]	1	0.400	0.941	0.210	0.948	0.087	0.959
	2	0.368	0.943	0.199	0.949	0.079	0.960
	3	0.441	0.940	0.258	0.947	0.095	0.958
	4	0.417	0.941	0.224	0.948	0.094	0.958
				$k = 5$			
20 [50%]	1	3.412	0.903	0.991	0.920	0.227	0.946
	2	3.316	0.906	0.910	0.922	0.221	0.947
	3	3.764	0.898	1.072	0.917	0.249	0.944
	4	3.566	0.901	1.035	0.919	0.239	0.945
20 [80%]	1	3.106	0.911	0.886	0.930	0.205	0.949
	2	2.923	0.912	0.885	0.931	0.198	0.949
	3	3.212	0.907	0.906	0.925	0.219	0.947
	4	3.152	0.909	0.890	0.928	0.208	0.948
40 [50%]	1	2.339	0.916	0.868	0.936	0.188	0.951
	2	1.970	0.918	0.865	0.937	0.183	0.952
	3	2.871	0.914	0.882	0.931	0.192	0.950
	4	2.589	0.915	0.875	0.935	0.190	0.951
40 [80%]	1	1.556	0.923	0.839	0.938	0.165	0.953
	2	1.554	0.924	0.820	0.939	0.155	0.953
	3	1.728	0.919	0.860	0.937	0.179	0.952

(Continued)

Table 7 (continued)

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
	4	1.708	0.922	0.848	0.938	0.171	0.953
80 [50%]	1	1.060	0.928	0.697	0.941	0.141	0.955
	2	1.017	0.930	0.694	0.941	0.132	0.955
	3	1.184	0.925	0.809	0.939	0.148	0.954
	4	1.106	0.927	0.701	0.940	0.143	0.954
80 [80%]	1	0.710	0.932	0.583	0.943	0.094	0.957
	2	0.583	0.934	0.502	0.944	0.089	0.958
	3	0.970	0.931	0.648	0.942	0.121	0.956
	4	0.922	0.932	0.591	0.943	0.119	0.956

Table 8: The interval evaluations of $R(t)$

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
			$k = 2$				
20 [50%]	1	0.205	0.942	0.172	0.955	0.138	0.959
	2	0.215	0.941	0.176	0.954	0.140	0.959
	3	0.229	0.940	0.179	0.954	0.142	0.958
	4	0.291	0.937	0.190	0.953	0.144	0.958
20 [80%]	1	0.179	0.945	0.152	0.957	0.121	0.961
	2	0.190	0.944	0.155	0.957	0.125	0.961
	3	0.194	0.943	0.161	0.956	0.131	0.960
	4	0.196	0.943	0.162	0.956	0.132	0.960
40 [50%]	1	0.155	0.948	0.134	0.960	0.091	0.965
	2	0.157	0.948	0.143	0.959	0.101	0.964
	3	0.163	0.947	0.145	0.958	0.109	0.963
	4	0.171	0.946	0.148	0.958	0.116	0.962
40 [80%]	1	0.127	0.951	0.111	0.963	0.079	0.967
	2	0.133	0.950	0.117	0.963	0.082	0.967
	3	0.135	0.950	0.122	0.962	0.085	0.966
	4	0.139	0.949	0.130	0.961	0.089	0.966
80 [50%]	1	0.107	0.954	0.093	0.965	0.060	0.970
	2	0.111	0.953	0.099	0.965	0.068	0.969
	3	0.115	0.953	0.103	0.964	0.071	0.968
	4	0.122	0.952	0.107	0.964	0.075	0.968
80 [80%]	1	0.085	0.957	0.074	0.968	0.044	0.973
	2	0.089	0.956	0.082	0.967	0.049	0.972
	3	0.095	0.955	0.087	0.967	0.053	0.971
	4	0.099	0.955	0.091	0.966	0.056	0.971

(Continued)

