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GLOBAL POINT-SELECTION STRATEGY BASED ON THE MF-DISCREPANCY FOR STOCHASTIC RESPONSE ANALYSIS OF STRUCTURES WITH INDEPENDENT RANDOM VARIABLES

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Abstract. Randomness widely exists in engineering systems, and it is a great challenge to precisely analyze the stochastic response and dynamic reliability of engineering structures. Some methods based on the high-efficacy global point sets have been proved to be effective. Reasonably measuring and improving the efficacy of the global point sets are of great importance in order to further improve the accuracy of these methods. To measure the efficacy of a point set with an acceptable computational cost even for high-dimensional cases, a new MF-discrepancy is introduced in the present paper. An extended Koksma-Hlawka inequality for error estimation in terms of the MF-discrepancy is established first, providing theoretical basis for the proposal of both the MF-discrepancy and the MF-discrepancy based point-selection strategy. Then the analytic expression of the MF-discrepancy is presented, giving deeper insights into its quantitative properties, and the lowest bound of the MF-discrepancy is rigorously derived and proven as a result. A unified theoretical framework for the MF-discrepancy-based point-selection strategy, which can greatly reflect the physical properties of the system, is established. The widely applied GF-discrepancy-based point-selection method is included in the framework as a method focusing on the local minimization of the MF-discrepancy, and a novel point-selection method focusing on the global minimization of the MF-discrepancy is introduced. In the global point-selection method, the composition and the correlation rank of a point set are separated: the composition of each dimension is first

determined based on the lowest bound of MF-discrepancy, then the correlation rank is determined according to some certain rules and the final representative point set can be obtained. Several numerical examples are studied to verify the high efficacy of the MF-discrepancy-based point-selection strategy.

1 INTRODUCTION

It is necessary to take into account the widespread randomness in the engineering system during the evaluation of the performance for structures under disastrous actions. The stochastic response analysis of dynamic systems is a challenging problem [1]. Great endeavor has been made to cope with this challenge, and many classical methods were developed including the random simulation method [2-3], the random perturbation approach [4] and the orthogonal expansion theory [5-6]. However, it is still a difficult barrier to analyze the stochastic response of high-dimensional nonlinear systems.

To address the great challenge, some methods based on low-discrepancy point sets have been developed, such as some numerical solution methods in the point evolution path for the probability density evolution method (PDEM) [1]. The efficacy of the representative point set is of great significance for this kind of methods. To measure the uniformity and the efficacy of a given point set, the discrepancy [7-8], the F-discrepancy [9] and the extended F-discrepancy (EF-discrepancy) [10] were proposed. To bound the worst error of a representative point set in numerical integral, the Koksma-Hlawka inequality was first established in the discrepancy [7] and extended to the F-discrepancy and the EF-discrepancy with rigorous proof [10]. However, the computational efforts of these three discrepancies grow exponentially as the increase of the dimension, which is a NP-hard problem [8]. To reduce the computational efforts, the generalized F-discrepancy (GF-discrepancy) [10] was further proposed for point sets with assigned probabilities. The Koksma-Hlawka inequality was further extended to the GF-discrepancy according to the relationship between the EF-discrepancy and the GF-discrepancy [11].

A series of strategies for optimal point-selection were proposed based on the GF-discrepancy, e.g., the rearrangement approach aiming at reducing the GF-discrepancy, the two-step point-selection method based on the minimization of the GF-discrepancy [12] and the iterative screening-rearrangement method [13]. However, the application of the GF-discrepancy is only limited to the point sets with assigned probabilities, which is a great barrier to the application of the GF-discrepancy to optimal point-selection.

In the present paper, a new maximal marginal EF-discrepancy (MF-discrepancy for short) is outlined [14]. The quantitative properties of the MF-discrepancy are studied, and thereby a point-selection strategy based on the global minimization of the MF-discrepancy is introduced. The present paper is organized as follows. In Section 2, the definitions and error-estimate properties of the classical discrepancies, including the discrepancy, the F-discrepancy, the EF-discrepancy and the GF-discrepancy are reviewed first, then the MF-discrepancy is defined, and the Koksma-Hlawka inequality is extended to the MF-discrepancy. In Section 3, the analytical expression of the MF-discrepancy is given, and as a result, the lowest bound of the MF-discrepancy is revealed. In Section 4, an enhanced point-selection strategy for globally minimizing the MF-discrepancy with a unified theoretical framework is outlined. The efficiency of the method is demonstrated by several numerical examples in Section 5. Finally, in Section 6, concluding remarks and future work are demonstrated.

