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# Novel Parameterized Utility Function on Dual Hesitant Fuzzy Rough Sets and Its Application in Pattern Recognition

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Received: 5 January 2019; Accepted: 2 February 2019; Published: 19 February 2019

**Abstract:** Based on comparative studies on correlation coefficient theory and utility theory, a series of rules that utility functions on dual hesitant fuzzy rough sets (DHFRSs) should satisfy, and a kind of novel utility function on DHFRSs are proposed. The characteristic of the introduced utility function is a parameter, which is determined by decision-makers according to their experiences. By using the proposed utility function on DHFRSs, a novel dual hesitant fuzzy rough pattern recognition method is also proposed. Furthermore, this study also points out that the classical dual tool is suitable to cope with dynamic data in exploratory data analysis situations, while the newly proposed one is suitable to cope with static data in confirmatory data analysis situations. Finally, a medical diagnosis and a traffic engineering example are introduced to reveal the effectiveness of the newly proposed utility functions on DHFRSs.

**Keywords:** utility function; hesitant fuzzy rough set; correlation coefficient; traffic engineering; pattern recognition

## 1. Introduction

Fifty years ago, Zadeh [1] introduced the famous concept "fuzzy set". In fuzzy set theory, it is intermittently difficult for people to resolve the membership degree of an element of a set. To study this kind of fuzzy hesitant situations, Torra and Narukawa [2] and Torra [3] introduced a new branch of fuzzy sets, i.e., hesitant fuzzy sets (HFSs), where the membership of a target belonging to a concept is represented by a combination of different values. Practices show that HFS can describe hesitant phenomenon more comprehensively than other extensions of fuzzy set. Dual hesitant fuzzy sets (DHFSs) concept, proposed in Zhu et al. [4], is an integrated set encompassing fuzzy set, intuitionistic fuzzy set, HFS, and fuzzy multi-sets as special cases (for more details about intuitionistic set, please see, Atanassov [5]; for more details about multi-sets, please see, Miyamoto [6]). Like intuitionistic fuzzy set, a DHFS proposes the membership and the non-membership degrees of a target to a set. Up until now, there has been an expeditious growth of interest in both theory and application on DHFSs.

Rough sets, which was first introduced by Pawlak [7], are another kind of handy mathematical tool for coping with vague and uncertain information. In a rough set, approximation operators are usually defined as an equivalence relation. In 1982, some general rough set models were proposed based on Pawlak's work, and one of the most critical characteristics of them was that they all relaxed the restriction of the aforementioned equivalence relation. Here, it is worthy to mention that the study proposed in Yao [8], which proposed a study on three-way decision making models under a rough set environment, is very magnificent. Recently, many promising scientific regulations have been proposed on a novel rough set with two universes, such as Ma and Sun [9], Sun et al. [10], Yan et al. [11], Yang et al. [12], Yang et al. [13], etc.

In a fuzzy set, there is a degree of membership for any element to the given set; whereas a rough set is a formal approximation of a crisp set according to a couple of sets which provide the lower and the upper approximation of the original one. Obviously, fuzzy sets and rough sets are highly complementary. Dubois and Prade [14] first proposed the concept "fuzzy rough sets" to stretch the crisp rough set. From then on, many wonderful scientific research results on fuzzy rough sets have been proposed, such as Radzikowska and Kerre [15], Dai and Tian [16], Tiwari and Srivastava [17], etc. Very recently, motivated by generalizing the concept of fuzzy rough sets, Yang et al. [18] founded the combinations of the theories of HFS and rough set. Subsequently, by combining the dual hesitant fuzzy set (DHFS) and rough set theory, Zhang et al. [19] developed a rough set model called dual hesitant fuzzy rough sets (DHFRSs) over two universes. They also built up a general shell frame of the decision making methods based upon the DHFRSs of two universes. Besides, Zhang and Shu [20] proposed a general framework for the research of generalized interval-valued fuzzy rough sets using axiomatic techniques.

Fuzzy sets and rough sets have a great role in multiple attribute decision making (MCDM) and pattern recognitions filed. On this point, some impressing works have been paid attention upon. For example, Roy et al. [21] used a novel combinative distance-based assessment method to handle MCDM problems. Vasiljevic et al. [22] presented novel methodology for a logistic center location analysis based on GIS and SWOT analyses under rough environments. Chatterjee [23] used rough multi-attribute ideal-real comparative analysis to evaluate the environmental performance of suppliers for each evaluation criterion. Meanwhile, Pamucar et al. [24] presented a new approach to the treatment of uncertainty and imprecision in the MCDM based on interval rough numbers. Pamucar et al. [25] presented a multi-criteria model for evaluating the quality of university websites' integration of interval rough AHP and interval rough multi-attributive border approximation area comparison methods. Pamučar et al. [26] presents a new approach for dealing with uncertainty by using interval-valued fuzzy-rough numbers.

Motivated by the works mentioned above, this study explores the DHFRSs from the viewpoint of utility. In the field of statistics, data analysis is sometimes divided into descriptive statistics analysis, exploratory data analysis, and confirmatory data analysis [27]. At present, the DHFRSs have been studied from the viewpoint of exploratory data analysis. This study is of confirmatory data analysis by using utility on DHFRSs. It is noteworthy that in this manuscript DHFRSs were studied from the viewpoint of confirmatory data analysis for the first time. In this study, a series of rules that the utility function on DHFRSs should satisfy were proposed, and a novel utility function on DHFRSs was proposed too. The characteristic of the introduced utility function was a parameter, which was determined by decision-makers according to their experiences. Furthermore, by using the novel utility function on DHFRSs, a dual hesitant fuzzy rough pattern recognition method was also introduced. The rest of this study was organized as follows. Section 2 reviews the concepts and the fundamental properties of HFSs, DHFSs, and DHFRSs. Section 3 proposes an analysis on DHFRSs, a series of rules that the utility function on DHFRSs should satisfy, a novel utility function on DHFRSs, and a pattern recognition approach in dual hesitant rough settings. Section 4 introduces a numerical case to illustrate the usefulness of the presented utility function, and presents sensitivity analysis [28]. Section 5 ends the study with some conclusive remarks.