Table 8 (continued)

nFP%	Plan	95% ACI		95% BCI Prior [A]		95% BCI Prior [B]	
			$k = 5$				
20 [50%]	1	0.191	0.945	0.134	0.958	0.088	0.961
	2	0.204	0.944	0.137	0.957	0.089	0.961
	3	0.213	0.943	0.144	0.957	0.093	0.960
	4	0.279	0.940	0.159	0.956	0.105	0.959
20 [80%]	1	0.167	0.948	0.121	0.960	0.072	0.963
	2	0.173	0.947	0.125	0.960	0.075	0.963
	3	0.180	0.946	0.129	0.959	0.078	0.962
	4	0.186	0.946	0.130	0.959	0.082	0.962
40 [50%]	1	0.131	0.951	0.109	0.963	0.063	0.967
	2	0.149	0.951	0.110	0.962	0.066	0.966
	3	0.157	0.950	0.114	0.961	0.067	0.965
	4	0.161	0.949	0.118	0.961	0.068	0.964
40 [80%]	1	0.108	0.954	0.090	0.966	0.054	0.969
	2	0.110	0.953	0.097	0.966	0.055	0.968
	3	0.115	0.953	0.102	0.965	0.056	0.968
	4	0.122	0.952	0.107	0.964	0.057	0.968
80 [50%]	1	0.080	0.957	0.071	0.968	0.046	0.972
	2	0.085	0.956	0.074	0.968	0.051	0.971
	3	0.091	0.956	0.077	0.967	0.052	0.970
	4	0.102	0.955	0.083	0.967	0.053	0.970
80 [80%]	1	0.063	0.960	0.053	0.971	0.037	0.975
	2	0.072	0.959	0.059	0.970	0.039	0.974
	3	0.076	0.958	0.063	0.970	0.040	0.973
	4	0.079	0.958	0.069	0.969	0.042	0.973

Table 9: The interval evaluations of $h(t)$

nFP%	Plan	95% ACI		95% BCI Prior [A]		95% BCI Prior [B]	
			$k = 2$				
20 [50%]	1	1.276	0.918	0.595	0.932	0.269	0.943
	2	1.154	0.921	0.557	0.934	0.245	0.944
	3	1.018	0.922	0.536	0.935	0.229	0.945
	4	0.904	0.923	0.526	0.936	0.216	0.947
20 [80%]	1	0.895	0.925	0.522	0.937	0.210	0.947
	2	0.871	0.926	0.512	0.937	0.199	0.948
	3	0.819	0.927	0.505	0.938	0.181	0.948

(Continued)

Table 9 (continued)

*n*FP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
	4	0.798	0.928	0.493	0.939	0.178	0.949
40 [50%]	1	0.792	0.928	0.484	0.940	0.169	0.950
	2	0.789	0.929	0.468	0.941	0.167	0.950
	3	0.734	0.930	0.456	0.942	0.166	0.951
	4	0.607	0.932	0.455	0.942	0.145	0.952
40 [80%]	1	0.568	0.935	0.449	0.943	0.140	0.953
	2	0.549	0.936	0.443	0.943	0.136	0.954
	3	0.537	0.937	0.423	0.944	0.134	0.954
	4	0.534	0.937	0.415	0.945	0.129	0.955
80 [50%]	1	0.521	0.938	0.402	0.945	0.126	0.956
	2	0.484	0.940	0.370	0.946	0.120	0.956
	3	0.420	0.943	0.369	0.947	0.117	0.957
	4	0.383	0.945	0.366	0.947	0.112	0.957
80 [80%]	1	0.376	0.946	0.346	0.948	0.110	0.958
	2	0.372	0.946	0.339	0.949	0.103	0.958
	3	0.368	0.947	0.312	0.950	0.091	0.959
	4	0.351	0.948	0.286	0.951	0.088	0.960
				$k = 5$			
20 [50%]	1	1.978	0.906	0.622	0.929	0.286	0.941
	2	1.864	0.909	0.587	0.931	0.251	0.943
	3	1.755	0.910	0.562	0.932	0.242	0.944
	4	1.724	0.911	0.555	0.933	0.232	0.945
20 [80%]	1	1.607	0.913	0.544	0.934	0.220	0.946
	2	1.538	0.914	0.536	0.934	0.210	0.947
	3	1.462	0.915	0.512	0.935	0.197	0.948
	4	1.397	0.916	0.510	0.936	0.190	0.948
40 [50%]	1	1.368	0.917	0.499	0.937	0.185	0.949
	2	1.331	0.918	0.493	0.938	0.182	0.949
	3	1.123	0.919	0.491	0.938	0.181	0.950
	4	0.855	0.921	0.484	0.939	0.174	0.951
40 [80%]	1	0.797	0.923	0.471	0.940	0.172	0.952
	2	0.771	0.924	0.467	0.940	0.169	0.953
	3	0.769	0.925	0.460	0.941	0.159	0.953
	4	0.768	0.925	0.417	0.942	0.155	0.954
80 [50%]	1	0.709	0.926	0.402	0.942	0.153	0.954
	2	0.687	0.928	0.392	0.943	0.151	0.954
	3	0.613	0.930	0.384	0.944	0.147	0.955
	4	0.513	0.933	0.378	0.944	0.143	0.956
80 [80%]	1	0.495	0.934	0.364	0.945	0.131	0.957

