14th World Congress on Computational Mechanics (WCCM)
ECCOMAS Congress 2020)
Virtual Congress: 11-–15 January 2021
F. Chinesta, R. Abgrall, O. Allix and M. Kaliske (Eds)

RECTANGULAR ISOTROPIC KIRCHHOFF-LOVE PLATE ON AN ELASTIC FOUNDATION UNDER THE ACTION OF UNSTEADY ELASTIC DIFFUSION PERTURBATIONS

ZEMSKOV V. ANDREI[1,2], TARLAKOVSKII V. DMITRY[2,1]

[1] Moscow Aviation Institute (National Research Institute)
125993, Moscow, Volokolamskoe shosse, 4, Russia
azemskov1975@mail.ru

[2] Research Institute of Mechanics, Lomonosov Moscow State University
119192, Moscow, Michurinsky prospekt 1, Russia
tdvhome@mail.ru

Key words: elastic diffusion, Laplace transform, Fourier series, Green's function, Kirchhoff-Love plate, elastic foundation, unsteady problem

Abstract. We study unsteady elastic diffusion vibrations of a freely supported rectangular isotropic Kirchhoff-Love plate on an elastic foundation, which is under the action of a distributed transverse load. A model that describes coupled elastic diffusion processes in multicomponent continuum is used for the mathematical problem formulation. The longitudinal and transverse vibrations equations of a rectangular isotropic Kirchhoff-Love plate with diffusion were obtained from the model using the d'Alembert variational principle.

The problem solution of unsteady elastic diffusion plate vibrations is sought in integral form. The bulk Green's functions are the kernels of the integral representations. To find the Green's functions, we used the Laplace transform in time and the expansion into double trigonometric Fourier series in spatial coordinates. Green's functions in the image domain are represented in the form of rational functions depends on the Laplace transform parameter. The transition to the original domain is done analytically through residues and tables of operational calculus. The bulk Green's functions analytical expressions are obtained.

Using a two-component continuum, a numerical study of unsteady mechanical and diffusion fields interaction is done for an isotropic plate. The solution is presented in analytical form, as well as in the form of three-dimensional graphs of the displacement fields and concentration increments on time and coordinates.

1 INTRODUCTION

A review of modern publications shows that the issues related to the study of the interaction of different physical nature fields is relevant. In particular, the load-carrying capacity of individual structural elements such as beams, plates and shells can be influenced by diffusion processes arising from mechanical loads. These issues are discussed in the articles [1, 2], where the influence of diffusion processes on the bearing capacity of a shallow transversally isotropic shell is investigated. The publications [3, 4, 5]

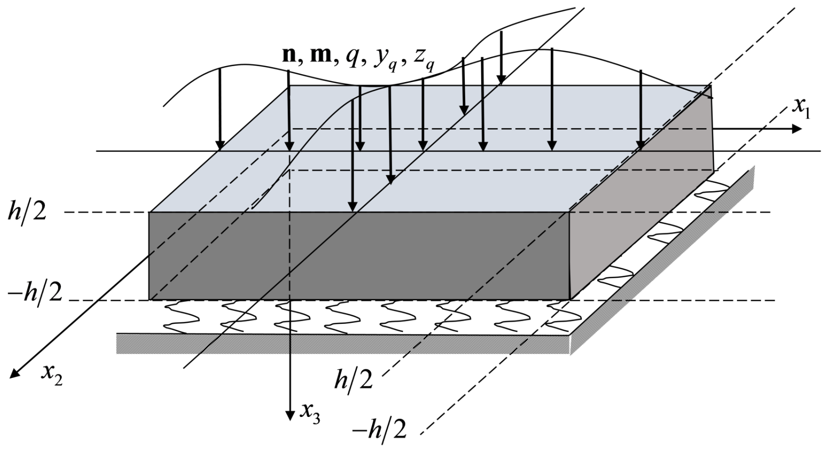

Figure 1: Forces and moments acting upon the plate

are devoted to the study of mechanodiffusion processes in plates. The calculation of an elastic spherical shells with diffusion is considered in [6].

It should be noted that all these problems are solved in a stationary formulation. The problems statements of unsteady elastic diffusion vibrations of beams and plates and methods for their solution are absent in the publications known to date.

This article considers the effects of the interaction of mechanical and diffusion fields in a Kirchhoff-Love plate on an elastic foundation. A mathematical model of plate elastic diffusion vibrations is obtained based on variational principles as well as the well-known plate theory relations presented in the works [7, 8, 9, 10].

2 GENERAL ESPECIFICATIONS

The problem of unsteady elastic diffusion vibrations of a rectangular Kirchhoff-Love plate on an elastic foundation is considered. A diagram of the applied forces and bending moments, as well as the orientation of the axes of a rectangular Cartesian coordinate system is shown in the figure 1.

Here $n = \{n_1, n_2\}$ is a longitudinal load density; $m = \{m_1, m_2\}$ is a moment density (bending moment per unit surface); q is a transverse load density; y_q, z_q are linear distribution of diffusion volume source.

For the problem formulation, we use the coupled N-component elastic diffusion continuum model in a rectangular Cartesian coordinate system, which has the next form [11, 12, 13]:

$$\ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i,\ \dot{\eta}^{(q)} = -\frac{\partial J_i^{(q)}}{\partial x_i} + Y^{(q)},\ \eta^{(N+1)} = -\sum_{q=1}^{N} \eta^{(q)}\ \left(q = \overline{1,N}\right). \tag{1}$$

where σ_{ij} and $J_i^{(q)}$ are the stress tensor and the diffusion flux vector components respectively, which are defined as follows $\left(anindexq = \overline{1,N}\right)$:

$$\sigma_{ij} = C_{ijkl}\frac{\partial u_k}{\partial x_l} - \sum_{q=1}^{N} \alpha_{ij}^{(q)} \eta^{(q)},\ J_i^{(q)} + \tau_q \dot{J}_i^{(q)} = -\sum_{t=1}^{N} D_{ij}^{(q)} g^{(qt)} \frac{\partial \eta^{(t)}}{\partial x_j} + \Lambda_{ijkl}^{(q)} \frac{\partial^2 u_k}{\partial x_j \partial x_l}. \tag{2}$$

