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A residual correction method based on finite calculus

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Abstract *In this paper, a residual correction method based upon an extension of the finite calculus concept is presented. The method is described and applied to the solution of a scalar convection-diffusion problem and the problem of elasticity at the incompressible or quasi-incompressible limit. The formulation permits the use of equal interpolation for displacements and pressure on linear triangles and tetrahedra as well as any low order element type. To add additional stability in the solution, pressure gradient corrections are introduced as suggested from developments of sub-scale methods. Numerical examples are included to demonstrate the performance of the method when applied to typical test problems.*

1. Introduction

In a series of papers, Oñate *et al.* have introduced the concept of finite increment calculus (or finite calculus) method for providing a stabilization which is very effective (Oñate, 1998, 2000; Oñate and Arraez, 2002; Oñate and Garcia, 2001; Oñate and Manzan, 1999; Oñate *et al.*, 1998, 2001). What we shall show in this paper is the problem of incompressible flow or elasticity which can reproduce very simply the ideas used in other stabilization processes already published or those recently in the process of introduction.

The concept of the finite calculus is to start the approximation from equations or a series of equations in which the size of the domain in which balance (or equilibrium) equations are established is finite and of the order of the finite spacing that will be used in the numerical solution. This physical concept is equivalent to the attempt of a discrete solution of a system of

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equations that is of the form in which not only the original residual appears but also the derivatives of this residual, multiplied by the size of the increment used in the numerical process (which we will call h), appears. Thus, a very general statement can be made purely algebraically.

When using a numerical process for obtaining solution to a differential equation, it can be written in a much abbreviated form as

$$r^{\alpha} = 0 \tag{1}$$

where α may stand for one if a single equation is considered or in a more general system lists the whole set of governing equations being used. In the finite calculus, we solve a modified system of differential equations written as

$$r^{\alpha} - \frac{1}{2} h_i \frac{\partial r^{\alpha}}{\partial x_i} = 0. \tag{2}$$

Equation (2) can be obtained for problems in mechanics by invoking the balance of fluxes (or equilibrium of forces) in a domain of finite size and retaining higher order terms in the Taylor series expansions for approximating the different balance terms.

Clearly, the system (2) if solved by a numerical process in which h_i are characteristic lengths which are constantly being reduced to make the solution more accurate, such as the sizes of finite element or a finite difference increments, the solution will converge to the exact one given by equation (1). As h_i tends to zero the solution will not be polluted by any other terms that frequently occur in procedures using least squares where spurious solutions sometimes occur.

It is at this stage important to observe that not all equations have to include additional terms involving h_i as some of these where no stability problems arise can simply be omitted without destroying the general procedure. Indeed, the intelligent use of such omission should be applied by the investigator when dealing with a particular set of equations.

In Section 2, equation (2) is extended to a more general form of a residual correction method. We then consider the application of the extended process to the convection-diffusion equation with a scalar variable and show that it mimics the well known characteristic based Galerkin (CBS) or streamline upwinding process (Zienkiewicz and Taylor, 2000a).

We next consider the application of the process to linear elasticity in which incompressibility or near (quasi) incompressibility occurs. The case with full incompressibility is identical to the problem of Stokes flow and thus, the description is applicable to either process. It is well known that unless special precautions are taken the (near) incompressibility term, in numerical approximations in which displacement (velocity) and pressure are independently approximated, can lead to stability difficulties, with the well known Babuška-Brezzi limitations applicable (Babuška, 1973; Brezzi, 1974) or alternatively the mixed patch test (Zienkiewicz and Taylor, 1997, 2000b; Zienkiewicz *et al.*, 1986) not being passed.

The application of a finite calculus process is introduced and shown to give alternative solution, including one which leads to the Brezzi-Pitkäranta (Brezzi and Pitkäranta, 1984) or Galerkin least squares (Hughes *et al.*, 1989) result of adding a Laplacian term as well as an *orthogonal sub-scale* form (Codina, 2000; Codina and Blasco, 2000). These are obtained with very little manipulation. Indeed, once we derive these forms it will be clear that they are equally applicable once certain coefficients are identified.

The more complex process of time stepping introduced by Chorin (1967, 1968) and later generalized by Zienkiewicz and Codina (1995) and Zienkiewicz *et al.* (1995) can obviously be developed but here we shall delay its introduction to a later work.

2. Theory

As noted earlier, the solution of problems posed as differential equations is here expressed in a residual form as

$$r^{\alpha} = 0; \quad \alpha = 1, 2, \ldots, N \tag{3}$$

where α denotes the particular equation we consider and N is the total number of equations defining the problem.

Generally, exact solutions to the equations are not available and numerical solutions are introduced to obtain approximate solutions. Here, we consider discrete methods in which the solution is defined at a set of points which are connected by a finite element method, a finite difference method, or any method in which some measure of their separation is defined by a parameter h for each point or element.

For this case, starting from the set of modified equation (2) and after some manipulations we can extend the residual form to incorporate additional terms which vanish for an exact solution but give beneficial properties to the discrete formulation (Oñate *et al.*, 2002). In the present work, we express the extended form as

$$\hat{r}^{\alpha} = r^{\alpha} + \gamma_i^{\alpha\beta} h \frac{\partial r^{\beta}}{\partial x_i} = 0 \tag{4}$$

where $i = 1, \ldots, d$, with d the space dimension of the problem being solved and a repeated index implies summation over the range of the index. Here, h is a single characteristic length and the $\gamma_i^{\alpha\beta}$ are parameters which must be dimensionally consistent with the equation being solved (α) and the modifier equation (β) added. For the case where $\alpha = \beta$ the parameter $\gamma_i^{\alpha\beta}$ is dimensionless.

The specification of a problem is completed by adding appropriate boundary (and for time dependent problems, initial) conditions which give a well posed form.

Here, we shall only consider a weak (weighted residual) form to construct approximate solutions. Accordingly, for equation (3) we use the solution form

$$G(R^{\alpha}, r^{\alpha}) = \int_{\Omega} R^{\alpha} r^{\alpha} \, \mathrm{d}\Omega = 0 \tag{5}$$

where R^{α} is an arbitrary conjugate variable to each equation. For equation (4) the solution form

$$G(\hat{R}^{\alpha}, \hat{r}^{\alpha}) = \int_{\Omega} \hat{R}^{\alpha} \hat{r}^{\alpha} \, \mathrm{d}\Omega = 0 \tag{6}$$

is used in which again $\hat{R}^{\alpha}$ is the conjugate variable to each equation. Often it is sufficient to let $R^{\alpha} = \hat{R}^{\alpha}$. Integration of some terms by parts in the above forms permits some boundary conditions to be explicitly added to the weak form (Neumann conditions) whereas some must be included explicitly in the trial functions used to construct the approximate solution (Dirichlet conditions). As shown by Oñate (1998), it is expedient in constructing solutions to equation (6) to express the flux type boundary conditions in extended form also.

To make the above notions clear we consider some example problems.

2.1 Scalar advection-diffusion equation

We first consider the scalar advection-diffusion equation given by

$$r = -a_j \frac{\partial \phi}{\partial x_j} + \frac{\partial}{\partial x_j}\left[k \frac{\partial \phi}{\partial x_j}\right] + Q = 0 \quad \text{in } \Omega \tag{7}$$

in which k is a positive diffusion parameter, a_j are components of a specified vector, Q is a given source term, ϕ is the dependent variable and Ω is the domain of the problem. The boundary conditions are given in terms of either Dirichlet,

$$\phi = \bar{\phi} \quad \text{on } \Gamma_1 \tag{8}$$

or Neumann,

$$q_n = -n_j k \frac{\partial \phi}{\partial x_j} = n_j q_j = \bar{q}_n \quad \text{on } \Gamma_2 \tag{9}$$

types where n_j are components of an outward pointing normal to the boundary and q_j are components of the *diffusive flux*.

2.1.1 Residual equation for solution. Here, there is only a single equation and we may write equation (4) in the simpler form

$$\hat{r} = r + \gamma_i h \frac{\partial r}{\partial x_i} = 0 \tag{10}$$

indicating that only d parameters need to be defined. A possible definition is to let

$$\gamma_i = \gamma \frac{a_i}{|a|} \tag{11}$$

where $|a| = \sqrt{a_i a_i}$ and γ is a parameter to be selected.