(Continued)

Table 9 (continued)

nFP%	Plan	95% ACI		95% BCI			
				Prior [A]		Prior [B]	
	2	0.486	0.934	0.349	0.946	0.119	0.957
	3	0.442	0.935	0.339	0.947	0.110	0.958
	4	0.436	0.936	0.325	0.947	0.103	0.959

We present the following findings on the highest level of CPs and the lowest levels of RMSE, MAB, and AIL values:

- Every estimate provided for σ, μ, $R(t)$, or $h(t)$ behaves suitably.
- Every estimate behaves suitably as n(or PF%) grows. As $\sum_{i=1}^{r} S_i$ decreases, we also reach a similar observation.
- With respect to the interval estimates', we discover that the bottom limits of every unknown parameter are consistently positive. Additionally, we have seen that the calculated interval estimates of $R(t)$ fall inside the interval of (0, 1).
- When comparing the two prior sets, A and B, on the Bayes evaluations, it is noted that the estimates derived from the latter are superior to those derived from the former.
- As k grows, we have noted that:
  - The calculated RMSE and MAB values of σ and $R(t)$ decrease.
  - The RMSEs and MABs of μ and $h(t)$ calculated from the likelihood approach increase while those calculated from the Bayes' approach decrease.
  - The AILs of σ and $R(t)$ decrease while those of μ and $h(t)$ increase.
  - The CPs of σ and $R(t)$ increase while those of μ and $h(t)$ decrease.
- Comparing the proposed progressive plans 1, 2, 3, and 4, we have noted that:
  - The estimation results of σ or $R(t)$ become satisfactorily based on Plan-1 (when the live items are removed uniformly) than others.
  - The estimation results of $\mu(h(t))$ perform better based on Plan-2 (Plan-4) (when the live items are removed at the first (last) stage) than others.
- Owing to the gamma knowledge of σ and μ, the Bayes estimates of σ, μ, $R(t)$, or $h(t)$ perform better than alternative estimates. Likewise, the same conclusion is drawn about credible intervals.
- Overall, the Bayes method via MH methodology is recommended for estimating the MKR parameters σ, μ, $R(t)$, or $h(t)$ in the presence of data collected via the proposed strategy.

5 Optimal Progressive Censoring Plan

In this section, we discuss several optimality criteria for selecting the best progressive censoring plan. The statistical literature has recently focused on finding the optimal censoring scheme, as evidenced by the works of Balakrishnan and Aggarwala [26] and Pradhan and Kundu [27]. The following discussion covers some of the optimality criteria used in this context. In practice, our goal is to choose the censoring scheme that provides the most information about the unknown parameters.

Table 10 presents commonly used optimal criteria to help us select the best progressive censoring strategy.

