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Multivalued Fixed Point Methods for Sequential Coupled Caputo-Type Fractional Differential Inclusions with Coupled Boundary Conditions

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ABSTRACT: We study a coupled system of sequential Caputo-type fractional differential inclusions with two-point integral boundary conditions. Using fixed-point techniques for multivalued operators, existence results are established under Carathéodory-type and Lipschitz-type assumptions on the multifunctions. The obtained results extend several known solvability criteria for fractional differential inclusions with nonlocal boundary conditions. An example is included to illustrate the applicability of the theoretical results.

KEYWORDS: Fractional differential inclusions; sequential caputo derivative; coupled systems; carathéodory multifunctions; lipschitz multifunctions; fixed point theorems

MSC: 34A60; 34A08; 47H04; 47H10; 26A33

1 Introduction

Fractional calculus has attracted considerable attention due to its ability to describe memory and hereditary effects arising in physical, biological, and engineering systems. Fractional differential equations provide an effective framework for modeling complex phenomena that cannot be adequately represented by classical integer-order models. Owing to their applications in viscoelasticity, control theory, signal processing, fluid dynamics, and biological processes, the qualitative analysis of fractional differential equations has become an active research area. Fundamental contributions to this field can be found in [1–3].

In many practical models, the nonlinear term may be uncertain or nonunique. To address such situations, fractional differential inclusions extend fractional differential equations by replacing the nonlinear term with a set-valued map. This approach enables the incorporation of uncertainties, discontinuities, and multiple responses. Classical references on multivalued analysis include [4–8].

Recently, considerable attention has been devoted to fractional differential inclusions with integral, nonlocal, and multipoint boundary conditions [9–12]. Such boundary conditions arise naturally in many applications and often provide more realistic descriptions of underlying phenomena. Moreover, coupled systems of fractional differential equations and inclusions have received increasing interest due to their capability to model interacting dynamical processes [13–16].

Several existence results for fractional differential inclusions have been established using multivalued fixed-point methods. Ahmad et al. [17] investigated sequential fractional differential inclusions with multipoint boundary conditions, while Obukhovskii et al. [14] studied fractional inclusions in Banach spaces. Wang et al. [13] considered affine periodic boundary conditions, and Haddouchi and Samei [18] analyzed sequential fractional problems with nonlocal boundary conditions. More recently, Abbas et al. [19] examined fractional differential inclusions subject to multipoint boundary constraints.

Motivated by recent developments in fractional calculus and differential inclusions, we consider the coupled system of sequential fractional differential inclusions

$$\begin{cases} ({}^{C}D^{\alpha} + \lambda_1 {}^{C}D^{\alpha-1})\chi(\xi) \in \mathcal{F}(\xi, \chi(\xi), \Omega(\xi)), \quad \xi \in [0, \mathcal{T}], \quad 1 < \alpha \le 2, \\ ({}^{C}D^{\beta} + \lambda_2 {}^{C}D^{\beta-1})\Omega(\xi) \in \mathcal{G}(\xi, \chi(\xi), \Omega(\xi)), \quad \xi \in [0, \mathcal{T}], \quad 1 < \beta \le 2, \end{cases} \tag{1}$$

where $\mathcal{T} > 0$ is fixed. The corresponding boundary specifications are imposed through the following coupled integral conditions:

$$\begin{cases} \int_0^{\mathcal{T}} \chi'(\xi)\, d\xi = \zeta\big(\Omega'(\mu)\big), \quad \mu \in [0, \mathcal{T}], \\ \int_0^{\mathcal{T}} \Omega'(\xi)\, d\xi = \eta\big(\chi'(\rho)\big), \quad \rho \in [0, \mathcal{T}], \qquad \chi(0) = 0, \qquad \Omega(0) = 0. \end{cases} \tag{2}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denote the Caputo fractional derivatives of orders α and β, respectively, while ${}^{C}D^{\alpha-1}$ and ${}^{C}D^{\beta-1}$ denote the corresponding lower-order Caputo derivatives. The parameters $\lambda_1, \lambda_2 \in \mathbb{R}$ are prescribed constants and $\mathcal{F}, \mathcal{G}$ are set-valued maps from $[0, \mathcal{T}] \times \mathbb{R}^2$ into $\mathcal{P}(\mathbb{R})$. while $\zeta, \eta \in \mathbb{R}$ are given constants representing the coupling parameters in the boundary conditions.

The main objective of this paper is to establish existence results for system (1) and (2) under Carathéodory-type and Lipschitz-type assumptions on the multifunctions $\mathcal{F}$ and $\mathcal{G}$. By applying fixed-point techniques for multivalued operators, solvability criteria are obtained in both convex-valued and nonconvex-valued settings. An example is presented to illustrate the applicability of the theoretical results.

The paper is organized as follows. Section 2 contains the necessary preliminaries. Section 3 presents the main existence results. Section 4 provides an illustrative example, and the final section concludes the paper.

2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. We denote by $\mathfrak{C}(\mathbb{X})$ and $\mathfrak{K}_c(\mathbb{X})$ the families of all nonempty closed subsets and all nonempty compact convex subsets of $\mathbb{X}$, respectively. Standard notions of upper semicontinuity and complete continuity for multifunctions are used throughout the paper (see [5,20]).

Let $\mathfrak{F} : [0, \tau] \times \mathbb{R}^2 \to \mathfrak{K}_c(\mathbb{R})$.

Then $\mathfrak{F}$ is called an L^1–Carathéodory multifunction if

(i) for every $(u, v) \in \mathbb{R}^2$, the map $t \mapsto \mathfrak{F}(t, u, v)$ is measurable;

(ii) for a.e. $t \in [0, \tau]$, the map $(u, v) \mapsto \mathfrak{F}(t, u, v)$ is u.s.c.;

(iii) for each $R > 0$, there exists $\zeta_R \in L^1([0, \tau], \mathbb{R}^+)$ such that

$$\sup\{|w| : w \in \mathfrak{F}(t, u, v)\} \le \zeta_R(t)$$

for all $|u|, |v| \le R$ and a.e. $t \in [0, \tau]$.

Lemma 1 (Kakutani Nonlinear Alternative [5])**:** *Let $\mathbb{D}$ be a closed convex subset of a Banach space $\mathbb{M}$ and let $\mathbb{U}$ be an open subset of $\mathbb{D}$ with $0 \in \mathbb{U}$. If $\mathcal{T} : \overline{\mathbb{U}} \to \mathfrak{K}_c(\mathbb{D})$ is compact and u.s.c., then either*

(a) $\mathcal{T}$ has a fixed point in $\overline{\mathbb{U}}$, or
(b) there exist $u \in \partial\mathbb{U}$ and $\mu \in (0,1)$ such that $u \in \mu\mathcal{T}(u)$.

Lemma 2 (Covitz–Nadler Fixed Point Theorem [21])**:** *Let $(\mathbb{X}, d)$ be a complete metric space and let $\Psi : \mathbb{X} \to \mathfrak{C}(\mathbb{X})$ be a multivalued contraction. Then Ψ possesses a fixed point.*

Remark 1: *The assumptions imposed on the multifunctions $\mathcal{F}$ and $\mathcal{G}$ play a crucial role in the existence analysis. The Carathéodory-type conditions guarantee the measurability of the multifunctions with respect to the independent variable and their upper semicontinuity with respect to the state variables. These properties are essential for the existence of measurable selections and for establishing the closed-graph property of the associated multivalued operator. Furthermore, the growth conditions ensure boundedness and relative compactness of the operator, which are required in the application of the Kakutani nonlinear alternative.*

In the Lipschitz framework, the imposed Lipschitz bounds are used to derive a contraction estimate with respect to the Hausdorff metric, thereby allowing the application of the Covitz–Nadler fixed-point theorem. Although these assumptions are sufficient for the techniques employed in this paper, weaker conditions may also yield solvability results through alternative approaches, such as measures of noncompactness, condensing operators, or generalized multivalued fixed-point theorems. The investigation of such weaker assumptions is left for future research.