Here, the dots denote are time derivatives. All quantities in (1) and (2) are dimensionless. For them the following notation is used

$$
\begin{gathered}
x_i=\frac{x_i^*}{l},\ u_i=\frac{u_i^*}{l},\ \tau=\frac{ct}{l},\ C_{ijkl}=\frac{C_{ijkl}^*}{C_{1111}^*},\ C^2=\frac{C_{1111}^*}{\rho},\ \alpha_{ij}^{(q)}=\frac{\alpha_{ij}^{*(q)}}{C_{1111}^*},\ l_m=\frac{l_m^*}{l},\\
D_{ij}^{(q)}=\frac{D_{ij}^{*(q)}}{Cl},\ \Lambda_{ijkl}^{(q)}=\frac{m^{(q)}D_{ij}^{*(q)}\alpha_{kl}^{*(q)}n_0^{(q)}}{\rho RT_0Cl},\ F_i=\frac{\rho lF_i^*}{C_{1111}^*},\ Y^{(q)}=\frac{lY^{*(q)}}{C},\ h=\frac{h^*}{l},\ \tau_q=\frac{c\tau^{(q)}}{l},
\end{gathered}
\tag{3}
$$

where t is time; x_i^* are the rectangular Cartesian coordinates; u_i^* are the displacement vector components; l is the characteristic linear size of the problem; l_1^*, l_2^* are the dimensions of the plate; h^* is the thickness of the plate; $\eta^{(q)}=n^{(q)}-n_0^{(q)}$ is the concentration increment of q-th component in the multicomponent continuum; $n^{(q)}$ and $n_0^{(q)}$ are the actual and initial concentrations; C_{ijkl}^* are elastic constant tensor components; ρ is the medium density; $\alpha_{ij}^{*(q)}$ are coefficients characterizing the medium volumetric changes due to diffusion; $D_{ij}^{*(q)}$ are the self-diffusion coefficients; R is the universal gas constant; T_0 is the medium temperature; $m^{(q)}$ is the molar mass of q-th component; F_i^* and $Y^{(q)}$ are the mechanical and diffusive bulk perturbations; $g^{(qt)}$ is the Darken thermodynamical factor; $\tau^{(q)}$ is the relaxation time of diffusion perturbations.

To construct equations for the bending of the plate, we turn to the variational formulation of the problem. According to the d'Alembert variational principle, the relations (1) – (3) can be written in the form [10]

$$
\begin{gathered}
\int_G\left(\ddot{u}_i-\frac{\partial\sigma_{ij}}{\partial x_j}-F_i\right)\delta u_i dG+\sum_{q=1}^{N}\int_G\left(1+\tau_q\frac{\partial}{\partial\tau}\right)\left(\dot{\eta}^{(q)}+\frac{\partial J_i^{(q)}}{\partial x_i}-Y^{(q)}\right)\delta\eta^{(q)}dG+\\
+\iint_{\Pi_\sigma}\left(\sigma_{ij}n_j-P_i\right)\delta u_i dS+\sum_{q=1}^{N}\iint_{\Pi_J}\left(J_i^{(q)}+\tau_q\dot{J}_i^{(q)}-I_i^{(q)}\right)n_i\delta\eta^{(q)}dS=0.
\end{gathered}
\tag{4}
$$

Here δu_i and $\delta\eta^{(q)}$ are virtual displacements and concentration increments; P_i and $I_i^{(q)}$ are surface disturbances; G is the problem solution domain; n_i are components of the outer normal unit vector to ∂G, $\partial G=\Pi_J\bigcup\Pi_\sigma$.

Further, we assume that:

1. The problem solution domain is rectangular parallelepiped $G=D\times[-h/2,h/2]$, where $D=[0,l_1]\times[0,l_2]$ is a rectangular domain in the plate middle surface $x_3=0$, $\Gamma=\partial D$ is boundary domain (fig. 1).
2. Plate surface is $\Pi=\Pi_-\cup\Pi_+\cup\Pi_b$, where Π_- is the bottom surface corresponding to $x_3=-h/2$, Π_+ is the top surface corresponding to $x_3=h/2$, $\Pi_b=\Pi_{11}\cup\Pi_{21}\cup\Pi_{12}\cup\Pi_{22}$ is the lateral surface. Here, lateral surfaces Π_{1k} corresponding to $x_k=0$, the surfaces Π_{2k} corresponding to $x_k=l_k$, $k=1,2$. It is assumed that the bottom/top surface is free of mechanical loads

$$
\sigma_{ij}n_j\big|_{\Pi_-}=\sigma_{ij}n_j\big|_{\Pi_+}=0. \tag{5}
$$

We will also assume that mass transfer through the bottom/top surface is absent

$$J_i^{(q)}\Big|_{\Pi_-} = J_i^{(q)}\Big|_{\Pi_+} = 0.$$

3. The plate material is a homogeneous isotropic continuum

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),\ \Lambda^{(q)}_{\alpha\alpha\beta\beta} = \Lambda_q,\ \alpha^{(q)}_{\alpha\alpha} = \alpha_q,\ D^{(q)}_{\alpha\alpha} = D_q.$$

   Here λ and μ are Lame coefficients, δ_{ij} are Kronecker symbol. Due to (3) $\lambda + 2\mu = 1$.