2 MAXIMAL MARGINAL EF-DISCREPANCY

The efficacy of a representative point set can be captured by the accuracy of the probability distribution for the mapped stochastic responses. However, the selection of a representative point set is a process operating in the random variable space without the data of the corresponding responses. Trying to catch the efficacy of a representative point set in the random variable space, several discrepancies was proposed successively.

The discrepancy was first proposed to measure the uniformity of a point set corresponding to random variables with uniform distribution [7-8]. Denote the point set with n points as $\mathcal{P}_n$ and the random variables as $\boldsymbol{X} = \{X_1, X_2, \ldots, X_s\}^{\mathrm{T}}$, then the definition of the discrepancy can be written as

$$D(\mathcal{P}_n) = \sup_{\boldsymbol{x}\in \mathrm{C}^s} \left| \frac{N(\mathcal{P}_n, [0,\boldsymbol{x}))}{n} - V([0,\boldsymbol{x})) \right| \tag{1}$$

in which $N(\mathcal{P}_n, [0,\boldsymbol{x}))$ is the number of points scattered in the hyper-rectangle $[0,\boldsymbol{x})$, and $V([0,\boldsymbol{x}))$ is the volume of the hyper-rectangle $[0,\boldsymbol{x})$.

Then, considering the nonuniformly distributed random variables, the F-discrepancy was further proposed [9]. The point sets measured by the discrepancy and the F-discrepancy all hold equal weights. While adopting unequal weights may bring about more information and precision, the extended F-discrepancy (EF-discrepancy) was proposed to measure the efficacy of point sets with unequal weights [10].

$$D_{\mathrm{EF}}(\mathcal{P}_n) = \sup_{\boldsymbol{x}\in \mathbb{R}^s} \left| F_{\mathrm{e}}(\boldsymbol{x}, \mathcal{P}_n) - F(\boldsymbol{x}) \right| \tag{2}$$

of which the empirical CDF $F_{\mathrm{e}}(\boldsymbol{x}, \mathcal{P}_n)$ is extended to

$$F_{\mathrm{e}}(\boldsymbol{x}, \mathcal{P}_n) = \sum_{q=1}^{n} p_q \cdot I(\boldsymbol{x}_q \le \boldsymbol{x}) \tag{3}$$

where p_q's are the weights of the representative points satisfying $0 < p_q < 1$ and $\sum_{q=1}^{n} p_q = 1$, and $I(\cdot)$ is the indicator function. If $p_q = 1/n$ holds for $q = 1, 2, \ldots, n$, the EF-discrepancy is degraded to the F-discrepancy.

If the point set $\mathcal{P}_n$ is adopted to the numerical integral, then the celebrated Koksma-Hlawka inequality gives an error bound for any function with bounded variation. The Koksma-Hlawka inequality was given as [7]

$$\left| \int_{\mathrm{C}^s} f(\boldsymbol{x}) \mathrm{d}x - \frac{1}{n} \sum_{q=1}^{n} f(\boldsymbol{x}_q) \right| \le D(\mathcal{P}_n) \cdot \mathrm{TV}(f) \tag{4}$$

where $\mathrm{TV}(f)$ is the total variation of the function f, proposed to measure the irregularity of the hypersurface $f(\boldsymbol{x})$, written as

$$\mathrm{TV}\left(f\right)=\sum_{\alpha_1+\cdots+\alpha_s=1}^{s}\int_0^1\cdots\int_0^1\left|\frac{\partial^{\alpha_1+\cdots+\alpha_s}f\left(\boldsymbol{x}\right)}{\partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s}}\right|\mathrm{d}x_1\cdots\mathrm{d}x_s \tag{5}$$

The Koksma-Hlawka inequality was further extended to the form of the F-discrepancy [9] and the form of the EF-discrepancy [10] with rigorous proof.

The discrepancy, the F-discrepancy and the EF-discrepancy can measure the efficacy of a representative point set in the random variable space and give the worst error bound for numerical integration. However, the computational effort of these discrepancies grows exponentially with the increase of dimension, meaning that they are NP-hard problems [10]. To avoid the computational complexity, the generalized F-discrepancy (GF-discrepancy), of which the computational effort grows linearly with the increase of dimension, was proposed [10]. The GF-discrepancy is the maximum of marginal EF-discrepancy, but it is only for representative point sets with assigned probabilities [15] which greatly limit the application of the GF-discrepancy in guiding optimal point-selection.