## 2. Preliminaries

### 2.1. The Notion of DHFS

**Definition 1** ([4]). *Let U be a given fixed set, then a DHFS $\mathbb{D}$ on U is defined as:*

$$\mathbb{D} = \{\langle x, h_{\mathbb{D}}(x), g_{\mathbb{D}}(x)\rangle x \in U\}, \tag{1}$$

*in which $h_{\mathbb{D}}(x)$ and $g_{\mathbb{D}}(x)$ are two sets of some values in $[0,1]$, standing for the possible membership and non-membership degrees of the element $x \in U$ to the set $\mathbb{D}$, respectively, satisfying $0 \le \gamma,\ \eta \le 1$, and $0 \le \gamma^+ + \eta^+ \le 1$ for all $x \in U$, $\gamma \in h_{\mathbb{D}}(x)$, $\eta \in g_{\mathbb{D}}(x)$, $\gamma^+ \in h^+_{\mathbb{D}}(x) = \cup_{\gamma \in h_{\mathbb{D}}(x)} max\{\gamma\}$, $\eta^+ \in g^+_{\mathbb{D}}(x) = \cup_{\eta \in g_{\mathbb{D}}(x)} max\{\eta\}$. Besides, $d(x) = (h_{\mathbb{D}}(x), g_{\mathbb{D}}(x))$ is named a DHF element(DHFE), and is denoted as $d = (h, g)$. Additionally, the set of all DHFSs on U is symbolized as DHF $(U)$.*

**Definition 2** ([19]). *Let U be a finite and non-empty discourse domain. For any $\mathbb{A}, \mathbb{B} \in DHF(U)$, then the complement of $\mathbb{A}$ (which is denoted by $\mathbb{A}^c$), the union of $\mathbb{A}$ and $\mathbb{B}$ (which is denoted by $\mathbb{A} \Cup \mathbb{B}$), and the intersection of $\mathbb{A}$ and $\mathbb{B}$( which is denoted by $\mathbb{A} \Cap \mathbb{B}$) are defined by*

$$\mathbb{A}^c = \{\langle x, g_{\mathbb{A}}(x), h_{\mathbb{A}}(x)\rangle x \in U\} = \{\langle x, \{g^{\sigma(t)}_{\mathbb{A}}(x)\}, \{h^{\sigma(s)}_{\mathbb{A}}(x)\}|x \in U\}, \tag{2}$$

$$\mathbb{A} \Cup \mathbb{B} = \{\langle x, \{h^{\sigma(s)}_{\mathbb{A}}(x) \vee h^{\sigma(s)}_{\mathbb{B}}(x)\}, \{g^{\sigma(t)}_{\mathbb{A}}(x) \wedge g^{\sigma(t)}_{\mathbb{B}}(x)\}\rangle|x \in U, 1 \le s \le k, 1 \le t \le l\}, \tag{3}$$

$$\mathbb{A} \Cap \mathbb{B} = \{\langle x, \{h^{\sigma(s)}_{\mathbb{A}}(x) \wedge h^{\sigma(s)}_{\mathbb{B}}(x)\}, \{g^{\sigma(t)}_{\mathbb{A}}(x) \vee g^{\sigma(t)}_{\mathbb{B}}(x)\}\rangle|x \in U, 1 \le s \le k, 1 \le t \le l\}, \tag{4}$$

*respectively, where $g^{\sigma(t)}_{\mathbb{A}}(x)$ represents the tth largest value in $g_{\mathbb{A}}(x)$, $h^{\sigma(s)}_{\mathbb{A}}(x)$ represents the sth largest value in $h_{\mathbb{A}}(x)$, $k = max\{l(h_{\mathbb{A}}(x)), l(h_{\mathbb{B}}(x))\}$, and $l = max\{l(g_{\mathbb{A}}(x)), l(g_{\mathbb{B}}(x))\}$. When $l(h_{\mathbb{A}}(x)) \ne l(h_{\mathbb{B}}(x))$ or $l(g_{\mathbb{A}}(x)) \ne l(g_{\mathbb{B}}(x))$, the existed methods usually extend the shorter set by adding any value in it.*

**Definition 3** ([4]). *Let $d_i = (h_{d_i}, g_{d_i})(i = 1,2)$ be any two given DHF elements. $s(d_i) = s(h_{d_i}) - s(g_{d_i}) = \frac{\sum \gamma \in h_{d_i} \gamma}{l(h_{d_i})} - \frac{\sum \gamma \in g_{d_i} \gamma}{l(g_{d_i})}$ is termed as the score function of $d_i$, and $p(d_i) = s(h_{d_i}) + s(g_{d_i}) = \frac{\sum \gamma \in h_{d_i} \gamma}{l(h_{d_i})} + \frac{\sum \gamma \in g_{d_i} \gamma}{l(g_{d_i})}$ is termed as the accuracy function of $d_i$, where $l(h_{d_i})$ and $l(g_{d_i})$ are the cardinality numbers of $h_{d_i}$ and $g_{d_i}$, respectively. When $s_{d_1} > s_{d_2}$, it is considered that $d_1 \succ d_2$; when $s_{d_1} = s_{d_2}$, $p_{d_1} = p_{d_2}$ it is considered that $d_1 \equiv d_2$; when $s_{d_1} = s_{d_2}$, $p_{d_1} > p_{d_2}$, it is considered that $d_1 \prec d_2$.*

### 2.2. The Notion of DHFRS

**Definition 4** ([18]). *Let U be a finite and non-empty universe domain. A hesitant fuzzy relation $\mathcal{R}$ on U is represented as a hesitant fuzzy subset where $\mathcal{R} \in HF(U \times U)$ and $\mathcal{R} = \{\langle (x,y), h_{\mathcal{R}}(x,y)|(x,y) \in U \times U\}$. For all $(x,y) \in U \times U$, $h_{\mathcal{R}}(x,y)$ is a set of the values in $[0,1]$, which is denoted as the possible membership degrees of the relationships between x and y.*

**Definition 5** ([19]). *Let U, V be two finite and non-empty universes. A DHF subset $\mathbb{R}$ of the universe $U \times V$ is termed as a DHF relation from U to V , namely, $\mathbb{R}$ is given by $\mathbb{R} = \{\langle (x,y), h_{\mathbb{R}}(x,y), g_{\mathbb{R}}(x,y)|(x,y) \in U \times V\}$, where $h_{\mathbb{R}}, g_{\mathbb{R}} : U \times V \to [0,1]$ are two sets where their elements are valued in $[0,1]$, expressing the possible membership and non-membership degrees of the relationship between x and y, respectively, in which $0 \le \gamma, \eta \le 1$ and $0 \le \gamma^+, \eta^+ \le 1$. Besides, for all $(x,y) \in U \times V$ , $\gamma \in h_{\mathbb{R}}(x,y), \eta \in g_{\mathbb{R}}(x,y)$, it gets that $\gamma^+ \in h^+_{\mathbb{R}}(x,y) = \bigcup_{\gamma \in h_{\mathbb{R}}(x,y)} max\{\gamma\}$,$\eta^+ \in g^+_{\mathbb{R}}(x,y) = \bigcup_{\eta \in g_{\mathbb{R}}(x,y)} max\{\eta\}$. In particular, if $U = V$ , $\mathbb{R}$ is termed as a DHF relation on U.*

