

The role of Uncoupled Eulerian-Lagrangian approach in UNDEX investigations for ship-like structures

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ABSTRACT

Understanding the dynamic response of submerged structures under pressure loads is crucial for naval shipbuilding and offshore engineering. Among these loads, underwater explosions (UNDEX) are particularly critical, directly affecting the safety and performance of structures such as submarines, drilling platforms, and military vessels. Advancements in numerical methods have made simulations the primary tool for addressing UNDEX events. Numerical approaches for UNDEX typically employ coupled or decoupled techniques. Coupled methods resolve the fluid-structure interaction (FSI) problem by directly and simultaneously modelling the interaction between the explosion-induced pressure wave and the ship's dynamic response. Uncoupled (or decoupled) methods use a 2-step numerical procedure considering before the time-domain loads on a rigid structure and then their structural dynamic effect. Coupled models are highly accurate for UNDEX scenarios but time-consuming, while decoupled models offer computational efficiency and easier implementation but with less predictive accuracy due to the lack of consideration for FSI. Despite its importance, there is a lack of comprehensive studies comparing coupled and uncoupled approaches for UNDEX modelling. This work addresses this gap by evaluating the structural response of a floating ship-like structure under an UNDEX scenario using the two approaches. It employs a Coupled Eulerian-Lagrangian (CEL) method to account for FSI, and a two-step Uncoupled Eulerian-Lagrangian (UEL) approach for the decoupled analysis, excluding seabed or surface reflection effects. The study highlights the key results of both strategies, comparing their advantages, limitations, and applications in marine and naval engineering.

Keywords: underwater explosion; fluid-structure interaction; numerical simulation; ship-like structure; uncoupled approach; coupled approach.

1 INTRODUCTION

Comprehending the dynamic behaviour of submerged structures under pressure is crucial in naval and offshore engineering. Underwater explosions (UNDEX) are of significant concern as they impact the safety of submarines, drilling platforms, and combat ships.

A typical UNDEX event causes three damage mechanisms depending on the hull's distance [1–3]: initial shock waves with high velocity and pressure, low-frequency pressure waves from gas bubble pulsation, and high-speed water jets from bubble collapse striking the structure.

These mechanisms lead to highly complex phenomena involving fluid-structure interaction (FSI), significant deformations, fracturing, and material nonlinearity under high strain rates and temperatures [4]. Unlike air explosions, UNDEX-induced FSI triggers interconnected non-trivial effects such as cavitation, reflection, and absorption.

Three main approaches are employed to study UNDEX and its effects [5,6]: experimental testing, analytical modelling, and numerical simulations. These methods focus on understanding both the characteristics of UNDEX loading and the nonlinear dynamic responses of structures.

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Experimental investigations into UNDEX field are challenging due to the event's complex and hazardous dynamics [7,8]. High costs and logistical difficulties make full-scale testing prohibitive [6,9], and experimental data of scaled prototypes are rarely published, limiting insights advances into UNDEX phenomena [10].

Analytical models play a key role in providing theoretical insights into UNDEX effects but are constrained by simplifications and approximations that may overlook the full complexity of real-world events [11].

Advancements in numerical methods and computing power have made numerical simulations a powerful alternative for studying UNDEX on high-performance computers [12,13]. Software such as LS-DYNA, ABAQUS, ANSYS, and MSC Dytran are widely used to predict transient UNDEX loading and structural responses of marine and offshore structures [14,15].

Several numerical approaches are available in the literature to simulate UNDEX loads, and the main distinction lies in the approach to solve the FSI problem, which can be coupled or uncoupled. Coupled methodologies solve the interaction between the explosion-induced pressure wave and the ship hull's dynamic response within a single computational framework, enabling simultaneous solutions of both fluid and structural domains [16], considering FSI effects. In contrast, uncoupled (or decoupled) methodologies do not directly model the FSI but involve a two-step process, addressing the fluid and structural domains sequentially [17].

The most used strategy for modelling FSI in UNDEX problems is the Coupled Eulerian-Lagrangian (CEL) technique. The CEL approach combines the strengths of Eulerian and Lagrangian theories for modelling fluid and structure, respectively, and enabling their continuous interaction [18].