2.1.2 Weak form of residual equation for numerical solution. Since there is only one equation, the arbitrary conjugate variable is related to ϕ and here we let

$$\hat{R} = \delta\phi \tag{12}$$

and a weak form for the extended problem is given as

$$G(\delta\phi, \hat{r}) = \int_\Omega \delta\phi \left[r + \gamma h \frac{a_i}{|a|} \frac{\partial r}{\partial x_i} \right] \mathrm{d}\Omega = 0 \tag{13}$$

Introducing the definition of r and integrating the diffusion and γ terms by parts yields

$$\begin{aligned} G(\delta\phi, \hat{r}) = & \int_\Omega \left[\frac{\partial \delta\phi}{\partial x_j} k \frac{\partial \phi}{\partial x_j} - \delta\phi a_i \frac{\partial \phi}{\partial x_i} \right] \mathrm{d}\Omega - \int_\Omega \frac{\partial \delta\phi}{\partial x_i} \left[\gamma h \frac{a_i}{|a|} r \right] \mathrm{d}\Omega \\ & + \int_\Gamma \delta\phi \left[-n_i k \frac{\partial \phi}{\partial x_i} + n_i \gamma h \frac{a_i}{|a|} r \right] \mathrm{d}\Gamma = 0 \end{aligned} \tag{14}$$

In Oñate (1998), it is shown that the modified form of the Neumann boundary condition given by equation (14) can be anticipated *a priori* by using the concept of balance in a domain of finite size next to a boundary segment.The integrand of the boundary integral may be written as

$$-n_i k \frac{\partial \phi}{\partial x_i} + n_i \gamma h \frac{a_i}{|a|} r = q_n + \gamma h \frac{a_n}{|a|} r \tag{15}$$

where $a_n = n_i a_i$ is the component of the vector a that is normal to the boundary. Here it is convenient to split the boundary into Γ_1 and Γ_2 and to consider an extended form of the Neumann boundary condition given by Oñate (1998).

$$q_n + \gamma h \frac{a_n}{|a|} r = \bar{q} \quad \text{on } \Gamma_2 \tag{16}$$

together with $\delta\phi = 0$ on Γ_1. With these additions the weak form becomes

$$G(\delta\phi,\hat{r}) = \int_{\Omega}\left[\frac{\partial\,\delta\phi}{\partial x_i}k\frac{\partial\phi}{\partial x_i} - \delta\phi a_i\frac{\partial\phi}{\partial x_i}\right]\mathrm{d}\Omega - \int_{\Omega}\frac{\partial\,\delta\phi}{\partial x_i}\left[\gamma h\frac{a_i}{|a|}r\right]\mathrm{d}\Omega + \int_{\Gamma_2}\delta\phi\bar{q}_n\,\mathrm{d}\Gamma = 0 \tag{17}$$

It appears from the above construction that C_1 continuous interpolation is required in an approximate solution, whereas, in a normal Galerkin solution process, only C_0 interpolation is necessary. In the following, we shall assume only C_0 interpolation and ignore interface boundary terms (we could also argue that these terms will in fact tend to zero with h or based on a residual which is zero for an exact solution).

It is instructive at this point to consider the form of the γ term. To this end we note that

$$\frac{\partial\,\delta\phi}{\partial x_i}\frac{a_i}{|a|}r = \frac{\partial\,\delta\phi}{\partial x_i}\frac{a_i}{|a|}\left(-a_j\frac{\partial\phi}{\partial x_j} + \frac{\partial}{\partial x_j}\left[k\frac{\partial\phi}{\partial x_j}\right] + Q\right) \tag{18}$$

For the case of linear interpolation of ϕ and $\delta\phi$ on triangles in two dimensions or tetrahedra in three dimensions, the second derivative terms vanish within an element and the added second term reduces to

$$\frac{\partial\,\delta\phi}{\partial x_i}\frac{a_i}{|a|}r \approx \frac{\partial\,\delta\phi}{\partial x_i}\frac{a_i}{|a|}\left(-a_j\frac{\partial\phi}{\partial x_j} + Q\right) \tag{19}$$

the first term on the right side being a conventional *streamline* diffusion type term and the latter a modification to any source term Q. The above is a well known process which mimics both the CBS (Zienkiewicz and Codina, 1995; Zienkiewicz *et al.*, 1995) or the streamline upwind Petrov Galerkin (SUPG) (Brooks and Hughes, 1982) methods. Note that with the algebra used the value of γ must be negative to *add* diffusion to the discrete problem (Oñate, 1998). Of course the optimal choice of γ has to be decided.

Different alternatives for computing γ taking advantage of the form of the modified governing equation (10) are given by Oñate (1998, 2000), Oñate and Arraez (2002), Oñate and Garcia (2001), Oñate and Manzan (1999) and Oñate *et al.* (1998). In practice the value of γ generally follows that deduced in the mid 1970s (Christie *et al.*, 1976; Zienkiewicz *et al.*, 1977).

2.2 Linear elasticity: incompressible limit behavior

As a second example, we consider the problem of linear isotropic elasticity in which limits approach near incompressible behavior. As it is customary for this class of problems we first split the stress σ_{ij} as

$$\sigma_{ij} = s_{ij} + \delta_{ij} p \quad \text{where} \quad p = \frac{1}{3}\sigma_{ii} \tag{20}$$

In this form, s_{ij} define deviator stresses and p is the mean stress (pressure). Similarly, we split the strain ε_{ij} as

$$\varepsilon_{ij} = e_{ij} + \frac{1}{3}\delta_{ij}\varepsilon_{\mathrm{v}} \quad \text{where} \quad \varepsilon_{\mathrm{v}} = \varepsilon_{ii} \tag{21}$$

In this form, e_{ij} are deviator strains and ε_{v} is the (small strain) volume change. Using the above splits, we have the following systems:

Momentum equations. Ignoring effects of inertial loading, the balance of linear momentum describe the equilibrium behavior and are given by

$$\frac{\partial s_{ji}}{\partial x_j} + \frac{\partial p}{\partial x_i} + b_i = 0 \tag{22}$$

where b_i are body forces per unit of volume.

Strain-displacement equations. The standard form for the infinitesimal strain-displacement relations is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{23}$$

in which u_i are components of the displacement vector. The strain-displacement relations for the split form are given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij}\frac{2}{3}\frac{\partial u_k}{\partial x_k}\right) \tag{24}$$

for the deviator (isochoric) part and

$$\varepsilon_{\mathrm{v}} = \frac{\partial u_k}{\partial x_k} \tag{25}$$

for the volumetric part.

For an incompressible material, ε_{v} will be zero and for a near (quasi) incompressible material it will be very small relative to the deviatoric part.

Stress-strain (constitutive) equations. The stress-strain relations for an isotropic linearly elastic material may be written as

$$s_{ij} = 2Ge_{ij} \tag{26}$$

for the deviatoric part and

$$p = K\varepsilon_{\mathrm{v}} \tag{27}$$

for the volumetric part. The two parameters G and K denote the shear modulus and bulk modulus, respectively. Note that the volumetric equation also may be written as

$$\frac{1}{K}p = \varepsilon_{\mathrm{v}} \tag{28}$$

and thus incompressible behavior implies $K = \infty$. Near or quasi incompressible behavior is characterized by $K/G \gg 1$. Using the definitions

$$G = \frac{E}{2(1+\nu)} \quad \text{and} \quad K = \frac{E}{3(1-2\nu)} \tag{29}$$

near incompressible behavior is also given by $\nu \to 1/2$.

Boundary conditions. The boundary conditions are given by specified displacements

$$u_i = \bar{u}_i \quad \text{on } \Gamma_1 \tag{30}$$

and specified traction

$$t_i = n_j \sigma_{ji} = \bar{t}_i \quad \text{on } \Gamma_2 \tag{31}$$

2.2.1 Residual equations for solution. The system of residual equations which we propose to solve for three-dimensional applications are:

(1) the reduced set obtained by substituting the constitutive equation for deviatoric stresses into the strain-displacement equations and then into the momentum equation; and

(2) the volumetric strain-displacement equation substituted into the constitutive equation for pressure-volume effects.