4. Transverse deflections are considered small. Then the linearization of the unknown quantities with respect to the variable x_3 will have the form [7, 8, 9, 10]

$$\begin{gathered} u_1(x_1,x_2,x_3,\tau) = u(x_1,x_2,\tau) - x_3\chi_1(x_1,x_2,\tau), \\ u_2(x_1,x_2,\tau) = v(x_1,x_2,\tau) - x_3\chi_1(x_1,x_2,\tau), \\ u_3(x_1,x_2,\tau) = w(x_1,x_2,\tau) + x_3\psi(x_1,x_2,\tau), \\ \eta^{(q)} = N_q(x_1,x_2,\tau) + x_3 H_q(x_1,x_2,\tau). \end{gathered} \tag{6}$$

5. We also consider that a straight fiber normal to the middle surface after deformation also remains straight and normal to the middle surface (the Kirchhoff plate theory). We assume that there are no deformations along the axis Ox_3

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = \psi = 0 \ \Rightarrow \psi = 0,$$

$$\begin{cases} \varepsilon_{13} = \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) = -\chi_1 + \dfrac{\partial w}{\partial x_1} = 0 \\ \varepsilon_{23} = \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) = -\chi_2 + \dfrac{\partial w}{\partial x_2} = 0 \end{cases} \Rightarrow \chi_k = \frac{\partial w}{\partial x_k},\ k = 1,2.$$

   Therefore, the equalities (6) are written as follows:

$$u_1 = u - x_3\frac{\partial w}{\partial x_1},\ u_2 = v - x_3\frac{\partial w}{\partial x_2},\ u_3 = w,\ \eta^{(q)} = N_q + x_3 H_q. \tag{7}$$

   Then, the components of the stress tensor and the diffusion flux vector will have the form

$$\begin{gathered} \sigma_{11} = \frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + \lambda\left(\frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2}\right) - \sum_{q=1}^{N}\alpha_q\left(N_q + x_3 H_q\right), \\ \sigma_{22} = \lambda\left(\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2}\right) + \frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2} - \sum_{q=1}^{N}\alpha_q\left(N_q + x_3 H_q\right), \end{gathered}$$

$$\sigma_{33} = \lambda\left(\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2}\right) + \lambda\left(\frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2}\right) - \sum_{q=1}^{N}\alpha_q H_q,$$

$$\sigma_{12} = \mu\left(\frac{\partial u}{\partial x_2} - 2x_3\frac{\partial^2 w}{\partial x_1 \partial x_2} + \frac{\partial v}{\partial x_1}\right),$$
$$\sigma_{13} = \mu\left(-\frac{\partial w}{\partial x_1} + \frac{\partial w}{\partial x_1}\right) = 0,\ \sigma_{23} = \mu\left(-\frac{\partial w}{\partial x_2} + \frac{\partial w}{\partial x_2}\right) = 0, \tag{8}$$

$$J_1^{(q)} + \tau_q \dot{J}_1^{(q)} = -D_q\left(\frac{\partial N_q}{\partial x_1} + x_3\frac{\partial H_q}{\partial x_1}\right) + \Lambda_q\left(\frac{\partial^2 u}{\partial x_1^2} - x_3\frac{\partial^3 w}{\partial x_1^3}\right) + \Lambda_q\left(\frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3\frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right),$$
$$J_2^{(q)} + \tau_q \dot{J}_2^{(q)} = -D_q\left(\frac{\partial N_q}{\partial x_2} + x_3\frac{\partial H_q}{\partial x_2}\right) + \Lambda_q\left(\frac{\partial^2 u}{\partial x_1 \partial x_2} - x_3\frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right) + \Lambda_q\left(\frac{\partial^2 v}{\partial x_2^2} - x_3\frac{\partial^3 w}{\partial x_2^3}\right),$$
$$J_3^{(q)} + \tau_q \dot{J}_3^{(q)} = -D_q H_q - \Lambda_q\frac{\partial^2 w}{\partial x_1^2} - \Lambda_q\frac{\partial^2 w}{\partial x_2^2},\ \left(q = \overline{1,N}\right).$$

6. A linear transverse load-deflection relationship is presumed (Winkler Model) [14]

$$q = \widetilde{q} - c_w w,\ c_w = \frac{c_w^* l}{C_{1111}^*}. \tag{9}$$

where c_w^* is the foundation modulus.

3 The equations of elastic diffusion vibrations of the Kirchhoff-Love plate on an elastic foundation

Substituting the equalities (6) – (9) into the variational equation (4), we obtain the following boundary value problems:

- plate longitudinal deformations problem

$$\ddot{u} = \frac{\partial^2 u}{\partial x_1^2} + \mu\frac{\partial^2 u}{\partial x_2^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x_1 \partial x_2} - \sum_{q=1}^{N}\alpha_q\frac{\partial N_q}{\partial x_1} + \frac{n_1}{h},$$
$$\ddot{v} = (\lambda + \mu)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_2^2} + \mu\frac{\partial^2 v}{\partial x_1^2} - \sum_{q=1}^{N}\alpha_q\frac{\partial N_q}{\partial x_2} - \frac{n_2}{h},$$
$$\dot{N}_q + \tau_q \ddot{N}_q = D_q\left(\frac{\partial^2 N_q}{\partial x_1^2} + \frac{\partial^2 N_q}{\partial x_2^2}\right) - \Lambda_q\left(\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 v}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 v}{\partial x_2^3}\right) + \frac{y_q}{h};$$

- plate deflections problem

$$\frac{\partial^2 \ddot{w}}{\partial x_1^2} + \frac{\partial^2 \ddot{w}}{\partial x_2^2} - \frac{12}{h^2}\ddot{w} = \frac{\partial^4 w}{\partial x_2^4} + \frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{12}{h^3}c_w w +$$
$$+ \sum_{q=1}^{N}\alpha_q\left(\frac{\partial^2 H_q}{\partial x_2^2} + \frac{\partial^2 H_q}{\partial x_1^2}\right) - \frac{12}{h^3}\left(\frac{\partial m_2}{\partial x_2} + \frac{\partial m_1}{\partial x_1} + q\right), \tag{10}$$
$$\dot{H}_q + \tau_q \ddot{H}_q = D_q\left(\frac{\partial^2 H_q}{\partial x_1^2} + \frac{\partial^2 H_q}{\partial x_2^2}\right) + \Lambda_q\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4}\right) + \frac{12}{h^3}z_q.$$

