An implicit modified Lax-Wendroff scheme for irregular 2D space regions

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Abstract

Nowadays, there are many modelling problems which involve the shallow water equations for irregular domains. One of the most important blocks for solving these equations is the advection equation. In this paper, we present the implementation of an implicit modified Lax-Wendroff scheme in order to approximate the solution of the advection equation in some irregular domains in the plane, using a general finite difference method on structured convex grids.

Keywords: Finite difference, Structured grid, Advection equation, Lax-Wendroff

1. Introduction

Let us consider the problem of obtaining the finite difference approximation to the solution to the advection problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \qquad \Omega \times [0, T], \quad a, b \in I\!R,$$

$$u(x, y, 0) = g(x, y),$$

$$u(x, y, t)|_{S_1} = h(x, y, t),$$

where Ω is a simply connected planar domain and $\partial\Omega$ is a positively oriented Jordan polygon, such that $\partial\Omega = S_1 \cup S_2$ and S_1 and S_2 are connected sets (Fig. 1). In this paper, the goal is to approximate the solution u in non-rectangular and non-symmetrical regions Ω, where the classical stability analysis is not longer applicable. It is important to emphasize that for this kind of regions is necessary to generate convex grids and we do it by minimizing an appropriate functional; for instance, the harmonic and area functionals [1, 2] implemented in the UNAMALLA software [4], which can be used to mesh a wide variety of simply connected domains in the plane.

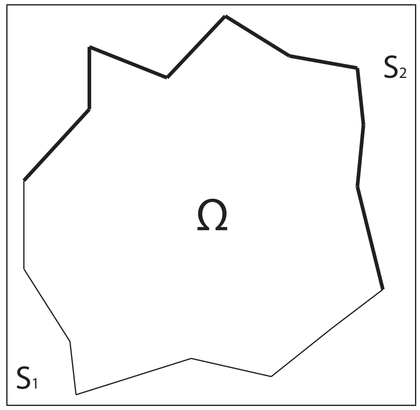

Figure 1: An example of an Ω domain with $\partial\Omega = S_1 \cup S_2$

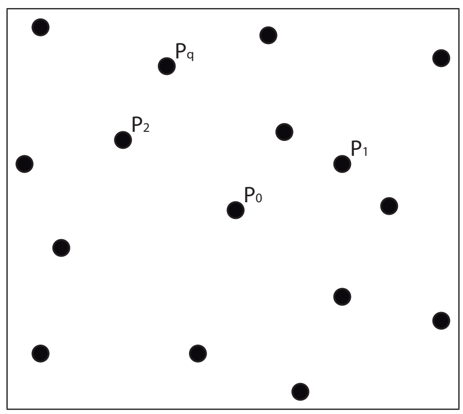

Figure 2: Arbitrary distribution of p_0 and its neighbors.

2. Proposed Lax-Wendroff scheme

To approximate the second order linear operator

$$Lu = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu \tag{1}$$

at the point p_0 using approximations to the values of u at some neighbors $p_1, \ldots, p_q$ of p_0 (fig. 2), we use a finite difference scheme in p_0, which can be written as the linear combination

$$L_0 = \Gamma_0 u(p_0) + \Gamma_1 u(p_1) + ... + \Gamma_q u(p_q). \tag{2}$$

A scheme is consistent if

$$[Lu]_{p_0} - L_0 \to 0$$

as $p_1, \ldots, p_q \to p_0$ [3]. Using the first six terms of the Taylor expansion of the previous expression, up to second order, the consistency condition yields the matrix system

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ 0 & \Delta x_1 & ... & \Delta x_q \\ 0 & \Delta y_1 & ... & \Delta y_q \\ 0 & (\Delta x_1)^2 & ... & (\Delta x_q)^2 \\ 0 & \Delta x_1 \Delta y_1 & ... & \Delta x_q \Delta y_q \\ 0 & (\Delta y_1)^2 & ... & (\Delta y_q)^2 \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ . \\ . \\ . \\ \Gamma_q \end{pmatrix} = \begin{pmatrix} F(p_0) \\ D(p_0) \\ E(p_0) \\ 2A(p_0) \\ B(p_0) \\ 2C(p_0) \end{pmatrix}, \tag{3}$$

where Δx and Δy are the horizontal and vertical space steps respectively.