Accordingly, we will solve the resulting set of equations given by:

$$\begin{aligned}
r^1 &= \frac{\partial s_{j1}(u_k)}{\partial x_j} + \frac{\partial p}{\partial x_1} + b_1 = 0 \\
r^2 &= \frac{\partial s_{j2}(u_k)}{\partial x_j} + \frac{\partial p}{\partial x_2} + b_2 = 0 \\
r^3 &= \frac{\partial s_{j3}(u_k)}{\partial x_j} + \frac{\partial p}{\partial x_3} + b_3 = 0 \\
r^4 &= \frac{\partial u_i}{\partial x_i} - \frac{1}{K}p = 0
\end{aligned} \tag{32}$$

In a two-dimensional plane strain or axisymmetric problem we ignore the third equation and adjust ranges of the indices i, j, $k = 1, 2$. Plane stress gives no difficulties for any range of properties and may be considered using a pure displacement approach. In the above, we use the notation

$$s_{ij}(u_k) = G\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \delta_{ij}\frac{2}{3}\frac{\partial u_k}{\partial x_k}\right] \tag{33}$$

to denote the effects of substitutions for the deviatoric behavior. We observe that equation (32) permits consideration of a solution when the bulk modulus K is infinite (the incompressible limit). As is well known, only elements which pass the Brezzi-Babuška (Babuška, 1973; Brezzi, 1974) or mixed patch test (Zienkiewicz *et al.*, 1986) lead to stable solutions and for low order elements these always involve different interpolation for u_k and p. However, the use of the modified residual equations will allow accurate solution to be achieved even when the displacements u_k and pressure p are approximated within finite elements with linear interpolation on triangles or tetrahedra.

Before starting the solution process, we note that the momentum residuals r^{α} for $\alpha = 1, 2, 3$ have dimensions F/L^3, whereas the constitutive residual r^4 is dimensionless. Thus, whenever we have a $\gamma_i^{\alpha\beta}$ in which $\alpha \neq \beta$ we must include terms which restore correct dimension to the equations. This has been noted previously for other approaches (Brezzi and Pitkäranta, 1984; Zienkiewicz and Taylor, 2000b).

A simple construction for the $\hat{r}^{\alpha}$ is to set all $\gamma_i^{\alpha\beta}$ to zero except for γ_1^{41}, γ_2^{42} and γ_3^{43}, which we will set as

$$\gamma_1^{41} = \gamma_2^{42} = \gamma_3^{43} = \gamma\frac{h}{2G} \tag{34}$$

in which γ is a single scalar variable to be chosen and the factor h/G is chosen for dimensional considerations (we use G as it remains bounded for all values of K computed for $0 < \nu \leq 0.5$). Thus, the residual equations we will use in our approximation are given by

$$\hat{r}^{\alpha} = r^{\alpha} \quad \text{for} \;\; \alpha = 1, 2, 3 \tag{35}$$

and

$$\hat{r}^4 = r^4 + \gamma\frac{h^2}{2G}\frac{\partial r^i}{\partial x_i} \tag{36}$$

The modified pressure-volumetric strain equation (36) can also be derived by manipulating the original finite calculus forms of the equilibrium and pressure constitutive equations given by equation (2) (Oñate *et al.*, 2002).

We expand the divergence of the first three residuals to give

$$\frac{\partial r^i}{\partial x_i} = \frac{\partial^2 s_{ji}}{\partial x_i \partial x_j} + \frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial b_i}{\partial x_i} \tag{37}$$

At this point there are different options we can pursue. The first is similar to that introduced by Brezzi and Pitkäranta (1984) and extended to elasticity by Zienkiewicz and Taylor (2000b). In this approach we note (for constant G and K) that

$$\begin{aligned} \frac{\partial^2 s_{ji}}{\partial x_i \partial x_j} &= G\left(\frac{\partial^3 u_j}{\partial x_i \partial x_i \partial x_j} + \frac{\partial^3 u_i}{\partial x_j \partial x_i \partial x_j} - \frac{2}{3}\delta_{ij}\frac{\partial^3 u_k}{\partial x_k \partial x_i \partial x_j}\right) \\ &= \frac{4G}{3}\frac{\partial^2}{\partial x_i \partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) \\ &= \frac{4G}{3K}\frac{\partial^2 p}{\partial x_i \partial x_i} \end{aligned} \tag{38}$$

which permits the derivative of the residual to be written as

$$\frac{\partial r^i}{\partial x_i} = \left(1 + \frac{4G}{3K}\right)\frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial b_i}{\partial x_i} \tag{39}$$

Thus, we have the fourth extended equation given by

$$\hat{r}^4 = r^4 + \gamma\frac{h^2}{2G}\left[\left(1 + \frac{4G}{3K}\right)\frac{\partial^2 p}{\partial x_i \partial x_i} + \frac{\partial b_i}{\partial x_i}\right] \tag{40}$$

An alternative approach to equation (40) represents the deviatoric and body force terms in the momentum equation by a new parameter set Π_i in which

$$\Pi_i = \frac{\partial s_{ji}(u_k)}{\partial x_j} + b_i = -\frac{\partial p}{\partial x_i} \tag{41}$$

To use this approach, we must introduce the additional set of residual equations

$$\begin{aligned} r^5 &= \frac{\partial p}{\partial x_1} + \Pi_1 = 0 \\ r^6 &= \frac{\partial p}{\partial x_2} + \Pi_2 = 0 \\ r^7 &= \frac{\partial p}{\partial x_3} + \Pi_3 = 0 \end{aligned} \tag{42}$$

The introduction of Π was deduced from an orthogonal sub-scale approach by Codina (2000) and Codina and Blasco (2000) and later employed by Chiumenti *et al.* (2002a, b) for elasticity and elasto-plasticity problems.

2.2.2 Weak form of residual equations for numerical solution. A Galerkin form to the equations may be constructed as

$$G(\hat{R}^\alpha, \hat{r}^\alpha) = -\int_\Omega R^i\, r^i \,\mathrm{d}\Omega + \int_\Omega \hat{R}^4 \hat{r}^4 \,\mathrm{d}\Omega = 0 \tag{43}$$

The first integral defines the statement on the momentum equation which we treat without modification and select

$$\hat{R}^i = \delta u_i \quad \text{for} \quad i = 1, \ldots, d. \tag{44}$$

The second integral is the constitutive equation for pressure-volume which we consider modified to the extended form and set

$$R^4 = \delta p. \tag{45}$$

For the present, we do not consider which extended form will be used, however, we note that introduction of Π_i will obviously require additional terms in equation (43).

After substitution of equation (32) and performing an integration by parts, the momentum terms become

$$\begin{aligned} G(\delta u_i, r^i) = & \int_\Omega \frac{1}{2}\left(\frac{\partial\, \delta u_i}{\partial x_j} + \frac{\partial\, \delta u_j}{\partial x_i}\right)[s_{ij}(u_k) + \delta_{ij} p] \,\mathrm{d}\Omega \\ & - \int_\Omega \delta u_i\, b_i \,\mathrm{d}\Omega - \int_\Gamma \delta u_i t_i \,\mathrm{d}\Gamma \end{aligned} \tag{46}$$

which we recognize as a standard Galerkin or virtual work form of the momentum equations.