Lemma 3: *The boundary value problem admits a unique solution for every pair $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}([0, \mathcal{T}], \mathcal{R})$.*

$$\begin{cases} ({}^c\mathcal{D}^{\alpha} + \lambda_1 {}^c\mathcal{D}^{\alpha-1})\chi(\xi) = \mathcal{H}_1(\xi), \\ ({}^c\mathcal{D}^{\beta} + \lambda_2 {}^c\mathcal{D}^{\beta-1})\Omega(\xi) = \mathcal{H}_2(\xi), \\ \int_0^{\mathcal{T}} \chi'(\xi)\, d\xi = \zeta\big(\Omega'(\mu)\big), \mu \in [0, \mathcal{T}], \int_0^{\mathcal{T}} \Omega'(\xi)\, d\xi = \eta\big(\chi'(\rho)\big), \rho \in [0, \mathcal{T}], \qquad \chi(0) = 0, \qquad \Omega(0) = 0. \end{cases} \tag{3}$$

is equivalent to the system of integral equations

$$\begin{aligned} &\chi(\xi) \\ =&\frac{(1 - e^{-\lambda_1 \xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(- \eta\lambda_1 \int_0^{\rho} e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho) + \int_0^{\mathcal{T}} \Big[\lambda_2 \int_0^{\xi} e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]\, d\xi \\ &- \int_0^{\mathcal{T}} \Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big] d\xi\Big) + \mathcal{B}_2\Big(- \zeta\lambda_2 \int_0^{\mu} e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu) \\ &+ \int_0^{\mathcal{T}} \Big[\lambda_1 \int_0^{\xi} e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]\, d\xi - \int_0^{\mathcal{T}} \Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big] d\xi\Big)\Bigg] + \int_0^{\xi} e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta, \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
&\Omega(\xi) \qquad (5)\\
&=\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi\\
&\quad-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\
&\quad+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta,
\end{aligned}
$$

where

$$
\begin{cases}
\mathcal{I}_1=-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi,\\
\mathcal{I}_2=-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi,\\
\mathcal{A}_1=\int_0^{\mathcal{T}}e^{-\lambda_1\xi}d\xi,\ \mathcal{A}_2=\zeta e^{-\lambda_1\mu},\mathcal{B}_1=\eta e^{-\lambda\rho},\ \mathcal{B}_2=\int_0^{\mathcal{T}}e^{-\lambda_2\xi}d\xi,\Delta=\Big(-\mathcal{B}_1\mathcal{A}_2+\mathcal{A}_1\mathcal{B}_2\Big).
\end{cases} \tag{6}
$$

Proof: Consider the linear fractional system

$$({}^{C}\mathcal{D}^{\alpha}+\lambda_1{}^{C}\mathcal{D}^{\alpha-1})\chi(\xi)=\mathcal{H}_1(\xi),\xi\in(0,\mathcal{T}),({}^{C}\mathcal{D}^{\beta}+\lambda_2{}^{C}\mathcal{D}^{\beta-1})\Omega(\xi)=\mathcal{H}_2(\xi),\xi\in(0,\mathcal{T}), \tag{7}$$

where $\mathcal{H}_1,\mathcal{H}_2\in C([0,\mathcal{T}],\mathbb{R})$.

We first derive an equivalent integral representation of the above system. Applying the fractional integral operators $\mathcal{I}^{\alpha}$ and $\mathcal{I}^{\beta}$ to both equations of (7) and using the standard properties of the Caputo derivative, we obtain

$$\mathcal{I}^{\alpha}\big({}^{C}\mathcal{D}^{\alpha}\chi\big)+\lambda_1\mathcal{I}^{\alpha}\big({}^{C}\mathcal{D}^{\alpha-1}\chi\big)=\mathcal{I}^{\alpha}\mathcal{H}_1,\qquad \mathcal{I}^{\beta}\big({}^{C}\mathcal{D}^{\beta}\Omega\big)+\lambda_2\mathcal{I}^{\beta}\big({}^{C}\mathcal{D}^{\beta-1}\Omega\big)=\mathcal{I}^{\beta}\mathcal{H}_2.$$

Using the identities for Caputo derivatives of orders $1<\alpha\le 2, 1<\beta\le 2$, the above equations reduce to first-order linear Volterra-type equations. Solving them by the method of variation of constants yields

$$\chi(\xi)=\mathfrak{c}_0e^{-\lambda_1\xi}+\frac{\mathfrak{c}_1}{\lambda_1}\big(1-e^{-\lambda_1\xi}\big)+\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)\,d\vartheta, \tag{8}$$

$$\Omega(\xi)=\mathfrak{d}_0e^{-\lambda_2\xi}+\frac{\mathfrak{d}_1}{\lambda_2}\big(1-e^{-\lambda_2\xi}\big)+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)\,d\vartheta, \tag{9}$$

where $\mathfrak{c}_0,\mathfrak{c}_1,\mathfrak{d}_0,\mathfrak{d}_1\in\mathbb{R}$ are arbitrary constants.

Next, differentiating (8) with respect to ξ and applying Leibniz's rule for differentiation under the integral sign, we obtain

$$\chi'(\xi)=\mathfrak{c}_1e^{-\lambda_1\xi}-\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)\,d\vartheta+\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi), \tag{10}$$

$$\Omega'(\xi)=\mathfrak{d}_1e^{-\lambda_2\xi}-\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)\,d\vartheta+\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi). \tag{11}$$

The conditions $\chi(0) = 0, \Omega(0) = 0$, immediately imply $\mathfrak{c}_0 = 0, \mathfrak{d}_0 = 0$. Substituting (10) into the nonlocal coupled boundary conditions

$$\int_0^{\mathcal{T}} \chi'(\xi)\, d\xi = \zeta\big(\Omega'(\mu)\big), \int_0^{\mathcal{T}} \Omega'(\xi)\, d\xi = \eta\big(\chi'(\rho)\big),$$

we obtain a linear algebraic system involving the unknown constants $\mathfrak{c}_1$ and $\mathfrak{d}_1$:

$$\mathcal{A}_1\mathfrak{c}_1 - \mathcal{A}_2\mathfrak{d}_1 = \mathcal{I}_1, -\mathcal{B}_1\mathfrak{c}_1 + \mathcal{B}_2\mathfrak{d}_1 = \mathcal{I}_2,$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{I}_i$ $(i = 1, 2)$ are given by the expressions introduced in the statement of the lemma.

The determinant of the coefficient matrix is

$$\Delta = \mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1.$$

Since $\Delta \neq 0$, the above system admits a unique solution. Applying Cramer's rule yields

$$\mathfrak{c}_1 = \frac{\mathcal{A}_2\mathcal{I}_2 + \mathcal{B}_2\mathcal{I}_1}{\Delta}, \qquad \mathfrak{d}_1 = \frac{\mathcal{B}_1\mathcal{I}_1 + \mathcal{A}_1\mathcal{I}_2}{\Delta}.$$

Finally, substituting these values into (8) produces the required integral representation of the solution pair (χ, Ω). Hence, the proof is complete. □

3 Main Results

Let us define the space $\Theta_1 = \big\{\chi(\xi) \mid \chi(\xi) \in C\big([0, \mathcal{T}], \mathbb{R}\big)\big\}$, endowed with the norm $\|\chi\| = \max_{\xi\in[0,\mathcal{T}]} |\chi(\xi)|$. Clearly, $(\Theta_1, \|\cdot\|)$ is a Banach space. Similarly, define the space $\Omega = \big\{\Omega(\xi) \;:\; \Omega \in C\big([0, \mathcal{T}]\big)\big\}$, equipped with the norm $\|\Omega\| = \max_{\xi\in[0,\mathcal{T}]} |\Omega(\xi)|$.

