Volumetric Constraint Models for Anisotropic Elastic Solids

by

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**Summary**

We study three “incompressibility flavors” of linearly-elastic anisotropic solids that exhibit volumetric constraints: isochoric, hydroisochoric and rigidtropic. An isochoric material deforms without volume change under any stress system. An hydroisochoric material does so under hydrostatic stress. A rigidtropic material undergoes zero deformations under a certain stress pattern. Whereas the three models coalesce for isotropic materials, important differences appear for anisotropic behavior. We find that isochoric and hydroisochoric models under certain conditions may be hampered by unstable physical behavior. Rigidtropic models can represent semistable physical materials of arbitrary anisotropy while including isochoric and hydroisochoric behavior as special cases.

**Keywords:** linear elasticity, solid, anisotropy, isotropy, rigidtropy, incompressibility, isochoric, hydroisochoric, volumetric constraints, stability, material, constitutive model, compliance.

1. **Introduction**

An incompressible linearly-elastic isotropic solid does not deform under hydrostatic stress. It does not change volume under pressure. Since deviatoric and volumetric deformations uncouple, no volume change occurs under any stress state. The three volumetric constraints just stated coalesce, and it is sufficient to qualify the material as incompressible.

A more careful study is necessary for anisotropic materials. In the present Note we examine three volumetric constraint models of a linearly elastic anisotropic solid. The following definitions are used for that examination.

A material is called **rigidtropic** if it does not deform (i.e., experiences zero strains) under a specific stress pattern, which is a null eigenvector of the strain-stress (compliance) matrix. The term “rigidtropic” is used in the sense of “rigidity in a certain way” as defined by that eigenvector.

A material is called **isochoric** if it does not change volume under any applied stress system [1, Sec. 77]. Alternatively: the volumetric strain is zero under any stress state.

A material is called **hydroisochoric** if it is isochoric under hydrostatic stress. Isochoric materials are hydroisochoric but the converse is not necessarily true.
entries semistable case it will be assumed that C

Here and strains e

The eigenvalues of C are γ_i for i = 1, 2...6, with v_i being the corresponding eigenvector normalized to length \sqrt{3}. (This nonstandard normalization simplifies linking up to the hydrostatic stress vector in Sections 4ff.) Accordingly the spectral decomposition is

\[ C = \frac{1}{3} \sum_{i=1}^{6} \gamma_i v_i v_i^T, \quad v_i^T v_j = 3\delta_{ij}, \]  

(3)

where δ_{ij} is the Kronecker delta. The eigenvalues will be arranged so that γ_1 = γ_{min} is the algebraically smallest one and γ_6 = γ_{max} the maximum. For stable or semistable models, γ_1 ≥ 0 and γ_j > 0 for j = 2, ... 6.

If γ_1 = 0 the material is rigidtropic according to the definition given in the Introduction, with v_1 defining the corresponding stress pattern. The volumetric strain is e_v = e_{11} + e_{22} + e_{33}. Isochoric behavior is mathematically characterized by e_v = 0 under any σ. Hydroisochoric behavior means that e_v = 0 under \( \sigma_p = p \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}^T \) for any p. These constraints are mathematically expressed in terms of C as follows.

Rigidtropic: \( \gamma_1 = 0, \quad \gamma_i > 0, \quad i = 2, \ldots 6. \)
hydroisochoric: \( C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23} = 0. \)
Isochoric: \( C_{1j} + C_{2j} + C_{3j} = 0, \quad j = 1, 2, 3. \)
Diagonal compliances are often known reliably from extensional and torsion tests. Off diagonal entries are typically less amenable to accurate measurement. Volumetric constraints, for example on volume change, are checked with triaxial tests. In any case, such constraints may be satisfied only approximately. Reference [2] discusses projection and scaling techniques for finding a “reference model” that satisfies constraints accurately while removing spurious instabilities due to experimental noise.

3. Examples

The following examples of compliance matrices pertain to an orthotropic material with the \{x_i\} aligned with the principal material axes. The diagonal entries are kept the same. The three nonzero off-diagonal entries are adjusted to meet the definitions (4).

Rigidtropic:

\[
C_{rig} = \begin{bmatrix}
1 & -3/8 & -3/16 & 0 & 0 & 0 \\
-3/8 & 1/4 & -1/48 & 0 & 0 & 0 \\
-3/16 & -1/48 & 1/9 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix} = \frac{1}{144}
\begin{bmatrix}
144 & -54 & -27 & 0 & 0 & 0 \\
-54 & 36 & -3 & 0 & 0 & 0 \\
-27 & -3 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 288 & 0 & 0 \\
0 & 0 & 0 & 0 & 720 & 0 \\
0 & 0 & 0 & 0 & 0 & 432 \\
\end{bmatrix}
\]

(5)

Eigenvalues: \[5, 3, 2, 1.181238, 0.180074, 0\]. The compliance matrix is semistable. The null eigenvector defining the rigid mode is \(v_1 = \sqrt{\frac{53}{35}}[1/2, 5/6, 1, 0, 0, 0]^T\).