Treating the second integral, we have the form

$$G(\delta p, \hat{r}^4) = \int_\Omega \delta p \left[\frac{\partial u_i}{\partial x_i} - \frac{1}{K} p\right] \mathrm{d}\Omega + \int_\Omega \frac{\gamma h^2}{2G}\, \delta p \,\frac{\partial r^i}{\partial x^i} \,\mathrm{d}\Omega \tag{47}$$

At this point we may either substitute using equation (39) or introduce the sub-scale terms Π_i. Introducing the Brezzi and Pitkäranta form and integrating by parts we obtain

$$
\begin{aligned}
G(\delta p, \hat{r}^4) = & \int_\Omega \delta p \left[\frac{\partial u_i}{\partial x_i} - \frac{1}{K} p \right] \mathrm{d}\Omega \\
& - \int_\Omega \frac{\gamma h^2}{2G} \frac{\partial \delta p}{\partial x_i} \left[\left(1 + \frac{4G}{3K} \right) \frac{\partial p}{\partial x_i} + b_i \right] \mathrm{d}\Omega \\
& + \int_\Gamma \delta p \frac{\gamma h^2}{2G} n_i \left[\left(1 + \frac{4G}{3K} \right) \frac{\partial p}{\partial x_i} + b_i \right] \mathrm{d}\Gamma
\end{aligned} \tag{48}
$$

Usually the last term is ignored in numerical computations. This act must be based on arguments concerning h or r_i tending to zero as, in general, the multiplier term is not zero. However, some controversy surrounds the results obtained by ignoring the term (Pierre, 1988, 1995).

An alternative is to introduce the sub-scale approximation defining Π_i and directly integrate the residual term to obtain

$$
\begin{aligned}
G(\delta p, \hat{r}^4) = & \int_\Omega \delta p \left[\frac{\partial u_i}{\partial x_i} - \frac{1}{K} p \right] \mathrm{d}\Omega - \int_\Omega \frac{\gamma h^2}{2G} \frac{\partial \delta p}{\delta x_i} \left[\frac{\partial p}{\partial x_i} + \Pi_i \right] \mathrm{d}\Omega \\
& + \int_\Gamma \delta p \frac{\gamma h^2}{2G} n_i \left[\frac{\partial p}{\partial x_i} + \Pi_i \right] \mathrm{d}\Gamma
\end{aligned} \tag{49}
$$

In this form, the boundary integral again involves h and a residual which permits it to be dropped without recourse to material behavior. Indeed, we may introduce any constitutive equations for other material behavior (e.g. visco-elasticity, elasto-plasticity, etc.) without altering the definition of Π_i.

Combining the above two terms and introducing a split on the boundary for displacement and traction types we can write the final weak form for the solution of the elasticity problem as

$$
\begin{aligned}
G(\delta u, \delta p, \hat{r}^\alpha) = & \int_\Omega \frac{1}{2} \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) [s_{ij}(u_k) + \delta_{ij} p] \, \mathrm{d}\Omega \\
& - \int_\Omega \delta u_i \, b_i \, \mathrm{d}\Omega - \int_{\Gamma_2} \delta u_i \bar{t}_i \, \mathrm{d}\Gamma \\
& + \int_\Omega \delta p \left[\frac{\partial u_i}{\partial x_i} - \frac{1}{K} p \right] \mathrm{d}\Omega - \int_\Omega \frac{\gamma h^2}{2G} \frac{\partial \delta p}{\partial x_i} r^i \, \mathrm{d}\Omega = 0
\end{aligned} \tag{50}
$$

where r^i is either deduced from equation (39) or is given by the Π_i form of the momentum equations expressed in equation (42). In the latter case, it is necessary to introduce a weighted residual form for the equations to provide a solution for the added variables. Two forms have been proposed for this projection. The first based on the orthogonal sub-scale argument adds the equations as

$$G = \int_{\Omega} \frac{\gamma h^2}{2G} \left[\delta\Pi_1 r^5 + \delta\Pi_2 r^6 + \delta\Pi_3 r^7 \right] \mathrm{d}\Omega = 0 \tag{51}$$

where we have set $R^{i+4} = \delta\Pi_i$. The second form merely uses

$$G = \int_{\Omega} \left[\delta\Pi_1 r^5 + \delta\Pi_2 r^6 + \delta\Pi_3 r^7 \right] \mathrm{d}\Omega = 0 \tag{52}$$

thus avoiding the γ term completely. For the elasticity problem the form given by equation (51) leads to fully symmetric equations which is an advantage if the equations are solved as one set. If, however, a split of the equations is used and the solution for equation (50) is first obtained with the Π_i set at their previous value (i.e. zero at the first iteration) followed by solution of either equation (51) or (52) little is gained by introducing the γ weighting.

Note that a typical term in equation (51) or (52) is given by

$$\delta\Pi_1 r^5 = \delta\Pi_1 \left[\frac{\partial p}{\partial x_1} + \Pi_1 \right] \tag{53}$$

Thus, treating R^5 as an arbitrary (virtual) Π_1 will lead to a mass type matrix in the numerical solution process (Brooks and Hughes, 1982; Codina, 2000; Oñate *et al.*, 2002). The idea of projecting values by what is essentially a least squares method is not new. In the context of stress projections it was used by Brauchli and Oden (1971). Cantin *et al.* (1978) and Loubignac *et al.* (1978) used a similar approach to build improved stress results from a displacement solution and in a mixed approximation and solution context this was extended by Zienkiewicz *et al.* (1985a, b). As noted earlier, the approximate recovery of the pressure gradients is not unique and other schemes can be considered (e.g. a superconvergent patch recovery (SPR) projection as introduced by Zienkiewicz and Zhu (1992).

3. Numerical examples

The above formulation has been implemented in the general purpose finite element program *FEAP* which is described by Taylor, (2003) and Zienkiewicz and Taylor (2000b, c). This formulation permits solutions using either the fully monolithic algorithm in which the displacements, pressures and pressure gradient projections are computed simultaneously or using a split algorithm in which the displacements and pressures are computed separately from the pressure projection values and iterative improvements are used (Oñate *et al.*, 2002).

3.1 Cook problem

We first consider the problem shown in Figure 1 known as the Cook problem. The original problem was given as a plane stress problem; however, to test the performance of elements in a quasi incompressible state it has been given as a plane strain problem. The properties are taken as shear modulus with a value of $G = 375$ and Poisson ratio with a value of $\nu = 0.4999$.

The remaining geometric factors are shown in Figure 1. Meshes of quadrilaterals and linear triangles are used with 2, 5, 10, 20 and 50 elements on each side of the mesh. The meshes for five elements per side are shown in Figure 2. To illustrate the effects of locking for this problem we also consider

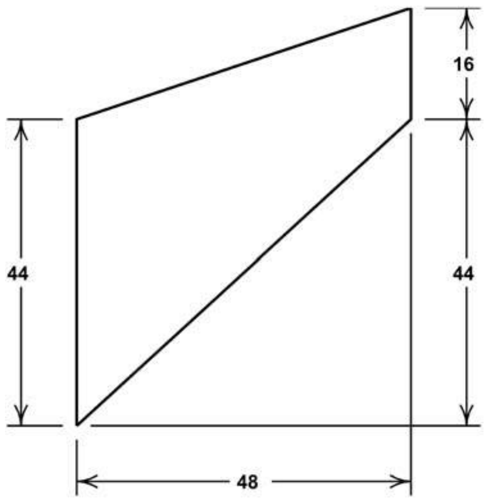


Figure 1.
Cook problem description

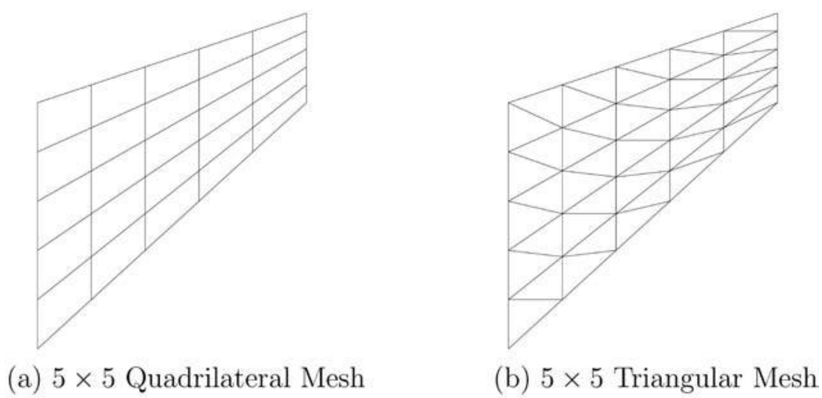


Figure 2.
Cook problem meshes: plane strain

standard displacement models employing linear triangles and four-node quadrilateral elements.