The product space $\Theta_1 \times \Theta_1$, endowed with the norm $\|(\chi, \Omega)\| = \|\chi\| + \|\Omega\|$, is also a Banach space. For each pair $(\chi, \Omega) \in \Theta_1 \times \Theta_1$, we define the sets of selections associated with the multivalued maps $\mathcal{F}$ and $\mathcal{G}$ as

$$\mathcal{V}_{\mathcal{F}(\chi,\Omega)} = \Big\{\mathcal{H}_1 \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}) \;:\; \mathcal{H}_1(\xi) \in \mathcal{F}\big(\xi, \chi(\xi), \Omega(\xi)\big),\ \text{a.e. } \xi \in \mathcal{J}\Big\}, \tag{12}$$

$$\mathcal{V}_{\mathcal{G}(\chi,\Omega)} = \Big\{\mathcal{H}_2 \in \mathcal{L}^1(\mathcal{J}, \mathbb{R}) \;:\; \mathcal{H}_2(\xi) \in \mathcal{G}\big(\xi, \chi(\xi), \Omega(\xi)\big),\ \text{a.e. } \xi \in \mathcal{J}\Big\}. \tag{13}$$

Let $(\chi, \Omega) \in \Theta_1 \times \Theta_1$. For each pair of measurable selections $\mathcal{H}_1(\xi) \in \mathcal{F}(\xi, \chi(\xi), \Omega(\xi)), \mathcal{H}_2(\xi) \in \mathcal{G}(\xi, \chi(\xi), \Omega(\xi))$, the corresponding integral representation determines a pair of functions. This observation motivates the definition of the multivalued operator Λ, whose fixed points coincide with solutions of problem (1) and (2).

By applying Lemma 3, we define the multi-valued operators $\Lambda_1, \Lambda_2 : \Theta_1 \times \Theta_1 \longrightarrow \mathscr{W}(\Theta_1 \times \Theta_1)$ by

$$\Lambda_1(\chi, \Omega)(\xi) = \{\mathfrak{g}_1 \in \Theta_1 \times \Theta_1 \;:\; \exists\, \mathcal{H}_1 \in \mathcal{V}_{\mathcal{F}(\chi,\Omega)}, \mathcal{H}_2 \in \mathcal{V}_{\mathcal{G}(\chi,\Omega)}\ \ \forall\ \mathfrak{g}_1(\chi, \Omega)(\xi) = \mathcal{P}_1(\xi, \chi, \Omega),\ \forall\ \xi \in \mathcal{J}\}, \tag{14}$$

$$\Lambda_2(\chi, \Omega)(\xi) = \{\mathfrak{g}_2 \in \Theta_1 \times \Theta_1 \;:\; \exists\, \mathcal{H}_1 \in \mathcal{V}_{\mathcal{F}(\chi,\Omega)}, \mathcal{H}_2 \in \mathcal{V}_{\mathcal{G}(\chi,\Omega)}\ \ \forall\ \mathfrak{g}_2(\chi, \Omega)(\xi) = \mathcal{P}_2(\xi, \chi, \Omega),\ \forall\ \xi \in \mathcal{J}\}, \tag{15}$$

where

$$
\begin{aligned}
&\mathcal{P}_1(\chi,\Omega)(\xi)\\
=&\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\xi)\Big]d\xi\Big)\\
&+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\xi)\Big]d\xi\Big)\Bigg]\\
&+\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{P}_2(\chi,\Omega)(\xi)\\
=&\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\xi)\Big]d\xi\Big)\\
&+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}(\mathcal{F}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta\Big]d\xi-\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\xi)\Big]d\xi\Big)\Bigg]\\
&+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}(\mathcal{G}(\vartheta,\chi(\vartheta),\Omega(\vartheta)))(\vartheta)d\vartheta, \qquad (17)
\end{aligned}
$$

We now define a multivalued operator $\Lambda:\Theta_1\times\Theta_1\to\mathscr{W}(\Theta_1\times\Theta_1)$ by $\Lambda(\chi,\Omega)(\xi)=(\Lambda_1(\chi,\Omega)(\xi),\Lambda_2(\chi,\Omega)(\xi))$ where Λ_1 and Λ_2 are given by (14) and (15), respectively. For notational simplicity in the forthcoming arguments, we employ the abbreviations

$$
\begin{aligned}
\Psi_1=&\frac{1-e^{-\lambda_1\xi}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg)+\eta\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\Bigg]+\mathcal{B}_2\Bigg[\lambda_1\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\frac{(1-e^{-\lambda_1\mathcal{T}})}{\lambda_1}\Bigg)\\
&+\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\Bigg]\Bigg\}+\Bigg(\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\frac{(1-e^{-\lambda_1\xi})}{\lambda_1}\Bigg). \qquad (18)\\
\Psi_2=&\frac{1-e^{-\lambda_1\xi}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[\lambda_2\Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\frac{(1-e^{-\lambda_2\mathcal{T}})}{\lambda_2}\Bigg)+\Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\Bigg)\Bigg]-\mathcal{B}_2\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg)+\zeta\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\Bigg)\Bigg]\Bigg\} \qquad (19)\\
\Psi_3=&\frac{1-e^{-\lambda_2\xi}}{\lambda_2\Delta}\Bigg\{\mathcal{B}_1\Bigg[\lambda_1\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\frac{(1-e^{-\lambda_1\mathcal{T}})}{\lambda_1}\Bigg)+\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\Bigg]+\mathcal{A}_1\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg)+\eta\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\Bigg] \qquad (20)\\
\Psi_4=&\frac{1-e^{-\lambda_2\xi}}{\lambda_2\Delta}\Bigg\{\mathcal{B}_1\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg)+\zeta\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\Bigg)\Bigg]\\
&+\mathcal{A}_1\Bigg[\lambda_2\Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\frac{(1-e^{-\lambda_2\mathcal{T}})}{\lambda_2}\Bigg)+\Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\Bigg)\Bigg]+\Bigg(\frac{\xi^{\beta-1}}{\Gamma(\beta)}\frac{(1-e^{-\lambda_2\xi})}{\lambda_2}\Bigg). \qquad (21)
\end{aligned}
$$

3.1 The Carathéodory Case

In this subsection, we establish an existence result for the coupled fractional inclusion system when the multifunctions $\mathcal{F}$ and $\mathcal{G}$ are convex-valued. To proceed with the analysis, we impose the following hypotheses.

Throughout the sequel, the following conditions will be assumed.

($\mathcal{W}_1$)The multifunctions $\mathcal{F},\mathcal{G} : \mathcal{J}\times\mathbb{R}^2 \to \mathscr{W}(\mathbb{R})$ are L^1-Carathéodory and possess convex values.

($\mathcal{W}_2$)There exist continuous and nondecreasing functions $\varphi_1,\varphi_2,\kappa_1,\kappa_2 : [0,\infty)\to[0,\infty)$, together with functions $l_1, l_2 \in C(\mathcal{J},\mathbb{R}^+)$, such that
$\|\mathcal{F}(\xi,\chi,\Omega)\|_{\mathscr{W}} = \sup\{|\mathcal{H}_1| : \mathcal{H}_1\in\mathcal{F}(\xi,\chi,\Omega)\} \le l_1(\xi)\big[\varphi_1(\|\chi\|)+\kappa_1(\|\Omega\|)\big]$, for every $(\xi,\chi,\Omega)\in\mathcal{J}\times\mathbb{R}^2$, and
$\|\mathcal{G}(\xi,\chi,\Omega)\|_{\mathscr{W}} = \sup\{|\mathcal{H}_2| : \mathcal{H}_2\in\mathcal{G}(\xi,\chi,\Omega)\} \le l_2(\xi)\big[\varphi_2(\|\chi\|)+\kappa_2(\|\Omega\|)\big]$, for every $(\xi,\chi,\Omega)\in\mathcal{J}\times\mathbb{R}^2$.