Despite extensive research on ship hull responses to UNDEX loading, coupled methods like CEL remain time-consuming due to their computational complexity [3]. For example, simulating a whole ship exposed to UNDEX can take days on high-performance computers, as naval platforms are large and require massive Eulerian volumes for water simulation. Additionally, managing the interaction between Eulerian and Lagrangian phases is computationally challenging [1].

To tackle the limitation of CEL technique, uncoupled methodologies can be adopted, even if they do not directly model the FSI. Uncoupled Eulerian-Lagrangian (UEL) is a common decoupled strategy in the literature, commonly applied in air blast simulations. The UEL approach follows a two-step process: first, it maps the blast pressure applied to the structure, and then evaluates the structural dynamic response, ignoring FSI effects. Although this strategy does not account for FSI, which is crucial in UNDEX cases, it has the significant advantage of providing results very quickly without the excessive computational cost of coupled analyses.

In the available literature there are no comparison between results obtained with coupled and uncoupled methodologies in UNDEX simulations, even if the possible usage of a computationally efficient methodology should be of paramount importance for the design of naval and offshore structures. Thus, the aim of this work is the comparison of CEL and UEL approaches for an UNDEX scenario and the evaluation of the potential of UEL method in providing quick results for UNDEX problem. A floating double-bottom structure subjected to near-field UNDEX is considered as a case study.

This work is organised as follows. Section 2 describes the numerical framework. Section 3 introduces the case study considered in this work. Section 4 discusses the results of the numerical simulations, gives insights into the comparison between CEL and UEL and proposing a way to effectively exploit UEL advantages. Finally, Section 5 draws out the conclusions from this work and presents possible future developments.

2 MODELLING APPROACHES

In this Section, the CEL and the UEL approaches used to simulate the UNDEX and its effects on the double-bottom structure under investigation are briefly described. This section is structured to provide a general overview of the two methods and their ability to describe underwater extreme loading conditions in a numerical framework.

2.1 Coupled Eulerian-Lagrangian strategy

The CEL approach integrates Eulerian and Lagrangian mesh formulations within a single simulation framework [17]. The Eulerian formulation is specifically designed to describe fluids and is characterized by fixed nodes in space and the use of advection algorithms to enable material flow through the mesh. Conversely,

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the Lagrangian formulation is primarily utilized to model solid bodies with defined shapes and the mesh can deform under pressure loads, making them ideal for capturing the deformation of structural components. In software like MSC Dytran, also the detonation can be properly modelled, and the behaviour of the explosive material is typically modelled by employing the Jones-Wilkins-Lee (JWL) equation of state (EOS) described in Eq. (1):

$$P = A \cdot \left(1 - \frac{\omega}{R_1 \cdot V}\right) \cdot e^{-R_1 \cdot V} + B \cdot \left(1 - \frac{\omega}{R_2 \cdot V}\right) \cdot e^{-R_2 \cdot V} + \frac{\omega \cdot e}{V}, \quad (1)$$

where A, B, R_1, R_2 and ω are material constant obtained from experimental data and e is defined as the internal energy per unit mass [19].

The detonation of the underwater charge material produces a shock wave that propagates through the surrounding water medium, which must be explicitly modelled to ensure the accurate simulation of underwater blast wave propagation. Focusing specifically on explosions occurring in water, the surrounding medium can be represented using a polynomial EOS [20], that relates the pressure in the fluid to the acoustic condensation μ and the specific internal energy e . When $\mu > 0$ (compression):

$$P = a_1 \cdot \mu + a_2 \cdot \mu^2 + a_3 \cdot \mu^3 + (b_0 + b_1 \cdot \mu + b_2 \cdot \mu^2) \cdot \rho_0 \cdot e, \quad (2)$$

while for $\mu < 0$ (tension):

$$PP = a_1 \cdot \mu + (b_0 + b_1 \cdot \mu) \cdot \rho_0 \cdot e, \quad (3)$$

where P is the pressure, $\mu = \eta - 1$, $\eta = \rho/\rho_0$, ρ_0 is the reference density, ρ is the whole material density, a_1, a_2, a_3, b_0, b_1 and b_2 are eulerian fluid constants. Finally, e represents del specific internal energy per unit mass.

Also, the air above the free surface is modelled using a Eulerian domain in which the ideal gas model expressed by the following EOS is implemented [21]:

$$p = (\gamma - 1) \cdot \rho \cdot e, \quad (4)$$

where ρ is the air density, γ the heat capacities of the gas and e is the e specific internal energy of air.