The displacements on the left boundary are fully restrained and a uniformly distributed shear load with unit intensity is applied to the right boundary.

In Figure 3, we present results for the vertical displacement at the top of the right boundary. The left figure presents results for the standard displacement formulation (Q1), the mixed Q1P0 element described by Simo *et al.* (1985) and the enhanced assumed strain element (Q1Enh) given by Simo and Rifai (1990). In the right part of the figure, we present results for triangles using the standard displacement model (T1), the mixed displacement pressure

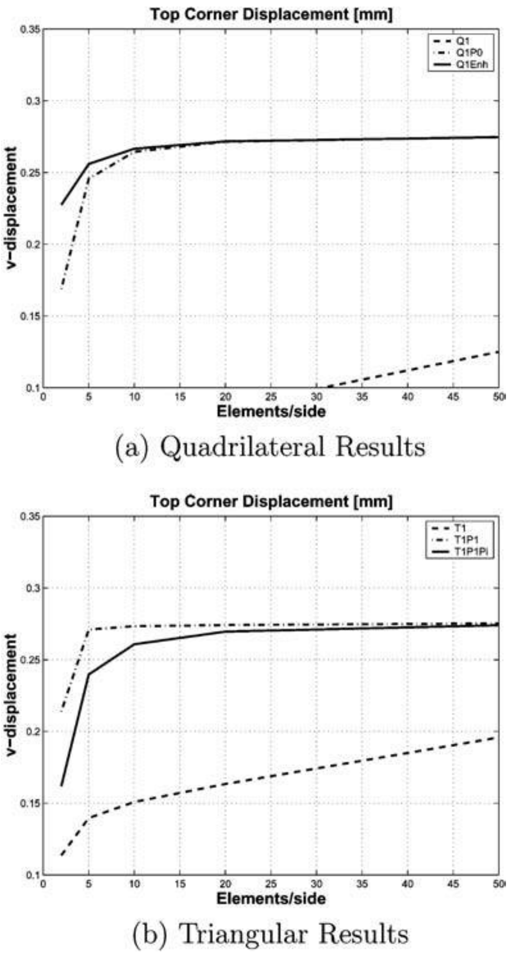


Figure 3.
Cook problem: vertical displacement at top corner

form (T1P1) and the new form (T1P1Pi). The results for the T1P1 element where both displacements and pressure are defined using continuous linear interpolation on each triangle are in fact identical to many other forms (Zienkiewicz and Taylor, 2000b) and occur as the first solution in the iterative split form. It is evident that adding the pressure projection does not improve the accuracy for the displacement at this point. Indeed, the effect is more evident in the pressure distribution as we shall show in subsequent examples.

In Figure 4, we compare the results for the Q1Enh and T1P1Pi elements. It is remarkable that the triangle performs quite well compared with the quadrilateral element, which in fact is excellent in both quasi-incompressible and bending applications.

In Figures 5 and 6, we present the same type of results as just described for the horizontal displacement at the top. In addition, in Figures 7-10, we show results for the bottom corner displacement components. Generally, behavior of results are similar to those at the top. However, we note that the displacements at these two points converge in a more monotonic manner using the formulation including the pressure projection variables Π_i.

3.2 Tension strip with slot

The second problem considered is a confined strip with a central slot with circular ends. The problem is loaded by a uniform axial load at the top and bottom and has geometric properties as shown in Figure 11. The material properties are: Young's modulus (E) of 24 and Poisson ratio (ν) of 0. 49999995. The analysis is performed in plane strain to again give a quasi-incompressible response. This problem has no infinite stress at any point and thus provides a check on the triangular elements to perform well over the entire problem.

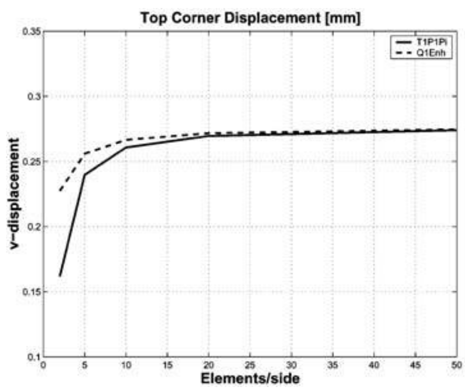
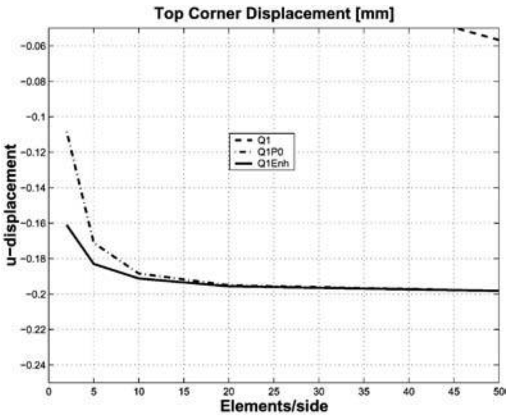
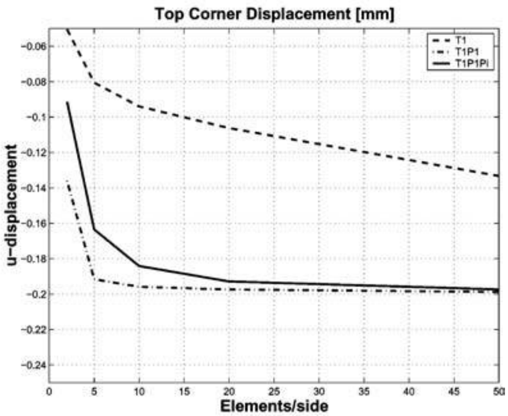


Figure 4.
Cook problem: vertical displacement at top corner



(a) Quadrilateral Results



(b) Triangular Results

Figure 5.
Cook problem: horizontal displacement at top corner

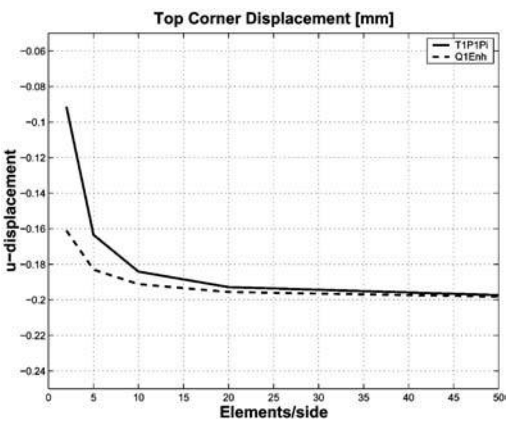
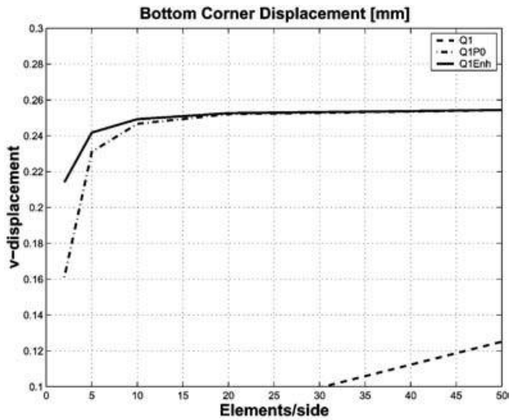
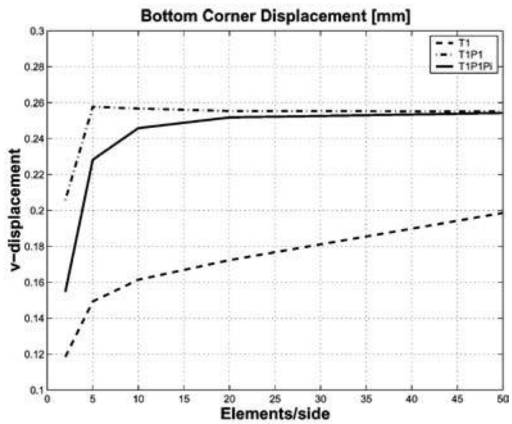