($\mathcal{W}_3$)There exists a constant $\mathcal{Z}>0$ satisfying $\frac{\mathcal{Z}}{(\Psi_1+\Psi_3)\|l_1\|(\varphi_1(\mathcal{Z})+\kappa_1(\mathcal{Z}))+(\Psi_2+\Psi_4)\|l_2\|(\varphi_2(\mathcal{Z})+\kappa_2(\mathcal{Z}))} > 1$, where the quantities Ψ_1,Ψ_2,Ψ_3, and Ψ_4 are given by (18)–(21).

Theorem 1: *Assume that conditions* ($\mathcal{W}_1$)–($\mathcal{W}_3$) *are fulfilled. Then the coupled system* (1) *and* (2) *possesses at least one solution on the interval* $\mathcal{J}$.

Proof: Let the multivalued operators $\Lambda_1,\Lambda_2 : \Theta_1\times\Theta_1\to\mathscr{W}(\Theta_1\times\Theta_1)$ be defined by (14) and (15), respectively. By virtue of assumption ($\mathcal{W}_1$), the multifunctions $\mathcal{F}$ and $\mathcal{G}$ are L^1-Carathéodory with convex values. Consequently, for every pair $(\chi,\Omega)\in\Theta_1\times\Theta_1$, the corresponding selection sets $\mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$ are nonempty. Therefore, for any fixed $(\chi,\Omega)\in\Theta_1\times\Theta_1$, one can choose $\mathcal{H}_1\in\mathcal{V}_{\mathcal{F}(\chi,\Omega)}and\mathcal{H}_2\in\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$. Using these selections in the definitions of Λ_1 and Λ_2, we obtain

$$\begin{aligned}\mathfrak{g}_1(\chi,\Omega)(\xi) &= \frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi\\&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)\\&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)\Bigg]+\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta,\end{aligned}\tag{22}$$

and

$$\begin{aligned}\mathfrak{g}_2(\chi,\Omega)(\xi) &= \frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi\\&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)\Bigg]+\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta,\end{aligned}\tag{23}$$

where $\mathfrak{g}_1\in\Lambda_1(\chi,\Omega)$ and $\mathfrak{g}_2\in\Lambda_2(\chi,\Omega)$. Consequently, $(\mathfrak{g}_1,\mathfrak{g}_2)\in\Lambda(\chi,\Omega)$.

Our next objective is to verify that the multivalued operator Λ satisfies the assumptions of the Leray–Schauder nonlinear alternative. The proof is divided into several steps. As a first step, we show that $\Lambda(\chi,\Omega)$ is convex for every $(\chi,\Omega)\in\Theta_1\times\Theta_1$. To this end, let $(\mathfrak{g}_i,\hat{\mathfrak{g}}_i)\in(\Lambda_1,\Lambda_2), i=1,2$. Then there exist selections $\mathcal{H}_{1i}\in\mathcal{V}_{\mathcal{F}(\chi,\Omega)},\mathcal{H}_{2i}\in\mathcal{V}_{\mathcal{G}(\chi,\Omega)}, i=1,2$, such that, for every $\xi\in\mathcal{J}$, the following relations hold:

$$\begin{aligned}\mathfrak{g}_i(\xi) &= \frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\vartheta)d\vartheta\Big]d\xi\\&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\mu)\\&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\vartheta)d\vartheta,\end{aligned}\tag{24}$$

and

$$\hat{\mathfrak{g}}_i(\xi) = \frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu} e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\mu) + \int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi} e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\vartheta)d\vartheta\Big]d\xi$$
$$+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\xi)\Big]d\xi\Big) + \mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho} e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1i}(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)$$
$$+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi} e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\xi)\Big]d\xi\Big)\Bigg] + \int_0^{\xi} e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2i}(\vartheta)d\vartheta. \tag{25}$$

For an arbitrary $\nu \in [0,\mathcal{T}]$ and each $\xi \in \mathcal{J}$, we obtain

$$\Big[\nu\mathfrak{g}_1 + (1-\nu)\mathfrak{g}_2\Big](\xi)$$
$$=\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho} e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(\vartheta)+(1-\nu)\mathcal{H}_{12}(\vartheta)](\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(\vartheta)+(1-\nu)\mathcal{H}_{12}(\vartheta)](\rho)$$
$$+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi} e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(\vartheta)+(1-\nu)\mathcal{H}_{22}(\vartheta)](\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(\vartheta)+(1-\nu)\mathcal{H}_{22}(\vartheta)](\xi)\Big]d\xi\Big)$$
$$+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu} e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(\vartheta)+(1-\nu)\mathcal{H}_{22}(\vartheta)](\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}[\nu\mathcal{H}_{21}(\vartheta)+(1-\nu)\mathcal{H}_{22}(\vartheta)](\mu)$$
$$+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi} e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(\vartheta)+(1-\nu)\mathcal{H}_{12}(\vartheta)](\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(\vartheta)+(1-\nu)\mathcal{H}_{12}(\vartheta)](\xi)\Big]d\xi\Big)\Bigg]$$
$$+\int_0^{\xi} e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}[\nu\mathcal{H}_{11}(\vartheta)+(1-\nu)\mathcal{H}_{12}(\vartheta)](\vartheta)d\vartheta. \tag{26}$$

Since $\mathcal{F}$ and $\mathcal{G}$ have convex values, it follows that the corresponding selection sets $\mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$ are also convex. Therefore, $\nu\mathfrak{g}_1 + (1-\nu)\mathfrak{g}_2 \in \Lambda_1$, $\nu\hat{\mathfrak{g}_1} + (1-\nu)\hat{\mathfrak{g}_2} \in \Lambda_2$. Consequently, $\nu(\mathfrak{g}_1, \hat{\mathfrak{g}_1}) + (1-\nu)(\mathfrak{g}_2, \hat{\mathfrak{g}_2}) \in \Lambda$, which establishes the convexity of $\Lambda(\chi,\Omega)$.

Next, we verify that Λ maps bounded subsets of $\Theta_1\times\Theta_1$ into bounded subsets of the same space. Let $\mathcal{B}_{\mathfrak{r}} = \big\{(\chi,\Omega)\in\Theta_1\times\Theta_1 : \|\chi,\Omega\| \le \mathfrak{r}\big\}$, where $\mathfrak{r} > 0$ is fixed. Clearly, $\mathcal{B}_{\mathfrak{r}}$ is a bounded subset of $\Theta_1\times\Theta_1$. For any $(\chi,\Omega)\in\mathcal{B}_{\mathfrak{r}}$, choose $\mathcal{H}_1 \in \mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{H}_2 \in \mathcal{V}_{\mathcal{G}(\chi,\Omega)}$. Then, from the definitions of the operators under consideration, we obtain