When modelling the surrounding medium (water and air), flow-out boundary condition needs to be specified at the external surfaces of the model to avoid unwanted reflections.

In the present research, thermal effect is not considered due to the adiabaticity of the explosion phenomenon [19], and no failure model is adopted for simplicity since the explosive scenario was designed to do not exceed the material's breaking stresses.

The structural domain is modelled using the Lagrangian formulation. The deformation of the structure can be determined by solving the structural problem with suitable force and displacement boundary conditions.

A specific coupling algorithm is used to transfers loads from the Eulerian to the Lagrangian domain and vice versa, solving FSI simultaneously. This enables the inclusion of advanced and complex phenomena, such as the mutual influence between structure and fluid and the cavitation [1]. However, this also led to the main drawback of the CEL approach, a significant demand for computational resources.

2.2 Uncoupled Eulerian-Lagrangian strategy

The UEL approach employs a two-step process: i) evaluation of the pressure field due to UNDEX considering the structure as rigid; ii) evaluation of the structural dynamic response subjected to the calculated pressure load while neglecting FSI effects.

In the first stage of the UEL method, the generation and propagation of the blast wave in the surrounding medium are simulated using the Eulerian mesh formulation. This step focuses on mapping the pressure load acting on the target structure and the implemented EOS are the same as for the CEL approach (Eq. 1, 2, 3, 4). The second stage of the UEL method involves using the pressure time history obtained in the first step as input for a pure Lagrangian analysis, considering the structure deformable.

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An important aspect of UEL analysis for submerged and floating structures lies in the challenge of accounting for the added mass effect, caused by the mass of water in contact with the structure and not intrinsically considered as in the CEL approach.

In practical naval applications, three main strategies can be adopted to account for this effect, with increasing precision: i) increment the density of the considered material (most simple and rough method); ii) employ numerical technique such as lumped masses connected to the wet surface with Multi-Point Constraint (MPC); iii) explicitly model the volume of water around the structure (*i.e.*, perform a CEL simulation to solve FSI but without the need to calculate the propagation of the shock wave). Added mass contribution should be considered within the 2°step of UEL strategy, where only the structure is present and subjected to the pressure field mapped in the 1°step onto the rigid target body.

To summarise, in Fig. 1 a schematic representation and comparison between CEL and UEL approaches is reported.

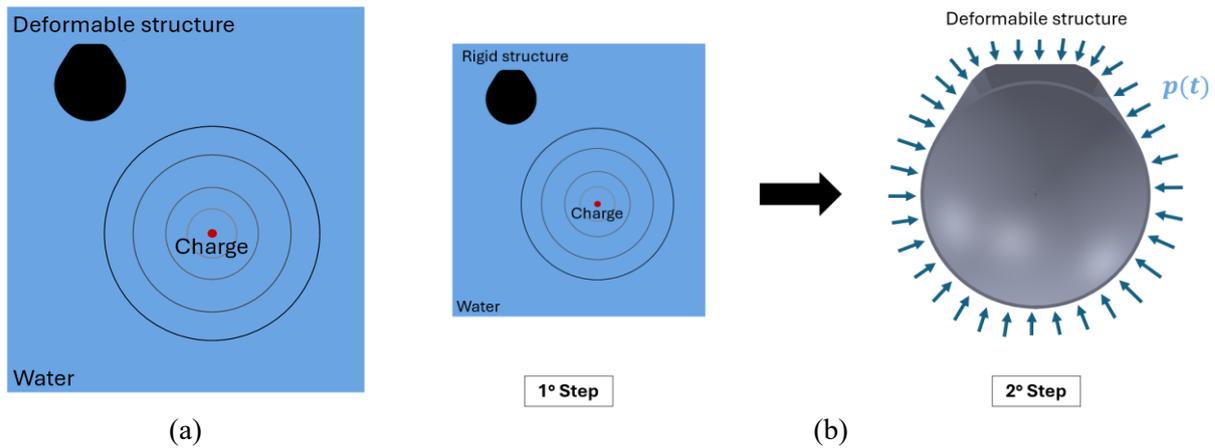


Figure 1: Scheme of modelling framework of a submarine section subjected to UNDEX and modelled with (a) CEL approach and (b) UEL methodology.

3 CASE STUDY

In this section the considered UNDEX scenario is illustrated and the numerical setup of the two approaches (CEL and UEL) is described with reference to the MSC Dytran software package.