One must note that the system of equations (3) has 6 equations and $q+1$ unknowns, so it is often not well-determined and has to be solved using a least square approach. This can be done in a non-iterative way by considering only the last 5 equations of the system (3) in a first step

$$\begin{pmatrix} \Delta x_1 & ... & \Delta x_q \\ \Delta y_1 & ... & \Delta y_q \\ (\Delta x_1)^2 & ... & (\Delta x_q)^2 \\ \Delta x_1 \Delta y_1 & ... & \Delta x_q \Delta y_q \\ (\Delta y_1)^2 & ... & (\Delta y_q)^2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ . \\ . \\ . \\ \Gamma_q \end{pmatrix} = \begin{pmatrix} D(p_0) \\ E(p_0) \\ 2A(p_0) \\ B(p_0) \\ 2C(p_0) \end{pmatrix} \tag{4}$$

and calculating the Cholesky factorization of its normal equations to obtain the $\Gamma_1, \ldots, \Gamma_q$ values. After that, in a second step, in order to determine the Γ_0 value, the first equation of (3)

$$\Gamma_0 = -\Gamma_1 ... - \Gamma_q + F(p_0) \tag{5}$$

is used.
In this way, the necessary set of coefficients required to define the scheme in (2) is easily calculated (See [7] for further details). To apply this approximation to the Lax-Wendroff scheme for the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0, \tag{6}$$

first we use the chain rule and we obtain

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + 2ab\frac{\partial^2 u}{\partial x \partial y} + b^2\frac{\partial^2 u}{\partial y^2};$$

finally, we substitute these partial derivatives into the Taylor expansion

$$u(x, t+\Delta t) = u(x,t) + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(\Delta t)^2 + ...$$

which yields (see [8])

$$\begin{aligned} & u(x, t+\Delta t) = u(x,t) \\ + & \left[-\Delta t\left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right) + \frac{(\Delta t)^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2} + 2ab\frac{\partial^2 u}{\partial x \partial y} + b^2\frac{\partial^2 u}{\partial y^2}\right)\right] + ... \end{aligned}$$

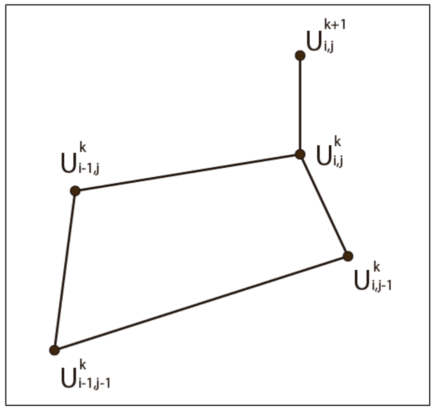

Figure 3: Upwind patch

One must note that the scheme defined by equations (4) and (5) can be applied to the linear operator

$$-\Delta t\left(a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}\right)+\frac{(\Delta t)^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2}+2ab\frac{\partial^2 u}{\partial x\partial y}+b^2\frac{\partial^2 u}{\partial y^2}\right).$$

The obtained coefficients Γ define the modified Lax-Wendroff scheme

$$u_0^{k+1}=u_0^k+\sum_{l=0}^{q}\Gamma_l u_l^k, \tag{7}$$

where k represents the time level, $u_i^k=u(p_i,t_k)$ and $t_k=t_0+k\Delta t$.
Scheme (7) can be used both in structured grids and non-structured grids. In this paper we are interested in the first ones to take advantage of the logical structure of the indices of the grid points in $G=\{p_{i,j}|1\leq i\leq m, 1\leq j\leq n\}$, thus we can use an upwind-like stencil (fig. 3) to obtain at the every interior grid node $p_{i,j}$