$$\begin{aligned}
|\mathfrak{g}_1(\chi,\Omega)(\xi)| \le & \frac{1-e^{-\lambda_1\xi}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[-\eta\lambda_1\Bigg(\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\frac{(1-e^{-\lambda_1\rho})}{\lambda_1}\Bigg) + \eta\frac{\rho^{\alpha-1}}{\Gamma(\alpha)}\Bigg] + \mathcal{B}_2\Bigg[\lambda_1\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\frac{(1-e^{-\lambda_1\mathcal{T}})}{\lambda_1}\Bigg) \\
&+\Bigg(\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}\Bigg)\Bigg]\Bigg\}\|l_1\|(\gamma_1(\mathfrak{r})+\kappa_1(\mathfrak{r})) + \Bigg(\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\frac{(1-e^{-\lambda_1\xi})}{\lambda_1}\Bigg)\|l_1\|(\gamma_1(\mathfrak{r})+\kappa_1(\mathfrak{r})) \\
&+\frac{1-e^{-\lambda_1\xi}}{\lambda_1\Delta}\Bigg\{\mathcal{A}_2\Bigg[\lambda_2\Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\frac{(1-e^{-\lambda_2\mathcal{T}})}{\lambda_2}\Bigg) + \Bigg(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\Bigg)\Bigg] \\
&+\mathcal{B}_2\Bigg[\zeta\lambda_2\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\frac{(1-e^{-\lambda_2\mu})}{\lambda_2}\Bigg) + \zeta\Bigg(\frac{\mu^{\beta-1}}{\Gamma(\beta)}\Bigg)\Bigg]\Bigg\}\|l_2\|(\gamma_2(\mathfrak{r})+\kappa_2(\mathfrak{r})), \\
\le & \Psi_1\|l_1\|(\gamma_1(\mathfrak{r})+\kappa_1(\mathfrak{r})) + \Psi_2\|l_2\|(\gamma_2(\mathfrak{r})+\kappa_2(\mathfrak{r})),
\end{aligned}$$

In a similar manner, one obtains

$$|\mathfrak{g}_2(\chi,\Omega)(\xi)| \le \Psi_3\|l_1\|\big(\varphi_1(\mathfrak{r})+\kappa_1(\mathfrak{r})\big) + \Psi_4\|l_2\|\big(\varphi_2(\mathfrak{r})+\kappa_2(\mathfrak{r})\big).$$

Combining the above estimates, we arrive at

$$\|\mathfrak{g}_1,\mathfrak{g}_2\| = \|\mathfrak{g}_1(\chi,\Omega)\| + \|\mathfrak{g}_2(\chi,\Omega)\| \le (\Psi_1+\Psi_3)\|l_1\|\big(\varphi_1(\mathfrak{r})+\kappa_1(\mathfrak{r})\big) + (\Psi_2+\Psi_4)\|l_2\|\big(\varphi_2(\mathfrak{r})+\kappa_2(\mathfrak{r})\big) =: l,$$

where l is a positive constant independent of $(\chi,\Omega)\in\mathcal{B}_{\mathfrak{r}}$. Therefore, $\Lambda(\mathcal{B}_{\mathfrak{r}})$ is bounded in $\Theta_1\times\Theta_1$.

We now verify the equicontinuity of the operator Λ. Let $\xi_1,\xi_2\in\mathcal{J}$ with $\xi_1<\xi_2$. For arbitrary selections $\mathcal{H}_1\in\mathcal{V}_{\mathcal{F}(\chi,\Omega)}, \mathcal{H}_2\in\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$, it follows from the definitions of Λ_1 and Λ_2 that

$$
\begin{aligned}
&|\mathfrak{g}_1(\chi,\Omega)(\xi_2)-\mathfrak{g}_1(\chi,\Omega)(\xi_1)|\\
\leq&\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)\Bigg]\\
&+\left|\int_0^{\xi_2}e^{-\lambda_1(\xi_2-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta-\int_0^{\xi_1}e^{-\lambda_1(\xi_1-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\right|\to 0,
\end{aligned}
$$

we deduce that $|\mathfrak{g}_1(\chi,\Omega)(\xi_2)-\mathfrak{g}_1(\chi,\Omega)(\xi_1)|\to 0$, as $\xi_2\to\xi_1$. Similarly, we conclude that $|\mathfrak{g}_2(\chi,\Omega)(\xi_2)-\mathfrak{g}_2(\chi,\Omega)(\xi_1)|\to 0$ as $\xi_2\to\xi_1$. We deduce that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are equi-continuous operators and so the operator $\mathfrak{g}$ also equi-continuous. From, the foregoing arguments, it follows that the operator $\Lambda(\chi,\Omega)$ is equi-continuous, and so completely by the Ascoli-Arzelá theorem.

In our next step, we show that the operator $\Lambda(\chi,\Omega)$ has a closed graph. Let $(\chi_n,\Omega_n)\to(\chi_*,\Omega_*)$, $(\mathfrak{g}_n,\hat{\mathfrak{g}_n})\in\Lambda(\chi_n,\Omega_n)$ and $(\mathfrak{g}_n,\hat{\mathfrak{g}_n})\to(\mathfrak{g}_*,\hat{\mathfrak{g}_*})$, then we need to show $(\mathfrak{g}_*,\hat{\mathfrak{g}_*})\in\Lambda(\chi_*,\Omega_*)$.

Observe that $(\mathfrak{g}_n,\hat{\mathfrak{g}_n})\in\Lambda(\chi_n,\Omega_n)$ implies that there exist $\mathcal{H}_{1n}\in\mathcal{V}_{\mathcal{F}(\chi_n,\Omega_n)}$, and $\mathcal{H}_{2n}\in\mathcal{V}_{\mathcal{G}(\chi_n,\Omega_n)}$ such that

$$
\begin{aligned}
&\mathfrak{g}_n(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{\mathfrak{g}}_n(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta,
\end{aligned}
$$

Let $\Pi_1, \Pi_2 : \mathcal{L}^1(\mathcal{J}, \Theta_1 \times \Theta_1) \longrightarrow \mathcal{C}(\mathcal{J}, \Theta_1 \times \Theta_1)$ be continuous linear operators defined as follows:

$$\begin{aligned}
&\Pi_1(\chi, \Omega)(\xi) \\
=&\frac{(1-e^{-\lambda_1 \xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho) + \int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi \\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big) + \mathcal{B}_2\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu) \\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)\Bigg] + \int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\Pi_2(\chi, \Omega)(\xi) \\
=&\frac{(1-e^{-\lambda_2 \xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu) + \int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi \\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big) + \mathcal{A}_1\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho) \\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)\Bigg] + \int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta,
\end{aligned} \tag{28}$$

By Lemma 1, the operator $(\Pi_1, \Pi_2) \circ (\mathcal{V}_{\mathcal{F}}, \mathcal{V}_{\mathcal{G}})$ has a closed graph. Moreover, for each $n \in \mathbb{N}$, $(\mathfrak{g}_n, \hat{\mathfrak{g}_n}) \in (\Pi_1, \Pi_2) \circ (\mathcal{V}_{\mathcal{F},(\chi_n,\Omega_n)}, \mathcal{V}_{\mathcal{G},(\chi_n,\Omega_n)})$. Since $(\chi_n, \Omega_n) \to (\chi_*, \Omega_*)$ and $(\mathfrak{g}_n, \hat{\mathfrak{g}_n}) \to (\mathfrak{g}_*, \hat{\mathfrak{g}}_*)$, the closedness of the graph implies the existence of sequences $\mathcal{H}_{1n} \in \mathcal{V}_{\mathcal{F}(\chi_n,\Omega_n)}$ and $\mathcal{H}_{2n} \in \mathcal{V}_{\mathcal{G}(\chi_n,\Omega_n)}$ such that

$$\begin{aligned}
&\mathfrak{g}_*(\chi_*, \Omega_*)(\xi) \\
=&\frac{(1-e^{-\lambda_1 \xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\rho) + \int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\vartheta)d\vartheta\Big]d\xi \\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\xi)\Big]d\xi\Big) + \mathcal{B}_2\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\mu) \\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\xi)\Big]d\xi\Big)\Bigg] + \int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\vartheta)d\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\mathfrak{g}}_*(\chi_*, \Omega_*)(\xi) \\
=&\frac{(1-e^{-\lambda_2 \xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\vartheta)d\vartheta + \zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\mu) + \int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\vartheta)d\vartheta\Big]d\xi \\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\xi)\Big]d\xi\Big) + \mathcal{A}_1\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1*}(\vartheta)d\vartheta + \eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho) \\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\vartheta)d\vartheta\Big]d\xi + \int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\xi)\Big]d\xi\Big)\Bigg] + \int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2*}(\vartheta)d\vartheta,
\end{aligned}$$

i.e., $(\mathfrak{g}_n, \mathfrak{g}_n) \in \Lambda(\chi_*, \Omega_*)$.