Figure 6.
Cook problem: horizontal displacement at top corner



(a) Quadrilateral Results



(b) Triangular Results

Figure 7.
Cook problem: vertical displacement at bottom corner

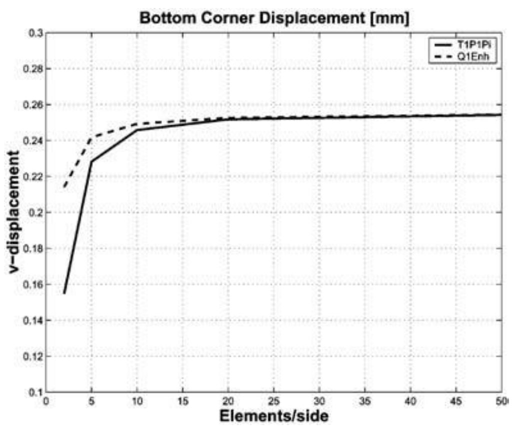
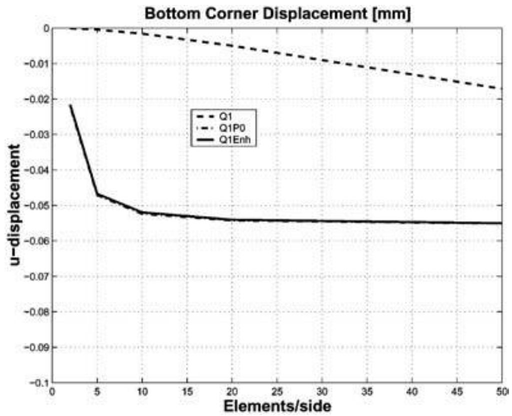
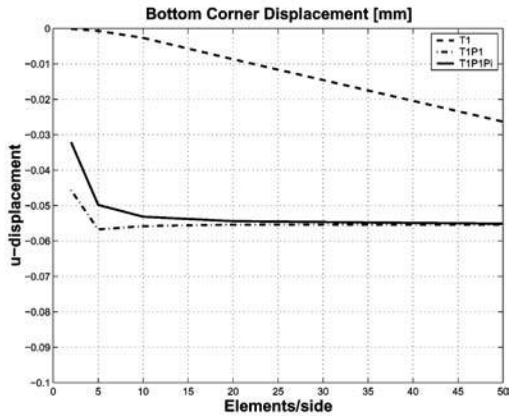


Figure 8.
Cook problem: vertical displacement at bottom corner



(a) Quadrilateral Results



(b) Triangular Results

Figure 9.
Cook problem: horizontal displacement at bottom corner

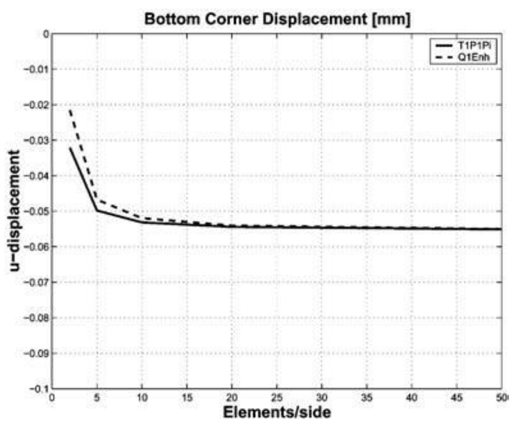


Figure 10.
Cook problem: horizontal displacement at bottom corner

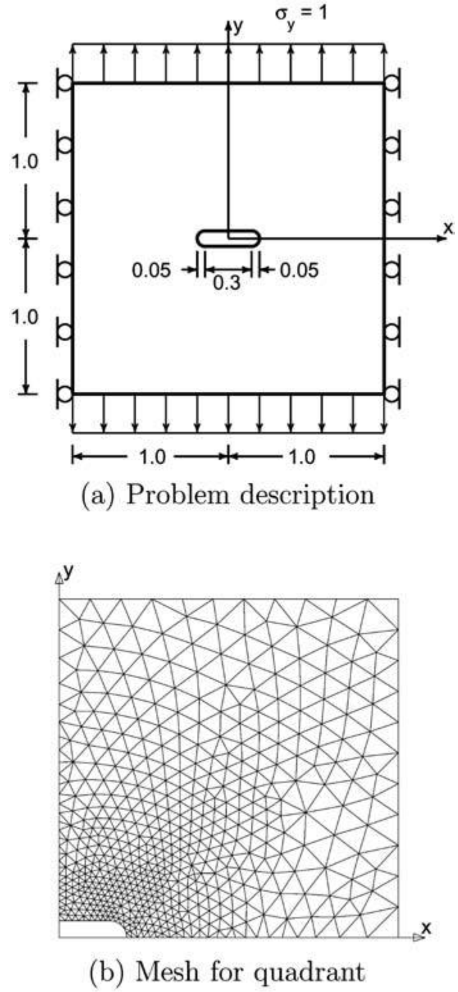
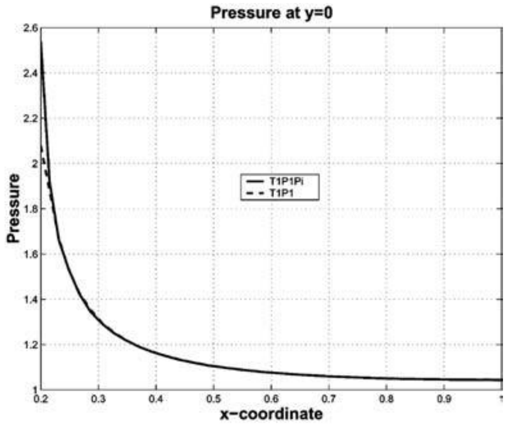


Figure 11.
Region and mesh used for slotted tension strip

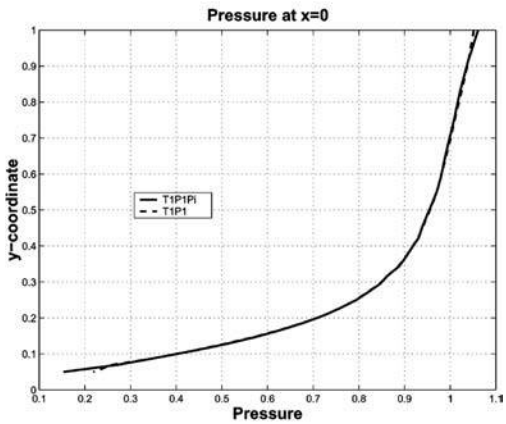
The distribution of pressure along the axes is presented in Figure 12 for cases with the Π_i (T1P1Pi) and for the Brezzi-Pitkäranta form (T1P1). Both are free of oscillations, however, the form which includes the Π_i modification produces answers which are in closer agreement with other methods (Zienkiewicz and Taylor, 2000b).

3.3 Driven cavity

We consider the problem of a square region in which the top is displaced horizontally by a unit displacement. This is a standard problem in fluid mechanics known as the *driven cavity problem* in which a velocity is specified at



(a) Pressure on x-axis



(b) Pressure on y-axis

Figure 12.
Pressure distribution on axes of slotted tension strip

the top instead of a displacement (Oñate and Garcia, 2001). The cavity has a unit length for both the horizontal and vertical sides. The material properties are Young's modulus, E, of 3 and Poisson ratio, ν, of 0.49999995. This leads to shear modulus of near unity, a ratio of the bulk to shear modulus (K/G) approximately equal to 10^7 and, thus, represents a quasi-incompressible solid. To represent the driven cavity, all boundaries have zero displacement in both the normal and tangential directions except the top face which has a unit horizontal displacement at all nodes except the corner ones which are set to zero.

For the solution reported here, the pressure at the center of the bottom face is set to zero. The problem is solved using different meshes ranging from ten elements per side to 40 elements per side. The 10×10 mesh is shown in Figure 13.

To illustrate the effects of the *sub-scale* Π_i terms, we plot the pressure along the horizontal centerline in Figure 14. The curve labeled T1P1 ignores the Π terms in all equations and thus leads to a form which includes only the displacement $\mathbf{u}$ and pressure p at each node. In this form, the equations are identical to that introduced originally by Brezzi and Pitkäranta (1984).

