Hybrid Fractional Coupled Systems with Generalized Caputo Derivatives

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Abstract

This paper establishes a comprehensive theoretical framework for a novel coupled system of nonlinear hybrid fractional differential equations involving generalized Caputo derivatives. The system's hybrid nature, coupled with the generality of the fractional operators, allows for modeling complex interdependent processes with memory effects that cannot be adequately captured by existing models.

Using Krasnoselskii's fixed point theorem, we prove the existence of at least one solution under explicit coupling conditions. Under appropriate Lipschitz conditions, we establish uniqueness via Banach's contraction principle, deriving a quantitative condition involving the fractional orders, generalization parameters, and Lipschitz constants. We also conduct a rigorous analysis of Ulam-Hyers and Ulam-Hyers-Rassias stability, obtaining explicit stability constants that provide quantitative bounds on how perturbations in the input affect the solution.

The theoretical results are validated through three carefully constructed numerical examples with explicit parameter values demonstrating existence, uniqueness, and Ulam-Hyers stability. A parameter sensitivity analysis confirms the robustness of the uniqueness condition across variations in fractional orders, generalization parameters, and interval lengths. The paper concludes with a discussion of limitations and directions for future research, including extensions to higher fractional orders, delay and impulsive effects, and numerical methods.

Keywords: Hybrid fractional differential equations, Generalized Caputo derivative, Coupled system, Fixed point theorems, Ulam-Hyers stability

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1 Introduction

Fractional calculus, the branch of mathematical analysis that generalizes integer-order derivatives and integrals to arbitrary real or complex orders, has transcended its purely theoretical origins to become an indispensable tool for modeling complex phenomena across science and engineering [1–5]. Unlike their classical counterparts, fractional-order operators are inherently non-local, endowing them with a powerful "memory effect" that makes them exceptionally well-suited for describing systems with long-range temporal dependencies and spatial interactions. This property has led to groundbreaking applications in viscoelasticity [6], bioengineering [7], control theory [8], signal processing [9], and financial modeling [10]. Recent advances have further extended these applications to autonomous systems, where fractional-order type-3 fuzzy control has demonstrated remarkable performance in path-tracking tasks for driverless cars, leveraging the memory-dependent nature of fractional operators to handle parametric uncertainties and unknown disturbances [11].

Within this expansive field, the study of fractional differential equations (FDEs) constitutes a central and rapidly evolving area of research. A significant advancement has been the introduction of generalized fractional operators, which incorporate an additional parameter that offers greater flexibility and a more nuanced description of complex processes. Among these, the generalized Caputo derivative, as defined by Katugampola [12], has garnered considerable attention. It generalizes both the classical Caputo and the Caputo-Hadamard derivatives, providing a unified framework that can be tailored to specific applications by adjusting its parameters [13, 14]. The versatility of this operator has recently been demonstrated in various contexts, including fractional iterative differential equations with boundary conditions via the μ-Caputo fractional derivative [15], and numerical investigations of coupled modified Korteweg-de Vries systems [16]. Furthermore, the ϑ-Caputo framework has proven particularly effective in modeling biological phenomena, such as cholera dynamics, where memory effects and environmental feedback play crucial roles [17].

Parallel to the development of generalized operators, the investigation of coupled systems of FDEs has emerged as a critical frontier. Many real-world processes are not isolated but are interdependent—think of predator-prey dynamics in ecology, the interaction of diseases in epidemiology, or synchronized oscillators in physics. Coupled systems of FDEs provide the mathematical language to model such intricate interactions where the evolution of one state variable is intrinsically linked to the others, and where memory effects play a crucial role in the system's collective behavior [18, 19]. The analysis of such systems continues to evolve, with recent studies addressing weakly coupled systems of subdiffusion equations that reveal fundamentally distinct decay behaviors compared to single equations [20], and coupled fractional stochastic differential equations that capture randomness and sudden fluctuations in real-world phenomena [21, 22]. The stabilization of coupled PDE-ODE systems with input saturation has also been investigated, providing insights into controllability and semi-global stabilization under constraints [23].

A further layer of complexity and practical relevance is added by the concept of hybrid fractional differential equations. A hybrid system is characterized by the perturbation of the differential equation by a function of the unknown solution, often separated into linear and nonlinear components. These systems, particularly those with perturbations of the first type (where the function of the solution is added to the unknown function inside the derivative), can model phenomena where a system's state is subject to abrupt changes or is influenced by an external force that depends on its current state [24, 25]. The hybrid structure introduces significant analytical challenges, as the standard theory for standard FDEs often cannot be directly applied. Recent developments have addressed these challenges through various approaches, including the analysis of fractional hybrid systems with impulsive effects, where Dhage's fixed point theorem is extended to establish existence and Ulam-Hyers stability under explicit conditions [26–29]. Hybrid stochastic fractional differential equations incorporating neutral effects and switching mechanisms have also been investigated within the Atangana-Baleanu-Caputo framework, capturing the joint influence of randomness, history, and abrupt transitions [30]. Variable-order fractional delay differential equations with integral boundary conditions further extend the modeling capabilities for systems with evolving memory [31,32].

The stability analysis of differential equations is paramount for understanding the long-term behavior of dynamical systems under perturbations. In this context, Ulam-Hyers (U-H) stability and its generalizations have become a focal point of modern analysis. This form of stability guarantees that an approximate solution of an equation is always close to an exact solution, a property of immense value in numerical analysis and applications where models are inherently approximate [33–37]. Establishing U-H stability for nonlinear fractional systems, especially coupled and hybrid ones, requires sophisticated techniques, often involving Gronwall-type inequalities and careful fixed-point arguments. The recent literature has witnessed significant progress in this direction, with studies on Ulam-type stability for fractional integro-delay differential equations via the ψ-Hilfer operator [38,39], and comprehensive stability analyses for variable-order fractional systems with multiple delays [31]. These developments provide a rich foundation for extending stability theory to more complex coupled hybrid configurations.

While significant progress has been made on each of these fronts individually—generalized operators, coupled systems, hybrid equations, and stability theory—their intersection remains a largely underexplored territory. The existing literature is rich with studies on standard coupled FDEs [40–42] and hybrid FDEs in the Riemann-Liouville or Caputo sense [43, 44]. However, the analysis of coupled systems of nonlinear hybrid FDEs within the framework of generalized Caputo derivatives presents a formidable and novel challenge. This gap in the literature represents a significant opportunity to develop a more unified and powerful theoretical framework. Recent works have begun to address related aspects, such as fractional iterative differential equations with μ-Caputo derivatives [15] and coupled fractional stochastic systems [21], but a comprehensive treatment of coupled hybrid systems with generalized Caputo derivatives remains absent.

The novelty and originality of this article is multi-faceted. First, we investigate, for the first time, a coupled system of nonlinear hybrid FDEs involving generalized Caputo derivatives. The system's hybrid nature, characterized by linear perturbations of the first type, coupled with the generality of the fractional operators (parameters $\alpha, \beta, \rho, \sigma$), allows for modeling a vast array of complex, interdependent processes with memory effects that cannot be adequately captured by existing models. Second, we provide a simultaneous and rigorous investigation of three fundamental properties—existence, uniqueness, and Ulam-Hyers stability—for this complex system. Many studies focus on one or two of these aspects; our work delivers a complete theoretical treatment. Third, the proofs are constructive, leveraging a combination of advanced fixed-point theorems. We employ Krasnoselskii's fixed point theorem to establish existence, which is particularly adept at handling the hybrid structure's decomposition into contractive and compact parts. We then use Banach's contraction principle to prove uniqueness under appropriate Lipschitz conditions. This dual approach allows us to obtain strong results under different sets of assumptions. Finally, our stability analysis is not merely qualitative but quantitative. We derive explicit formulas for the Ulam-Hyers stability constants, which is a significant strength for practical applications and numerical error estimation. It is important to emphasize that this work concerns fractional dynamical systems in the context of differential equations and their stability analysis, and does not involve guided wave propagation, surface waves, or any related physical wave phenomena.

In [45], the existence and uniqueness of solutions to nonlinear hybrid fractional differential equations within the Atangana-Baleanu-Caputo framework were investigated. Their analysis incorporates both linear and nonlinear perturbations.

$$\begin{cases} {}^{ABC}D_a^{\alpha,\rho}[x(t)\,\psi(t,x(t),y(t)) - \eta(t,x(t),y(t))] = \theta(t,x(t),y(t)), t \in J = [0,T],\ T > 0, \\ x(0)\psi(0,x(0),y(0)) + \mu\, x(T)\psi(T,x(T),y(T)) = \eta(0,x(0),y(0)) + \mu\,\eta(T,x(T),y(T)) + \beta, \end{cases} \tag{1}$$

where ${}^{ABC}D_a^{\alpha,\rho}$ denotes the Atangana–Baleanu–Caputo fractional derivative of order α and type ρ, $\psi, \eta, \theta \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ are given functions, and $\mu, \beta \in \mathbb{R}$ with $\mu \neq 1$.