Table 10: Some practical censorship plan optimal criteria

Criterion	Procedure
$\mathcal{O}_1$	Max trace $[\mathbf{J}(\cdot)]$
$\mathcal{O}_2$	Min trace $[\mathbf{J}^{-1}(\cdot)]$
$\mathcal{O}_3$	Min det $[\mathbf{J}^{-1}(\cdot)]$
$\mathcal{O}_4$	Min Var $(\log(\hat{y}_p))$, $0 < p < 1$

In terms of criterion $\mathcal{O}_1$, our objective is to maximize the trace of the observed Fisher information matrix, $\mathbf{J}(\cdot)$. Additionally, for criteria $\mathcal{O}_2$ and $\mathcal{O}_3$, we aim to minimize the determinant and trace of $\mathbf{J}^{-1}(\cdot)$, respectively. It is evident that criterion $\mathcal{O}_4$, which depends on the value of p, seeks to minimize the variance of the logarithmic MLE of the p-th quantile. Therefore, for the MKR distribution, the logarithm of $\hat{y}_p$ is given by

$$\log(\hat{y}_p) = \frac{1}{2}\log\left\{\frac{\log[1+(-\log(1-p))^{1/\hat{\sigma}}]}{\hat{\mu}}\right\}, 0 < p < 1.$$

By using the delta method, we can estimate the variance of $\log(\hat{y}_p)$.

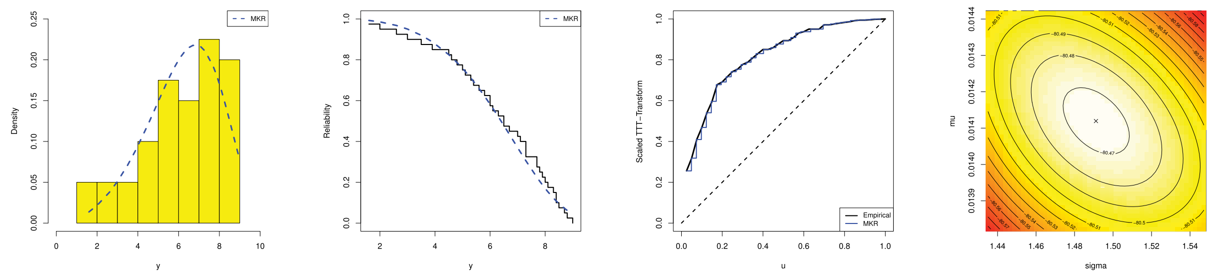
6 Data Analysis

In this part, we will analyze one real dataset from the engineering field to demonstrate how the suggested estimators may be applied in reality. In this example, the time-to-failure (10^3) of forty sets of turbocharged engines in one kind of diesel engine will be discussed; see Table 11. This set od data was originally provided by Xu et al. [28] and rediscussed later by Guerra et al. [29] and Mohammed et al. [30].

Table 11: Lifetime points of 40 turbochargers

1.6	2.0	2.6	3.0	3.5	3.9	4.5	4.6	4.8	5.0
5.1	5.3	5.4	5.6	5.8	6.0	6.0	6.1	6.3	6.5
6.5	6.7	7.0	7.1	7.3	7.3	7.3	7.7	7.7	7.8
7.9	8.0	8.1	8.3	8.4	8.4	8.5	8.7	8.8	9.0

Now, before going to evaluate the novel theoretical point and interval results of σ, μ, $R(t)$, and $h(t)$, we checked the validity of the MKR model for the turbochargers data set. So, the MLEs $\hat{\sigma}$ and $\hat{\mu}$ (with their standard errors (St.Errs)) as well as the Kolmogorov-Smirnov (K-S) statistic (with its p-value) are computed. As a result, we have the MLE(St.Err) of σ is 1.4911(0.2076) and of μ is 0.0141(0.0011). Further, the K-S(p-value) is 0.1005(0.8131). The K-S result indicates that the MKR model fits the turbocharger data quite well. In Fig. 2, using the entire turbocharger data, we display four visual tools of the present fitting, called (i) fitted/empirical MKR density line, (ii) fitted/empirical MKR reliability line, (iii) total time on test (TTT) plot, and (iv) contour of the log-likelihood. As we anticipated, Fig. 2 supports the same fitting results presented in Table 2, indicates that the turbocharger data exhibits an increasing failure rate, and shows that the estimates $\hat{\sigma} \equiv 1.4911$ and $\hat{\mu} \equiv 0.0141$ existed and are unique.