The unsteady bending mathematical model of a simply supported plate on an elastic foundation under a distributed load action is described by the equations (10), which are supplemented by zero initial conditions and boundary conditions, which are also obtained from the variational equation (4)

$$\begin{gathered}\left.\left(\frac{\partial^2 w}{\partial x_1^2}+\lambda\frac{\partial^2 w}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q H_q\right)\right|_{x_1=0}=0,\ \left.\left(\frac{\partial^2 w}{\partial x_1^2}+\lambda\frac{\partial^2 w}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q H_q\right)\right|_{x_1=l_1}=0,\\ \left.\left(\lambda\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q H_q\right)\right|_{x_2=0}=0,\ \left.\left(\lambda\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q H_q\right)\right|_{x_2=l_2}=0;\end{gathered} \tag{11}$$

$$w|_{x_1=0}=0,\ w|_{x_1=l_1}=0,\ w|_{x_2=0}=0,\ w|_{x_2=l_2}=0; \tag{12}$$

$$H_q\big|_{x_1=0}=0,\ H_q\big|_{x_1=l_1}=0,\ H_q\big|_{x_2=0}=0,\ H_q\big|_{x_2=l_2}=0. \tag{13}$$

4 Solution method

The solutions of the problem (10) – (13) under the action of distributed disturbances $F_1(x_1,x_2,\tau)=-12(\widetilde{q}+divm)/h^3$ and $F_{q+1}(x_1,x_2,\tau)=12z^{(q)}/h^3$ are sought in integral form ($q=\overline{1,N+1}$):

$$\begin{gathered}w(x_1,x_2,\tau)=\sum_{k=1}^{N+1}\int_0^{\tau}\int_0^{l_1}\int_0^{l_2}G_{1k}(x_1,x_2,\xi,\zeta,\tau-t)F_k(\xi,\zeta,t)\,d\xi d\zeta dt,\\ H_q(x_1,x_2,\tau)=\sum_{k=1}^{N+2}\int_0^{\tau}\int_0^{l_1}\int_0^{l_2}G_{q+1,k}(x_1,x_2,\xi,\zeta,\tau-t)F_k(\xi,\zeta,t)\,d\xi d\zeta dt,\end{gathered} \tag{14}$$

where G_{mk} are the bulk Greens functions, which satisfy the equations

$$\begin{gathered}\frac{\partial\ddot{G}_{1k}}{\partial x_1^2}+\frac{\partial\ddot{G}_{1k}}{\partial x_2^2}-\frac{12}{h^2}\ddot{G}_{1k}=\frac{\partial^4 G_{1k}}{\partial x_2^4}+\frac{\partial^4 G_{1k}}{\partial x_1^4}+2\frac{\partial^4 G_{1k}}{\partial x_1^2\partial x_2^2}+\frac{12}{h^3}c_w G_{1k}+\\ +\sum_{q=1}^{N}\alpha_q\left(\frac{\partial^2 G_{q+1,k}}{\partial x_2^2}+\frac{\partial^2 G_{q+1,k}}{\partial x_1^2}\right)-\delta_{1k}\delta(x_1-\xi)\,\delta(x_2-\zeta)\,\delta(\tau),\\ \dot{G}_{q+1,k}+\tau_q\ddot{G}_{q+1,k}=D_q\left(\frac{\partial^2 G_{q+1,k}}{\partial x_1^2}+\frac{\partial^2 G_{q+1,k}}{\partial x_2^2}\right)+\Lambda_q\left(\frac{\partial^4 G_{1k}}{\partial x_1^4}+2\frac{\partial^4 G_{1k}}{\partial x_1^2\partial x_2^2}+\frac{\partial^4 G_{1k}}{\partial x_2^4}\right)+\\ +\delta_{q+1,k}\delta(x_1-\xi)\,\delta(x_2-\zeta)\,\delta(\tau)\end{gathered} \tag{15}$$

and homogeneous boundary conditions

$$\begin{gathered}\left.\left(\frac{\partial^2 G_{1k}}{\partial x_1^2}+\lambda\frac{\partial^2 G_{1k}}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q G_{q+1,k}\right)\right|_{x_1=0}=0,\ \left.\left(\frac{\partial^2 G_{1k}}{\partial x_1^2}+\lambda\frac{\partial^2 G_{1k}}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q G_{q+1,k}\right)\right|_{x_1=l_1}=0,\\ \left.\left(\lambda\frac{\partial^2 G_{1k}}{\partial x_1^2}+\frac{\partial^2 G_{1k}}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q G_{q+1,k}\right)\right|_{x_2=0}=0,\ \left.\left(\lambda\frac{\partial^2 G_{1k}}{\partial x_1^2}+\frac{\partial^2 G_{1k}}{\partial x_2^2}+\sum_{q=1}^{N}\alpha_q G_{q+1,k}\right)\right|_{x_2=l_2}=0,\end{gathered}$$

$$G_{1k}|_{x_1=0}=0,\ G_{1k}|_{x_1=l_1}=0,\ G_{1k}|_{x_2=0}=0,\ G_{1k}|_{x_2=l_2}=0, \tag{16}$$

$$G_{q+1,k}\big|_{x_1=0}=0,\ G_{q+1,k}\big|_{x_1=l_1}=0,\ G_{q+1,k}\big|_{x_2=0}=0,\ G_{q+1,k}\big|_{x_2=l_2}=0.$$

To find the Green's functions G_{ik} the expanding into double trigonometric Fourier series in spatial coordinate and the Laplace transform in time are use. As a result, problem (15), (16) is reduced to the following system of linear algebraic equations (s is Laplace transform parameter, $\lambda_n=\pi n/l_1$, $\mu_m=\pi m/l_2$)

$$\begin{gathered}k_1(\nu_{nm},s)\,G_{1k}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)-\sum_{q=1}^{N}\alpha_q\nu_{nm}^2G_{q+1,k}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\delta_{1k}\sin\mu_m\zeta\sin\lambda_n\xi,\\ -\Lambda_q\nu_{nm}^4G_{1k}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)+k_{q+1}(\nu_{nm},s)\,G_{q+1,k}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\delta_{q+1,k}\sin\mu_m\zeta\sin\lambda_n\xi.\end{gathered}\tag{17}$$