Most recently, a maximal marginal EF-discrepancy (MF-discrepancy) is proposed [14]

$$D_{\mathrm{MF}}\left(\mathcal{P}_n\right)=\max_{1\le i\le s}\left\{\sup\left|F_{\mathrm{e},i}\left(x,\mathcal{P}_n\right)-F_i\left(x\right)\right|\right\} \tag{6}$$

in which the empirical marginal CDF are also expressed as:

$$F_{\mathrm{e},i}\left(x\right)=\sum_{q=1}^{n}p_q\cdot I\left(x_{q,i}\le x\right) \tag{7}$$

It is worth to highlight that the weights of all the points, denoted by $p_q(q=1,2,\cdots,n)$ in Eq. (7), are unrestricted in the definition of the MF-discrepancy.

The intrinsic relevance between the MF-discrepancy and the EF-discrepancy was studied [14]

$$D_{\mathrm{MF}}\left(\mathcal{P}_n\right)\le D_{\mathrm{EF}}\left(\mathcal{P}_n\right)\le(4-\frac{3}{s})D_{\mathrm{MF}}\left(\mathcal{P}_n\right) \tag{8}$$

As a result, the extended Koksma-Hlawka inequality in terms of the MF-discrepancy can be given as [14]

$$\begin{aligned}\left|\int_{\Omega_X}f\left(\boldsymbol{x}\right)\rho\left(\boldsymbol{x}\right)\mathrm{d}x-\sum_{q=1}^{n}p_q f\left(\boldsymbol{x}_q\right)\right|&\le(4-\frac{3}{s})D_{\mathrm{MF}}\left(\mathcal{P}_n\right)\cdot\mathrm{TV}\left(f\right)\\&\le 4D_{\mathrm{MF}}\left(\mathcal{P}_n\right)\cdot\mathrm{TV}\left(f\right)\end{aligned} \tag{9}$$

which shows a similar pattern to the well-known GF-discrepancy [11]. Eq. (9) provides the theoretical basis for the optimal point-selection based on the MF-discrepancy.

It is noted that the MF-discrepancy can reflect the uniformity among different dimensions only when the determination of point weights can capture the interdimensional correlation. The measure of the MF-discrepancy for the efficacy of a point set will not be so efficient if the weights are independent to the interdimensional correlation.

3 LOWEST BOUND OF MF-DISCREPANCY

The expression of the MF-discrepancy is given to further study the lowest bound of the MF-discrepancy for an s-dimensional point set $\mathcal{P}_n = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$. The point-number in the point set is n, and the weight of each point $\boldsymbol{x}_q\ (q = 1, ..., n)$ is p_q. Denote the i-th dimensional marginal CDF of the point x_q as $u_{q,i}$, i.e., $u_{q,i} = F_i(x_{q,i})\,(q = 1, ..., n; i = 1, ..., s)$, and the set of marginal CDF values is denoted as $\mathcal{U}_n = \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\}$. For each dimension, ordering the marginal CDF values $\{u_{1,i}, u_{2,i}, ..., u_{n,i}\}\ (i = 1, 2, ..., s)$ ascendingly yields $\{v_{1,i}, v_{2,i}, ..., v_{n,i}\}\ (i = 1, 2, ..., s)$. Let $v_{0,i} = 0$ and $v_{n+1,i} = 1$, then it holds that $0 = v_{0,i} \le v_{1,i} \le \cdots \le v_{n,i} \le v_{n+1,i} = 1$. The point weight corresponding to $v_{j,i}$ is denoted as $p_{j,i}$. The sum of the first k weights of $v_{\cdot,i}$ from $p_{1,i}$ to $p_{k,i}$, is denoted by $s_{k,j}$, i.e., $s_{k,i} = \sum_{1 \le j \le k} p_{j,i}$. Let $a_{0,i} = 0\ (i = 1, 2, ..., s)$, then the MF-discrepancy of the point set $\mathcal{P}_n$ yields

$$\begin{aligned} D_{\mathrm{MF}}(\mathcal{P}_n) &= \max_{1 \le i \le s}\left[D_{\mathrm{F},i}(\mathcal{P}_n)\right] = \max_{1 \le i \le s}\left[D_{\mathrm{F},i}(\mathcal{U}_n)\right] \\ &= \max_{1 \le i \le s}\left[\max_{0 \le k \le n}\left(\sup_{v_{k,i} < \alpha \le v_{k+1,i}} \left|s_{k,i} - \alpha\right|\right)\right] = \max_{1 \le i \le s}\left[\max_{1 \le k \le n}\left(\left|s_{k,i} - v_{k,i}\right|, \left|s_{k-1,i} - v_{k,i}\right|\right)\right] \\ &= \frac{1}{2}\max_{1 \le q \le n}(p_q) + \max_{1 \le i \le s}\left[\max_{1 \le k \le n}\left(\left|v_{k,i} - \frac{s_{k,i} + s_{k-1,i}}{2}\right|\right)\right] \end{aligned} \tag{10}$$