(a) Density (b) Reliability (c) TTT (d) Contour

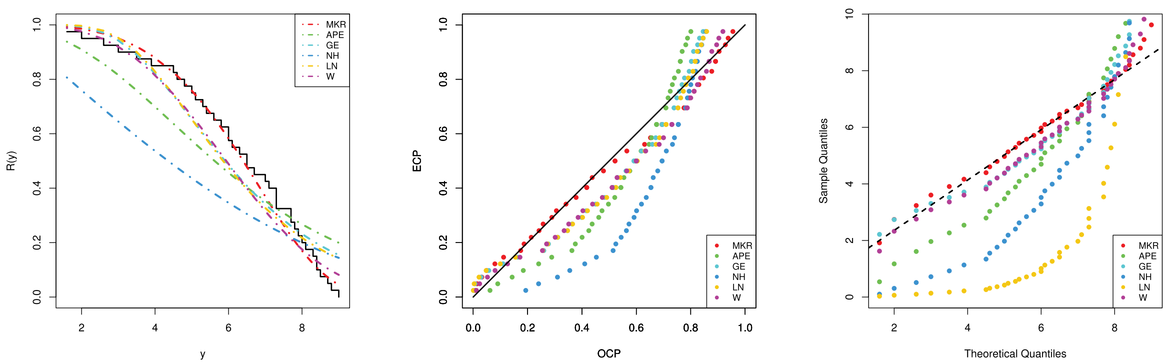
Figure 2: Four visual goodness-of-fit diagrams of the MKR model from turbocharger data

Before proceeding further, the adequacy of the proposed MKR model is examined using the complete turbocharger dataset by comparing its fit with five well-known lifetime distributions from the literature, namely: (1) alpha-power exponential (APE(σ, μ)), (2) generalized exponential (GE(σ, μ)), (3) Nadarajah–Haghighi (NH(σ, μ)), (4) log-normal (LN(σ, μ)), and (5) Weibull (W(σ, μ)) distributions. To determine the most appropriate model, six model selection criteria are employed: (i) negative log-likelihood (NL), (ii) Akaike (A), (iii) Bayesian (B), (iv) consistent Akaike (CA), (v) Hannan–Quinn (HQ), and (vi) K–S statistic along with its associated p-value. Table 12 reports the maximum likelihood estimates (with their standard errors) of σ and μ, along with the corresponding values of all model selection criteria. It is evident from Table 12 that the proposed MKR distribution provides the best fit to the turbocharger data, as it yields the lowest values for all information criteria and the highest p-value among the competing models.

Table 12: Summary fit of the MKR and its competitors from turbocharger data

Model	MLE (σ)		MLE (μ)		NL	A	B	CA	HQ	K-S	
	Est.	St.Err	Est.	St.Err						Distance	p-value
MKR	1.4915	0.2075	0.0141	0.0011	80.467	164.933	168.311	165.257	166.154	0.1006	0.8131
APE	89.314	51.798	0.3353	0.0335	94.934	193.868	197.246	194.193	195.090	0.2132	0.0527
GE	9.5148	2.8962	0.4498	0.0578	90.143	184.285	187.663	184.610	185.507	0.1542	0.2975
NH	28.216	15.983	0.0043	0.0024	102.12	208.250	211.628	208.574	209.471	0.3646	0.0048
LN	1.7668	0.0633	0.4006	0.0448	90.840	185.680	189.058	186.004	186.901	0.1498	0.3307
W	3.0917	0.1451	0.0028	0.0007	83.810	171.620	174.998	171.945	172.842	0.1361	0.4493

Furthermore, Fig. 3 illustrates several graphical diagnostics employed to assess the goodness of fit of the proposed MKR model in comparison with its competing lifetime distributions for the turbocharger dataset. These diagnostics include (i) plots of empirical and fitted reliability functions, and (ii) probability-probability (P–P) and quantile-quantile (Q–Q) plots. Collectively, the visual evidence presented in Fig. 3a–c supports the numerical findings, confirming that the MKR model provides an excellent fit and captures the underlying structure of the turbocharger failure data more accurately than the alternative models.