Here

$$k_1(\nu_{nm},s)=s^2\left(\nu_{nm}^2+\frac{12}{h^2}\right)+\nu_{nm}^4+\frac{12}{h^3}c_w,\ k_{q+1}(\nu_{nm},s)=s+\tau_qs^2+D_q\nu_{nm}^2,\ \nu_{nm}^2=\lambda_n^2+\mu_m^2,$$

$$G_{ik}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2}G_{ik}^{L}(x_1,x_2,\xi,\zeta,s)\sin\lambda_nx_1\sin\mu_mx_2dx_2dx_1,$$

$$G_{ik}^{L}(x_1,x_2,\xi,\zeta,s)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}G_{ik}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)\sin\lambda_nx_1\sin\mu_mx_2.\tag{18}$$

The solution of the system (17) has the form ($q,p=\overline{1,N}$)

$$\begin{gathered}G_{11}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\frac{\Pi(\nu_{nm},s)}{P(\nu_{nm},s)}\sin\mu_m\zeta\sin\lambda_n\xi,\\ G_{1,q+1}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\frac{\nu_{nm}^2\alpha_q\Pi_q(\nu_{nm},s)}{P(\nu_{nm},s)}\sin\mu_m\zeta\sin\lambda_n\xi,\\ G_{q+1,1}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\frac{\Lambda_q\nu_{nm}^4\Pi(\nu_{nm},s)}{Q_q(\nu_{nm},s)}\sin\mu_m\zeta\sin\lambda_n\xi,\\ G_{q+1,p+1}^{Lss}(\lambda_n,\mu_m,\xi,\zeta,s)=\frac{4}{l_1l_2}\left[\frac{\delta_{qp}}{k_{q+1}(\nu_{nm},s)}+\frac{\Lambda_q\nu_{nm}^6\alpha_p\Pi_p(\nu_{nm},s)}{Q_q(\nu_{nm},s)}\right]\sin\mu_m\zeta\sin\lambda_n\xi,\end{gathered}\tag{19}$$

where

$$\Pi(\nu_{nm},s)=\prod_{j=1}^{N}k_{j+1}(\nu_{nm},s),\ \Pi_j(\nu_{nm},s)=\prod_{r=1,r\neq j}^{N}k_{r+1}(\nu_{nm},s),$$

$$\begin{gathered}P(\nu_{nm},s)=k_1(\nu_{nm},s)\,\Pi(\nu_{nm},s)-\nu_{nm}^6\sum_{j=1}^{N}\alpha_j\Lambda_j\Pi_j(\nu_{nm},s),\\ Q_q(\nu_{nm},s)=k_q(\nu_{nm},s)\,P(\nu_{nm},s).\end{gathered}\tag{20}$$

The originals in equalities (19) are calculated on the base of residues and the tables of operational calculus (prime denotes derivative with respect to parameter s) [15]

$$G_{1k}^{ss}(\lambda_n,\mu_m,\xi,\zeta,\tau)=\frac{4}{l_1 l_2}\sin\mu_m\zeta\sin\lambda_n\xi\sum_{l=1}^{2N+2}A_{1k}^{(l)}(\nu_{nm},s_l(\nu_{nm}))\exp(s_l(\nu_{nm})\tau),$$
$$G_{1,q+1}^{ss}(\lambda_n,\mu_m,\xi,\zeta,\tau)=\frac{4}{l_1 l_2}\sin\mu_m\zeta\sin\lambda_n\xi\sum_{l=1}^{2N+2}A_{q+1,k}^{(l)}(\nu_{nm},s_l(\nu_{nm}))\exp(s_l(\nu_{nm})\tau),$$
$$G_{q+1,1}^{ss}(\lambda_n,\mu_m,\xi,\zeta,\tau)=\frac{4}{l_1 l_2}\sin\mu_m\zeta\sin\lambda_n\xi\sum_{l=1}^{2N+4}A_{q+1,1}^{(l)}(\nu_{nm},s_l(\nu_{nm}))\exp(s_l(\nu_{nm})\tau),$$
$$G_{q+1,p+1}^{ss}(\lambda_n,\mu_m,\xi,\zeta,\tau)=\frac{4}{l_1 l_2}\sum_{l=1}^{2}\frac{\exp(\chi_l(\nu_{nm})\tau)}{k'_{q+1}(\nu_{nm},s)}\sin\mu_m\zeta\sin\lambda_n\xi+ \tag{21}$$
$$+\frac{4}{l_1 l_2}\sin\mu_m\zeta\sin\lambda_n\xi\sum_{l=1}^{2N+4}A_{q+1,p+1}^{(l)}(\nu_{nm},s_l(\nu_{nm}))\exp(s_l(\nu_{nm})\tau),$$

where

$$A_{1k}^{(l)}(\nu_{nm},s_l(\nu_{nm}))=\frac{\Pi(\nu_{nm},s_l(\nu_{nm}))}{P'(\nu_{nm},s_l(\nu_{nm}))},$$
$$A_{q+1,k}^{(l)}(\nu_{nm},s_l(\nu_{nm}))=\frac{\nu_{nm}^2\alpha_q\Pi_q(\nu_{nm},s_l(\nu_{nm}))}{P'(\nu_{nm},s_l(\nu_{nm}))},$$
$$A_{q+1,1}^{(l)}(\nu_{nm},s_l(\nu_{nm}))=\frac{\Lambda_q\nu_{nm}^4\Pi(\nu_{nm},s_l(\nu_{nm}))}{Q'_q(\nu_{nm},s_l(\nu_{nm}))}, \tag{22}$$
$$A_{q+1,p+1}^{(l)}(\nu_{nm},s_l(\nu_{nm}))=\frac{\Lambda_q\nu_{nm}^6\alpha_p\Pi_p(\nu_{nm},s_l(\nu_{nm}))}{Q'_q(\nu_{nm},s_l(\nu_{nm}))}.$$