In [46], the authors focused on establishing the existence and uniqueness of solutions for a specific set of Caputo–Fabrizio fractional differential equations under periodic boundary conditions (PBCs).

$$ {}^{CF}D_0^r u(t) + {}^{CF}D_0^\alpha u(t) - \lambda u(t) = f(t,u(t)), \tag{2}$$

where

$$r := \alpha + 1, \quad 0 < \alpha < 1,\ \lambda > 0,\ t \in [0,T],$$

with

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{3}$$

Specifically, we consider the following coupled system of nonlinear hybrid fractional differential equations with generalized Caputo derivatives. For $t \in J = [a,T]$, with initial conditions $x(a) = x_0$, $y(a) = y_0$, the system is defined as:

$$\begin{cases} {}^{C}D_a^{\alpha,\rho}\left[x(t) - h_1(t,x(t),y(t))\right] = F_1(t,x(t),y(t)), \\ {}^{C}D_a^{\beta,\sigma}\left[y(t) - h_2(t,x(t),y(t))\right] = F_2(t,x(t),y(t)), \end{cases}$$

where $\alpha, \beta \in (0,1]$ are the fractional orders, $\rho, \sigma > 0$ are the generalization parameters, ${}^{C}D_a^{\alpha,\rho}$ denotes the generalized Caputo derivative, and $F_1, F_2, h_1, h_2 : J \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

The primary objectives of this paper are to establish an equivalent system of nonlinear integral equations, which serves as the foundation for our fixed-point analysis; to prove the existence of at least one solution for the system using Krasnoselskii's fixed point theorem; to establish sufficient conditions for the existence of a unique solution via Banach's contraction mapping principle; to conduct a rigorous analysis of Ulam-Hyers and Ulam-Hyers-Rassias stability for the proposed system, providing explicit stability constants; and to validate the theoretical findings through carefully constructed numerical examples that demonstrate the applicability of each main result.

The remainder of this paper is structured as follows: Section 2 revisits essential preliminaries and definitions, including a dedicated subsection on modelling assumptions and a comprehensive notation table. Section 3 contains an auxiliary lemma converting the differential system into an integral form, accompanied by a schematic workflow diagram. Our main results on existence, uniqueness, and stability are presented in Section 4, with remarks on coupling effects and comparisons with existing literature. Section 5 provides illustrative examples with detailed parameter justification, sensitivity analysis, and graphical summaries. Section 6 concludes the paper with final remarks, explicit quantitative contributions, limitations of the theoretical framework, and suggestions for future research.

2 Preliminaries

This section is devoted to presenting the fundamental definitions, notations, and preliminary results that will be essential for establishing our main theorems. Throughout this work, let $a \geq 0$ and $J = [a, T]$ be a closed interval with $T > a$. We denote by $\mathcal{C} = C(J, \mathbb{R})$ the Banach space of all continuous functions defined on J endowed with the supremum norm $\|x\|_\infty = \sup_{t \in J} |x(t)|$. For our coupled system, the appropriate Banach space will be the product space $\mathcal{B} = \mathcal{C} \times \mathcal{C}$ equipped with the norm $\|(x, y)\|_{\mathcal{B}} = \|x\|_\infty + \|y\|_\infty$.

2.1 Functional Framework

The analysis of the coupled system (2) is conducted within the Banach space $\mathcal{B} = \mathcal{C} \times \mathcal{C}$ with the norm defined above. The nonlinear functions $F_1, F_2, h_1, h_2 : J \times \mathbb{R}^2 \to \mathbb{R}$ are assumed to satisfy certain conditions (continuity, Lipschitz, or boundedness) which will be stated explicitly in Section 4 where they are employed in the fixed-point arguments. These assumptions are standard in the literature on fractional differential systems and ensure the applicability of Krasnoselskii's fixed point theorem, Banach's contraction principle, and the Ulam-Hyers stability analysis presented in Section 4.

We now recall the definitions of the generalized fractional operators.

Definition 1. .[Generalized Fractional Integral [12]] Let $\alpha > 0$, $\rho > 0$, and $f \in \mathcal{C}$. The generalized fractional integral of order α of the function f is defined by

$$I_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} f(s)\, ds, \quad t > a.$$

Definition 2. .[Generalized Caputo Fractional Derivative [12]] Let $\alpha \in (0, 1]$, $\rho > 0$, and f be a differentiable function on J. The generalized Caputo fractional derivative of order α of the function f is defined by

$${}^C D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{1}{(t^\rho - s^\rho)^\alpha} f'(s)\, ds, \quad t > a.$$

Remark 1. It is crucial to note that for a constant function c, the generalized Caputo derivative satisfies ${}^C D_{a^+}^{\alpha,\rho} c = 0$, a property that aligns with its physical interpretation and simplifies the analysis of hybrid systems.

Remark 2. .[Comparison with Other Fractional Operators] The generalized Caputo derivative employed in this work offers several advantages and limitations compared to other fractional operators:

- **Compared to Riemann-Liouville:** The Caputo formulation (including its generalization) allows for initial conditions expressed in terms of integer-order derivatives, which have clear physical interpretations. This makes it more suitable for modeling real-world phenomena where initial states are measured directly [1].
- **Compared to Hadamard:** The generalized Caputo derivative is related to Caputo-Hadamard-type operators through logarithmic-type kernels, providing a unified framework that encompasses both power-law and logarithmic memory structures [13].
- **Compared to Atangana-Baleanu:** The generalized Caputo derivative employs a power-law kernel that is singular at the endpoint, whereas the Atangana-Baleanu operator uses a non-singular Mittag-Leffler kernel. The singular kernel in the generalized Caputo setting is advantageous for modeling phenomena with strong memory near the origin but may be less suitable for processes requiring smooth kernel behavior [14].
- **Limitations:** The generalized Caputo derivative assumes differentiability of the function in the classical sense, which may be restrictive for highly irregular functions. Additionally, the parameter ρ introduces an extra degree of freedom that requires careful calibration in applications. For processes with variable memory order, variable-order fractional operators may be more appropriate [31].

The following lemmas establish the composition properties of these operators, which are fundamental for converting the differential problem into an integral equation.

Lemma 1 ([13])**.** *Let $\alpha \in (0,1]$, $\rho > 0$, and $f \in \mathcal{C}$. Then, the following identity holds:*

$$ {}^{C}D^{\alpha,\rho}_{a^+}\left(I^{\alpha,\rho}_{a^+} f(t)\right) = f(t). $$

Lemma 2 ([13])**.** *Let $\alpha \in (0,1]$, $\rho > 0$, and f be a differentiable function. Then, the following identity holds:*

$$ I^{\alpha,\rho}_{a^+}\left({}^{C}D^{\alpha,\rho}_{a^+} f(t)\right) = f(t) - f(a). $$

The proofs of our main results will rely on the following celebrated fixed point theorems.

Lemma 3 (Krasnoselskii's Fixed Point Theorem [1])**.** *Let E be a Banach space and $B \subset E$ be a nonempty, closed, convex, and bounded set. Let $\mathcal{P}, \mathcal{Q} : B \to E$ be two operators satisfying the following conditions:*

1. *$\mathcal{P}x + \mathcal{Q}y \in B$ for all $x, y \in B$;*
2. *$\mathcal{P}$ is a contraction mapping;*
3. *$\mathcal{Q}$ is continuous and compact.*

Then, there exists at least one fixed point $z \in B$ such that $z = \mathcal{P}z + \mathcal{Q}z$.

Lemma 4 (Banach Contraction Mapping Principle [1])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping, i.e., there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$. Then, T has a unique fixed point in X.*

Finally, we state the core definitions related to the stability analysis of our system.

Definition 3 (Ulam-Hyers Stability [8])**.** The coupled system (1) is Ulam-Hyers (U-H) stable if there exist real numbers $C_1, C_2 > 0$ such that for every $\epsilon_1, \epsilon_2 > 0$ and for every pair of functions $(x, y) \in \mathcal{B}$ satisfying the inequalities

$$\left| {}^C D_{a^+}^{\alpha,\rho} \left[x(t) - h_1(t, x(t), y(t))\right] - F_1(t, x(t), y(t)) \right| \leq \epsilon_1,$$

$$\left| {}^C D_{a^+}^{\beta,\sigma} \left[y(t) - h_2(t, x(t), y(t))\right] - F_2(t, x(t), y(t)) \right| \leq \epsilon_2,$$

there exists a unique solution $(x^*, y^*) \in \mathcal{B}$ of the system (1) with

$$\|(x, y) - (x^*, y^*)\|_{\mathcal{B}} \leq (C_1\epsilon_1 + C_2\epsilon_2).$$

2.2 Notation Summary

To assist the reader in navigating the multiple parameters and operators used throughout this manuscript, we provide the following notation table:

Symbol	Meaning
$\alpha, \beta \in (0, 1]$	Fractional orders of the generalized Caputo derivatives
$\rho, \sigma > 0$	Generalization parameters in the fractional operators
$J = [a, T]$	Interval of interest with $a \geq 0$, $T > a$
$\mathcal{C} = C(J, \mathbb{R})$	Banach space of continuous functions with supremum norm
$\mathcal{B} = \mathcal{C} \times \mathcal{C}$	Product Banach space for coupled system
$\|(x, y)\|_{\mathcal{B}} = \|x\|_\infty + \|y\|_\infty$	Norm on the product space
${}^C D_a^{\alpha,\rho}$	Generalized Caputo fractional derivative of order α with parameter ρ
$I_a^{\alpha,\rho}$	Generalized fractional integral of order α with parameter ρ
$H_1 = x_0 - h_1(a, x_0, y_0)$	Constant arising from initial condition in the first equation
$H_2 = y_0 - h_2(a, x_0, y_0)$	Constant arising from initial condition in the second equation
$L_{F_1}, L_{F_2}, L_{h_1}, L_{h_2}$	Lipschitz constants for the nonlinear functions (defined in (A2))
$M_{F_1}, M_{F_2}, M_{h_1}, M_{h_2}$	Boundedness constants for the nonlinear functions (defined in (A3))
$\Phi_1 = \frac{M_{F_1}(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}$	Integral bound for the first equation
$\Phi_2 = \frac{M_{F_2}(T^\sigma - a^\sigma)^\beta}{\sigma^\beta \Gamma(\beta+1)}$	Integral bound for the second equation
Λ	Contraction constant for uniqueness theorem (defined in Theorem 2)
C_1, C_2	Ulam-Hyers stability constants (defined in Theorem 3)

3 An Auxiliary Result

The foundation of our analysis relies on converting the coupled system of hybrid fractional differential equations into an equivalent system of nonlinear integral equations. This transformation is crucial as it allows us to reformulate the problem within the framework of fixed point theory, enabling the use of powerful topological and analytical methods to establish existence, uniqueness, and stability results. The overall workflow, from the differential system to the integral system and the subsequent fixed-point analysis, is illustrated schematically in Figure 1.

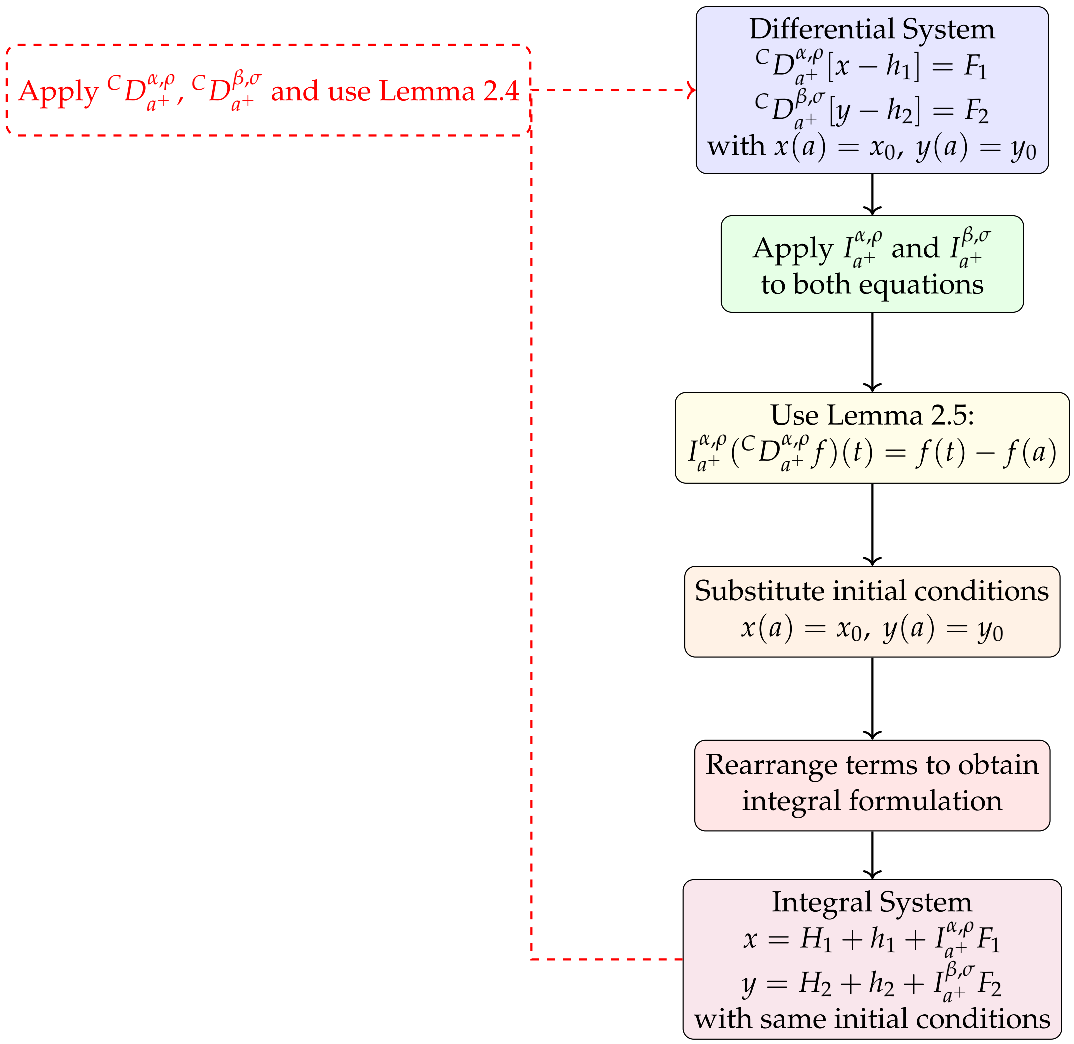

Figure 1: Workflow for converting the differential system into an equivalent integral system and vice versa

Lemma 5. *Let $\alpha,\beta\in(0,1]$, $\rho,\sigma>0$, and $a\geq 0$. Assume that $F_1,F_2,h_1,h_2 : J\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions. Then, a pair of functions $(x,y)\in\mathcal{B}$ is a solution of the coupled system of hybrid fractional differential equations*

$$\begin{cases} {}^{C}D_{a^+}^{\alpha,\rho}\left[x(t)-h_1(t,x(t),y(t))\right]=F_1(t,x(t),y(t)), \\ {}^{C}D_{a^+}^{\beta,\sigma}\left[y(t)-h_2(t,x(t),y(t))\right]=F_2(t,x(t),y(t)), \end{cases} \quad t\in J \tag{4}$$

satisfying the initial conditions

$$x(a)=x_0,\quad y(a)=y_0,$$

if and only if *it satisfies the following system of coupled nonlinear integral equations:*

$$\begin{cases} x(t) = H_1 + h_1(t, x(t), y(t)) + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)), \\ y(t) = H_2 + h_2(t, x(t), y(t)) + I_{a^+}^{\beta,\sigma} F_2(t, x(t), y(t)), \end{cases} \quad t \in J \tag{5}$$

with the same initial conditions $x(a) = x_0$*,* $y(a) = y_0$*, where the constants* H_1 *and* H_2 *are given by*

$$H_1 = x_0 - h_1(a, x_0, y_0), \quad H_2 = y_0 - h_2(a, x_0, y_0).$$

Proof. We prove the equivalence in two directions.

(⇒) Differential → Integral: Assume $(x, y) \in \mathcal{B}$ is a solution of the system of fractional differential equations (4) satisfying the initial conditions $x(a) = x_0$, $y(a) = y_0$.
Step 1: Apply the fractional integral to the first equation. Applying the generalized fractional integral operator $I_{a^+}^{\alpha,\rho}$ to both sides of the first equation in (4) yields:

$$I_{a^+}^{\alpha,\rho} \left({}^C D_{a^+}^{\alpha,\rho} \left[x(t) - h_1(t, x(t), y(t)) \right] \right) = I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)).$$

Step 2: Use Lemma 2 to simplify the left-hand side. By Lemma 2, which states that $I_{a^+}^{\alpha,\rho} \left({}^C D_{a^+}^{\alpha,\rho} f(t) \right) = f(t) - f(a)$ for differentiable functions f, we obtain:

$$[x(t) - h_1(t, x(t), y(t))] - [x(a) - h_1(a, x(a), y(a))] = I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)).$$

Step 3: Incorporate the initial conditions. Using the given initial conditions $x(a) = x_0$ and $y(a) = y_0$, this simplifies to:

$$x(t) - h_1(t, x(t), y(t)) = x_0 - h_1(a, x_0, y_0) + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)).$$

Step 4: Rearrange to obtain the integral form. Rearranging terms and substituting $H_1 = x_0 - h_1(a, x_0, y_0)$ yields the first integral equation in (5):

$$x(t) = H_1 + h_1(t, x(t), y(t)) + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)).$$

Step 5: Repeat for the second equation. A completely analogous process, applying $I_{a^+}^{\beta,\sigma}$ to the second equation in (4) and using Lemma 1 with the initial conditions, leads to:

$$y(t) = H_2 + h_2(t, x(t), y(t)) + I_{a^+}^{\beta,\sigma} F_2(t, x(t), y(t)),$$

where $H_2 = y_0 - h_2(a, x_0, y_0)$. Thus, (x, y) satisfies the integral system (5) with the prescribed initial conditions.