(a) RF (b) PP (c) QQ

Figure 3: Fitting diagrams for the MKR model and its competitors from turbocharger data

Now, to explain the acquired estimates from the turbocharger data, we randomly group the full turbocharger data into twenty groups, each containing $k = 2$ items; see Table 13. In this table, each bold number represents the first-failure item in each group. Next, by taking $r = 10$, different PFFC samples are generated; see Table 14. The provided point estimates (with St.Errs) and the 95% ACI/BCI estimates (with widths) of σ, μ, $R(t)$, and $h(t)$ (at $t = 5$) are derived from Table 15. The MCMC and BCI estimates of all unknown values are obtained using noninformative priors, where $M = 50,000$ and $B = 10,000$. The obtained estimation findings (in Table 15) of σ, μ, $R(t)$, and $h(t)$ created using frequentist (or Bayes) method are very close to each other.

Table 13: Random grouping for the turbocharger data

Group	1	2	3	4	5	6	7	8	9	10
Data	3.5	6.0	7.0	4.8	5.4	7.1	7.7	8.3	3.9	8.1
	5.3	8.8	1.6	6.5	8.5	4.6	2.6	5.6	9.0	6.5
Group	11	12	13	14	15	16	17	18	19	20
Data	4.5	8.0	5.8	5.1	5.0	7.8	8.4	7.7	7.3	2.0
	7.3	6.0	7.3	8.4	6.7	3.0	6.3	8.7	6.1	7.9

Table 14: Various PFFC samples from the turbocharger data

Sample	S	PFFC Data
$\mathbb{S}[1]$	$(1^{[10]})$	1.6, 3.0, 3.5, 4.8, 5.1, 5.6, 5.8, 6.1, 6.3, 6.5
$\mathbb{S}[2]$	$(5^{[2]}, 0^{[8]})$	1.6, 2.6, 3.0, 4.5, 4.6, 5.0, 5.1, 5.4, 5.8, 6.0
$\mathbb{S}[3]$	$(0^{[4]}, 5^{[2]}, 0^{[4]})$	1.6, 2.0, 2.6, 3.0, 3.5, 4.5, 4.8, 5.4, 5.8, 6.1
$\mathbb{S}[4]$	$(0^{[8]}, 5^{[2]})$	1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.6

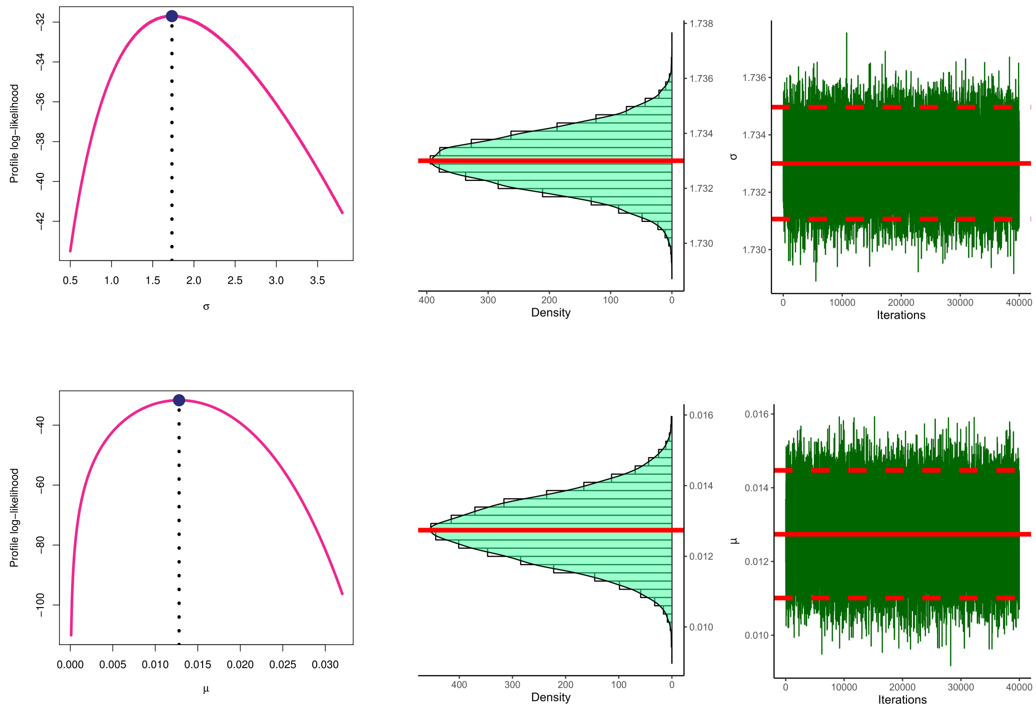
Table 15: Estimates of σ, μ, $R(t)$, and $h(t)$ from turbocharger data