3.1 UNDEX scenario and target structure

The considered scenario involves the detonation of a 0.18 kg submerged spherical TNT explosive charge with a diameter of 0.06 m. The charge is laced at a stand-off distance of 1 m from a floating double-bottom structure (Fig. 2b), as schematically shown in Fig. 2a. According to the classification of UNDEX phenomenon as in contact or non-contact, the proposed scenario belongs to the first one [2], and only the effects of the primary shock wave is considered.

The examined structure has dimensions of 1.75 m in length, 1.05 m in width, and 0.30 m in height. The distance between the girders is 0.35 m, which is also the spacing between the floors. Based on Archimedes' principle, the calculated draft T is 0.24 m. The thicknesses of the structural elements are reported in Tab. 1. The stiffeners are not modelled for simplicity, but this does not compromise the validity of the approach and of the comparative analysis between the different methodologies. The total weight of the structure is 310.9 kg, and the Q235 steel is used, as commonly employed in naval and marine constructions, with the mechanical characteristics reported in Tab. 2.

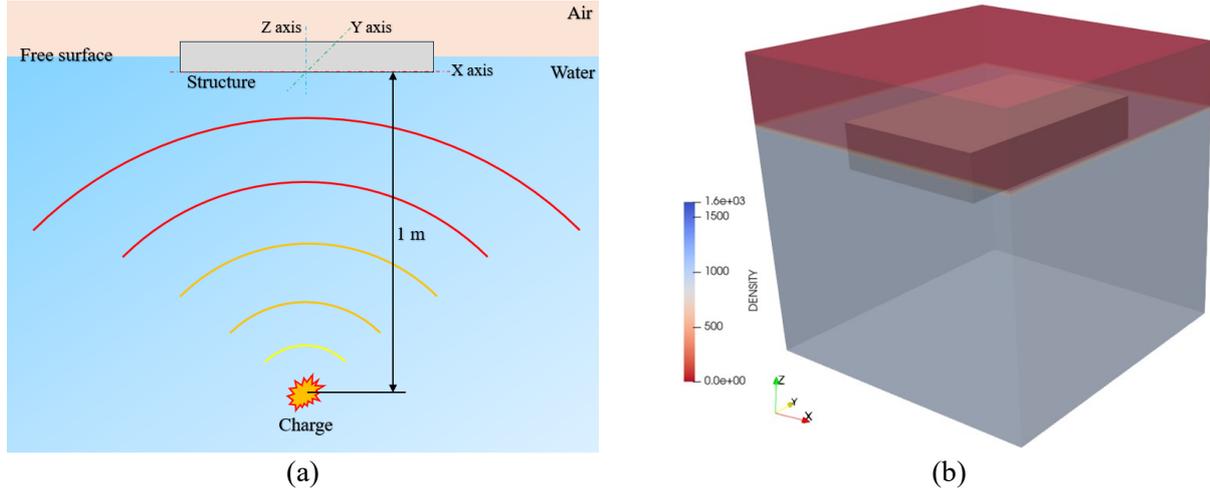


Figure 2: Features of the scenario considered: (a) relative position between structure and charge and (b) 3-D numerical model with the structure inside the Eulerian domain, with the density pattern (kg/m^3).

Table 1: Dimensions of the grillage's structural elements.

Structural Element	Type	Thickness (mm)	Structural Element	Type	Thickness (mm)
Outer bottom	Plate	10	Girder	Plate	6
Inner bottom	Plate	7.5	Floor	Plate	6

3.2 Numerical modelling

The Eulerian domain presents the dimensions of $2.0 \text{ m} \cdot 2.0 \text{ m} \cdot 2.0 \text{ m}$ and contains the Lagrangian structure subjected to UNDEX as shown in Fig. 2b.

The Eulerian domain is characterized by two main portions: the water (1.35 m thick) and the air above the free surface (0.55 m thick), while the air inside the double-bottom is not considered.

The Eulerian domain is discretized through a hexahedral mesh with a base size of 15 mm, chosen to properly model the peak pressure value, according to Cole's formulation [8]. The Lagrangian domain (*i.e.*, the double-bottom structure) is discretized using a shell mesh with a base size of 0.02 m (Fig. 2b).