$$u_{i,j}^{k+1}=u_{i,j}^k+\Gamma_0 u_{i,j}^k+\Gamma_1 u_{i-1,j}^k+\Gamma_2 u_{i,j-1}^k+\Gamma_3 u_{i-1,j-1}^k, \tag{8}$$

here, $u_{i,j}^k$ is the approximation of u at the grid point i,j and the time level k, where $1<i\leq m$, $1<j\leq n$, $0<k$.
The scheme defined by (8) is an explicit Lax-Wendroff scheme for the advection equation. Several examples of the use of this scheme were presented in [6], and different tests have shown that there is a strong dependence on the geometry of the domain to obtain stability, and for very irregular regions the scheme is conditionally stable. This is the main motivation to look for a Crank-Nicolson-like scheme.
Extending the scheme defined in (8) and adding a λ parameter to involve 2 different time levels we obtain

$$u_0^{k+1}=u_0^k+\lambda\left[\sum_{l=0}^{q}\Gamma_l u_l^k\right]+(1-\lambda)\left[\sum_{l=0}^{q}\Gamma_l u_l^{k+1}\right]. \tag{9}$$

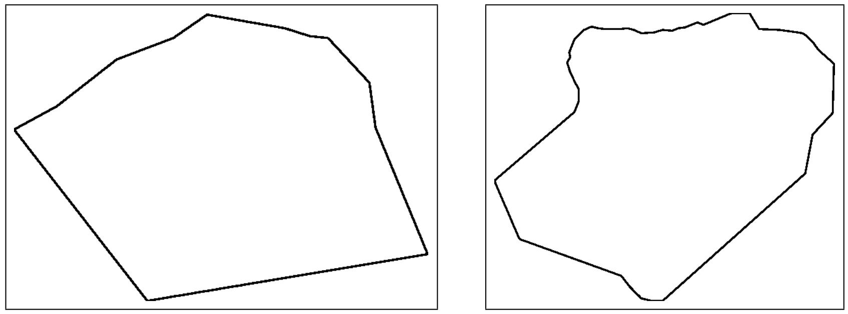

Figure 4: Selected regions for the numerical tests. CAB and HA.

Next, applying the same stencil idea used in the last scheme, we define an upwind patch (fig. 3) to get

$$\begin{aligned} u_{i,j}^{k+1} &= u_{i,j}^{k} + \lambda[\Gamma_0 u_{i,j}^{k} + \Gamma_1 u_{i-1,j}^{k} + \Gamma_2 u_{i,j-1}^{k} + \Gamma_3 u_{i-1,j-1}^{k}] + \\ & \quad (1-\lambda)[\Gamma_0 u_{i,j}^{k+1} + \Gamma_1 u_{i-1,j}^{k+1} + \Gamma_2 u_{i,j-1}^{k+1} + \Gamma_3 u_{i-1,j-1}^{k+1}] \end{aligned}$$

which, in aims of simplicity, can rewriten as

$$u_{i,j}^{k+1} = [\eta + \beta u_{i-1,j}^{k+1} + \gamma u_{i-1,j-1}^{k+1} + \delta u_{i,j-1}^{k+1}]/\alpha \tag{10}$$

where

$$\eta = \lambda[\Gamma_1 u_{i-1,j}^{k} + \Gamma_2 u_{i-1,j-1}^{k} + \Gamma_3 u_{i,j-1}^{k}] + u_{i,j}^{k}(1 + \lambda\Gamma_0)$$

and

$$\begin{aligned} \alpha &= 1 + (\lambda - 1)\Gamma_0 \\ \beta &= (1-\lambda)\Gamma_1 \\ \gamma &= (1-\lambda)\Gamma_2 \\ \delta &= (1-\lambda)\Gamma_3. \end{aligned}$$