Sample	Par.	MLE		MCMC		95% ACI			95% BCI		
		Est.	St.Err	Est.	St.Err	Lower	Upper	Width	Lower	Upper	Width
$\mathbb{S}[1]$	σ	1.7330	0.4465	1.7330	0.0010	0.8580	2.6080	1.7501	1.7311	1.7350	0.0039
	μ	0.0128	0.0024	0.0127	0.0009	0.0082	0.0174	0.0092	0.0110	0.0145	0.0035
	$R(t)$	0.8315	0.0553	0.8324	0.0214	0.7231	0.9399	0.2168	0.7906	0.8739	0.0833
	$h(t)$	0.1495	0.0484	0.1490	0.0224	0.0545	0.2444	0.1899	0.1062	0.1933	0.0871
$\mathbb{S}[2]$	σ	1.7775	0.4065	1.7775	0.0010	0.9807	2.5742	1.5935	1.7755	1.7794	0.0039
	μ	0.0210	0.0028	0.0209	0.0009	0.0154	0.0265	0.0112	0.0191	0.0228	0.0037
	$R(t)$	0.5972	0.0962	0.5985	0.0315	0.4087	0.7857	0.3770	0.5353	0.6588	0.1235
	$h(t)$	0.4709	0.1734	0.4705	0.0535	0.1311	0.8107	0.6796	0.3677	0.5763	0.2087
$\mathbb{S}[3]$	σ	1.5511	0.3604	1.5511	0.0010	0.8447	2.2576	1.4129	1.5492	1.5531	0.0039
	μ	0.0180	0.0035	0.0179	0.0010	0.0112	0.0248	0.0135	0.0161	0.0198	0.0037
	$R(t)$	0.6599	0.0856	0.6612	0.0278	0.4920	0.8277	0.3356	0.6058	0.7147	0.1088
	$h(t)$	0.3203	0.1362	0.3196	0.0360	0.0533	0.5873	0.5340	0.2505	0.3912	0.1407
$\mathbb{S}[4]$	σ	1.1789	0.3145	1.1789	0.0010	0.5624	1.7954	1.2330	1.1770	1.1809	0.0039
	μ	0.0123	0.0039	0.0122	0.0009	0.0047	0.0199	0.0153	0.0104	0.0141	0.0037
	$R(t)$	0.7412	0.0698	0.7428	0.0234	0.6043	0.8780	0.2736	0.6963	0.7879	0.0916
	$h(t)$	0.1641	0.0728	0.1632	0.0191	0.0214	0.3067	0.2853	0.1262	0.2009	0.0747

Notably, the narrower BCIs in Table 15, despite being based on weakly informative priors, indicate a regularizing effect that enhances the stability of the Bayesian estimates. This stands in contrast to the wider ACIs, which may reflect inefficiencies in asymptotic approximations under finite-sample conditions, thus requiring cautious interpretation of the corresponding precision measures. For further methodological insights and related discussions, the reader is referred to Zhang and Taflanidis [31].