The entire model is subjected to a gravitational load, proper non-reflecting boundary conditions are imposed to the Eulerian domain, and the analyses are performed for a total time of 0.015 seconds. Hydrostatic pressure is set to properly initialize the Eulerian domain.

As regards the boundary conditions for the structure, the case with the structure constrained along the perimeter of the outer bottom surface is considered, to measure the exact difference between CEL and UEL under the same boundary conditions.

About the plasticity behaviour, the Johnson-Cook law is used (Eq. (5)) and the parameters considered are reported in Tab. 2.

$$\sigma_Y = \left[A + B \cdot (\varepsilon_e^{pl})^n \right] \cdot \left[1 + C \cdot \ln(\dot{\varepsilon}_e^{pl} / \dot{\varepsilon}_0) \right] \cdot \left[1 - \alpha^m \right], \quad (6)$$

where A is the yield stress, B is the strain hardening coefficient, n is the strain hardening index, C is the strain-rate parameter, ε_e^{pl} is the equivalent or effective plastic strain, α is the non-dimensional temperature, m is the thermal softening coefficient, $\dot{\varepsilon}_0$ is the reference strain rate (measured per unit time) and $\dot{\varepsilon}_e^{pl}$ is the equivalent plastic strain rate.

As already introduced in Section 2.1, all the Eulerian materials considered inside the model (charge, water and air) are modelled using a specific EOS and the specific parameters considered in the analyses are reported in Tab. 2.

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Table 2: Lagrangian (double-bottom structure) and Eulerian (water, air and charge) material properties used to assess the numerical model.

Material	MSC DYTRAN model	Input parameters
Naval mild steel	Elasto-plastic	$\rho = 7850[kg/m^3], E = 207 \cdot 10^9[Pa], \nu = 0.3[-],$ $A = 235 \cdot 10^9[Pa], B = 899.7 \cdot 10^9[Pa], m = 0[-],$ $n = 0.940[-], C = 0.0391[-], \dot{\epsilon}_0 = 1[-]$
Water	Polynomial EOS	$\rho = 1025[kg/m^3], K = 2.2 \cdot 10^9[Pa],$ $e = 83950[J/kg], a_1 = 2.314 \cdot 10^9[Pa],$ $a_2 = 6.561 \cdot 10^9[Pa], a_3 = 1.126 \cdot 10^9[Pa],$ $b_0 = 0.4934[-], b_1 = 1.3937, b_2 = 0[-]$
TNT charge	JWL EOS	$\rho = 1630[kg/m^3], \gamma = 1.4 [-], W = 1.56[kg],$ $e = 4.76 \cdot 10^6[kJ/kg], A = 3.7 \cdot 10^{11}[-],$ $B = 2.23 \cdot 10^9[-], R_1 = 4.15[-],$ $R_2 = 0.95[-], \omega = 0.3[-]$
Air above the free surface	Ideal gas EOS	$\rho = 1.225[kg/m^3], \gamma = 1.4 [-],$ $e = 2.14 \cdot 10^5[J/kg], R = 287[J/kg \cdot K]$

Referring to the CEL approach, the structure is considered deformable, thus, as previously explained, a plasticity model (Eq. 5) is adopted to proper model both elastic and plastic deformation. The setting parameters are reported in Tab 2. A proper coupling surface is generated between the fluid and the structure to proper model FSI.

For the UEL analyses, the first step is set as for the CEL framework, but with a rigid structure. The second step of UEL approach involves a dynamic structural response analysis subjected to the pressure loads previously calculated, with the structure meshed as above reported and without considering the Eulerian domain.

As regards the added mass contribution, in the presented work the following strategy has been considered: added mass considered in the 2° step setting up a numerical model with only the structure and a lumped mass connected to the wet surface of the structure with MPC. In this case, the added mass is commonly calculated as the 150% of the ship's displacement. The performed numerical analyses with the different constraining conditions and implemented approaches are summarized in Tab. 3.

Table 3: Types of numerical analysis performed with MSC Dytran suite.

N°	Analysis type
1	CEL with BCs
2	UEL 1°step with BCs
3	UEL 2°step with BCs considering added mass contribution

Each analysis is performed in a workstation with 128 GB of RAM and an Intel® Core™ i9-13900K 3.00 GHz CPU, considering 20 cores.