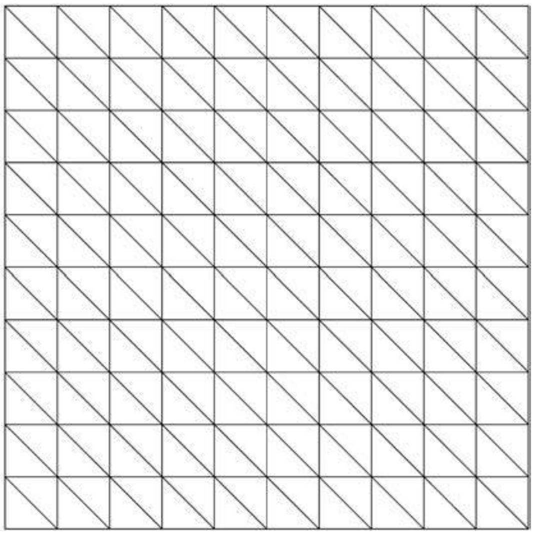


Figure 13.
The 10×10 mesh of triangles

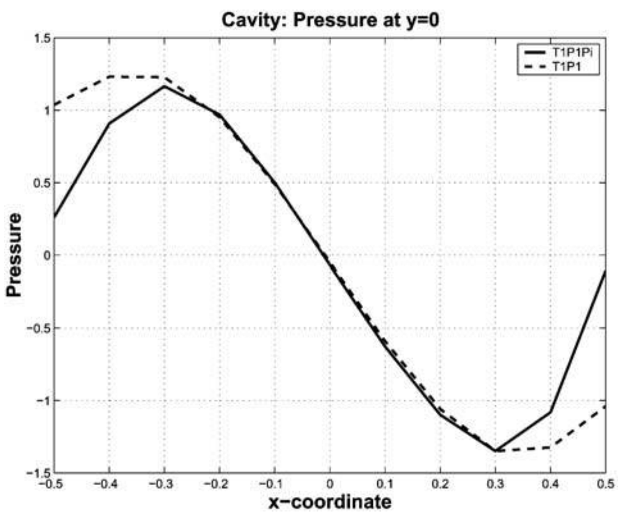


Figure 14.
Pressure on horizontal centerline for 10×10 mesh

In Figure 15 we show the same result for a 40×40 mesh. These results have been computed with the monolithic algorithm in which all five degrees of freedom (u_1, u_2, P, Π_1 and Π_2) are included in a single solution. The value for stabilization array is set as $\boldsymbol{\tau} = \gamma h^2/(2G)\mathbf{I}$ where h^2 is taken as twice the area of each triangle and $\gamma = 0.75$ as given by a finite calculus derivation (Oñate *et al.*, 2002).

The above solution was repeated using the iterative solution in which the displacement and pressure are solved separately from the Π variables. Figures 16 and 17 show the results for iterations 1, 3, and 5. The result for the

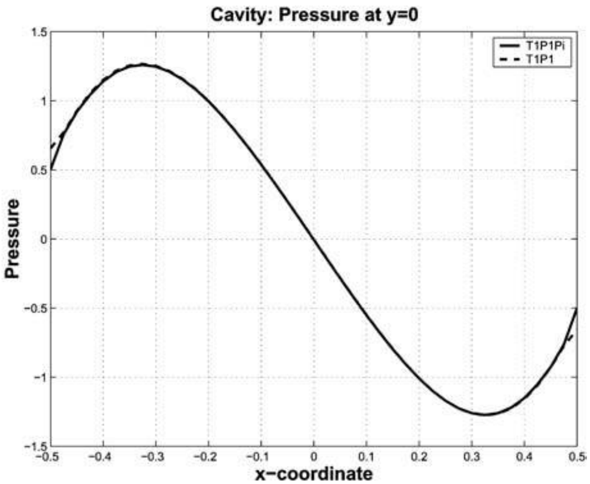


Figure 15.
Pressure on horizontal centerline for 40×40 mesh

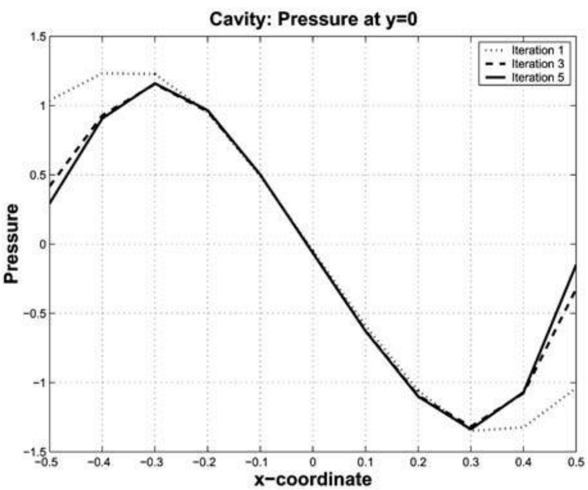


Figure 16.
Pressure on horizontal centerline for 10×10 mesh and iterative solution

10×10 mesh was iterated to convergence and the behavior for the results at the upper left corner are shown in Figure 18 (note that Π is multiplied by 1,000 to permit graphical display).

Figure 19 shows the sensitivity of the pressure at the mid height for different values of the parameter γ. It is evident that much less dependence on this value results from the addition of the added Π_i stabilization terms.

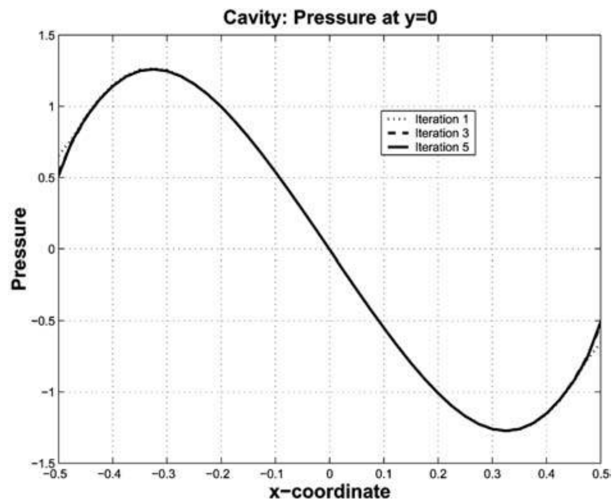


Figure 17.
Pressure on horizontal centerline for 40×40 mesh and iterative solution

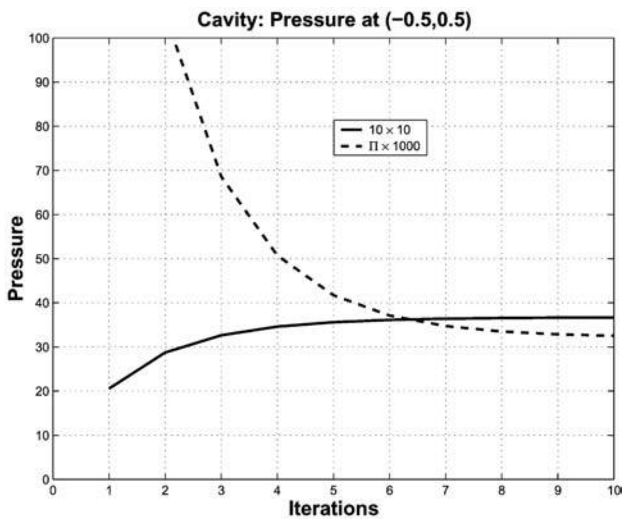
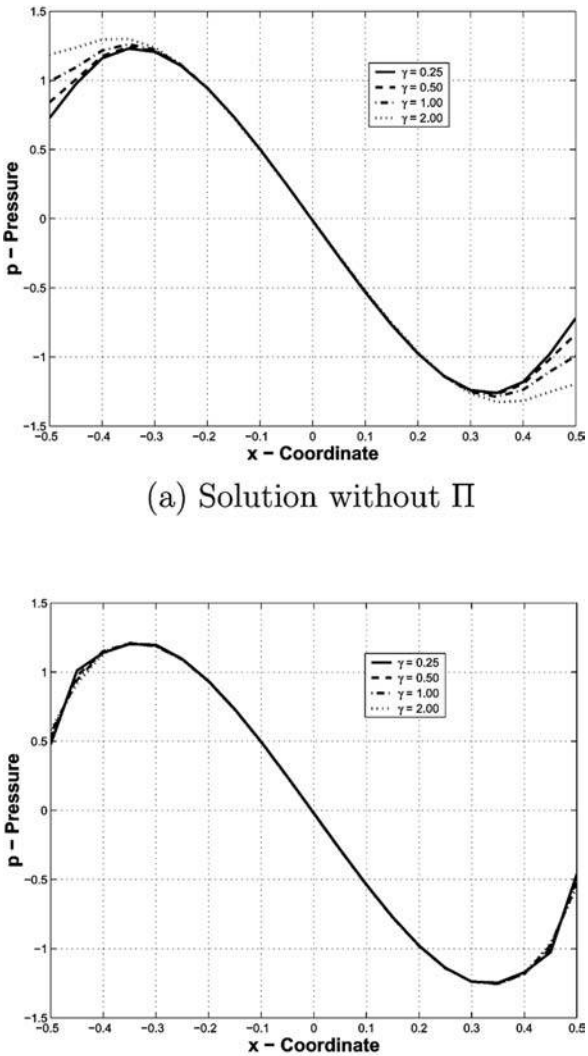