It remains to establish suitable a priori bounds for possible solutions. Let $(\chi, \Omega) \in \Psi\Lambda(\chi, \Omega)$ for some $\Psi \in [0, \mathcal{T}]$. By the definition of the operator Λ, there exist selections $\mathcal{H}_1 \in \mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{H}_2 \in \mathcal{V}_{\mathcal{G}(\chi,\Omega)}$ for each $\xi \in [0, \mathcal{T}]$, we obtain

$$\|\chi\| \le \Psi_1\|l_1\|(\varphi_1(\|\chi\|) + \kappa_1(\|\Omega\|)) + \Psi_1\|l_2\|(\varphi_2(\|\chi\|) + \kappa_2(\|\Omega\|)),$$
$$\|\Omega\| \le \Psi_3\|l_1\|(\varphi_1(\|\chi\|) + \kappa_1(\|\Omega\|)) + \Psi_4\|l_2\|(\varphi_2(\|\chi\|) + \kappa_2(\|\Omega\|)).$$

For each $\xi \in \mathcal{J}$, we achieve

$$\|\chi, \Omega\| = \|\chi\| + \|\Omega\| \le (\Psi_1 + \Psi_3)\|l_1\|(\varphi_1(\|\chi\|) + \kappa_1(\|\Omega\|)) + (\Psi_2 + \Psi_4)\|l_2\|(\varphi_2(\|\chi\|) + \kappa_2(\|\|)),$$

which implies that

$$\frac{\|(\chi, \Omega)\|}{(\Psi_1 + \Psi_3)\|l_1\|(\varphi_1(\|\chi\|) + \kappa_1(\|\Omega\|)) + (\Psi_2 + \Psi_4)\|l_2\|(\varphi_2(\|\chi\|) + \kappa_2(\|\|))} \le 1.$$

According to $(\mathcal{W}_3)$, $\mathcal{Z}$ exists such that $\|(\chi, \Omega)\| \ne \mathcal{Z}$. Let us $\mathcal{E} = \{(\chi, \Omega) \in \Theta_1 \times \Theta_1 : \|(\chi, \Omega)\| < \mathcal{Z}\}$. Note that the operator $\Lambda : \overline{\mathcal{E}} \to \mathscr{W}_{cv,cp}(\Theta_1) \times \mathscr{W}_{cv,cp}(\Theta_1)$ is completely continuous and upper semi continuous. From the choice of $\mathcal{E}$, there is no $(\chi, \Omega) \in \partial\mathcal{E}$ such that $(\chi, \Omega) \in \Psi\Lambda(\chi, \Omega)$ for some $\Psi \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [22], we deduce that Λ has a fixed point $(\chi, \Omega) \in \overline{\mathcal{E}}$, which is a solution of the problem (2). This completes the proof. □

3.2 The Lipschitz Case

This subsection is devoted to the study of problem (1) when the multivalued mappings $\mathcal{F}$ and $\mathcal{G}$ are permitted to have nonconvex-valued ranges.

For the next Theorem we use the following assumptions:

$(\mathcal{W}_4)$ $\mathcal{F}, \mathcal{G} : \mathcal{J} \times \mathbb{R}^2 \to \mathscr{W}_{cp}(\mathbb{R})$ are such that $\mathcal{F}(\cdot, \chi, \Omega) : \mathcal{J} \to \mathscr{W}_{cp}(\mathbb{R}^2)$ and $\mathcal{G}(\cdot, \chi, \Omega) : \mathcal{J} \to \mathscr{W}_{cp}(\mathbb{R}^2)$ are measurable for each $\chi, \Omega \in \mathbb{R}$.

$(\mathcal{W}_5)$ $\Pi_d(\mathcal{F}(\xi, \chi, \Omega), \mathcal{F}(\xi, \hat{\chi}, \hat{\Omega})) \le s_1(\xi)(|\chi - \hat{\chi}| + |\Omega - \hat{\Omega}|)$,
$\Pi_d(\mathcal{G}(\xi, \chi, \Omega), \mathcal{G}(\xi, \hat{\chi}, \hat{\Omega})) \le s_2(\xi)(|\chi - \hat{\chi}| + |\Omega - \hat{\Omega}|)$,
where, $\forall\ \xi \in \mathcal{J}$ and $\chi, \Omega, \hat{\chi}, \hat{\Omega} \in \mathbb{R}$ with $s_1, s_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+)$ and $d(0, \mathcal{F}(\xi, 0, 0)) \le s_1(\xi)$, $d(0, \mathcal{G}(\xi, 0, 0)) \le s_2(\xi)$ $\forall\ \xi \in \mathcal{J}$.

Theorem 2: *Suppose that assumptions $(\mathcal{W}_4)$ and $(\mathcal{W}_5)$ are satisfied. Then the coupled system (1)–(2) admits at least one solution on $\mathcal{J}$ provided that*

$$(\Psi_1 + \Psi_3)\|s_1\| + (\Psi_2 + \Psi_4)\|s_2\| < 1, \tag{29}$$

where the constants Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are defined in (18)–(21).

Proof: Under hypothesis $(\mathcal{W}_4)$, the sets $\mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$ are nonempty for every $(\chi, \Omega) \in \Theta_1 \times \Theta_1$. Moreover, by Theorem III.6 of [23], the multifunctions admit measurable selections. Our aim is to verify that the operator Λ fulfills the assumptions of the Covitz–Nadler fixed point theorem [21], which will subsequently yield the existence of a solution to the boundary value problem. We first show that $\Lambda(\chi, \Omega) \in \mathscr{W}_{cl}(\Theta_1) \times \mathscr{W}_{cl}(\Theta_1)$ for every $(\chi, \Omega) \in \Theta_1 \times \Theta_1$. Let $(\mathfrak{g}_n, \hat{\mathfrak{g}}_n) \in \Lambda(\chi_n, \Omega_n)$ be a sequence such that $(\mathfrak{g}_n, \hat{\mathfrak{g}}_n) \longrightarrow (\mathfrak{g}, \hat{\mathfrak{g}})$ in $\Theta_1 \times \Theta_1$. Since $\Theta_1 \times \Theta_1$ is complete, $(\mathfrak{g}, \hat{\mathfrak{g}}) \in \Theta_1 \times \Theta_1$. Furthermore, for each $n \in \mathbb{N}$, there exist $\mathcal{H}_{1n} \in \mathcal{V}_{\mathcal{F}(\chi_n,\Omega_n)}$ and $\mathcal{H}_{2n} \in \mathcal{V}_{\mathcal{G}(\chi_n,\Omega_n)}$ such that

$$
\begin{aligned}
&\mathfrak{g}_n(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{\mathfrak{g}}_n(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{1n}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_{1}(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{2n}(\vartheta)d\vartheta,
\end{aligned}
$$