4 RESULTS AND DISCUSSION

To compare the results obtained from the different numerical analysis resumed in Tab. 3, the vertical displacement calculated in point A (Fig. 5a) at the final time step of the simulations (0.015 s) is chosen as reference. The vertical displacement contour plots for the cases of Tab. 3 are reported in Fig. 3. In this figure, a vertical section of the double-bottom structure positioned at its mid-length is considered for improved visualization. In Tab. 4, the recorded displacements in point A are reported.

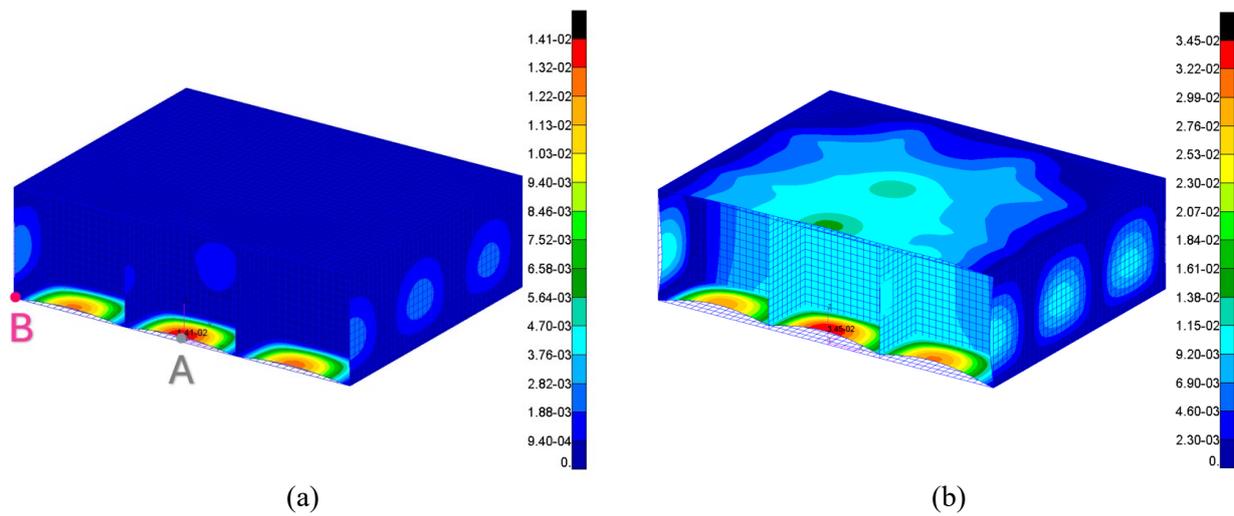


Figure 3: Vertical displacement plot contours (in m): (a) CEL with BCs and (b) UEL 2°step with BCs considering added mass contribution.

Table 4: Comparison between the vertical displacement recorded in Point A of the structure and analysis time in the different simulations of Tab. 3.

N°	Analysis type	Displacement point A (m)	Time
1	CEL with BCs	0.014	1 h 22 min
3	UEL 1°step + 2°step	0.034	1 h 12 min

The results reported in Fig. 3 and Tab. 4 indicate that the UEL procedure is more demanding for the structure from a structural point of view, as the recorded displacements are approximately 143% larger than in the CEL procedure with the contribution of the added mass. The discrepancy between the calculated deformations with CEL and UEL approaches can be attributed to FSI effects which are neglected by the UEL approach, leading to an overestimated pressure trend in the first step of the UEL approach. Overall, FSI tends to mitigate and reduce the load experienced by the structure subjected to UNDEX load.

Moving to the comparison of the time required for the analysis (Tab. 4), the CEL procedure involve a computational cost of about 33% higher than the UEL one.

5 CONCLUSIONS

This work compares the usage of CEL and UEL simulations in studying UNDEX and their effect on floating structures. This comparison is not yet presented in the available literature, despite the importance for naval and offshore engineering in the possibility to use UEL approaches and reduce the computational time of the simulations.

A rigorous qualitative and quantitative comparison of UEL and CEL approaches is performed in this work considering a simple double-bottom structure representative of a typical naval structure subjected to an UNDEX event.

The obtained results shows that UEL are effectively less cost demanding in terms of computational burden, but the obtained displacements are overestimated due to the non-proper evaluation of the pressure trend caused by the non-consideration of FSI in UEL, even if the added mass contribution is considered. Therefore, to obtain reliable results from UEL simulations, it is necessary to apply corrections to the results to bring them closer to the trustworthy coupled ones.

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