Then, the following inequality corresponding to the lowest bound of the MF-discrepancy can be obtained [14]

$$D_{\mathrm{MF}}(\mathcal{P}_n) \ge \frac{1}{2}\max_{1 \le q \le n}(p_q) \ge \frac{1}{2n} \tag{11}$$

in which $\max_{1 \le q \le n}(p_q)/2$ is the local minimum of the MF-discrepancy, and $1/(2n)$ is the global minimum of the MF-discrepancy. It can be seen in Eq. (10), when the local minimization of the MF-discrepancy is obtained, $v_{k,i} - (s_{k,i} + s_{k-1,i})/2$ holds for all $i\ (1 \le i \le s)$ and $k\ (1 \le k \le n)$, in other words, the empirical marginal CDF of every dimension intersect the exact marginal CDF at the midpoint of each probability increment. Further, if the probability increments are all equal to $1/n$, the global minimum $1/(2n)$ will be obtained. The local and global minimization of the MF-discrepancy for a point set are illustrated in Figure 1.

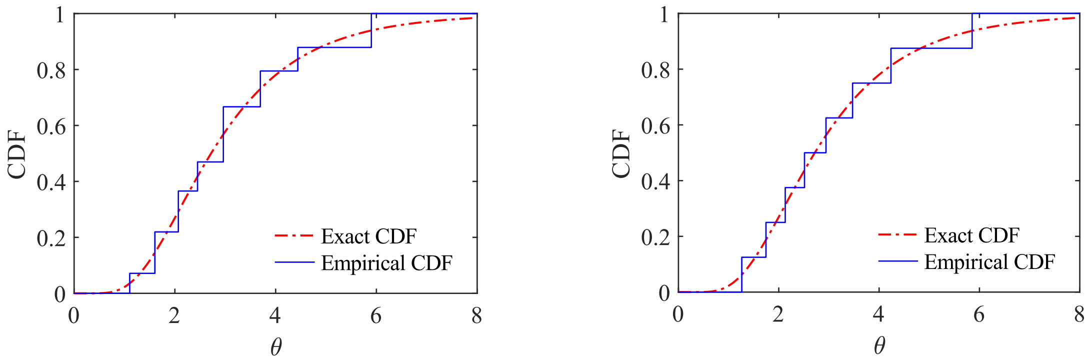
Based on the global minimum, two quantitative indicators are introduced to measure the goodness of the MF-discrepancy [14]. The first quantitative indicator is named as the goodness indicator, which is introduced to reduce, even almost exclude the influence of the point number n on the MF-discrepancy with a logarithmic expression, mapping the MF-discrepancy to the interval $[0,1]$

$$I_{\mathrm{MF}}(\mathcal{P}_n) = \log_{\frac{1}{2n}}\left[D_{\mathrm{MF}}(\mathcal{P}_n)\right] \tag{12}$$

Another indicator is named as the multiply-indicator, which is the quotient of $D_{\mathrm{MF}}(\mathcal{P}_n)$ divided by the global minimum $1/(2n)$

$$I_{\mathrm{M,MF}}(\mathcal{P}_n)=\frac{D_{\mathrm{MF}}(\mathcal{P}_n)}{1/(2n)}=2n\cdot D_{\mathrm{MF}}(\mathcal{P}_n) \tag{13}$$

The multiply-indicator $I_{\mathrm{M,MF}}$ holds a clear meaning and is linearly related to the MF-discrepancy. The above two indicators, i.e., the goodness indicator I_{MF} and the multiply-indicator $I_{\mathrm{M,MF}}$, both based on the global minimization of the MF-discrepancy, provides different perspectives on measuring and observing the MF-discrepancy.



(a) Local Minimization (b) Global Minimization

Figure 1: Local and global minimization of the MF-discrepancy

4 MF-DISCREPANCY-BASED STRATEGY FOR POINT SELECTION

An enhanced point-selection strategy with a unified theoretical framework for minimizing the MF-discrepancy is outlined based on the lowest bound of the MF-discrepancy.

The well-known point-selection method based on the minimization of the GF-discrepancy [12], i.e., two-step point-selection method, is indeed a point-selection strategy based on the local minimization of the MF-discrepancy (L-MMF), since the MF-discrepancy of the obtained point set is exactly $\max_{1\le q\le n}\left(p_q\right)/2$.

Based on the global minimization of the MF-discrepancy, a new point-selection strategy (G-MMF) is introduced [14]. It has been stated in Section 3 that two conditions should be satisfied to obtain the global minimization of the MF-discrepancy for a point set, i.e., the weights of the points are all $1/n$, and the marginal CDF of each dimension intersects with the empirical marginal CDF at the midpoint of each probability increment. These two conditions determine the coordinates of each component of the target point set. Then, to obtain the final point set, combination following some certain rules should be exerted. The point-selection algorithm contains the following steps.