Since both $\mathcal{F}$ and $\mathcal{G}$ take compact values, it is possible to extract subsequences of $\{\mathcal{H}_{1n}\}$ and $\{\mathcal{H}_{2n}\}$, still denoted by the same symbols, such that $\mathcal{H}_{1n}\to\mathcal{H}_1$ and $\mathcal{H}_{2n}\to\mathcal{H}_2$ in $\mathcal{L}^1(\mathcal{J},\mathbb{R})$ as $n\to\infty$. Consequently, $\mathcal{H}_1\in\mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{H}_2\in\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$. Hence, for every $\xi\in\mathcal{J}$, we obtain

$$
\begin{aligned}
&\mathfrak{g}_n(\chi_n,\Omega_n)(\xi)\to\mathfrak{g}(\chi,\Omega)(\xi)=\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)\\
&+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{\mathfrak{g}}_n(\chi_n,\Omega_n)(\xi)\to\hat{\mathfrak{g}}(\chi,\Omega)(\xi)\\
=&\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^{\mu}e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_2(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^{\xi}e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_1(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^{\rho}e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_1(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_2(\xi)\Big]d\xi\Big)\Bigg]+\int_0^{\xi}e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_2(\vartheta)d\vartheta,
\end{aligned}
$$

Consequently, $(\mathfrak{g},\hat{\mathfrak{g}})\in\Lambda$, and hence Λ possesses a closed graph. It remains to verify that Λ satisfies a Lipschitz-type condition. Specifically, we claim that there exists a constant $\hat{\rho}<1$ such that

$$\Pi_d\big(\Lambda(\chi,\Omega),\Lambda(\hat{\chi},\hat{\Omega})\big)\leq\hat{\rho}\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big)$$

for all $\chi,\hat{\chi},\Omega,\hat{\Omega}\in\Theta_1$. Let $(\chi,\hat{\chi}),(\Omega,\hat{\Omega})\in\Theta_1\times\Theta_1$ and $(\mathfrak{g}_1,\hat{\mathfrak{g}}_1)\in\Lambda(\chi,\Omega)$. Then there exists $\mathcal{H}_{11}\in\mathcal{V}_{\mathcal{F}(\chi,\Omega)}$ and $\mathcal{H}_{21}\in\mathcal{V}_{\mathcal{G}(\chi,\Omega)}$ such that, for each $\xi\in\mathcal{J}$, we have

$$\begin{aligned}
&\mathfrak{g}_1(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_1\xi})}{\Delta}\Bigg[\mathcal{A}_2\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\rho)+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\xi)\Big]d\xi\Big)+\mathcal{B}_2\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\mu)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\vartheta)d\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\mathfrak{g}}_1(\chi_n,\Omega_n)(\xi)\\
=&\frac{(1-e^{-\lambda_2\xi})}{\Delta}\Bigg[\mathcal{B}_1\Big(-\zeta\lambda_2\int_0^\mu e^{-\lambda_2(\mu-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\vartheta)d\vartheta+\zeta\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\mu)+\int_0^{\mathcal{T}}\Big[\lambda_1\int_0^\xi e^{-\lambda_1(\xi-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\vartheta)d\vartheta\Big]d\xi\\
&+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\xi)\Big]d\xi\Big)+\mathcal{A}_1\Big(-\eta\lambda_1\int_0^\rho e^{-\lambda_1(\rho-\vartheta)}\mathcal{I}^{\alpha-1}\mathcal{H}_{11}(\vartheta)d\vartheta+\eta\mathcal{I}^{\alpha-1}\mathcal{H}_1(\rho)\\
&+\int_0^{\mathcal{T}}\Big[\lambda_2\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\vartheta)d\vartheta\Big]d\xi+\int_0^{\mathcal{T}}\Big[\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\xi)\Big]d\xi\Big)\Bigg]+\int_0^\xi e^{-\lambda_2(\xi-\vartheta)}\mathcal{I}^{\beta-1}\mathcal{H}_{21}(\vartheta)d\vartheta,
\end{aligned}$$

Utilizing $(\mathcal{W}_5)$, we acquire

$$\begin{aligned}
&\Pi_d(\mathcal{F}(\xi,\chi,\Omega),\mathcal{F}(\xi,\hat{\chi},\hat{\Omega}))\leq s_1(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|),\\
&\Pi_d(\mathcal{G}(\xi,\chi,\Omega),\mathcal{G}(\xi,\hat{\chi},\hat{\Omega}))\leq s_2(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|).
\end{aligned}$$

So, $\exists\,\mathcal{H}_1\in\mathcal{F}(\xi,\chi(\xi),\Omega(\xi))$ and $\mathcal{H}_2\in\mathcal{G}(\xi,\chi(\xi),\Omega(\xi))$ such that

$$\begin{aligned}
&|\mathcal{H}_{11}(\xi)-\mathcal{H}_1|\leq s_1(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|),\\
&|\mathcal{H}_{21}(\xi)-\mathcal{H}_2|\leq s_2(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|).
\end{aligned}$$

Define $\mathcal{Q}_1,\mathcal{Q}_2:\mathcal{J}\to\mathscr{W}(\mathbb{R})$ by

$$\begin{aligned}
&\mathcal{Q}_1(\xi)=\{\mathcal{H}_1\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):\ s_1(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|)\},\\
&\mathcal{Q}_2(\xi)=\{\mathcal{H}_2\in\mathcal{L}^1(\mathcal{J},\mathbb{R}):\ s_2(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|)\}.
\end{aligned}$$

Since the multi-valued operator $\mathcal{Q}_1\cap\mathcal{F}(\xi,\chi(\xi),\Omega_1(\xi))$ and $\mathcal{Q}_2\cap\mathcal{G}(\xi,\chi(\xi),\Omega_1(\xi))$ are measurable (Proposition III.4 in [23]), there exists functions for $\mathcal{H}_{12}(\xi)$,$\mathcal{H}_{22}(\xi)$ which are measurable selections for $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and $\mathcal{H}_{12}(\xi)\in\mathcal{F}(\xi,\chi(\xi),\Omega_1(\xi))$, $\mathcal{H}_{22}(\xi)\in\mathcal{G}(\xi,\chi(\xi),\Omega_1(\xi))$ such that $\forall\ \xi\in\mathcal{J}$, we have

$$\begin{aligned}
&|\mathcal{H}_{11}(\xi)-\mathcal{H}_{12}(\xi)|\leq s_1(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|)\\
&|\mathcal{H}_{21}(\xi)-\mathcal{H}_{22}(\xi)|\leq s_2(\xi)(|\chi(\xi)-\hat{\chi}(\xi)|+|\Omega(\xi)-\hat{\Omega}(\xi)|).
\end{aligned}$$

Let

$$\|\mathfrak{g}_1(\chi,\Omega)-\mathfrak{g}_2(\chi,\Omega)\| \le \Psi_1\|s_1\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big)+\Psi_2\|s_2\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big).$$
$$\|\hat{\mathfrak{g}}_1(\chi,\Omega)-\hat{\mathfrak{g}}_2(\chi,\Omega)\| \le \Psi_3\|s_1\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big)+\Psi_4\|s_2\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big).$$

Combining the above estimates yields

$$\|(\mathfrak{g}_1,\hat{\mathfrak{g}}_1)-(\mathfrak{g}_2,\hat{\mathfrak{g}}_2)\| \le (\Psi_1+\Psi_3)\|s_1\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big)+(\Psi_2+\Psi_4)\|s_2\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big).$$

By reversing the roles of (χ,Ω) and $(\hat{\chi},\hat{\Omega})$, a similar estimate can be derived. Therefore,

$$\Pi_d\big(\Lambda(\chi,\Omega),\Lambda(\hat{\chi},\hat{\Omega})\big) \le (\Psi_1+\Psi_3)\|s_1\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big)+(\Psi_2+\Psi_4)\|s_2\|\big(\|\chi-\hat{\chi}\|+\|\Omega-\hat{\Omega}\|\big).$$