**Definition 6** ([19])**.** *Let $U$, $V$ be two non-empty and finite universes, $\mathbb{R}$ be a DHF relation from $U$ to $V$. The triple $(U, V, \mathbb{R})$ is termed as a DHF approximation space. For any $\mathbb{A} \in DHF(V)$, the lower and upper approximations of $\mathbb{A}$ with regard to $(U, V, \mathbb{R})$, suggested as $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two DHF sets of $U$ and are, respectively, specified as $\underline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\underline{\mathbb{R}}(\mathbb{A})}(x), g_{\underline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}$, $\overline{\mathbb{R}}(\mathbb{A}) = \{\langle x, h_{\overline{\mathbb{R}}(\mathbb{A})}(x), g_{\overline{\mathbb{R}}(\mathbb{A})}(x)\rangle | x \in U\}$, where $h_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \bigcap_{y \in V} \{g_{\mathbb{R}}(x, y) \cup h_{\mathbb{A}}(y)\}$, $g_{\underline{\mathbb{R}}(\mathbb{A})}(x) = \bigcup_{y \in V} \{h_{\mathbb{R}}(x, y) \cap g_{\mathbb{A}}(y)\}$, $h_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \bigcup_{y \in V} \{h_{\mathbb{R}}(x, y) \cap h_{\mathbb{A}}(y)\}$, $g_{\overline{\mathbb{R}}(\mathbb{A})}(x) = \bigcap_{y \in V} \{g_{\mathbb{R}}(x, y) \cup g_{\mathbb{A}}(y)\}$. Especially, $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ are, respectively, termed as the lower and upper approximations of $\mathbb{A}$ with regard to $(U, V, \mathbb{R})$. The pair $(\underline{\mathbb{R}}(\mathbb{A}), \overline{\mathbb{R}}(\mathbb{A}))$ is called the DHF rough set of $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$, and $\underline{\mathbb{R}}, \overline{\mathbb{R}} : DHF(V) \to DHF(U)$ are referred to as lower and upper DHF rough approximation operators, respectively.*

## 3. Main Results

### *3.1. Analysis on DHFRSs*

Usually, data is classified into three categories: descriptive data, exploratory data and confirmatory data. Specifically, descriptive statistics analysis mainly quantitatively describes or summarises the features of a collection of information; exploratory data analysis focuses on the detection of the new features in the data, while validation data analysis focuses on the verification or falsification of the existing hypothesis. In practice, the exploratory data analysis and the confirmatory data analysis have gotten more attention than descriptive statistics analysis. In most cases, the exploratory data analysis is done from the viewpoint of dynamic, while the exploratory data analysis is achieved from the viewpoint of static.

One of the most frequently used mathematical tools to deal with dual hesitant fuzzy rough information is the correlation coefficient theory. By definition, the correlation coefficient is a statistic value describing how closely two variables co-vary, and the correlation coefficient on DHFRSs attaches more importance on the tendency of the described object. Take medical diagnosis for example, when a person "has just begun to feel uncomfortable", the correlation coefficient between the symptom data of him and a kind of disease reflects the likelihood of the trend of him to suffer from the disease. Therefore, the correlation coefficient on DHFRSs is a kind of exploratory data which is studied from dynamic.

To analyze the DHFRSs from the viewpoint of static, this paper proposes a novel utility function on DHFRSs. Here, the meaning of utility is the quality of being of practical use (for more details about utility please see, Zhang and Xu [29]). By definition, the utility value of DHFRSs should be used to test whether measures of an object are consistent with a researcher's understanding of the nature of that construct, or whether the DHFRSs fit a threshold-based hypothesized measurement model. Thus, the utility value of a DHFRSs is a kind of confirmatory data. Take medical diagnosis for example, when a patient "is suffering from illness", the utility values of the symptom data of him to a disease reflects the degree of the severity of the disease, which is a static index. Therefore, in order to analyze the DHFRSs carefully, it is required to study the utility theory and the correlation coefficient theory into a unified framework. Under the guidance of this academic thought, a novel utility function on DHFRSs is proposed in the next subsection.

### *3.2. A Kind of Novel Utility Function on DHFRSs*

Firstly, the laws that a utility function should satisfy are proposed as follows.

**Theorem 1.** *Let $U = \{x_n, x_n, \cdots, x_n\}$ be a discrete universe of discourse, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ be three DHFSs on $U$ denoted as $\mathbb{A} = \{\langle x_i, h_{\mathbb{A}}(x_i), g_{\mathbb{A}}(x_i)\rangle | x_i \in U, i = 1, 2, \cdots, n\}$, $\mathbb{B} = \{\langle x_i, h_{\mathbb{B}}(x_i), g_{\mathbb{B}}(x_i)\rangle | x_i \in U, i = 1, 2, \cdots, n\}$, and $\mathbb{C} = \{\langle x_i, h_{\mathbb{C}}(x_i), g_{\mathbb{C}}(x_i)\rangle | x_i \in U, i = 1, 2, \cdots, n\}$, respectively. Let $E(\cdot)$ be a utility function on DHFRSs, then,*

*(i)* $0 \leq E_{\mathbb{A}}(x_i), E_{\mathbb{B}}(x_i), E_{\mathbb{C}}(x_i) \leq 1,\ for\ i \in \{1, 2, \cdots, n\}$;
*(ii)* $E_{\mathbb{A}}(x_i) = 0,\ iff\ h_{\mathbb{A}}(x_i) = \{0\},\ g_{\mathbb{A}}(x_i) = \{1\}$;
*(iii)* $E_{\mathbb{A}}(x_i) = 1,\ iff\ h_{\mathbb{A}}(x_i) = \{1\},\ g_{\mathbb{A}}(x_i) = \{0\}$;
*(iv)* $E_{\mathbb{A}}(x_i) \geq E_{\mathbb{B}}(x_i),\ if\ s(\mathbb{A}(x_i)) \geq s(\mathbb{B}(x_i)),\ p(\mathbb{A}(x_i)) \geq p(\mathbb{B}(x_i))$;
*(v)* $E_{\mathbb{A}}(x_i) \geq E_{\mathbb{C}}(x_i),\ if\ E_{\mathbb{A}}(x_i) \geq E_{\mathbb{B}}(x_i), E_{\mathbb{B}}(x_i) \geq E_{\mathbb{C}}(x_i)$,

*where $s(\cdot)$ and $p(\cdot)$ denotes the score and the accuracy functions on DHFRSs, which are generated by Definition 3.*

In Theorem 1, it is evident that the items $(i)$, $(ii)$, and $(iii)$ hold. For item $(iv)$, it is noteworthy that when $s(\mathbb{A}(x_i)) \geq s(\mathbb{B}(x_i))$, it holds that the score function value of $\mathbb{A}(x_i)$ is larger than that of $\mathbb{B}(x_i)$; meanwhile, when $p(\mathbb{A}(x_i)) \geq p(\mathbb{B}(x_i))$, it holds that the accuracy function value of $\mathbb{A}(x_i)$ is larger than which of $\mathbb{B}(x_i)$. In this situation, no matter from score or accuracy aspect, $\mathbb{A}(x_i)$ is prior to $\mathbb{B}(x_i)$. Therefore, it holds that $E_{\mathbb{A}}(x_i) \geq E_{\mathbb{B}}(x_i)$. For item $(v)$, it is noteworthy that for the same three objects, utility values are transferable when they are compared.