3. Numerical tests

For the numerical tests we have selected four different non-rectangular and non-symmetrical regions; they will be denoted as CAB, HA, MI and TATUS (Figures 4 and 5). All of them where scaled and shifted to lie in $[0,1] \times [0,1]$. For these regions grids with 41 point per side where generated in UNAMALLA [4] (Figures 6 and 7), by minimizing the functional

$$1/2(S_\omega(G) + L(G)).$$

The following functions were selected for the tests as the initial and boundary conditions in S_1

$$\text{CAB: } u(x,y,t) = 0.2e^{((-(x-0.5-0.1t)^2-(y-0.3-0.1t)^2)/.01)},$$

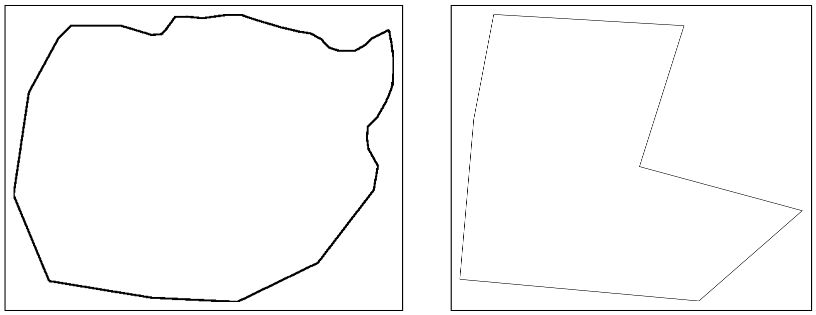

Figure 5: Selected regions for the numerical tests. MI and TATUS.

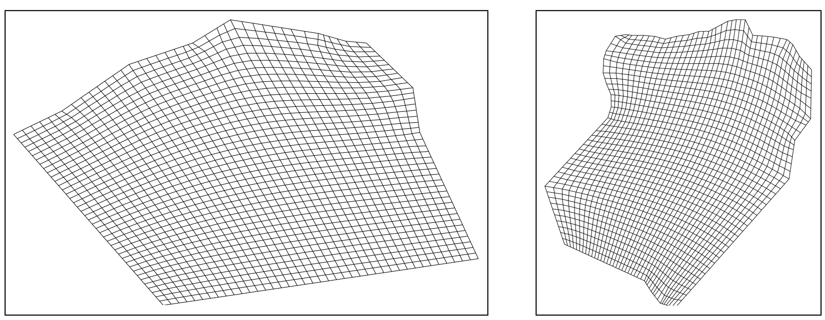

Figure 6: Grids with 41 points per side for CAB and HA.

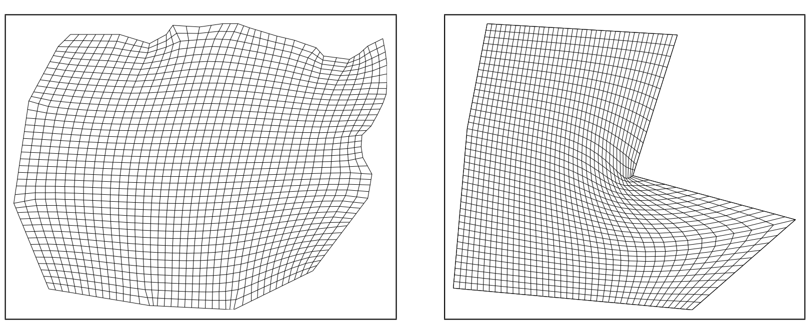

Figure 7: Grids with 41 points per side for MI and TATUS.

Time	$\lambda=0$	$\lambda=.25$	$\lambda=.5$	$\lambda=.75$
0.40	2.0829E-03	2.0818E-03	2.0688E-03	2.0557E-03
0.80	3.7533E-03	3.7515E-03	3.7302E-03	3.7088E-03
1.20	5.0482E-03	5.0463E-03	5.0199E-03	4.9935E-03
1.60	5.9022E-03	5.9010E-03	5.8720E-03	5.8428E-03
2.00	6.5273E-03	6.5268E-03	6.4951E-03	6.4633E-03

Table 1: Quadratic error for different λ values, CAB region.