First, according to the above two conditions, the components of each dimension, $\boldsymbol{z}_k=\left\{z_{1,k},z_{2,k},\ldots,z_{n,k}\right\}^{\mathrm{T}}$ $\left(k=1,\ldots,s\right)$, are determined by

$$z_{i,k} = F_k^{-1}\left(\frac{i}{n} - \frac{1}{2n}\right) \quad (i = 1, ..., n; k = 1, ..., s) \tag{14}$$

where $F_k^{-1}(\cdot)$ is the inverse CDF of the k -th dimension.

Then, combine the coordinates of each dimension and determine the final representative point set according to some certain rules. The weights of the points are all set to be $1/n$.

It is noted that the above two MF-discrepancy-based point-selection methods can both greatly reflect the physical properties of the system. In detail, the L-MMF makes the physical and probabilistic spatial partition rationally, and the G-MMF ensures uniform and not repetitive partition of the physical space.

5 NUMERICAL EXAMPLES

A 10-story shear frame is studied to verify the efficiency of the MF-discrepancy-based point-selection strategy, and then the point-selection strategy is adopted in the stochastic analysis of a 10-story reinforced concrete frame structure with random parameters by using the PDEM.

5.1 Example 1: A 10-story nonlinear shear frames with random parameters

To compare the differences in efficacy for different point-selection methods, a 2-norm-based relative error is defined as

$$e_2 = \frac{\|f(t) - f_0(t)\|_2}{\|f_0(t)\|_2} \tag{15}$$

where $f(t)$ is a function or a numerical result to be compared, for example, the mean and the standard deviation of several responses determined by a point set, and $f_0(t)$ is the corresponding baseline.

A 10-story shear frame is studied. From the bottom story to the top story, the mean values of masses, denoted by $m_i\,(1 \le i \le 10)$, are $120\,\mathrm{t}$, $120\,\mathrm{t}$, $120\,\mathrm{t}$, $130\,\mathrm{t}$, $110\,\mathrm{t}$, $110\,\mathrm{t}$, $100\,\mathrm{t}$, $110\,\mathrm{t}$, $110\,\mathrm{t}$, and $80\,\mathrm{t}$, respectively, and the mean values of the initial lateral story stiffness, denoted by $k_i\,(1 \le i \le 10)$, from the bottom to the top, are $1.8\times10^5\ \mathrm{kN/m}$, $1.8\times10^5\ \mathrm{kN/m}$, $1.8\times10^5\ \mathrm{kN/m}$, $1.8\times10^5\ \mathrm{kN/m}$, $1.7\times10^5\ \mathrm{kN/m}$, $1.7\times10^5\ \mathrm{kN/m}$, $1.7\times10^5\ \mathrm{kN/m}$, $1.7\times10^5\ \mathrm{kN/m}$, $1.6\times10^5\ \mathrm{kN/m}$, and $1.6\times10^5\ \mathrm{kN/m}$, respectively. The El Centro ground acceleration is adopted as the external excitation and the amplitude of the acceleration is 0.8 g. The Rayleigh damping is adopted, i.e., the damping matrix is expressed by $\mathbf{C} = a\mathbf{M} + b\mathbf{K}$, where $a = 0.01\,\mathrm{Hz}$ and $b = 0.005\,\mathrm{s}$. The Bouc-Wen model is adopted to characterize the nonlinearity of the restoring force.

There are 3 cases with 10, 20, and 60 random variables, respectively.

Case 1. 10 random variables including the lateral stiffness parameters of all the 10 stories.

Case 2. 20 random variables including the lateral stiffness parameters and the masses of all the 10 stories.

Case 3. 60 random variables including the lateral stiffness parameters, the masses, and the four parameters in the Bouc-Wen model of all 10 stories.