In the obtained equalities $s_j(\nu_{nm})$, $j=\overline{1,N+2}$ are zeros of the polynomial $P(\nu_{nm},s)$, $\chi_l(\nu_{nm})$ are additional zeros of the polynomial $Q_q(\nu_{nm},s)$

$$\chi_1(\nu_{nm})=s_{2N+3}(\nu_{nm})=\frac{-1-\sqrt{1-4\tau_q k_{q+2}(\nu_{nm},0)}}{2\tau_q},$$
$$\chi_2(\nu_{nm})=s_{2N+4}(\nu_{nm})=\frac{-1+\sqrt{1-4\tau_q k_{q+2}(\nu_{nm},0)}}{2\tau_q}.$$

5 Example

Take calculating the material with the following characteristics (duralumin, λ^* and μ^* are Lame coefficients) [16]:

$$\lambda^*=6.93\cdot10^{10}\frac{N}{m^2},\ \mu^*=2.56\cdot10^{10}\frac{N}{m^2},\ T_0=800K,\ \rho=2700\frac{kg}{m^3},\ h=10^{-3}m$$
$$\alpha_{11}^{*(1)}=\alpha_{22}^{*(1)}=1.55\cdot10^7\frac{J}{kg},\ \alpha_{11}^{*(2)}=\alpha_{22}^{*(2)}=6.14\cdot10^7\frac{J}{kg},$$
$$D_{11}^{*(1)}=D_{22}^{*(1)}=7.73\cdot10^{-14}\frac{m^2}{s},D_{11}^{*(2)}=D_{22}^{*(2)}=6.67\cdot10^{-14}\frac{m^2}{s},$$
$$n_0^{(1)}=0.95,\ n_0^{(2)}=0.05,\ m^{(1)}=0.027\frac{kg}{mol},\ m^{(2)}=0.064\frac{kg}{mol}.$$

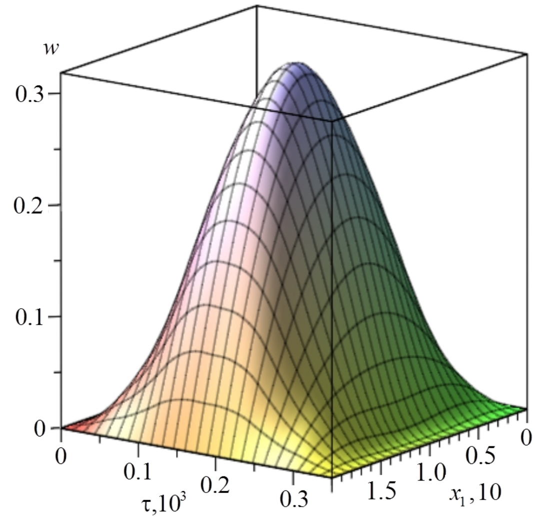

Figure 2: The plate deflections $w(x_1, l_2/2, \tau)$, $c_w^* l = 1.25 \cdot 10^7 N/m^2$

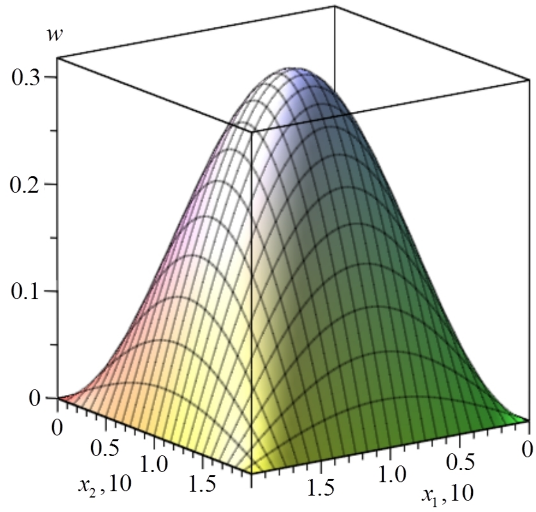

Figure 3: The plate deflections $w(x_1, l_2/2, \tau)$, $c_w^* = 0$

We assume that the plate is rectangular $l_1^* = 0.01\ m$, $l_2^* = 0.01\ m$: thickness $h^* = 0.001\ m$. We set the load in the form

$$F_1(x_1, x_2, \tau) = -\frac{12}{h^3}(\widetilde{q} + \mathrm{div} m) = H(\tau)\sin \pi x_2,\ F_{q+1}(x_1, x_2, \tau) = 0. \tag{23}$$

where $H(\tau)$ is the Heaviside function.

Calculating the convolutions (14) we get

$$\begin{aligned} w(x_1, x_2, \tau) &= \frac{2}{l_1}\sin\frac{\pi x_2}{l_2}\sum_{n=1}^{\infty}\sum_{l=1}^{2N+2} A_{11}^{(l)}(\nu_{n1}, s_l(\nu_{n1}))\frac{[\exp(s_l(\nu_{n1})\tau) - 1][1 - (-1)^n]}{\lambda_n s_l(\nu_{n1})}\sin \lambda_n x_1, \\ H_q(x_1, x_2, \tau) &= \frac{2}{l_1}\sin\frac{\pi x_2}{l_2}\sum_{n=1}^{\infty}\sum_{l=1}^{2N+4} A_{q+1,1}^{(l)}(\nu_{n1}, s_l(\nu_{n1}))\frac{[\exp(s_l(\nu_{n1})\tau) - 1][1 - (-1)^n]}{\lambda_n s_l(\nu_{n1})}\sin \lambda_n x_1. \end{aligned} \tag{24}$$

The calculation results are presented in Figures 2 – 7. Figure 2 shows the plate deflections on an elastic foundation. Figure 3 shows deflections in the absence of an elastic foundation.

Based on the performed calculations, it can be seen that the unsteady plate bending initiates diffusion flows in the plate. The Figure 4 shows the linear density of the aluminum concentration increment during elastic diffusion vibrations of the plate on an elastic foundation. In the Figure 5 is the same, but in the absence of an elastic foundation.

In Figures 6 – 9 are demonstrate the diffusion field effect on the displacement field.