Figure 18.
Convergence of pressure p and Π at upper left corner vs iteration number

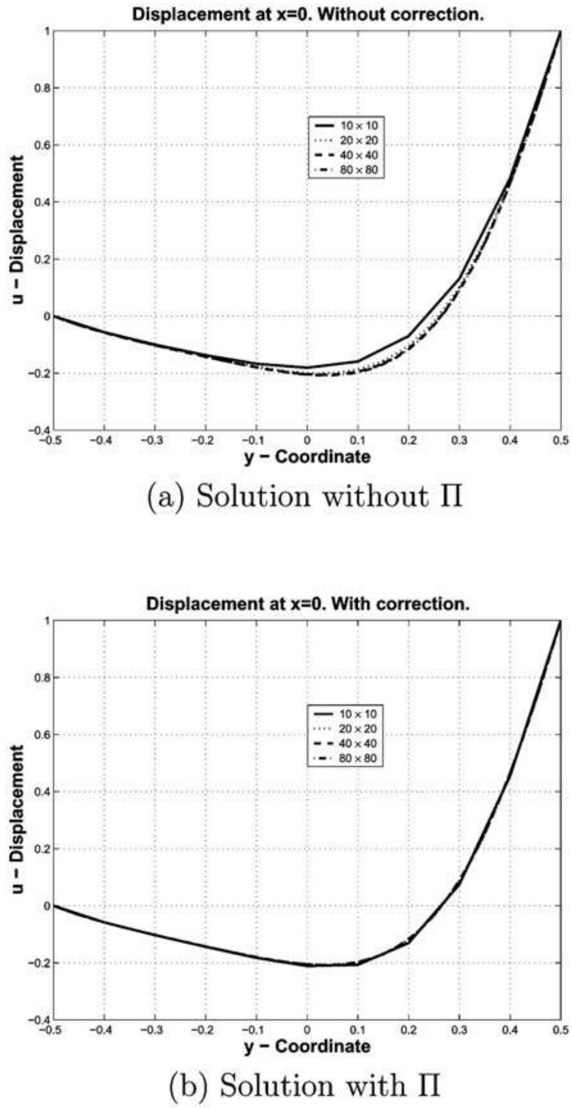


(a) Solution without Π

(b) Solution with Π

Figure 19.
Dependence of pressure on value of γ for 20×20 mesh

Finally, in Figures 20-22, the convergence of the centerline displacements and pressure are presented for different uniform mesh divisions with 10, 20, 40 and 80 elements per side. Again, it is evident that much less sensitivity results from the addition of the Π_i terms.



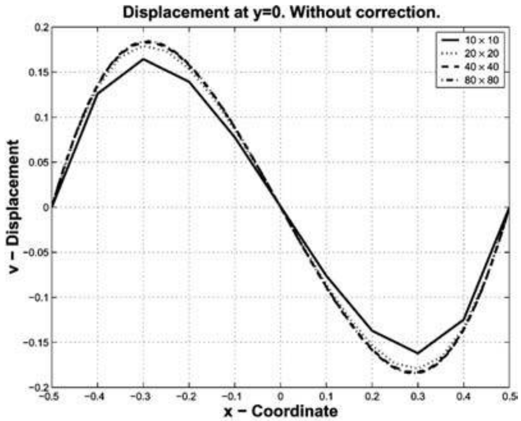
(a) Solution without Π

(b) Solution with Π

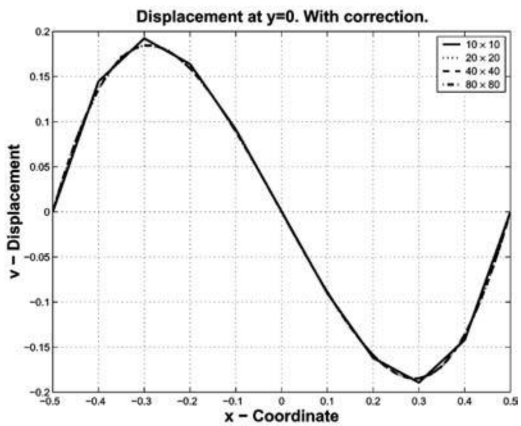
Figure 20.
Dependence of **u**-displacement at $x = 0$ with mesh subdivision

4. Concluding remarks

The concept of finite calculus which accepts that the governing equations in mechanics are satisfied in a domain of finite size, provides a natural procedure for introducing residual corrections into the discrete forms of the classical equations of the infinitesimal theory. This allows to derive new families of



(a) Solution without Π

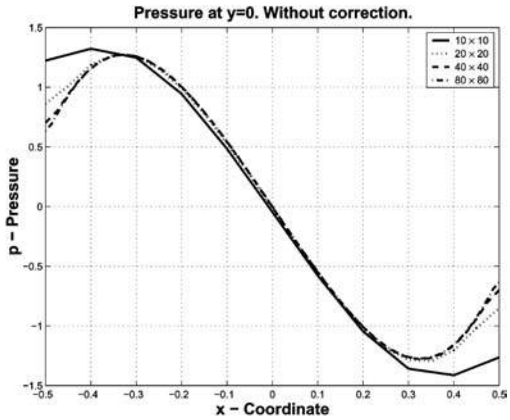


(b) Solution with Π

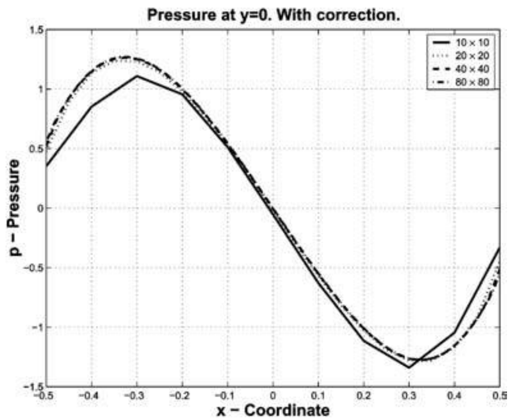
Figure 21.
Dependence of v-displacement at $y = 0$ with mesh subdivision

stabilized numerical methods using finite element, finite differences or, indeed, any other discretization procedure. In this paper, we have shown two examples of the derivation of residual corrected forms for the advective-diffusive equation and the incompressible and near (quasi) incompressible linear elasticity equations.

It is shown that for the elasticity problem, the finite calculus process results in the addition of a Laplacian of pressure terms and a projected gradient of pressure forms. The examples of application show that this form is essential to obtain accurate numerical results which converge in a



(a) Solution without Π



(b) Solution with Π

Figure 22.
Dependence of pressure at $y = 0$ with mesh subdivision

more monotonic manner and are less sensitive to the value of the stabilization parameters.

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