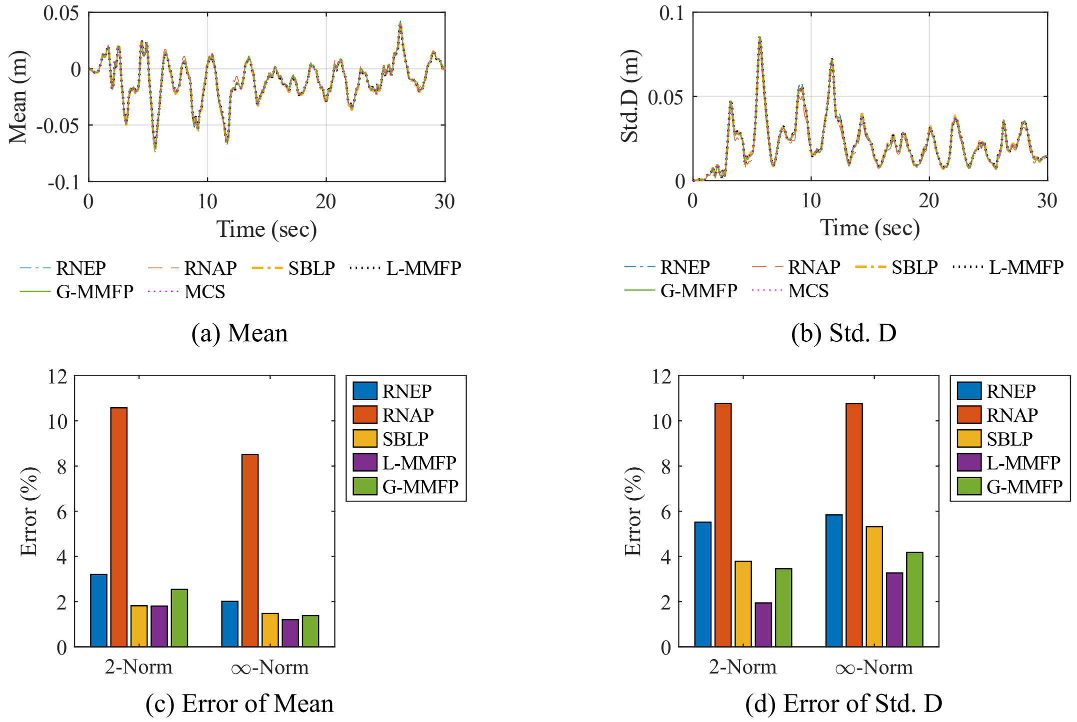


Figure 2: The mean, the standard deviation and the relative errors for the displacements of the top floor （Case 1. 10 random variables）

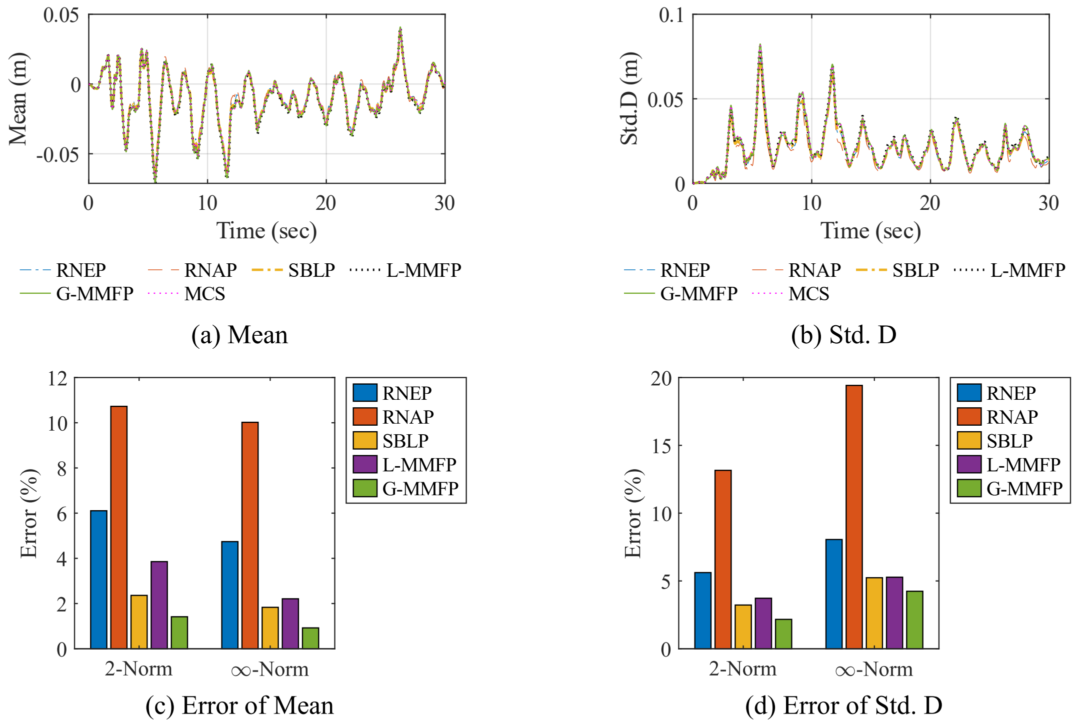
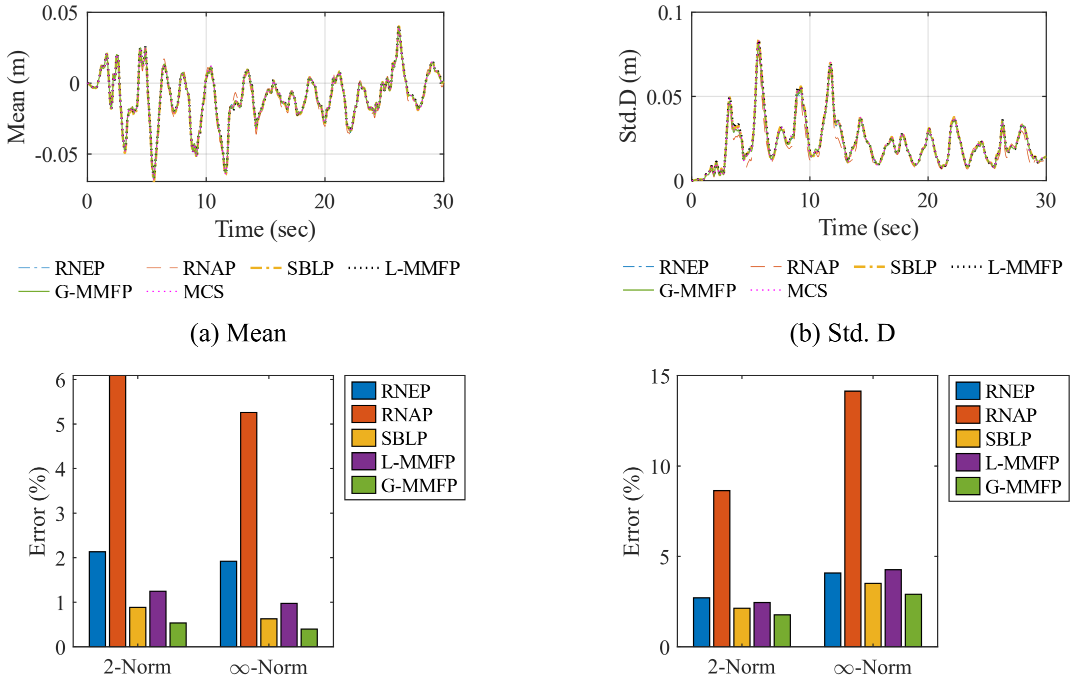