(⇐) Integral → Differential: Conversely, assume $(x, y) \in \mathcal{B}$ satisfies the integral system (5) together with the initial conditions $x(a) = x_0$, $y(a) = y_0$.
Step 1: Rewrite the first integral equation. From the first equation in (5), we have

$$x(t) - h_1(t, x(t), y(t)) = H_1 + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)).$$

Step 2: Apply the generalized Caputo derivative. Applying ${}^{C}D^{\alpha,\rho}_{a^+}$ to both sides yields

$$ {}^{C}D^{\alpha,\rho}_{a^+}[x(t) - h_1(t, x(t), y(t))] = {}^{C}D^{\alpha,\rho}_{a^+}[H_1] + {}^{C}D^{\alpha,\rho}_{a^+}\left(I^{\alpha,\rho}_{a^+}F_1(t, x(t), y(t))\right). $$

Step 3: Use the operator properties. Since H_1 is constant, Remark 2.3 gives ${}^{C}D^{\alpha,\rho}_{a^+}[H_1] = 0$. By Lemma 2.5,

$$ {}^{C}D^{\alpha,\rho}_{a^+}\left(I^{\alpha,\rho}_{a^+}F_1(\cdot)\right) = F_1(t, x(t), y(t)). $$

Therefore,

$$ {}^{C}D^{\alpha,\rho}_{a^+}[x(t) - h_1(t, x(t), y(t))] = F_1(t, x(t), y(t)), $$

which is precisely the first equation in (4).
Step 4: Repeat for the second equation. Similarly, from the second equation in (5),

$$ y(t) - h_2(t, x(t), y(t)) = H_2 + I^{\beta,\sigma}_{a^+}F_2(t, x(t), y(t)), $$

and applying ${}^{C}D^{\beta,\sigma}_{a^+}$ gives

$$ {}^{C}D^{\beta,\sigma}_{a^+}[y(t) - h_2(t, x(t), y(t))] = F_2(t, x(t), y(t)). $$

Step 5: Initial conditions. Since $x(a) = x_0$ and $y(a) = y_0$ by assumption, the pair (x, y) satisfies both the differential equations and the prescribed initial conditions. Hence, (x, y) is a solution of (4).

This completes the proof of the equivalence. □

4 Main Results

This section presents the core theoretical contributions of this work. We establish three fundamental properties of the coupled system (4): the existence of at least one solution, the uniqueness of such a solution under stronger conditions, and the Ulam–Hyers stability of the system. The proofs are constructive and detailed, leveraging the fixed point theorems stated in the Preliminaries and the equivalent integral formulation derived in Lemma 5.

To facilitate our analysis, we begin by stating the necessary hypotheses on the nonlinear functions F_1, F_2, h_1, and h_2.
Assumptions:

(A1) Continuity: The functions

$$ F_1, F_2, h_1, h_2 : J \times \mathbb{R}^2 \to \mathbb{R} $$

are jointly continuous.

(A2) Lipschitz conditions: There exist constants $L_{F_1}, L_{F_2}, L_{h_1}, L_{h_2} > 0$ such that for all $t \in J$ and all $u, v, x, y \in \mathbb{R}$,

$$ \begin{aligned} |F_1(t, x, y) - F_1(t, u, v)| &\le L_{F_1}(|x - u| + |y - v|), \\ |F_2(t, x, y) - F_2(t, u, v)| &\le L_{F_2}(|x - u| + |y - v|), \\ |h_1(t, x, y) - h_1(t, u, v)| &\le L_{h_1}(|x - u| + |y - v|), \\ |h_2(t, x, y) - h_2(t, u, v)| &\le L_{h_2}(|x - u| + |y - v|). \end{aligned} $$

(A3) Boundedness: There exist positive constants $M_{F_1}, M_{F_2}, M_{h_1}, M_{h_2}$ such that for all $(t, x, y) \in J \times \mathbb{R}^2$,

$$|F_1(t,x,y)| \leq M_{F_1}, \qquad |F_2(t,x,y)| \leq M_{F_2},$$
$$|h_1(t,x,y)| \leq M_{h_1}, \qquad |h_2(t,x,y)| \leq M_{h_2}.$$

We now proceed to the first main result.

Theorem 1 (Existence). *Assume that conditions (A1), (A2), and (A3) hold. If*

$$L_{h_1} + L_{h_2} < 1,$$

then the coupled system (4) *has at least one solution* $(x^*, y^*) \in \mathcal{B}$.

Proof. The proof employs Krasnoselskii's fixed point theorem (Lemma 3). Based on the equivalent integral system (5) from Lemma 5, define the operator $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{T}(x,y) = (\mathcal{T}_1(x,y), \mathcal{T}_2(x,y)),$$

where

$$\mathcal{T}_1(x,y)(t) = H_1 + h_1(t, x(t), y(t)) + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)),$$
$$\mathcal{T}_2(x,y)(t) = H_2 + h_2(t, x(t), y(t)) + I_{a^+}^{\beta,\sigma} F_2(t, x(t), y(t)).$$

It follows from the continuity of the functions and integral operators that $\mathcal{T}$ is well defined and maps $\mathcal{B}$ into itself.

We decompose $\mathcal{T}$ as

$$\mathcal{T} = \mathcal{P} + \mathcal{Q},$$

where

$$\mathcal{P}_1(x,y)(t) = h_1(t, x(t), y(t)), \qquad \mathcal{Q}_1(x,y)(t) = H_1 + I_{a^+}^{\alpha,\rho} F_1(t, x(t), y(t)),$$
$$\mathcal{P}_2(x,y)(t) = h_2(t, x(t), y(t)), \qquad \mathcal{Q}_2(x,y)(t) = H_2 + I_{a^+}^{\beta,\sigma} F_2(t, x(t), y(t)).$$

Intuition: The decomposition $\mathcal{T} = \mathcal{P} + \mathcal{Q}$ separates the hybrid contribution from the fractional memory contribution. The operator $\mathcal{P}$ captures the immediate state-dependent interaction through h_1 and h_2, while $\mathcal{Q}$ contains the fractional integral terms that encode memory. Krasnoselskii's theorem is particularly suitable in this setting because it combines a contractive component with a compact one.

Step 1: Define the set and show that $\mathcal{P}(x,y) + \mathcal{Q}(u,v) \in B_R$.

For the Banach space $\mathcal{B} = C(J, \mathbb{R}) \times C(J, \mathbb{R})$, we equip $\mathcal{B}$ with the norm

$$\|(x,y)\|_{\mathcal{B}} := \|x\|_\infty + \|y\|_\infty,$$

where $\|x\|_\infty = \sup_{t \in J} |x(t)|$.

Let $R > 0$ be chosen later, and define the closed ball

$$B_R = \{(x,y) \in \mathcal{B} : \|(x,y)\|_{\mathcal{B}} \leq R\},$$

which is nonempty, closed, bounded, and convex. For any $(x,y),(u,v) \in B_R$, we estimate $\|\mathcal{P}(x,y)+\mathcal{Q}(u,v)\|_{\mathcal{B}}$.

For the first component,

$$\begin{aligned}
&|\mathcal{P}_1(x,y)(t)+\mathcal{Q}_1(u,v)(t)| \\
&\leq |h_1(t,x(t),y(t))| + |H_1| + \left|I_{a^+}^{\alpha,\rho}F_1(t,u(t),v(t))\right| \\
&\leq M_{h_1} + |H_1| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}|F_1(s,u(s),v(s))|\,ds \\
&\leq M_{h_1} + |H_1| + \frac{M_{F_1}\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}\,ds.
\end{aligned}$$

Using the substitution $\tau = s^\rho$, so that $d\tau = \rho s^{\rho-1}ds$, the integral becomes

$$\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}\,ds = \frac{1}{\rho}\int_{a^\rho}^{t^\rho} \frac{d\tau}{(t^\rho-\tau)^{1-\alpha}} = \frac{(t^\rho - a^\rho)^\alpha}{\rho\alpha}.$$

Therefore,

$$|\mathcal{P}_1(x,y)(t)+\mathcal{Q}_1(u,v)(t)| \leq M_{h_1} + |H_1| + \frac{M_{F_1}(t^\rho - a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \leq M_{h_1} + |H_1| + \Psi_1,$$

where

$$\Psi_1 := \frac{M_{F_1}(T^\rho - a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}.$$

Hence,

$$\|\mathcal{P}_1(x,y)+\mathcal{Q}_1(u,v)\|_\infty \leq M_{h_1} + |H_1| + \Psi_1.$$

Similarly,

$$|\mathcal{P}_2(x,y)(t)+\mathcal{Q}_2(u,v)(t)| \leq M_{h_2} + |H_2| + \frac{M_{F_2}(t^\sigma - a^\sigma)^\beta}{\sigma^\beta\Gamma(\beta+1)} \leq M_{h_2} + |H_2| + \Psi_2,$$

where

$$\Psi_2 := \frac{M_{F_2}(T^\sigma - a^\sigma)^\beta}{\sigma^\beta\Gamma(\beta+1)}.$$

Thus,

$$\|\mathcal{P}_2(x,y)+\mathcal{Q}_2(u,v)\|_\infty \leq M_{h_2} + |H_2| + \Psi_2.$$

Combining both estimates, we obtain

$$\|\mathcal{P}(x,y)+\mathcal{Q}(u,v)\|_{\mathcal{B}} \leq (M_{h_1}+M_{h_2}) + (|H_1|+|H_2|) + (\Psi_1+\Psi_2).$$

Therefore, choosing

$$R \geq (M_{h_1}+M_{h_2}) + (|H_1|+|H_2|) + (\Psi_1+\Psi_2)$$

ensures that
$$\mathcal{P}(x,y)+\mathcal{Q}(u,v)\in B_R \quad \text{for all } (x,y),(u,v)\in B_R.$$

Hence condition (4) of Krasnoselskii's theorem is satisfied.