By Definition 6, for any given $\mathbb{A} \in DHF(V)$, the lower and upper approximations of $\mathbb{A}$ with regard to $(U, V, \mathbb{R})$ are two DHFSs of $U$. Therefore, for any given utility function of $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, it should also satisfy Theorem 1. Theoretically, there are many utility functions on the lower and upper approximations of DHFRSs which satisfy Theorem 1, and one of them is proposed as follows.

**Definition 7.** *Let $U$, $V$ be two given finite and non-empty universes, and $\mathbb{R}$ be a DHF relation from $U$ to $V$, and $U = \{x_1, x_2, \cdots, x_m\}$, $V = \{y_1, y_2, \cdots, y_n\}$. The triple $(U, V, \mathbb{R})$ is termed as a DHF approximation space. For any given $\mathbb{A} \in DHF(V)$, the lower and upper approximations of $\mathbb{A}$ with regard to $(U, V, \mathbb{R})$, denoted by $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$, are two DHFSs. Then a utility function of $\mathbb{A}$ with respect to $(U, V, \mathbb{R})$ is denoted as*

$$E_{\mathbb{A}}(x_i) = \frac{1}{2}\Big[1 + \frac{s(\underline{\mathbb{R}}(\mathbb{A})(x_i))}{2 - p(\underline{\mathbb{R}}(\mathbb{A})(x_i))}\Big]^{\lambda} \cdot \Big[1 + \frac{s(\overline{\mathbb{R}}(\mathbb{A})(x_i))}{2 - p(\overline{\mathbb{R}}(\mathbb{A})(x_i))}\Big]^{1-\lambda}, \tag{5}$$

*where $\lambda$ is given by the decision makers, $0 \leq \lambda \leq 1$, $i = 1, 2, \cdots, m$.*

Obviously, $E(\cdot)$ satisfies Theorem 1. To save space, the proof procedure is not proposed here.

### *3.3. Novel Dual Hesitant Fuzzy Rough Pattern Recognition Method*

This subsection presents a dual hesitant fuzzy rough pattern recognition method based on the proposed utility function. It is noteworthy that the decision approach on DHFRSs over two universes proposed by Zhang et al. [19] is very wonderful, where the two parameters $T_2$ and $T_3$ are obtained by using cut set theory. In their study, $T_2$ and $T_3$ are obtained from repeated measurements of the same object in a similar way. By comparing $T_2$ and $T_3$, the dependability and the reliability of the object is guaranteed. Here, in the novel dual hesitant fuzzy rough pattern recognition method, a utility function is used to describe the quality of the object, and a parameter $\lambda$ is used to measure the importance of the lower and upper approximations of the related HFS.

For convenience, the discussed pattern recognition problem is denoted as follows. Let the universe $U = \{x_1, x_2, \cdots, x_m\}$ be a given pattern set, the universe $V = \{y_1, y_2, \cdots, y_n\}$ be the given symptom set, and $R(x_i, y_j)$ be an intuitionistic fuzzy relation from $U$ to $V$. Assuming that there is an object $\mathbb{A}$ which has some symptoms in the universe $V$ which is described as $\mathbb{A} = \{\langle y_j, h_{\mathbb{A}}(y_j), g_{\mathbb{A}}(y_j)\rangle | y_j \in V\}$, where $h_{\mathbb{A}}(y_j)$ is a set of some different values in [0, 1], describing the possible membership degrees of $\mathbb{A}$ to the symptom $y_j$, $g_{\mathbb{A}}(y_j)$, which is a set of some different values in [0, 1] describing the possible non-membership degrees of $\mathbb{A}$ to the symptom $y_j$. Then, how does one find the optimal pattern in $U$ that $\mathbb{A}$ belongs to? To solve this problem, a novel dual hesitant fuzzy rough pattern recognition approach is proposed.

**Step 1** According to Definition 6, we consider the lower and upper approximations $\underline{\mathbb{R}}(\mathbb{A})$ and $\overline{\mathbb{R}}(\mathbb{A})$ of DHFSs $\mathbb{A}$ with regard to $(U, V, \mathbb{R})$.

**Step 2** By Equation (5), the utility value of $\mathbb{A}$ with respect to each $x_i (i = 1, 2, \cdots, m)$ is obtained as $E_{\mathbb{A}}(x_i)$. Then, a utility vector is obtained as $E_{\mathbb{A}} = (E_{\mathbb{A},\mathbb{R},x_1}(\lambda_1), E_{\mathbb{A},\mathbb{R},x_2}(\lambda_2), \cdots, E_{\mathbb{A},\mathbb{R},x_m}(\lambda_m))$. For convenience, we only consider the situation where all the $\lambda_i (i = 1, 2, \cdots, m)$ are equal.

**Step 3** For any given $\lambda^*$, an index $T_0$ is obtained as $T_0 = \{k | \max_{x_k \in U}\{E_{\mathbb{A},R,\lambda^*}(x_k)\}\}$, and we choose $x_k (k \in T_0)$ as the optimal pattern.

In the next section, a medical diagnosis problem, which is selected from Szmidt and Kacprzyk [30] and Zhang et al. [19], and a residential design problem, which is adopted from Tian et al. [31], are studied by using the proposed dual hesitant fuzzy rough pattern recognition method.

## 4. Illustrative Examples

### *4.1. Example 1*

To make a correct diagnosis for a given patient with a given values of symptoms, a medical knowledge base is necessary that involves elements described according as intuitionistic fuzzy sets. Let the universe $U = \{x_1, x_2, \cdots, x_5\}$ be a set of five diseases, where $x_1$ stands for "viral fever", $x_2$ stands for "malaria", $x_3$ stands for "typhoid", $x_4$ stands for "stomach problem", $x_5$ stands for "chest problem". Let $V = \{y_1, y_2, \cdots, y_5\}$ be five symptoms, where $y_1$ stands for "temperature", $y_2$ stands for "headache", $y_3$ stands for "stomach pain", $y_4$ stands for "cough", and $y_5$ stands for "chest pain", respectively. Assume that $R$ is a given medical knowledge statistic data of the relationship between the diseases $x_i (x_i \in U)$ and the symptoms $y_j (y_j \in V)$. $R$ is described by an intuitionistic fuzzy relation from $U$ to $V$, and the statistic data of $R$ is proposed by Szmidt and Kacprzyk (2002), whose details are shown in Table 1.