Figure 3: The mean, the standard deviation and the relative errors for the displacements of the top floor （Case 2. 20 random variables）



(a) Mean (b) Std. D

(c) Error of Mean (d) Error of Std. D

Figure 4: The mean, the standard deviation and the relative errors for the displacements of the top floor (Case 3. 60 random variables)

The number of points for the three cases are 200, 300, and 500, respectively. The L-MMF and the G-MMF are studied, and the point sets containing random points with equal weights (denoted as RNEP) and with assigned probabilities (denoted as RNAP), and the point set inversed from the Sobol' sets (denoted as SBLP) are also studied as references. The displacement of the top floor is concerned, and the results from 1×10^5 trials of Monte Carlo simulation are adopted as the baselines. The mean, the standard deviation, and their relative errors for the displacements of the top floor are shown in Figures 2-4.

It is shown that the method based on the local minimization of the MF-discrepancy is more efficient and robust in low-dimensional cases, while the method based on the global minimization of the MF-discrepancy performs best in high-dimensional cases.

5.2 Example 2: A 10-story reinforced concrete frame structure

The point-selection strategy G-MMF is then incorporated into the probability density evolution method (PDEM) for the stochastic response analysis of a 10-story reinforced concrete frame structure (as shown in Figure 5) modeled by the finite element method.