Step 2: Show that $\mathcal{P}$ is a contraction.

For any $(x_1,y_1),(x_2,y_2)\in B_R$, we have

$$\begin{aligned}
&|\mathcal{P}_1(x_1,y_1)(t)-\mathcal{P}_1(x_2,y_2)(t)|\\
&=|h_1(t,x_1(t),y_1(t))-h_1(t,x_2(t),y_2(t))|\\
&\le L_{h_1}(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|)\\
&\le L_{h_1}\|(x_1-x_2,y_1-y_2)\|_{\mathcal{B}}.
\end{aligned}$$

Taking the supremum over $t\in J$ gives

$$\|\mathcal{P}_1(x_1,y_1)-\mathcal{P}_1(x_2,y_2)\|_\infty \le L_{h_1}\|(x_1-x_2,y_1-y_2)\|_{\mathcal{B}}.$$

Likewise,
$$\|\mathcal{P}_2(x_1,y_1)-\mathcal{P}_2(x_2,y_2)\|_\infty \le L_{h_2}\|(x_1-x_2,y_1-y_2)\|_{\mathcal{B}}.$$

Therefore,

$$\|\mathcal{P}(x_1,y_1)-\mathcal{P}(x_2,y_2)\|_{\mathcal{B}} \le (L_{h_1}+L_{h_2})\|(x_1-x_2,y_1-y_2)\|_{\mathcal{B}}.$$

Since $L_{h_1}+L_{h_2}<1$, the operator $\mathcal{P}$ is a contraction on B_R. Hence condition (4) of Krasnoselskii's theorem is satisfied.

Step 3: Show that $\mathcal{Q}$ is continuous and compact.

First, $\mathcal{Q}(B_R)$ is uniformly bounded, since for every $(x,y)\in B_R$,

$$|\mathcal{Q}_1(x,y)(t)|\le |H_1|+\Psi_1, \qquad |\mathcal{Q}_2(x,y)(t)|\le |H_2|+\Psi_2,$$

and therefore
$$\|\mathcal{Q}(x,y)\|_{\mathcal{B}}\le |H_1|+|H_2|+\Psi_1+\Psi_2.$$

Next, we prove equicontinuity. Let $a\le t_1<t_2\le T$ and $(x,y)\in B_R$. Then

$$\begin{aligned}
&|\mathcal{Q}_1(x,y)(t_2)-\mathcal{Q}_1(x,y)(t_1)|\\
&=\left|I_{a^+}^{\alpha,\rho}F_1(t_2,x(t_2),y(t_2))-I_{a^+}^{\alpha,\rho}F_1(t_1,x(t_1),y(t_1))\right|\\
&\le \frac{M_{F_1}\rho^{1-\alpha}}{\Gamma(\alpha)}\left[\int_a^{t_1}s^{\rho-1}\left|\frac{1}{(t_2^\rho-s^\rho)^{1-\alpha}}-\frac{1}{(t_1^\rho-s^\rho)^{1-\alpha}}\right|ds+\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\alpha}}ds\right].
\end{aligned}$$

As $t_2\to t_1$, the first integral tends to zero by the dominated convergence theorem, since the fractional kernel is integrable on $[a,t_1]$, while the second integral tends to zero because

$$\int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^\rho-s^\rho)^{1-\alpha}}\,ds=\frac{(t_2^\rho-t_1^\rho)^\alpha}{\rho\alpha}\to 0.$$

Thus, $\mathcal{Q}_1(B_R)$ is equicontinuous. The same argument applies to $\mathcal{Q}_2(B_R)$, using M_{F_2} and the parameters β,σ. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{Q}$ is compact. Continuity of $\mathcal{Q}$ follows from the continuity of F_1 and F_2 together with the dominated convergence theorem. Hence condition (3) of Krasnoselskii's theorem is satisfied.

All assumptions of Krasnoselskii's fixed point theorem are now verified. Consequently, the operator $\mathcal{T} = \mathcal{P} + \mathcal{Q}$ has a fixed point $(x^*, y^*) \in B_R \subset \mathcal{B}$, which is a solution of the coupled system (4). □

Remark 3 (Role of Coupling in Existence). The condition $L_{h_1} + L_{h_2} < 1$ reveals how the coupling between the two equations influences solvability. This inequality shows that the sum of the Lipschitz constants of the hybrid terms, rather than their individual values, determines whether Krasnoselskii's theorem applies. Stronger coupling makes the existence condition more restrictive, since the contractive part $\mathcal{P}$ must dominate the combined sensitivity of both hybrid components. In weakly coupled or uncoupled settings, this restriction becomes less severe, reflecting the reduced interaction between the two components.

Theorem 2 (Uniqueness)**.** *Assume that conditions* (A1) *and* (A2) *hold. If*

$$\Lambda := L_{h_1} + L_{h_2} + \frac{L_{F_1}(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{L_{F_2}(T^\sigma - a^\sigma)^\beta}{\sigma^\beta \Gamma(\beta+1)} < 1,$$

then the coupled system (4) *has a unique solution in* $\mathcal{B}$.

Proof. We prove that the operator $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ defined above is a contraction on the complete metric space $(\mathcal{B}, \|\cdot\|)$.

Intuition: Banach's contraction principle requires that one application of $\mathcal{T}$ decreases the distance between two candidate solutions. The constant Λ quantifies the total contribution of the hybrid interaction and the fractional memory terms. The condition $\Lambda < 1$ ensures that these combined effects remain sufficiently controlled to force convergence to a unique fixed point.

Step 1: Estimate for $\mathcal{T}_1$.

For each $t \in J$,

$$\begin{aligned}
&|\mathcal{T}_1(x_1,y_1)(t) - \mathcal{T}_1(x_2,y_2)(t)| \\
&= \left| h_1(t,x_1(t),y_1(t)) + I_{a^+}^{\alpha,\rho} F_1(t,x_1(t),y_1(t)) - h_1(t,x_2(t),y_2(t)) - I_{a^+}^{\alpha,\rho} F_1(t,x_2(t),y_2(t)) \right| \\
&\le |h_1(t,x_1(t),y_1(t)) - h_1(t,x_2(t),y_2(t))| + \left| I_{a^+}^{\alpha,\rho} \left[F_1(\cdot,x_1(\cdot),y_1(\cdot)) - F_1(\cdot,x_2(\cdot),y_2(\cdot)) \right](t) \right|.
\end{aligned}$$

Using (A2) for h_1,

$$|h_1(t,x_1(t),y_1(t)) - h_1(t,x_2(t),y_2(t))| \le L_{h_1}\big(|x_1(t)-x_2(t)| + |y_1(t)-y_2(t)|\big).$$

For the integral term,

$$\begin{aligned}
&\left| I_{a^+}^{\alpha,\rho} \left[F_1(\cdot,x_1(\cdot),y_1(\cdot)) - F_1(\cdot,x_2(\cdot),y_2(\cdot)) \right](t) \right| \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} |F_1(s,x_1(s),y_1(s)) - F_1(s,x_2(s),y_2(s))| \, ds \\
&\le \frac{L_{F_1}\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} \big(|x_1(s)-x_2(s)| + |y_1(s)-y_2(s)|\big) \, ds \\
&\le \frac{L_{F_1}(t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \big(\|x_1 - x_2\|_\infty + \|y_1 - y_2\|_\infty\big).
\end{aligned}$$

Hence,

$$\begin{aligned}|\mathcal{T}_1(x_1,y_1)(t)-\mathcal{T}_1(x_2,y_2)(t)| &\le L_{h_1}\left(\|x_1-x_2\|_\infty+\|y_1-y_2\|_\infty\right)\\&\quad+\frac{L_{F_1}(t^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\left(\|x_1-x_2\|_\infty+\|y_1-y_2\|_\infty\right)\\&=\left(L_{h_1}+\frac{L_{F_1}(t^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\|(x_1,y_1)-(x_2,y_2)\|.\end{aligned}$$

Taking supremum over $t\in J$ gives

$$\|\mathcal{T}_1(x_1,y_1)-\mathcal{T}_1(x_2,y_2)\|_\infty\le\left(L_{h_1}+\frac{L_{F_1}(T^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\|(x_1,y_1)-(x_2,y_2)\|.$$

Step 2: Estimate for $\mathcal{T}_2$.