**Table 1.** Symptoms characteristic values for considered diagnoses.

| $R(x_i, y_j)$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $y_1$ | (0.4,0.0) | (0.7,0.0) | (0.3,0.3) | (0.1,0.7) | (0.1,0.8) |
| $y_2$ | (0.3,0.5) | (0.2,0.6) | (0.6,0.1) | (0.2,0.4) | (0.0,0.8) |
| $y_3$ | (0.1,0.7) | (0.0,0.9) | (0.2,0.7) | (0.8,0.0) | (0.2,0.8) |
| $y_4$ | (0.4,0.3) | (0.7,0.0) | (0.2,0.6) | (0.2,0.7) | (0.2,0.8) |
| $y_5$ | (0.1,0.7) | (0.1,0.8) | (0.1,0.9) | (0.2,0.7) | (0.8,0.1) |

Let $P = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4\}$ be a given set consisting of four different patients. Assume that every patient $\mathbb{A}_k (k = 1, 2, 3, 4)$ can see three different doctors, and each doctor provides the possible membership degree and non-membership degree to the symptoms of a patient. To carefully consider all the doctors' diagnostic results, the symptoms of each patient $\mathbb{A}_k (k = 1, 2, 3, 4)$ are described by a DHFS on the universe $V$. All symptoms of every $\mathbb{A}_k$ are proposed in a matrix as

$$C = \begin{array}{c} \\ \mathbb{A}_1 \\ \mathbb{A}_2 \\ \mathbb{A}_3 \\ \mathbb{A}_4 \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ (\{0.8,0.5,0.7\},\{0.2,0.1,0.1\}) & (\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}) & (\{0.5,0.4,0.3\},\{0.5,0.3,0.2\}) \\ (\{0.0,0.7,0.3\},\{0.3,0.2,0.2\}) & (\{0.4,0.2,0.6\},\{0.4,0.3,0.3\}) & (\{0.8,0.7,0.7\},\{0.2,0.1,0.0\}) \\ (\{0.8,0.2,0.4\},\{0.2,0.1,0.1\}) & (\{0.3,0.2,0.0\},\{0.6,0.5,0.5\}) & (\{0.3,0.4,0.4\},\{0.5,0.4,0.5\}) \\ (\{0.3,0.4,0.2\},\{0.6,0.4,0.6\}) & (\{0.3,0.2,0.1\},\{0.7,0.5,0.6\}) & (\{0.3,0.5,0.6\},\{0.3,0.4,0.4\}) \end{array}$$

$$\begin{array}{cc} y_4 & y_5 \\ (\{0.6,0.8,0.5\},\{0.2,0.2,0.1\}) & (\{0.1,0.2,0.4\},\{0.6,0.5,0.5\}) \\ (\{0.3,0.2,0.2\},\{0.7,0.6,0.5\}) & (\{0.3,0.4,0.4\},\{0.6,0.3,0.5\}) \\ (\{0.2,0.4,0.6\},\{0.4,0.4,0.3\}) & (\{0.7,0.8,0.9\},\{0.1,0.1,0.0\}) \\ (\{0.3,0.2,0.2\},\{0.7,0.6,0.6\}) & (\{0.6,0.5,0.3\},\{0.4,0.3,0.4\}) \end{array},$$

Then, how does one diagnose these patients? This problem was settled by using the novel dual hesitant fuzzy rough pattern recognition approach proposed in Section 3.

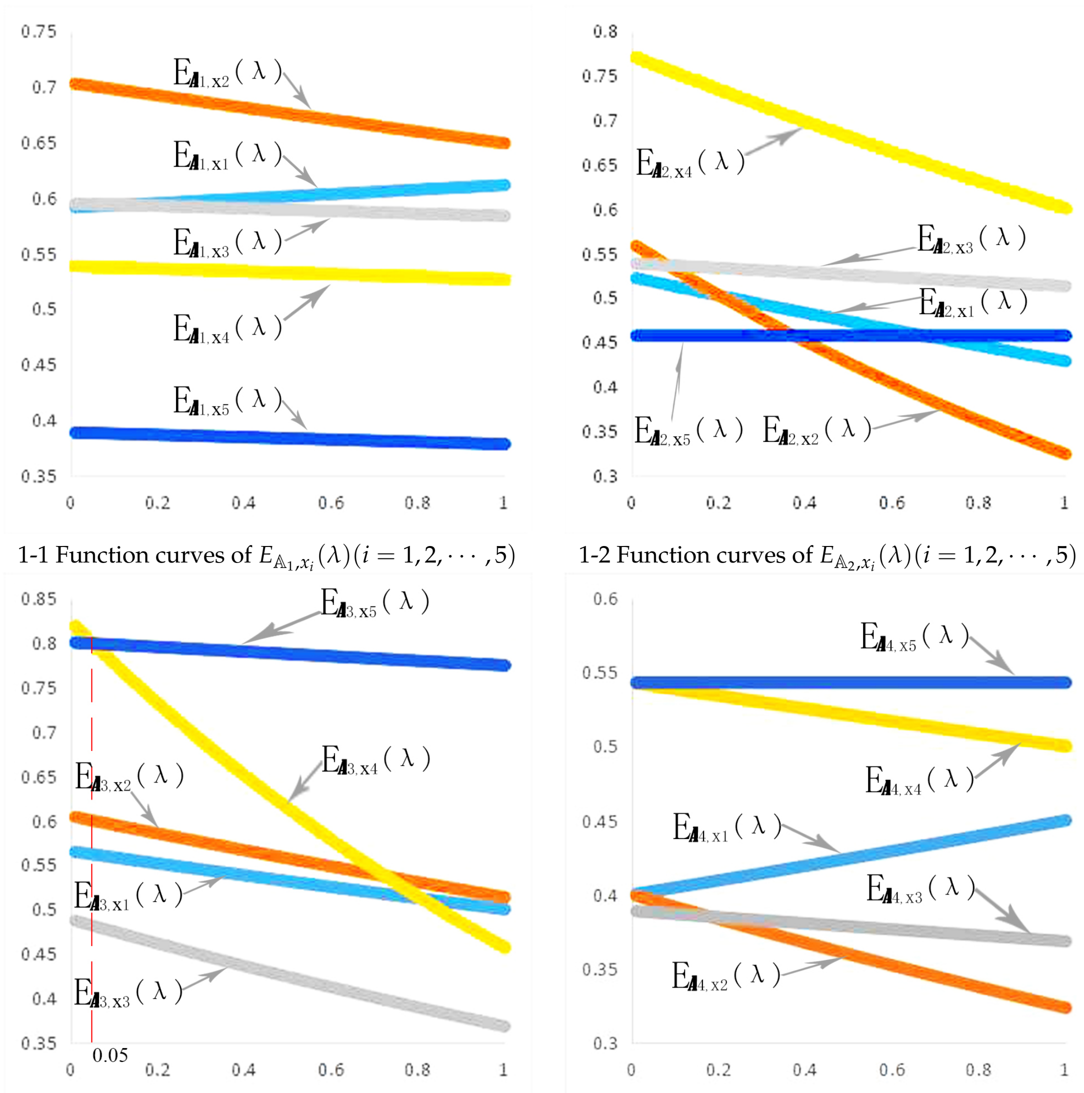
**Step 1** By Definition 6, both the lower and upper approximations of each $\mathbb{A}_k(i = 1,2,3,4)$ expressed in the function of $(U, V, \mathbb{R})$ can be presented as