The Figures 7 and 9 show the case when there is no elastic foundation. It is shown that, starting from a certain time, the elastic-diffusion vibrations begin to lag behind the elastic vibrations. In the presence of an elastic foundation, the influence of diffusion effects becomes negligible. The Figures 6, 8 show the

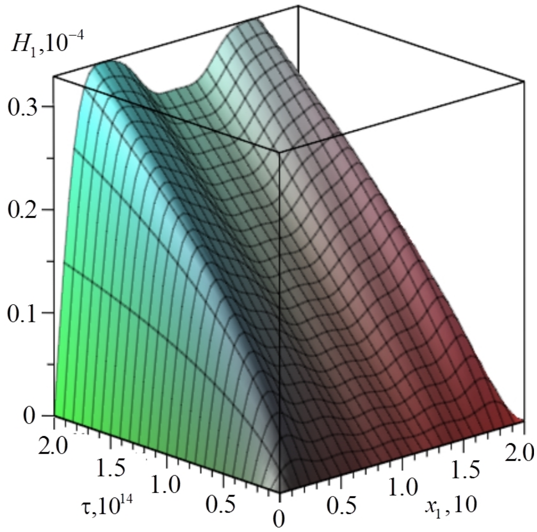

Figure 4: The concentration increment $H_1(x_1, l_2/2, \tau)$, $c_w^* l = 1.25 \cdot 10^7 N/m^2$

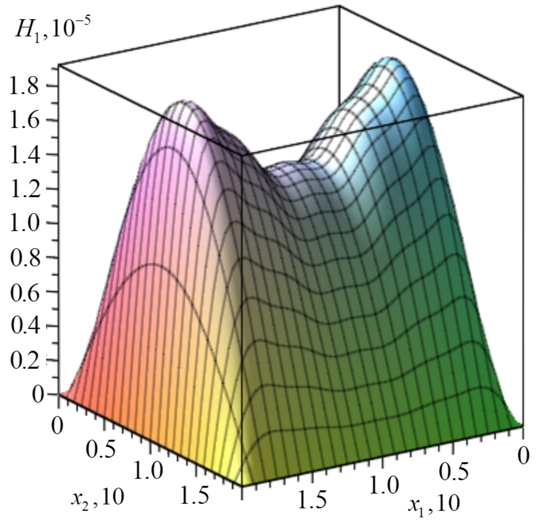

Figure 5: The concentration increment $H_1(x_1, l_2/2, \tau)$, $c_w^* = 0$

comparative results of elastic and elastic diffusion vibrations for the plate on an elastic foundation $c_w^* l = 1.25 \cdot 10^7 N/m^2$. At a sufficiently long time interval $\tau \sim 10^{12}$, the effect of diffusion on the displacement field was not found.

6 CONCLUSIONS

The unsteady vibrations mathematical model of a rectangular isotropic Kirchhoff-Love plate on an elastic foundation is constructed. The proposed model considers the effects of the interaction of mechanical and diffusion fields in continuum. An algorithm for bulk Green's functions constructing is proposed based on the use of the Laplace transform and expansions in trigonometric Fourier series.

By the example of plate bending under the action of a distributed mechanical load, the effect of interaction between mechanical and diffusion fields is demonstrated. It is shown that, on the one hand, unsteady bending initiates the process of mass transfer. On the other hand, diffusion effects the displacement field, which manifests itself in the form of a delay in mechanodiffusion oscillations with respect to purely mechanical ones. However, the presence of an elastic foundation significantly reduces the effect of diffusion on the displacement field. These results are presented in analytical and graphical forms.

REFERENCES

[1] Shvets R.N., Flyachok V.M.: Uravneniya mexanodiffuzii anizotropny'ch obolochek s uchetom poperechny'ch deformacij [The equations of mechanothermodiffusion of anisotropic shells taking account of transverse strains]. *Mat. Met. Fiz.-Mekh. Polya* (1984) 20: 54–61. [In Russian].

[2] Shvets R.N., Flyachok V.M. Variacionny'j podxod k resheniyu dinamicheskich zadach mexanotermodiffuzii anizotropny'ch obolochek [A variational approach to solving dynamic problems of mechanothermal diffusion of anisotropic shells]. *Mat. fiz. i nelinejnaya mexanika*. (1991) 16: 39–43. [In Russian].

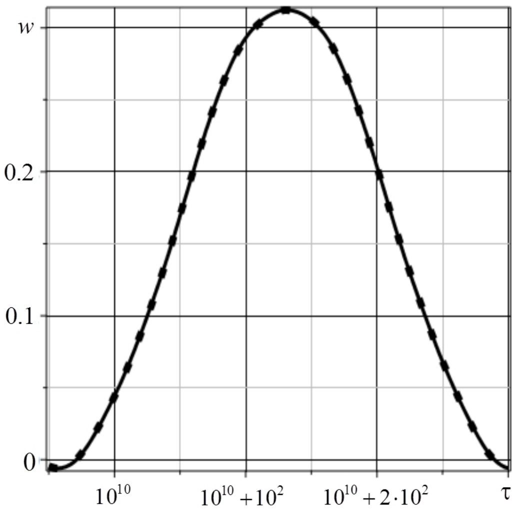

Figure 6: The plate deflections $w\left(\frac{l_1}{2},\frac{l_2}{2},\tau\right)$, $c_w^* l = 1.25\cdot 10^7 N/m^2$

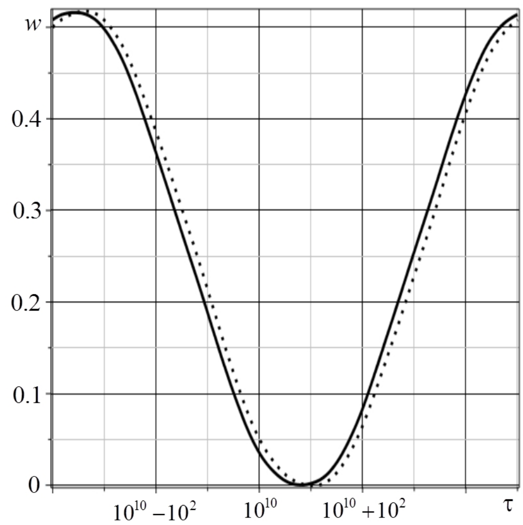

Figure 7: The plate deflections $w\left(\frac{l_1}{2},\frac{l_2}{2},\tau\right)$, $c_w^* = 0$

[3] Aouadi M. Miranville A. Smooth attractor for a nonlinear thermoelastic diffusion thin plate based on Gurtin–Pipkin's model. *Asymptotic Analysis*. (2015) **95**: 129–160.