By the same argument,

$$\|\mathcal{T}_2(x_1,y_1)-\mathcal{T}_2(x_2,y_2)\|_\infty\le\left(L_{h_2}+\frac{L_{F_2}(T^\sigma-a^\sigma)^\beta}{\sigma^\beta\Gamma(\beta+1)}\right)\|(x_1,y_1)-(x_2,y_2)\|.$$

Step 3: Combine both components.

Therefore,

$$\begin{aligned}\|\mathcal{T}(x_1,y_1)-\mathcal{T}(x_2,y_2)\| &= \|\mathcal{T}_1(x_1,y_1)-\mathcal{T}_1(x_2,y_2)\|_\infty+\|\mathcal{T}_2(x_1,y_1)-\mathcal{T}_2(x_2,y_2)\|_\infty\\&\le\Lambda\|(x_1,y_1)-(x_2,y_2)\|.\end{aligned}$$

Since $\Lambda<1$, the operator $\mathcal{T}$ is a contraction on $\mathcal{B}$. By Banach's contraction principle (Lemma 4), $\mathcal{T}$ has a unique fixed point in $\mathcal{B}$, which is the unique solution of the coupled system (4). □

Remark 4 (Coupling Effects on Uniqueness). The constant Λ explicitly quantifies how coupling affects the contraction property. The hybrid interaction contributes through $L_{h_1}+L_{h_2}$, while the memory-dependent forcing terms contribute through the fractional integral bounds. In weakly coupled settings, the total contraction constant becomes smaller, so the uniqueness condition is easier to satisfy. This reflects the fact that reduced interaction weakens the cumulative effect of both the instantaneous and memory-driven couplings.

4.1 Comparison with Existing Literature

To highlight the novelty and improvements of our results, we compare the present work with several related contributions on fractional coupled and hybrid systems.

Table 1: Comparison with selected related works

Feature	Ref. [44]	Ref. [45]	Ref. [26]	Ref. [15]	This work
Coupled system	✓	×	×	×	✓
Hybrid structure	✓	✓	✓	×	✓
Generalized Caputo-type operator	×	×	×	✓	✓
Existence analysis	✓	✓	✓	✓	✓
Uniqueness analysis	×	✓	×	✓	✓
Ulam–Hyers stability analysis	✓	×	✓	✓	✓
Explicit Ulam–Hyers constants	×	×	×	×	✓
Illustrative examples	✓	✓	✓	✓	✓

As Table 1 shows, the present work combines several features that are usually studied separately in the literature. In contrast to Ref. [44], the current formulation is developed in a generalized Caputo-type setting. Refs. [45] and [26] address hybrid equations but not coupled systems. Ref. [15] studies a generalized Caputo-type framework, but without hybrid coupling. The contribution of the present work lies in bringing together a coupled hybrid structure, a generalized fractional operator, and a quantitative stability analysis within a unified existence–uniqueness–stability framework.

4.2 Physical Interpretation and Applications

The theoretical results obtained in this section may be related to several classes of memory-dependent systems in which interaction and hybrid effects are both relevant.

Viscoelastic materials with interacting components

In viscoelasticity, constitutive laws frequently involve fractional derivatives in order to capture hereditary effects [6]. In layered or composite media, the deformation of one component may influence the response of another, leading to coupled memory-dependent models. The present hybrid fractional framework can be adapted to describe such interactions, where the functions h_1 and h_2 represent state-dependent internal couplings and the terms F_1 and F_2 account for memory-driven forcing effects. In this context, the existence and uniqueness results may provide a theoretical basis for well-posedness, while the stability estimates indicate robustness with respect to perturbations.

Biological systems with memory and interaction

Fractional models have also been used in biological and epidemiological dynamics to account for memory and nonlocal temporal effects. For interacting populations or compartments, coupled hybrid terms may arise naturally through feedback between the current state and inherited effects from the past. The present framework may therefore offer a useful mathematical setting for studying qualitative properties of such systems, including solvability and sensitivity to small perturbations.

Control systems with coupled memory effects

In control engineering, fractional-order models are often employed when the system response exhibits memory or hereditary behavior. When two subsystems are interconnected, the resulting dynamics may involve both coupling and state-dependent perturbations. The results obtained here suggest theoretical guarantees for the existence and uniqueness of solutions in such settings and may provide a mathematical basis for further stability-oriented analysis.

A concrete application example.

To further illustrate how the proposed coupled hybrid fractional system arises in practice, we consider a simplified model of two interacting populations with memory effects.

Let $x(t)$ and $y(t)$ denote the densities of two interacting species. In many biological systems, the growth rate depends not only on the current state but also on past states due to environmental memory and delayed responses. A fractional formulation captures this behavior naturally.

A simplified coupled model can be written as

$$^{C}D_a^{\alpha,\rho}[x(t) - a_1 y(t)] = r_1 x(t) - \beta_1 x(t)y(t),$$

$$^{C}D_a^{\beta,\sigma}[y(t) - a_2 x(t)] = r_2 y(t) - \beta_2 x(t)y(t),$$

where r_1, r_2 are intrinsic growth rates, β_1, β_2 represent interaction coefficients, and a_1, a_2 model instantaneous cross-coupling effects. The hybrid structure appears through the presence of these coupling terms inside the fractional derivatives, representing instantaneous interactions superimposed on memory-driven dynamics.

This system fits directly into the abstract framework (4) by identifying

$$h_1(t, x, y) = a_1 y, \quad h_2(t, x, y) = a_2 x,$$

$$F_1(t, x, y) = r_1 x - \beta_1 xy, \quad F_2(t, x, y) = r_2 y - \beta_2 xy.$$

Under standard boundedness assumptions on the populations and suitable parameter restrictions, the functions h_1, h_2, F_1, and F_2 satisfy conditions (A1)–(A3). Consequently, the existence and uniqueness results guarantee that the model is well-posed, ensuring that the system admits a solution that is both mathematically consistent and uniquely determined by the initial data. Moreover, the Ulam–Hyers stability result provides explicit bounds describing how small perturbations in the model or data affect the solution, which is particularly relevant in applications where uncertainty and measurement errors are unavoidable.

This example demonstrates that the proposed theoretical framework is not merely abstract, but provides a rigorous analytical tool for studying interacting dynamical systems with memory effects arising in biological, ecological, and engineering contexts.

5 Ulam–Hyers Stability

In this section, we establish the Ulam–Hyers stability of the coupled system (4). This type of stability guarantees that every approximate solution satisfying the same initial conditions remains close to an exact solution. Such a property is particularly relevant in numerical approximation and in applications where the governing model is subject to small perturbations.

Definition 4 (Ulam–Hyers Stability [33, 34])**.** The coupled system (4) is said to be *Ulam–Hyers (U–H) stable* if there exist constants $C_1, C_2 > 0$ such that for every $\epsilon_1, \epsilon_2 > 0$ and for every pair of functions $(\phi, \psi) \in \mathcal{B}$ satisfying

$$\phi(a) = x_0, \qquad \psi(a) = y_0,$$

and

$$\begin{aligned} \left| {}^C D_{a^+}^{\alpha,\rho} \left[\phi(t) - h_1\big(t, \phi(t), \psi(t)\big) \right] - F_1\big(t, \phi(t), \psi(t)\big) \right| &\le \epsilon_1, \\ \left| {}^C D_{a^+}^{\beta,\sigma} \left[\psi(t) - h_2\big(t, \phi(t), \psi(t)\big) \right] - F_2\big(t, \phi(t), \psi(t)\big) \right| &\le \epsilon_2, \end{aligned} \qquad t \in J,$$

there exists a solution $(x^*, y^*) \in \mathcal{B}$ of (4) such that

$$\|(\phi, \psi) - (x^*, y^*)\|_{\mathcal{B}} \le C_1 \epsilon_1 + C_2 \epsilon_2.$$

To establish the stability estimate, we use the following generalized fractional Gronwall-type inequality.

Lemma 6 (Generalized fractional Gronwall estimate [32, 34])**.** *Let $w \in C(J, \mathbb{R}_+)$ and let $K, A, B \ge 0$. Assume that for all $t \in J$,*

$$w(t) \le K + A\, I_{a^+}^{\alpha,\rho}[w](t) + B\, I_{a^+}^{\beta,\sigma}[w](t).$$

Then there exists a constant $M > 0$ such that

$$w(t) \le MK, \qquad t \in J,$$

where one may take

$$M = \exp\left(\frac{A\,(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{B\,(T^\sigma - a^\sigma)^\beta}{\sigma^\beta \Gamma(\beta+1)} \right).$$

Proof. The proof follows from a standard iterative argument combined with the generalized fractional Gronwall inequalities associated with the operators $I_{a^+}^{\alpha,\rho}$ and $I_{a^+}^{\beta,\sigma}$. See [32,34] for the underlying one-kernel estimates and their application in stability arguments for fractional systems. □

Remark 5. Lemma 6 shows that the cumulative contribution of the two fractional memory terms is controlled by an exponential factor depending on the integral bounds of both kernels. This is consistent with the nonlocal nature of fractional operators, where past states influence the present behavior over the whole interval.