$$C = \begin{array}{c} \\ \underline{\mathbb{R}}(\mathbb{A}_1) \\ \underline{\mathbb{R}}(\mathbb{A}_2) \\ \underline{\mathbb{R}}(\mathbb{A}_3) \\ \underline{\mathbb{R}}(\mathbb{A}_4) \\ \overline{\mathbb{R}}(\mathbb{A}_1) \\ \overline{\mathbb{R}}(\mathbb{A}_2) \\ \overline{\mathbb{R}}(\mathbb{A}_3) \\ \overline{\mathbb{R}}(\mathbb{A}_4) \end{array} \begin{array}{ccc} y_1 & y_2 & y_3 \\ (\{0.6,0.5,0.5\},\{0.3,0.3,0.2\}) & (\{0.6,0.6,0.5\},\{0.2,0.2,0.2\}) & (\{0.6,0.5,0.4\},\{0.4,0.3,0.2\}) \\ (\{0.3,0.3,0.0\},\{0.4,0.4,0.4\}) & (\{0.3,0.2,0.0\},\{0.7,0.6,0.5\}) & (\{0.6,0.3,0.2\},\{0.4,0.3,0.3\}) \\ (\{0.5,0.4,0.2\},\{0.4,0.4,0.3\}) & (\{0.6,0.4,0.2\},\{0.4,0.4,0.3\}) & (\{0.3,0.2,0.1\},\{0.6,0.5,0.5\}) \\ (\{0.3,0.3,0.2\},\{0.4,0.4,0.4\}) & (\{0.3,0.2,0.2\},\{0.7,0.6,0.6\}) & (\{0.3,0.2,0.1\},\{0.6,0.6,0.5\}) \\ (\{0.4,0.4,0.4\},\{0.2,0.1,0.1\}) & (\{0.7,0.7,0.5\},\{0.2,0.1,0.1\}) & (\{0.6,0.5,0.4\},\{0.3,0.3,0.2\}) \\ (\{0.4,0.3,0.2\},\{0.3,0.2,0.2\}) & (\{0.7,0.3,0.2\},\{0.3,0.2,0.2\}) & (\{0.6,0.4,0.2\},\{0.3,0.3,0.3\}) \\ (\{0.4,0.4,0.2\},\{0.2,0.1,0.1\}) & (\{0.7,0.4,0.2\},\{0.2,0.1,0.1\}) & (\{0.3,0.3,0.2\},\{0.3,0.3,0.3\}) \\ (\{0.4,0.3,0.2\},\{0.6,0.6,0.4\}) & (\{0.4,0.3,0.2\},\{0.6,0.6,0.4\}) & (\{0.3,0.3,0.2\},\{0.6,0.6,0.4\}) \end{array}$$

$$\begin{array}{cc} y_4 & y_5 \\ (\{0.5,0.4,0.3\},\{0.5,0.3,0.2\}) & (\{0.4,0.2,0.1\},\{0.6,0.5,0.5\}) \\ (\{0.6,0.4,0.4\},\{0.2,0.2,0.2\}) & (\{0.4,0.4,0.3\},\{0.6,0.5,0.3\}) \\ (\{0.4,0.4,0.3\},\{0.5,0.5,0.4\}) & (\{0.8,0.8,0.7\},\{0.2,0.2,0.2\}) \\ (\{0.4,0.4,0.3\},\{0.4,0.4,0.3\}) & (\{0.6,0.5,0.3\},\{0.4,0.4,0.3\}) \\ (\{0.5,0.4,0.3\},\{0.4,0.3,0.2\}) & (\{0.4,0.2,0.2\},\{0.6,0.5,0.5\}) \\ (\{0.8,0.7,0.7\},\{0.2,0.1,0.0\}) & (\{0.4,0.4,0.3\},\{0.6,0.5,0.3\}) \\ (\{0.4,0.4,0.3\},\{0.5,0.5,0.4\}) & (\{0.8,0.8,0.7\},\{0.1,0.1,0.0\}) \\ (\{0.6,0.5,0.3\},\{0.4,0.4,0.3\}) & (\{0.6,0.5,0.3\},\{0.4,0.4,0.3\}) \end{array},$$

**Step 2** By Equation (5), the utility values of all the $\mathbb{A}_k(k = 1,2,3,4)$ with respect to each $x_i(i = 1,2,\cdots,5)$ are obtained as $E_{\mathbb{A}_k} = (E_{\mathbb{A}_k,x_1}(\lambda_1), E_{\mathbb{A}_k,x_2}(\lambda_2), E_{\mathbb{A}_k,x_3}(\lambda_3), E_{\mathbb{A}_k,x_4}(\lambda_4), \cdots, E_{\mathbb{A}_k,x_5}(\lambda_5))$. For any given $k = 1,2,3,4$, $i = 1,2,3,4,5$, take $\lambda = [0, 0.01, 1]$, a set of values of $E_{\mathbb{A}_k,x_i}(\lambda)$ can be obtained. The sketch Maps of them are shown in Figure 1.

**Step 3** By Figure 1, one can get the following conclusions. $(i)$ For any $\lambda \in [0,1]$, $\mathbb{A}_1$ is sustaining from the disease "malaria $(x_2)$", $\mathbb{A}_2$ is sustaining from the disease "stomach problem $(x_4)$", and $\mathbb{A}_4$ is also sustaining from the disease "chest problem $(x_5)$". $(ii)$ For any $\lambda \in [0.05,1]$, $\mathbb{A}_3$ is sustaining from the disease "chest problem $(x_5)$", and for any $\lambda \in [0,0.05]$, it gets that $\mathbb{A}_3$ is sustaining from the disease "stomach problem$(x_4)$", and therefore, $\mathbb{A}_3$ needs some more high-technology inspections.




1-1 Function curves of $E_{\mathbb{A}_1,x_i}(\lambda)(i = 1,2,\cdots,5)$ 1-2 Function curves of $E_{\mathbb{A}_2,x_i}(\lambda)(i = 1,2,\cdots,5)$

1-3 Function curves of $E_{\mathbb{A}_3,x_i}(\lambda)(i = 1,2,\cdots,5)$ 1-4 Function curves of $E_{\mathbb{A}_4,x_i}(\lambda)(i = 1,2,\cdots,5)$

**Figure 1.** Function curves of $E_{\mathbb{A}_k,x_i}(\lambda)(k = 1,2,3,4; i = 1,2,\cdots,5)$.

It is noteworthy that [19] could solve this problem too, and the results were as follows: Patient $\mathbb{A}_1$ is suffering from the disease "typhoid" ($x_3$); Patient $\mathbb{A}_2$ is suffering from the disease "stomach problem" ($x_4$); patient $\mathbb{A}_3$ is suffering from the disease "chest problem" ($x_5$); patient $\mathbb{A}_4$ is also suffering from the disease "chest problem" ($x_5$).