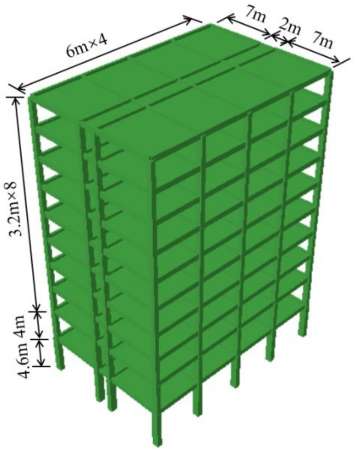
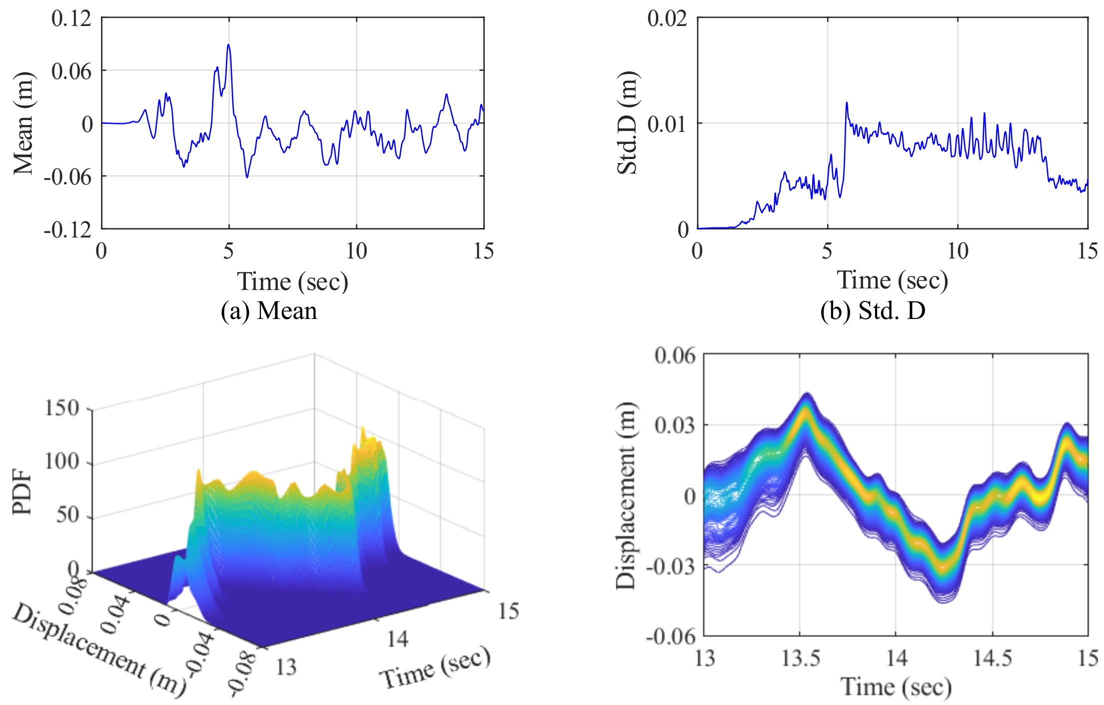


Figure 5: 10-story reinforced concrete frame structure



(a) Mean

(b) Std. D

(c) PDF evolution surface

(d) PDF evolution contour

Figure 6: The probability density evolution analysis for the displacement of the first floor

The Abaqus software is adopted to establish the finite-element model and calculate the results [16]. The beam elements are adopted in the beams and columns, while the multilayer shell elements are used to model the floor plates. There are 8680 elements in total including 4840 beam elements and 3840 shell elements, and 28260 degrees of freedom for the whole structure. The constitutive model in Chinese code for design of concrete [17] is adopted in beams and columns, and the elastic model for concrete is adopted in the floor plates. The simplified bilinear model for steel is adopted in beam, columns, and plates. The El Centro ground acceleration is applied on the base of the structure along the direction of the short side, and the amplitude is set to 1.6g.

The elastic modulus, the compressive strength and the density of the concrete, the elastic modulus of the steel bar, the live load and the wall load of all the ten stories, sixty physical quantities in total, are assumed to be independent random variables.

The PDEM is adopted to analyze the concerned stochastic displacement of the first floor. The analysis employs 500 representative points, with the probability of each point being determined through probability space partitioning in the PDEM framework. The analysis results are shown in Figure 6. It can be seen that adopting the point-selection method based on the global minimization of the MF-discrepancy in high-dimensional case, desirable stochastic response analytical results can be obtained by the PDEM.

6 CONCLUDING REMARKS

The optimal selection of the representative point set is of great importance in the methods of the stochastic response analysis based on the low-discrepancy point sets. In the present paper, a novel MF-discrepancy, i.e., maximal marginal EF-discrepancy, is first outlined, and as a result, an MF-discrepancy-based point-selection method is further introduced and verified in efficiency. The conclusions include:

(1) The MF-discrepancy is introduced, and the Koksma-Hlawka inequality gets an extended form, which can measure the worst error of numerical integral in the MF-discrepancy.

(2) The analytical expression of the MF-discrepancy is given, as a result, the lowest bound of the MF-discrepancy is revealed.

(3) A novel point-selection strategy is introduced based on the global minimization of the MF-discrepancy. The point-selection strategy is verified to be very efficient especially in high-dimensional cases by several numerical examples.

Some meaningful problems about the MF-discrepancy and the corresponding MF-discrepancy-based optimal point selection should be studied further, including, e.g., theoretical proof of the relationship between EF-discrepancy and the MF-discrepancy, an added screening criterion on the G-MMF, the MF-discrepancy-based point-selection for dependent random variables, etc.

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