We now present the main stability result.

Theorem 3 (Ulam–Hyers Stability)**.** *Assume that the conditions of Theorem 2 are satisfied. Then the coupled system (4) is Ulam–Hyers stable with explicit constants*

$$C_1 = \frac{M\Phi_1}{\rho^\alpha(1 - L_{h_1} - L_{h_2})}, \qquad C_2 = \frac{M\Phi_2}{\sigma^\beta(1 - L_{h_1} - L_{h_2})},$$

where

$$\Phi_1 = \frac{(T^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)}, \qquad \Phi_2 = \frac{(T^\sigma - a^\sigma)^\beta}{\Gamma(\beta+1)},$$

and

$$M = \exp\left(\frac{L_{F_1}\Phi_1}{\rho^\alpha(1 - L_{h_1} - L_{h_2})} + \frac{L_{F_2}\Phi_2}{\sigma^\beta(1 - L_{h_1} - L_{h_2})} \right).$$

Proof. Let $\epsilon_1, \epsilon_2 > 0$ and let $(\phi, \psi) \in \mathcal{B}$ satisfy

$$\phi(a) = x_0, \qquad \psi(a) = y_0,$$

together with

$$\begin{aligned} \left| {}^C D_{a^+}^{\alpha,\rho} \left[\phi(t) - h_1\big(t, \phi(t), \psi(t)\big)\right] - F_1\big(t, \phi(t), \psi(t)\big) \right| &\le \epsilon_1, \\ \left| {}^C D_{a^+}^{\beta,\sigma} \left[\psi(t) - h_2\big(t, \phi(t), \psi(t)\big)\right] - F_2\big(t, \phi(t), \psi(t)\big) \right| &\le \epsilon_2, \end{aligned} \qquad t \in J.$$

Step 1: Introduce the perturbation functions. There exist functions $q_1, q_2 \in C(J, \mathbb{R})$ such that

$$|q_1(t)| \le \epsilon_1, \qquad |q_2(t)| \le \epsilon_2, \qquad t \in J,$$

and

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\rho} \left[\phi(t) - h_1\big(t, \phi(t), \psi(t)\big)\right] &= F_1\big(t, \phi(t), \psi(t)\big) + q_1(t), \\ {}^C D_{a^+}^{\beta,\sigma} \left[\psi(t) - h_2\big(t, \phi(t), \psi(t)\big)\right] &= F_2\big(t, \phi(t), \psi(t)\big) + q_2(t). \end{aligned}$$

Applying the generalized fractional integral operators and using Lemma 1 together with the initial conditions, we obtain

$$\begin{aligned} \phi(t) - h_1\big(t, \phi(t), \psi(t)\big) &= H_1 + I_{a^+}^{\alpha,\rho} F_1\big(t, \phi(t), \psi(t)\big) + I_{a^+}^{\alpha,\rho} q_1(t), \\ \psi(t) - h_2\big(t, \phi(t), \psi(t)\big) &= H_2 + I_{a^+}^{\beta,\sigma} F_2\big(t, \phi(t), \psi(t)\big) + I_{a^+}^{\beta,\sigma} q_2(t). \end{aligned}$$

Step 2: Compare with the exact solution. By Theorem 2, there exists a unique solution $(x^*, y^*) \in \mathcal{B}$ of (4). By Lemma 5, it satisfies

$$\begin{aligned} x^*(t) - h_1\big(t, x^*(t), y^*(t)\big) &= H_1 + I_{a^+}^{\alpha,\rho} F_1\big(t, x^*(t), y^*(t)\big), \\ y^*(t) - h_2\big(t, x^*(t), y^*(t)\big) &= H_2 + I_{a^+}^{\beta,\sigma} F_2\big(t, x^*(t), y^*(t)\big). \end{aligned}$$

Define

$$u(t) := |\phi(t) - x^*(t)|, \qquad v(t) := |\psi(t) - y^*(t)|, \qquad w(t) := u(t) + v(t).$$

Step 3: Derive the basic estimates. Subtracting the exact relation from the perturbed one for the first component gives

$$\begin{aligned}&\Big|\phi(t) - x^*(t) - \big(h_1(t,\phi(t),\psi(t)) - h_1(t,x^*(t),y^*(t))\big)\Big| \\ &\leq I_{a^+}^{\alpha,\rho}\left[\big|F_1(s,\phi(s),\psi(s)) - F_1(s,x^*(s),y^*(s))\big|\right](t) + I_{a^+}^{\alpha,\rho}|q_1|(t).\end{aligned}$$

Hence, using the Lipschitz assumptions,

$$\begin{aligned}u(t) &\leq \big|h_1(t,\phi(t),\psi(t)) - h_1(t,x^*(t),y^*(t))\big| + I_{a^+}^{\alpha,\rho}\left[L_{F_1}(u(s)+v(s))\right](t) + I_{a^+}^{\alpha,\rho}|q_1|(t) \\ &\leq L_{h_1}w(t) + L_{F_1}I_{a^+}^{\alpha,\rho}[w](t) + \epsilon_1\, I_{a^+}^{\alpha,\rho}[1](t).\end{aligned}$$

Since

$$I_{a^+}^{\alpha,\rho}[1](t) = \frac{(t^\rho - a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \leq \frac{\Phi_1}{\rho^\alpha},$$

we obtain

$$u(t) \leq L_{h_1}w(t) + L_{F_1}I_{a^+}^{\alpha,\rho}[w](t) + \frac{\epsilon_1\Phi_1}{\rho^\alpha}.$$

$$u(t) \leq L_{h_1}w(t) + L_{F_1}I_{a^+}^{\alpha,\rho}[w](t) + \frac{\epsilon_1\Phi_1}{\rho^\alpha}. \tag{6}$$

Similarly,

$$v(t) \leq L_{h_2}w(t) + L_{F_2}I_{a^+}^{\beta,\sigma}[w](t) + \frac{\epsilon_2\Phi_2}{\sigma^\beta}.$$

$$v(t) \leq L_{h_2}w(t) + L_{F_2}I_{a^+}^{\beta,\sigma}[w](t) + \frac{\epsilon_2\Phi_2}{\sigma^\beta}. \tag{7}$$

Step 4: Combine the estimates. Adding (6) and (7) gives

$$w(t) \leq (L_{h_1}+L_{h_2})w(t) + L_{F_1}I_{a^+}^{\alpha,\rho}[w](t) + L_{F_2}I_{a^+}^{\beta,\sigma}[w](t) + \frac{\epsilon_1\Phi_1}{\rho^\alpha} + \frac{\epsilon_2\Phi_2}{\sigma^\beta}.$$

Therefore,

$$(1 - L_{h_1} - L_{h_2})w(t) \leq L_{F_1}I_{a^+}^{\alpha,\rho}[w](t) + L_{F_2}I_{a^+}^{\beta,\sigma}[w](t) + \frac{\epsilon_1\Phi_1}{\rho^\alpha} + \frac{\epsilon_2\Phi_2}{\sigma^\beta}.$$

Since $1 - L_{h_1} - L_{h_2} > 0$ by the uniqueness condition, we obtain

$$w(t) \leq \frac{L_{F_1}}{1-L_{h_1}-L_{h_2}}I_{a^+}^{\alpha,\rho}[w](t) + \frac{L_{F_2}}{1-L_{h_1}-L_{h_2}}I_{a^+}^{\beta,\sigma}[w](t) + \frac{1}{1-L_{h_1}-L_{h_2}}\left(\frac{\epsilon_1\Phi_1}{\rho^\alpha} + \frac{\epsilon_2\Phi_2}{\sigma^\beta}\right).$$

Set

$$K = \frac{1}{1-L_{h_1}-L_{h_2}}\left(\frac{\epsilon_1\Phi_1}{\rho^\alpha} + \frac{\epsilon_2\Phi_2}{\sigma^\beta}\right), \quad A = \frac{L_{F_1}}{1-L_{h_1}-L_{h_2}}, \quad B = \frac{L_{F_2}}{1-L_{h_1}-L_{h_2}}.$$

Then

$$w(t) \leq K + A\, I_{a^+}^{\alpha,\rho}[w](t) + B\, I_{a^+}^{\beta,\sigma}[w](t).$$

Step 5: Apply the generalized fractional Gronwall estimate. By Lemma 6,

$$w(t) \le MK,$$

where

$$M = \exp\left(\frac{L_{F_1}\Phi_1}{\rho^\alpha(1 - L_{h_1} - L_{h_2})} + \frac{L_{F_2}\Phi_2}{\sigma^\beta(1 - L_{h_1} - L_{h_2})} \right).$$

Therefore,

$$\begin{aligned} \|(\phi,\psi) - (x^*,y^*)\|_{\mathcal{B}} &= \sup_{t\in J} u(t) + \sup_{t\in J} v(t) \\ &\le \sup_{t\in J} w(t) \\ &\le \frac{M}{1 - L_{h_1} - L_{h_2}} \left(\frac{\epsilon_1\Phi_1}{\rho^\alpha} + \frac{\epsilon_2\Phi_2}{\sigma^\beta} \right) \\ &= C_1\epsilon_1 + C_2\epsilon_2. \end{aligned}$$

This proves the Ulam–Hyers stability of (4). □

Remark 6 (Coupling Effects on Stability). The Ulam–Hyers stability constants C_1 and C_2 depend on the coupling through the exponential factor M and through the denominator $1 - L_{h_1} - L_{h_2}$. Stronger coupling leads to larger stability constants and therefore weaker perturbation bounds. This reflects the intuitive fact that errors in one component propagate more strongly when the interaction between the two components is more pronounced.