In view of condition (29), the operator Λ is a contraction with respect to the Hausdorff metric. Hence, an application of the Covitz–Nadler fixed point theorem guarantees the existence of a fixed point

$$(\chi,\Omega)\in\Lambda(\chi,\Omega).$$

Therefore, (χ,Ω) is a solution of the boundary value problem (1) and (2). This completes the proof. □

Remark 2: *The assumptions (W_1)–(W_5) are fundamental to the proof of Theorem 2. Assumptions (W_1) and (W_2) ensure that the multifunctions $\mathcal{F}$ and $\mathcal{G}$ are measurable, upper semicontinuous, and have nonempty, compact, and convex values, guaranteeing the existence of measurable selections. Assumption (W_3) provides the required growth condition to establish boundedness and relative compactness of the solution operator. Assumption (W_4) ensures continuity with respect to the Hausdorff metric, while (W_5) yields the contraction condition $(\Psi_1+\Psi_3)\|s_1\|+(\Psi_2+\Psi_4)\|s_2\|<1$, which enables the application of the Covitz–Nadler fixed-point theorem to obtain the existence of a solution.*

4 An Application

Example 1: *Consider the coupled sequential fractional differential inclusion system*

$$\begin{cases} ({}^CD^{\alpha}+\lambda_1\,{}^CD^{\alpha-1})\chi(\xi)\in\mathcal{F}(\xi,\chi(\xi),\Omega(\xi)), & \xi\in[0,\mathcal{T}],\\ ({}^CD^{\beta}+\lambda_2\,{}^CD^{\beta-1})\Omega(\xi)\in\mathcal{G}(\xi,\chi(\xi),\Omega(\xi)), & \xi\in[0,\mathcal{T}],\\ \displaystyle\int_0^{\mathcal{T}}\chi'(\xi)\,d\xi=\zeta\big(\Omega'(\mu)\big),\qquad \int_0^{\mathcal{T}}\Omega'(\xi)\,d\xi=\eta\big(\chi'(\rho)\big),\qquad \chi(0)=0,\qquad \Omega(0)=0. \end{cases} \tag{30}$$

Choose $\alpha=1.5,\beta=1.7,\lambda_1=\lambda_2=0.5,\rho=0.9,\mu=0.7,\mathcal{T}=2$.

Substituting the above parameter values into the expressions of Ψ_1,Ψ_2,Ψ_3 and Ψ_4 given in Eq. (3.10), we obtain $\Psi_1=1.2642,\Psi_2=0.4228,\Psi_3=0.3188,\Psi_4=1.2642$.

Define the multi-functions

$$\mathcal{F}(\xi,\chi,\Omega)=\left[-\frac{1}{16}\frac{|\chi|}{1+|\chi|},\ \frac{1}{16}\frac{|\sin\Omega|}{1+|\sin\Omega|}\right],\qquad \mathcal{G}(\xi,\chi,\Omega)=\left[-\frac{1}{16}\frac{|\Omega|}{1+|\Omega|},\ \frac{1}{16}\frac{|\cos\chi|}{1+|\cos\chi|}\right].$$

We verify assumptions $(\mathcal{W}_1)$–$(\mathcal{W}_5)$.

Verification of $(\mathcal{W}_1)$. For every $(\xi,\chi,\Omega)\in[0,\mathcal{T}]\times\mathbb{R}^2$, the values of $\mathcal{F}$ and $\mathcal{G}$ are compact intervals in $\mathbb{R}$. Hence they are nonempty, closed and convex. Since the endpoint functions are continuous, $\mathcal{F}$ and $\mathcal{G}$ are measurable in ξ and upper semicontinuous with compact values.

Verification of $(\mathcal{W}_2)$. Since the functions $\frac{|x|}{1+|x|},\frac{|\sin x|}{1+|\sin x|},\frac{|\cos x|}{1+|\cos x|}$ are continuous on $\mathbb{R}$, the multifunctions $\mathcal{F}$ and $\mathcal{G}$ satisfy assumption $(\mathcal{W}_2)$.

Verification of $(\mathcal{W}_3)$. *Since*

$$0 \le \frac{|x|}{1+|x|} \le 1, \qquad 0 \le \frac{|\sin x|}{1+|\sin x|} \le 1, \qquad 0 \le \frac{|\cos x|}{1+|\cos x|} \le 1,$$

we have

$$\sup\{|v| : v \in \mathcal{F}(\xi,\chi,\Omega)\} \le \frac{1}{16}, \qquad \sup\{|w| : w \in \mathcal{G}(\xi,\chi,\Omega)\} \le \frac{1}{16}.$$

Therefore,

$$q_1(\xi) = q_2(\xi) = \frac{1}{16}, \qquad \xi \in [0,\mathcal{T}],$$

which are continuous and bounded on $[0,\mathcal{T}]$.

Verification of $(\mathcal{W}_4)$. *The function* $\phi(x) = \frac{|x|}{1+|x|}$ *is globally Lipschitz continuous with Lipschitz constant one since*

$$|\phi(x) - \phi(y)| \le |x-y|, \qquad x, y \in \mathbb{R}.$$

Moreover,

$$|\sin x - \sin y| \le |x-y|, \qquad |\cos x - \cos y| \le |x-y|.$$

Hence,

$$\Pi_d\Big(\mathcal{F}(\xi,\chi,\Omega), \mathcal{F}(\xi,\widehat{\chi},\widehat{\Omega})\Big) \le \frac{1}{16}\Big(|\chi-\widehat{\chi}| + |\Omega-\widehat{\Omega}|\Big), \qquad \Pi_d\Big(\mathcal{G}(\xi,\chi,\Omega), \mathcal{G}(\xi,\widehat{\chi},\widehat{\Omega})\Big) \le \frac{1}{16}\Big(|\chi-\widehat{\chi}| + |\Omega-\widehat{\Omega}|\Big).$$

Therefore, the Lipschitz constants are $s_1 = s_2 = \frac{1}{16}$, *and assumption* $(\mathcal{W}_4)$ *is satisfied.*

Verification of $(\mathcal{W}_5)$. *Using the computed values of* Ψ_1, Ψ_2, Ψ_3 *and* Ψ_4, *we obtain*

$$(\Psi_1 + \Psi_3)s_1 + (\Psi_2 + \Psi_4)s_2 = (1.2642 + 0.3188)\left(\frac{1}{16}\right) + (0.4228 + 1.2642)\left(\frac{1}{16}\right) = 0.2043 < 1.$$

Hence the contraction condition required in Theorem 2 is satisfied. Consequently, all assumptions $(\mathcal{W}_1)$–$(\mathcal{W}_5)$ *hold. Therefore, by Theorem 2, the coupled sequential fractional differential inclusion system* (30) *admits at least one solution pair* $(\chi,\Omega) \in C([0,\mathcal{T}],\mathbb{R}) \times C([0,\mathcal{T}],\mathbb{R})$.

5 Conclusion

In this paper, we investigated a coupled system of sequential fractional differential inclusions with two-point integral boundary conditions. By employing fixed-point methods for multivalued operators, we established existence results under both Carathéodory and Lipschitz assumptions on the multifunctions. The obtained results extend several related contributions in the literature and provide a framework for analyzing coupled fractional inclusion systems with nonlocal boundary conditions. An illustrative example was presented to verify the applicability of the theoretical findings.

Future research may focus on the study of uniqueness and various stability concepts, including Ulam–Hyers and Ulam–Hyers–Rassias stability, for similar classes of fractional inclusion systems. Other possible extensions include problems involving more general fractional operators, impulsive and stochastic effects, delay terms, higher-dimensional coupled systems, fractional differential inclusions, and more general nonlocal boundary conditions.

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