Hydroisochoric:

\[
C_{hyd} = \begin{bmatrix}
1 & -11/27 & -95/432 & 0 & 0 & 0 \\
-11/27 & 1/4 & -23/432 & 0 & 0 & 0 \\
-95/432 & -23/432 & 1/9 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix} = \frac{1}{432}
\begin{bmatrix}
432 & -176 & -95 & 0 & 0 & 0 \\
-176 & 108 & -23 & 0 & 0 & 0 \\
-95 & -23 & 48 & 0 & 0 & 0 \\
0 & 0 & 0 & 576 & 0 & 0 \\
0 & 0 & 0 & 0 & 1440 & 0 \\
0 & 0 & 0 & 0 & 0 & 864 \\
\end{bmatrix}
\]

(6)

Eigenvalues: \[5, 3, 2, 1.208689, 0.211580, -0.059158\]. The compliance matrix is unstable.

Isochoric:

\[
C_{iso} = \begin{bmatrix}
1 & -41/72 & -31/72 & 0 & 0 & 0 \\
-41/72 & 1/4 & 23/72 & 0 & 0 & 0 \\
-31/72 & 23/72 & 1/9 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix} = \frac{1}{144}
\begin{bmatrix}
144 & -82 & -62 & 0 & 0 & 0 \\
-82 & 36 & 46 & 0 & 0 & 0 \\
-62 & 46 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 288 & 0 & 0 \\
0 & 0 & 0 & 0 & 720 & 0 \\
0 & 0 & 0 & 0 & 0 & 432 \\
\end{bmatrix}
\]

(7)

Eigenvalues: \[5, 3, 2, 1.508781, 0, -0.147669\]. The compliance matrix is unstable.
4. Hydroisochoric Model

Assume that the material modeled by (2) is hydroisochoric. Consequently

\[ C \sigma_p = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{23} & C_{24} \\ C_{33} & C_{34} & C_{35} \end{bmatrix} \begin{bmatrix} p \\ p \end{bmatrix} = \begin{bmatrix} p(C_{11} + C_{12} + C_{13}) \\ p(C_{12} + C_{22} + C_{23}) \\ p(C_{13} + C_{23} + C_{33}) \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \end{bmatrix}, \]  

(8)

with \( e_v = e_{11} + e_{22} + e_{33} = p(C_{11} + C_{22} + C_{33} + 2C_{12} + 2C_{13} + 2C_{23}) = 0 \).

(The value of the shear strains is of no interest.) The complementary energy density produced by \( \sigma_p \) is

\[ U^p = \frac{1}{2} \sigma_p^T C \sigma_p = \frac{1}{2} p(e_{11} + e_{22} + e_{33}) = \frac{1}{2} p e_v = 0. \]  

(9)

But \( \gamma_p = U^p / (\sigma_p^T \sigma) = U^p / (3p^2) = 0 \) is the Rayleigh quotient of \( \sigma_p \) with \( C \). According to the Courant-Fisher theorem [2], \( \gamma_p \) must lie in the closed interval \([\gamma_{\min}, \gamma_{\max}]\):

\[ \gamma_1 \leq \gamma_p \leq 0 \leq \gamma_6 \]  

(10)

If \( \sigma_p \) is not an eigenvector of \( C \): \( C \sigma_p \neq 0 \), the leftmost equality in (10) is not possible. Consequently

\[ \gamma_1 < 0, \]  

(11)

and the model is unstable.

If \( C \sigma_p = 0 \) the sum of the first three columns (or rows) of \( C \) must vanish. The hydroisochoric model then coalesces with the isochoric one, which is analyzed next.