To verify the validity and uniqueness of the obtained classical estimates of σ and μ, using $\mathbb{S}[1]$ as an example, Fig. 4 confirms the existence and uniqueness of $\hat{\sigma}$ and $\hat{\mu}$. Fig. 4 also displays the density (along its Gaussian kernel) and trace plots based on their lasting 40,000 iterations from $\mathbb{S}[1]$ (as an example) in order to demonstrate the efficiency of the MCMC findings of σ and μ. It shows that the convergence is sufficient after 40,000 MCMC iterations, suggesting that the influence of the starting values may be effectively eliminated by eliminating the first 10,000 iterations. It also indicates that the estimates of σ or μ are fairly symmetrical.

To demonstrate the idea of picking the optimal progressive design from the turbocharger data, all optimum criteria $\mathcal{O}_i$ for $i = 1, 2, 3, 4$ are assessed for all $\mathbb{S}[i]$, $i = 1, 2, 3, 4$, samples. Table 16 revealed that the uniform censoring used in $\mathbb{S}[1]$ is the best based on $\mathcal{O}_i$ for $i = 1, 3, 4$, while the right censoring used in $\mathbb{S}[4]$ is the best based on $\mathcal{O}_2$ compared to others.



(a) Profile log-likelihood (b) Density/Trace

Figure 4: Plots of σ (top) and μ (bottom) from turbocharger data

Table 16: Optimal progressive designs from turbocharger data

Sample	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$		
$p \rightarrow$				0.3	0.6	0.9
$\mathbb{S}[1]$	287840.67	0.19933	6.92×10^{-7}	2.77×10^{-112}	6.35×10^{-59}	7.99×10^{-20}
$\mathbb{S}[2]$	130219.70	0.16526	1.27×10^{-6}	3.87×10^{-69}	1.71×10^{-36}	1.53×10^{-12}
$\mathbb{S}[3]$	117739.63	0.12993	1.11×10^{-6}	7.97×10^{-75}	8.04×10^{-38}	8.81×10^{-12}
$\mathbb{S}[4]$	126227.75	0.09895	7.84×10^{-7}	7.94×10^{-93}	4.46×10^{-42}	4.11×10^{-10}

7 Concluding Remarks

In this study, we have evaluated the modified Kies Rayleigh distribution using both classical and Bayesian estimation approaches. Our focus was on data collected through the progressive first-failure censoring plan. The reliability and hazard rate functions are estimated, along with the distribution scale and shape parameters, using both point and interval estimation techniques. The Bayes estimates are obtained by utilizing the Markov Chain Monte Carlo procedure with a normal proposal distribution, due to the complex form of the posterior distribution. A simulation study is conducted to compare the efficiency of the different estimates. The simulation results indicated that as the number of elements in each group increases, the root mean square error and mean absolute bias of the shape parameter and reliability function decrease, while they increase for the scale and hazard rate function. On the other hand, the average interval length for the shape parameter and reliability function decreases with an increase in the number of elements in each group. In conclusion, it can be observed that the Bayes estimates, whether point or interval, outperform the classical ones in the various scenarios considered. It was also noted that coverage percentages were often below the nominal level of 95%, especially in small sample sizes. For this purpose, to offer better performance for the proposed asymptotic intervals when small sample sizes are available, the bootstrapping procedures are recommended. We consider four optimality criteria to select the best progressive censoring plan. The suitability of the modified Kies Rayleigh distribution to model dataset from engineering field was investigated. Finally, we examine the optimality criteria based on the considered real dataset, using four progressive plans. In future research, the Bayesian analysis can be extended by incorporating asymmetric loss functions such as LINEX and general entropy loss functions. This extension would allow for more practically relevant estimators of the reliability and hazard rate functions, particularly in reliability contexts where the cost of overestimation and underestimation of product lifetime are not equivalent.

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Ethics Approval: Not applicable.

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