[4] Aouadi M. On thermoelastic diffusion thin plate theory. *Appl. Math. Mech. Engl. Ed.* (2015) **36**(5): 619–632.

[5] Aouadi M., Miranville A. Quasi-stability and global attractor in nonlinear thermoelastic diffusion plate with memory. *Evolution equations and control theory*. (2015) **4**(3): 241–263.

[6] Bhattacharya D., Kanoria M. The influence of two temperature generalized thermoelastic diffusion inside a spherical shell. *International Journal of Engineering and Technical Research (IJETR)*. (2014) **2**(5): 151–159.

[7] Tarlakovskii D.V., Zemskov A.V. An Elastodiffusive Orthotropic Euler-Bernoulli Beam with Considering Diffusion Flux Relaxation. *Math. Comput. Appl.* (2019) **24**(1)23. doi:10.3390/mca24010023.

[8] Afanasieva O.A., Zemskov A.V. Unsteady Elastic-Diffusion Oscillations of a Simply Supported Kirchhoff Plate Under the Distributed Transverse Load Action. *In: Gdoutos E., Konsta-Gdoutos M. (eds) Proceedings of the Third International Conference on Theoretical, Applied and Experimental Mechanics. ICTAEM 2020. Structural Integrity, vol 16. Springer, Cham* (2020): 181–186.

[9] Le K.C. *Vibrations of shells and rods*. Berlin, Springer Verlag, (1999), 425 P.

[10] Mikhailova E.Yu., Tarlakovskii D.V., Fedotenkov G.V. *Obshchaya teoriya uprugikh obolochek* [General theory of anisotropic shells]. Moscow, MAI, (2018) [In Russian].

[11] Igumnov L.À., Tarlakovskii D.V., Zemskov A.V. A two-dimensional nonstationary problem of elastic diffusion for an orthotropic one-component layer. *Lobachevskii Journal of Mathematics*. (2017) **38**(5): 808–817.

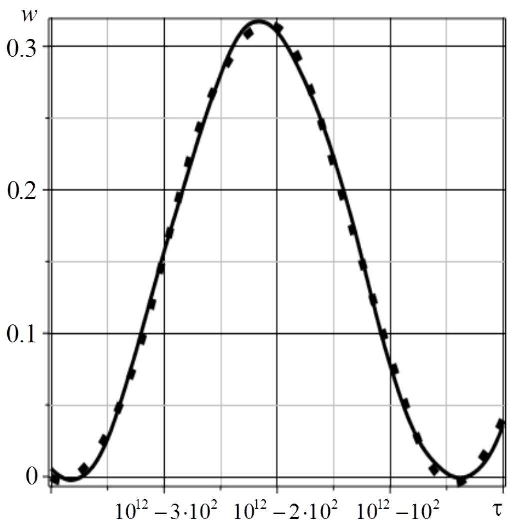

Figure 8: The plate deflections $w\left(\frac{l_1}{2}, \frac{l_2}{2}, \tau\right)$, $c_w^* l = 1.25 \cdot 10^7 N/m^2$

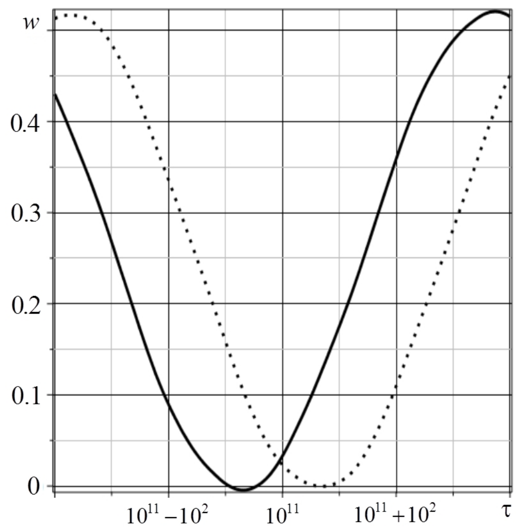

Figure 9: The plate deflections $w\left(\frac{l_1}{2}, \frac{l_2}{2}, \tau\right)$, $c_w^* = 0$

[12] Zemskov A.V., Tarlakovskiy D.V. Two-dimensional nonstationary problem elastic for diffusion an isotropic one-component layer. *Journal of Applied Mechanics and Technical Physics*. (2015) **56**(6): 1023–1030.

[13] Tarlakovskii D.V., Vestyak V.A., Zemskov A.V. Dynamic Processes in Thermoelectromagnetoelastic and Thermoelastodiffusive Media. *Encyclopedia of thermal stress, volume 2, Springer Dordrecht Heidelberg New York London, Springer reference*. (2014): 1064 – 1071.

[14] Shahin, N.A., Mbakisya O. Simply supported beam response on elastic foundation carrying repeated rolling concentrated load. *Journal of Engineering Science and Technology*. (2010) **5**(1): 52–66.

[15] Ditkin V.A., Prudnikov A.P. *Spravochnik po operacionnomu ischisleniyu* [Operational Calculus Handbook]. Moscow, Vy‘sshaya shkola, (1965) [In Russian].

[16] *Fizicheskie velichiny: Spravochnik* [Physical quantities: Handbook] / Babichev A.P., Babushkina N.A., Bratkovskij A.M. ets. Edited by Grigor‘ev I.S., Melixov I.Z. Moscow, Energoatomizdat, (1991) [In Russian].