The comparison shows that the pattern recognition results obtained by [19] and this study were basically the same, and they all could be provided to decision makers to support them.

### *4.2. Example 2*

The following studied example is adopted from Tian et al. [31], where a residential quarter was to be built following open management policies. For convenience, the studied quarter is denoted as $\mathbb{A}_0$. By investigation, our team finds that there were four types of layout which meet the requirements of the owners of the studied quarter. For convenience, they are denoted as $x_1$, $x_2$, $x_3$, $x_4$. Residents had certain routes to leave and back under certain residential area layout type, and the routes decided travel time, speed, distance and possible conflicts of their trips [32,33]. The routes in the open residential area were different from those in the gated residential area. Different routes brought out variation in travel time, speed, distance and conflicts, and the variation degree could be measured by delay time ($f_1$), traffic safety index ($f_2$), travel speed ($f_3$) and trip distance ($f_4$) [34,35]. Then, the four types of residential area layout were considered with four traffic impact indexes according to local conditions, and the dual hesitant fuzzy information of the four layout types under each index were obtained using the expert evaluation method, which was as follows.

$$R(x_i, y_j) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \left( \begin{array}{c} y_1 \\ \langle\{0.5,0.4,0.4,0.4,0.3\},\{0.4,0.4,0.4,0.3,0.3\}\rangle \\ \langle\{0.7,0.6,0.6,0.5,0.5\},\{0.3,0.3,0.2,0.2,0.2\}\rangle \\ \langle\{0.7,0.7,0.7,0.7,0.6\},\{0.3,0.2,0.1,0.1,0.1\}\rangle \\ \langle\{0.6,0.5,0.5,0.4,0.3\},\{0.4,0.4,0.3,0.3,0.2\}\rangle \\ y_2 \\ \langle\{0.5,0.5,0.4,0.4,0.2\},\{0.5,0.4,0.4,0.3,0.3\}\rangle \\ \langle\{0.7,0.7,0.7,0.6,0.6\},\{0.3,0.3,0.2,0.2,0.1\}\rangle \\ \langle\{0.6,0.5,0.5,0.4,0.3\},\{0.4,0.4,0.3,0.2,0.2\}\rangle \\ \langle\{0.6,0.5,0.5,0.5,0.5\},\{0.4,0.4,0.3,0.3,0.2\}\rangle \\ y_3 \\ \langle\{0.5,0.5,0.5,0.5,0.3\},\{0.5,0.4,0.4,0.4,0.2\}\rangle \\ \langle\{0.7,0.7,0.7,0.6,0.6\},\{0.3,0.2,0.2,0.1,0.1\}\rangle \\ \langle\{0.7,0.6,0.6,0.6,0.6\},\{0.3,0.2,0.2,0.2,0.1\}\rangle \\ \langle\{0.6,0.5,0.4,0.3,0.3\},\{0.4,0.4,0.3,0.3,0.2\}\rangle \\ y_4 \\ \langle\{0.5,0.5,0.4,0.3,0.2\},\{0.5,0.5,0.4,0.4,0.4\}\rangle \\ \langle\{0.7,0.6,0.5,0.5,0.4\},\{0.3,0.3,0.3,0.2,0.2\}\rangle \\ \langle\{0.6,0.6,0.6,0.5,0.5\},\{0.4,0.3,0.3,0.2,0.1\}\rangle \\ \langle\{0.5,0.4,0.4,0.2,0.2\},\{0.5,0.5,0.5,0.4,0.2\}\rangle \end{array} \right).$$

Meanwhile, since the owners of the studied residential quarter $\mathbb{A}_0$ always had more independence and given the special demand for the surrounding environment, an SP survey on owners was conducted and their fuzzy demand information was collected, and the results are denoted as

$$\begin{aligned} \mathbb{A}_0 = \ & \{\langle f_1, \{0.7,0.6,0.5\}, \{0.3,0.2,0.1\}\rangle,\ \langle f_2, \{0.7,0.6,0.6\}, \{0.3,0.3,0.2\}\rangle, \\ & \langle f_3, \{0.6,0.6,0.5\}, \{0.4,0.4,0.2\}\rangle,\ \langle f_4, \{0.5,0.5,0.4\}, \{0.2,0.2,0.1\}\rangle\}. \end{aligned}$$

How should one design the given residential under the aforementioned conditions? In the following, the problem was solved by using the novel dual hesitant fuzzy rough pattern recognition approach proposed in Section 3.

**Step 1** By Definition 5, the lower and upper approximations of each $\mathbb{A}_0$ with respect to $(U, V, \mathbb{R})$ are obtained as

$$\underline{\mathbb{R}}(\mathbb{A}_0) = \{\langle x_1, \{0.5, 0.5, 0.5, 0.5, 0.4\}, \{0.4, 0.4, 0.4, 0.3, 0.2\}\rangle,$$
$$\langle x_2, \{0.5, 0.5, 0.5, 0.5, 0.4\}, \{0.4, 0.4, 0.4, 0.4, 0.2\}\rangle,$$
$$\langle x_3, \{0.5, 0.5, 0.5, 0.5, 0.4\}, \{0.4, 0.4, 0.4, 0.4, 0.2\}\rangle,$$
$$\langle x_4, \{0.5, 0.5, 0.5, 0.5, 0.4\}, \{0.4, 0.4, 0.4, 0.3, 0.2\}\rangle\}$$
$$\overline{\mathbb{R}}(\mathbb{A}_0) = \{\langle x_1, \{0.5, 0.5, 0.5, 0.5, 0.3\}, \{0.4, 0.4, 0.4, 0.3, 0.2\}\rangle,$$
$$\langle x_2, \{0.7, 0.7, 0.7, 0.6, 0.6\}, \{0.3, 0.3, 0.3, 0.2, 0.2\}\rangle,$$
$$\langle x_3, \{0.7, 0.7, 0.7, 0.6, 0.5\}, \{0.3, 0.3, 0.3, 0.2, 0.1\}\rangle,$$
$$\langle x_4, \{0.6, 0.5, 0.5, 0.5, 0.5\}, \{0.4, 0.4, 0.3, 0.3, 0.2\}\rangle\}.$$

**Step 2** By Equation (4), one gets the utility values of $\mathbb{A}_0$ with respect to each $x_i (i = 1, 2, 3, 4)$ as

$$E_{\mathbb{A}_0,x_1}(\lambda) = \frac{1}{2} \cdot [1 + \frac{0.14}{2 - 0.82}]^{\lambda} \cdot [1 + \frac{0.12}{2 - 0.80}]^{1-\lambda},\ E_{\mathbb{A}_0,x_2}(\lambda) = \frac{1}{2} \cdot [1 + \frac{0.12}{2 - 0.84}]^{\lambda} \cdot [1 + \frac{0.40}{2 - 0.92}]^{1-\lambda},$$
$$E_{\mathbb{A}_0,x_3}(\lambda) = \frac{1}{2} \cdot [1 + \frac{0.12}{2 - 0.84}]^{\lambda} \cdot [1 + \frac{0.40}{2 - 0.88}]^{1-\lambda},\ E_{\mathbb{A}_0,x_4}(\lambda) = \frac{1}{2} \cdot [1 + \frac{0.14}{2 - 0.82}]^{\lambda} \cdot [1 + \frac{0.20}{2 - 0.84}]^{1-\lambda}.$$