Time	$\lambda=0$	$\lambda=.25$	$\lambda=.5$	$\lambda=.75$
0.40	2.3269E-03	2.3271E-03	8.5090E-03	2.3040E-03
0.80	4.2465E-03	4.2468E-03	4.2279E-03	4.2090E-03
1.20	5.8840E-03	5.8841E-03	5.8607E-03	5.8372E-03
1.60	7.3132E-03	7.3130E-03	7.2869E-03	7.2607E-03
2.00	8.5646E-03	8.5638E-03	8.5365E-03	8.5090E-03

Table 2: Quadratic error for different λ values, HA region.

$$\text{HA: } u(x,y,t) = 0.2e^{((-(x-0.45-0.1t)^2-(y-0.5-0.1t)^2)/.01)},$$

$$\text{MI: } u(x,y,t) = 0.2e^{((-(x-0.4-0.1t)^2-(y-0.25-0.1t)^2)/.01)},$$

$$\text{TATUS: } u(x,y,t) = 0.2e^{((-(x-0.15-0.1t)^2-(y-0.2-0.1t)^2)/.01)},$$

the time interval $[0,2]$ was uniformly subdivided in 2000 subintervals. Using these values, our Lax-Wendroff modified scheme was applied in the selected grids.

At the k-th time level, the values of the norm of the quadratic error can be computed as the grid function

$$\|e^k\|_2 = \sqrt{\sum_{i,j}(u_{i,j}^k - U_{i,j}^k)^2 \mathcal{A}_{i,j}} \quad ,$$

where $U_{i,j}^k$ y $u_{i,j}^k$ are the approximated and the exact values, respectively, of the solution computed at the i,j-th element, and $\mathcal{A}_{i,j}$ is the area of the polygon defined by $\{P_{i+1,j}, P_{i,j+1}, P_{i-1,j}, P_{i,j-1}\}$.

In tables (1), (2), (3) and (4) we present the quadratic errors at different time levels and λ values for the respective initial and boundary conditions.

Also, in order to show the computation cost of the method, in table (5) we present the average of the computational time cost for each region. All the test where realized in a Dell 14r with an Intel(R) Core(TM) i5-2410M Processor running at 2.30GHz, 4.00 GB of RAM memory, with Windows 7 Ultimate operative system.

Time	$\lambda = 0$	$\lambda = .25$	$\lambda = .5$	$\lambda = .75$
0.40	2.4336E-03	2.4325E-03	2.4223E-03	2.4122E-03
0.80	4.3139E-03	4.3130E-03	4.2958E-03	4.2786E-03
1.20	5.8041E-03	5.8043E-03	5.7817E-03	5.7591E-03
1.60	7.0281E-03	7.0296E-03	7.0028E-03	6.9759E-03
2.00	8.0712E-03	8.0739E-03	8.0437E-03	8.0134E-03

Table 3: Quadratic error for different λ values, MI region.

Time	$\lambda = 0$	$\lambda = .25$	$\lambda = .5$	$\lambda = .75$
0.40	1.2185E-03	4.9441E-03	4.9329E-03	1.2085E-03
0.80	2.3077E-03	2.3018E-03	2.2958E-03	2.2898E-03
1.20	3.2848E-03	3.2767E-03	3.2686E-03	3.2605E-03
1.60	4.1625E-03	4.1527E-03	4.1429E-03	4.1330E-03
2.00	4.9441E-03	4.9329E-03	4.9217E-03	4.9105E-03

Table 4: Quadratic error for different λ values, TATUS region.

Region	Average Time
CAB	5.545857s
HA	4.650228s
MI	4.974361s
TATUS	8.571698s

Table 5: Average of the computational time for each region with $\lambda = .5$.

4. Conclusions and future work

We can conclude that the proposed scheme is of very low computational cost and relatively easy to implement. However, as follows from the numerical tests, in the context of the discussed problem the geometry of the region is a very important and non trivial issue, and more experimental work is required in order to get a better understanding of the effects of the complex geometries on theses schemes.

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