5. Isochoric Model

The model is isochoric if the sum of the first three rows (or columns) of \( C \) is the null 6-vector. Equivalently \( \sigma_p \) is a null eigenvector of \( C \). The Rayleigh quotient test (10) does not offer sufficient information on stability and a deeper look at \( C \) is required. Nonetheless a sufficient criterion for instability can be derived by considering the upper 3 \times 3 principal minor \( \tilde{C} \). From the last of (4), \( \tilde{C} \) must have the form:

\[ \tilde{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{23} & C_{24} \\ C_{33} & C_{34} & C_{35} \end{bmatrix} \begin{bmatrix} C_{11} \frac{1}{2}(C_{33} - C_{11} - C_{22}) \frac{1}{2}(C_{22} - C_{11} - C_{33}) \frac{1}{2}(C_{11} - C_{22} - C_{33}) \\ \frac{1}{2}(C_{33} - C_{11} - C_{22}) C_{22} \frac{1}{2}(C_{11} - C_{22} - C_{33}) \frac{1}{2}(C_{22} - C_{11} - C_{33}) \frac{1}{2}(C_{11} - C_{22} - C_{33}) \frac{1}{2}(C_{22} - C_{11} - C_{33}) \frac{1}{2}(C_{11} - C_{22} - C_{33}) \end{bmatrix}. \]  

(12)

This matrix is singular. Taking \( \alpha = C_{11} / C_{22} \) and \( \beta = C_{11} / C_{33} \) for convenience, an eigenvalue analysis shows that \( \tilde{C} \) is indefinite if

\[ 2 \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) < 1 + \left( \frac{1}{\alpha} - \frac{1}{\beta} \right)^2, \]  

(13)

and is positive semidefinite if the inequality is reversed. If \( \tilde{C} \) is indefinite, so is \( C \) and the model is unstable. If \( \tilde{C} \) is semidefinite, an eigenvalue analysis of the complete \( C \) is required to decide on stability. The stability regions of \( \tilde{C} \) are shown in Figure 1, where “potentially semistable” indicates that confirmation by a analysis of the full \( C \) is required. An exception is an orthotropic material referred to principal material axes, in which case no further tests are necessary if \( C_{44}, C_{55} \) and \( C_{66} \) are positive.

Figure 1 illustrates that a wide range of diagonal compliances in \( \tilde{C} \) is detrimental to stability. For example if \( \alpha = \beta \), instability is guaranteed to happen for \( \alpha > 4 \).
6. Rigidtropic Model

If \( C \) is nonnegative with \( \nu_1 = 0 \) and \( w = v_1 \) is the only null eigenvector the material is rigidtropic under that stress mode. For an isotropic material \( w = [1 \ 1 \ 0 \ 0 \ 0 \ 0]^T = \alpha_p \), the hydrostatic stress mode. For an anisotropic material mode \( w \) generally will contain shear stresses. Introducing effective pressure as \( p = \frac{1}{3} w^T \alpha \) and effective volumetric strain as \( e_v = w^T \sigma \), the volumetric and deviatoric energies can be uncoupled [3].

If the rigid stress mode is \( \sigma_p \), rigidtropic reduces to isochoric. This inclusion is pictured in Figure 2.

7. Isotropic Material

If the solid is isotropic with elastic modulus \( E > 0 \) and Poisson’s ratio \( \nu \),

\[
C = \frac{1}{E} \begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
1 & -\nu & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2(1 + \nu) & 0 & 0 & 0 & 0 & 0 \\
2(1 + \nu) & 0 & 0 & 0 & 0 & 0 \\
\text{symm} & 2(1 + \nu) & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Under hydrostatic stress \( \sigma_p \), \( e_v = 3(1 - 2\nu)p/E \), which vanishes for \( \nu = \frac{1}{2} \). It is easy to verify that if \( \nu = \frac{1}{2} \), \( e_v = 0 \) for any \( \sigma \) and the material is isochoric. Furthermore \( \sigma_p \) is the only null eigenvector of \( C \). Consequently \( \gamma_p = \gamma_1 = 0 \) and \( C \) has no negative eigenvalues. The definitions of rigidtropic, incompressible and isochoric behavior coalesce for this model.

8. Conclusion

It remains to pin down the label “incompressible.” In continuum mechanics this term means that the stress is determined by the deformation history only up to a hydrostatic pressure or “extra stress” \( p \) [4, Sec. 30]. This is equivalent to what we call here the hydroisochoric model, which as previously
Figure 2. Schematic of inclusions between rigidtropic, isochoric and hydroisochoric models. The crosshatched area marks a singular $C$ matrix.

shown for semistable materials merges with the isochoric model. Restricting attention to the semistable case, the model nesting is:

$$\text{Isotropic semistable} \equiv \text{Hydroisochoric semistable} \equiv \text{Incompressible} \subset \text{Rigidtropic}. \quad (15)$$

These and related model inclusions are sketched in Figure 2. From a mathematical standpoint, the splitting techniques used for the rigidtropic model by Felippa and Oñate [3] apply equally to isochoric behavior, and no special distinction for the incompressible case needs to be made.

We do not consider here the comparatively rare case of a compliance matrix possessing two or more zero eigenvalues. For those the analysis is complicated by the appearance of a multidimensional null space. Such “multi-rigidtropic” models require separate treatment.

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References