**Step 3** For any given $\lambda \in [0, 1]$, $E_{\mathbb{A}_0,x_1}(\lambda)$, $E_{\mathbb{A}_0,x_2}(\lambda)$, $E_{\mathbb{A}_0,x_3}(\lambda)$ and $E_{\mathbb{A}_0,x_4}(\lambda)$ are compared, and the following conclusions are obtained: $(i)$ For any $\lambda \in [0, 0.9049]$, the studied residential area $\mathbb{A}_0$ should be built following the type $x_2$; $(ii)$ For any $\lambda \in [0.9049, 1]$, the studied residential area $\mathbb{A}_0$ should be built following the type $x_4$. For more details on $E_{\mathbb{A}_0,x_1}(\lambda)$, $E_{\mathbb{A}_0,x_2}(\lambda)$, $E_{\mathbb{A}_0,x_3}(\lambda)$ and $E_{\mathbb{A}_0,x_4}(\lambda)$ please see Figure 2, where Line 1 describes $E_{\mathbb{A}_0,x_1}(\lambda)$, Line 2 describes $E_{\mathbb{A}_0,x_2}(\lambda)$, Line 3 is describes $E_{\mathbb{A}_0,x_3}(\lambda)$, and Line 4 is describes $E_{\mathbb{A}_0,x_4}(\lambda)$.

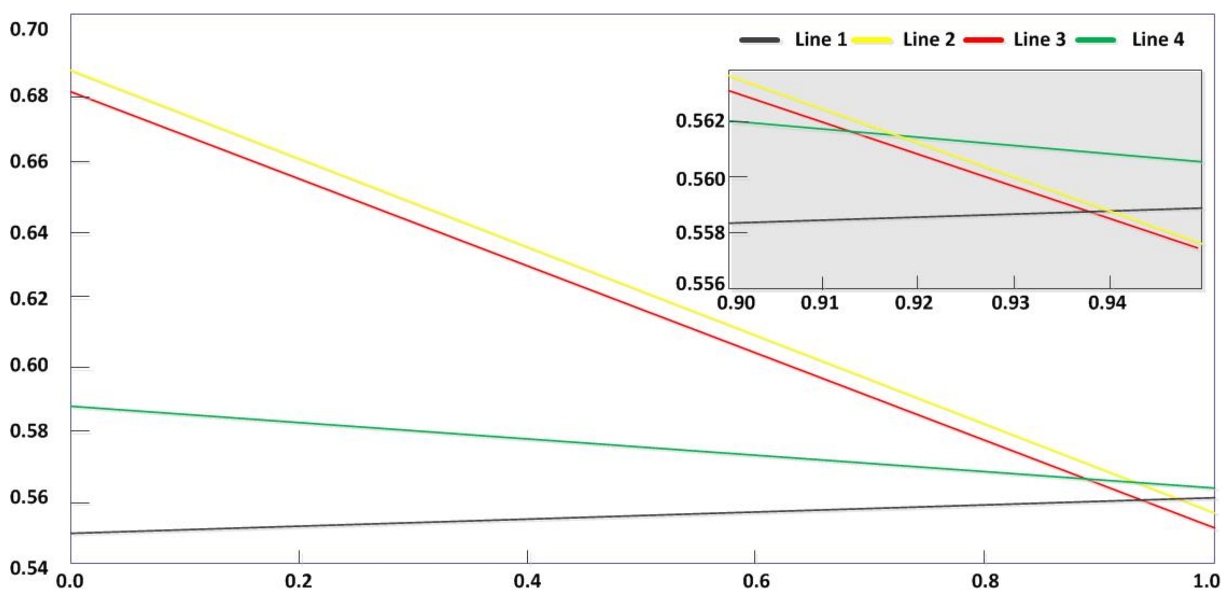



**Figure 2.** The sketch maps of $E_{\mathbb{A}_0,x_1}(\lambda)$, $E_{\mathbb{A}_0,x_2}(\lambda)$, $E_{\mathbb{A}_0,x_3}(\lambda)$ and $E_{\mathbb{A}_0,x_4}(\lambda)$.

It is noteworthy that the given model selection problem on this openness of residential area varied with the parameter $\lambda$. In this study, The bigger the parameter $\lambda$ was, the more conservative the decision-making process was; while the smaller the parameter $\lambda$ was, the more risky the decision-making process was.

It is also noteworthy that [19] caouldsolve this problem too, and their results was that $\mathbb{A}_0$ should be built following the type $x_2$. Since different principles are followed by our study and that of [19], they can complement each other when they are used.

## 5. Conclusions

In this study, dual hesitant fuzzy rough information is explored from the perspective of utility analysis. The main innovation points of this study are as follows.

(1) A series of laws that utility function of dual hesitant fuzzy rough set(DHFRS) should satisfy are proposed.

(2) By inductive and comparative studies, this study points out that the classical dual hesitant fuzzy rough pattern recognition approach, which is based on correlation coefficient theory, is suitable to deal with dynamic data in an exploratory data analysis situation, while the newly proposed one is suitable to deal with static data in a confirmatory data analysis situation.

(3) A novel utility function on DHFRSs is proposed. The main characteristics of the proposed utility function are that it has a parameter which is determined by decision-makers according to their experiences.

(4) Based on utility theory, a dual hesitant fuzzy rough pattern recognition method is proposed.

Just like the correlation coefficient theory, utility theory on DHFRSs also has its limitations. For example, it is not suitable to deal with dynamic data in exploratory data analysis situation. Therefore, to deal with dual hesitant fuzzy rough information scientifically, the decision makers should make their decisions according to the characteristics of the problem on hand. Meanwhile, the proposed utility function can not only be used for DHFRSs, they can also be used for dual hesitant pythagorean fuzzy sets [36]. In the future we will study the problem.

**Author Contributions:** Conceptualization, F.Z.; Data curation, Z.W., J.S., W.W. and X.T.; Investigation, Z.W. and X.T.; Methodology, F.Z. and J.S.; Project administration, F.Z.; Writing—original draft, J.S. and W.W.; Writing—review & editing, J.S. and W.W.

**Funding:** The work was supported in part by the project of National Natural Science Fund of China (No. 51508319), the research program of "the National Special Authorized Social Science Fund of China (No. 07@ZH005), the Nature and Science Fund from Zhejiang Province Ministry of Education (No. Y201327642)".

**Conflicts of Interest:** The authors declare no conflict of interest.

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