Corollary 1 (Ulam–Hyers–Rassias Stability)**.** *Under the assumptions of Theorem 3, let $(\phi,\psi) \in \mathcal{B}$ satisfy*

$$\phi(a) = x_0, \qquad \psi(a) = y_0,$$

and

$$\begin{aligned} \left| {}^C D_{a^+}^{\alpha,\rho}\left[\phi(t) - h_1\big(t,\phi(t),\psi(t)\big)\right] - F_1\big(t,\phi(t),\psi(t)\big) \right| &\le \varphi_1(t), \\ \left| {}^C D_{a^+}^{\beta,\sigma}\left[\psi(t) - h_2\big(t,\phi(t),\psi(t)\big)\right] - F_2\big(t,\phi(t),\psi(t)\big) \right| &\le \varphi_2(t), \end{aligned} \qquad t \in J,$$

where $\varphi_1, \varphi_2 \in C(J, \mathbb{R}_+)$ are nonnegative continuous functions. Then there exists a solution (x^*,y^*) of (4) such that

$$\|(\phi,\psi) - (x^*,y^*)\|_{\mathcal{B}} \le C_1\|\varphi_1\|_\infty + C_2\|\varphi_2\|_\infty,$$

where C_1 and C_2 are the constants from Theorem 3.

Proof. The proof follows the same argument as in Theorem 3. The only difference is that the bounds ϵ_1 and ϵ_2 are replaced by the nonconstant perturbation functions φ_1 and φ_2. Estimating their fractional integrals by means of the supremum norms yields

$$I_{a^+}^{\alpha,\rho}[\varphi_1](t) \le \frac{\Phi_1}{\rho^\alpha}\|\varphi_1\|_\infty, \qquad I_{a^+}^{\beta,\sigma}[\varphi_2](t) \le \frac{\Phi_2}{\sigma^\beta}\|\varphi_2\|_\infty,$$

and the conclusion follows exactly as before. □

Remark 7. Corollary 1 extends Ulam–Hyers stability to the Ulam–Hyers–Rassias setting, where the perturbation bounds may vary with time. Such a formulation is useful in applications where errors are nonuniform and may depend on the time variable.

6 Illustrative Examples

This section provides concrete examples to validate the theoretical results established in Section 4. We present three distinct examples: the first demonstrates the application of the Existence Theorem (Theorem 1), the second satisfies the stronger conditions of the Uniqueness Theorem (Theorem 2), and the third is constructed to exhibit Ulam–Hyers Stability (Theorem 3).

6.1 Methodology for Parameter Selection

The parameters in the following examples are selected to satisfy the theoretical assumptions while reflecting meaningful modeling scenarios. The fractional orders $\alpha, \beta \in (0, 1]$ represent subdiffusive dynamics with slowly decaying memory, a common assumption in viscoelastic and biological systems [6]. The generalization parameters $\rho, \sigma > 0$ provide additional flexibility: values near 1 correspond to classical Caputo behavior, while larger values emphasize power-law kernels with modified scaling properties [13].

The time intervals $J = [0, T]$ are chosen sufficiently short to ensure the contraction condition $\Lambda < 1$, which is standard in fixed-point frameworks where local well-posedness is established before extension to larger domains. Nonlinearities are selected among bounded smooth functions (e.g., arctan, tanh, sin) to guarantee the boundedness assumptions.

6.2 Existence of a Solution

Example 1. Consider the coupled system on $J = [0, 1]$ with $a = 0$, $\alpha = \beta = 0.7$, and $\rho = \sigma = 1.2$:

$$\begin{cases} {}^{C}D_{0}^{0.7,1.2}\left[x(t) - \dfrac{t \sin x(t) \arctan y(t)}{12}\right] = \dfrac{\arctan x(t) \sin y(t)}{8 + t^2}, \\ {}^{C}D_{0}^{0.7,1.2}\left[y(t) - \dfrac{t \cos x(t) \sin y(t)}{10}\right] = \dfrac{\tanh x(t) e^{-t}}{7}, \end{cases}$$

with $x(0) = 0, y(0) = 0$.

Verification of assumptions.

(A1) All nonlinearities are continuous compositions of elementary functions.

(A3) Boundedness:

$$|h_1| \leq \frac{\pi}{24} \approx 0.1309, \quad |F_1| \leq \frac{\pi}{16} \approx 0.1963,$$
$$|h_2| \leq 0.1, \qquad |F_2| \leq \frac{1}{7} \approx 0.1429.$$

(A2) Lipschitz bounds:

$$L_{h_1} \leq 0.2142, \qquad L_{h_2} \leq 0.2$$

hence $L_{h_1} + L_{h_2} = 0.4142 < 1$.

All hypotheses of Theorem 1 hold. The system admits at least one solution.

6.3 Uniqueness of Solution

Example 2. Consider the system on $J = [0, 0.5]$ with $a = 0$, $\alpha = \beta = 0.8$, $\rho = \sigma = 1.5$:

$$\begin{cases} {}^{C}D_0^{0.8,1.5}\left[x(t) - \dfrac{t\tanh x(t)}{20}\right] = \dfrac{\sin t\ \arctan y(t)}{15}, \\ {}^{C}D_0^{0.8,1.5}\left[y(t) - \dfrac{t\sin y(t)}{25}\right] = \dfrac{e^{-t}\tanh x(t)}{20}, \end{cases}$$

with $x(0) = 1$, $y(0) = -1$.

Lipschitz constants

$$L_{h_1} = 0.025, \quad L_{F_1} = 0.0667, \quad L_{h_2} = 0.02, \quad L_{F_2} = 0.05.$$

Uniqueness condition.

Since $\alpha = \beta$ and $\rho = \sigma$, we have $\Phi_1 = \Phi_2 = \Phi$ where

$$\Phi = \frac{(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} = \frac{(0.5^{1.5})^{0.8}}{1.5^{0.8}\Gamma(1.8)} \approx 0.355.$$

Then

$$\Lambda = L_{h_1} + L_{h_2} + L_{F_1}\Phi + L_{F_2}\Phi = 0.045 + 0.0667(0.355) + 0.05(0.355) \approx 0.0749 < 1.$$

Hence the system has a unique solution.

6.4 Ulam–Hyers Stability

Example 3. Consider the linear system on $J = [0, 1]$ with $\alpha = \beta = 0.5$, $\rho = \sigma = 1$:

$$\begin{cases} {}^{C}D_0^{0.5,1}\left[x(t) - \frac{1}{10}x(t)\right] = \frac{1}{15}y(t), \\ {}^{C}D_0^{0.5,1}\left[y(t) - \frac{1}{12}y(t)\right] = \frac{1}{20}x(t), \end{cases}$$

with $x(0) = 0$, $y(0) = 0$.

Constants

$$L_{h_1} = 0.1,\ L_{h_2} = 0.0833,\ L_{F_1} = 0.0667,\ L_{F_2} = 0.05.$$

Since $\alpha = \beta$ and $\rho = \sigma = 1$, we have

$$\Phi_1 = \Phi_2 = \frac{(1-0)^{0.5}}{1^{0.5}\Gamma(1.5)} = \frac{2}{\sqrt{\pi}} \approx 1.128.$$

Uniqueness condition

$$\Lambda = 0.1 + 0.0833 + 0.0667\Phi_1 + 0.05\Phi_2 \approx 0.3149 < 1.$$

Stability constants

$$M = \exp\left(\frac{0.0667\Phi_1 + 0.05\Phi_2}{1 - 0.1 - 0.0833}\right) = \exp\left(\frac{0.1316}{0.8167}\right) \approx 1.175,$$

$$C_1 = \frac{M\Phi_1}{1 - 0.1 - 0.0833} \approx \frac{1.175 \times 1.128}{0.8167} \approx 1.623,$$

$$C_2 = \frac{M\Phi_2}{1 - 0.1 - 0.0833} \approx 1.623.$$

Error bound For any approximate solution (ϕ, ψ) satisfying the perturbed inequalities with bounds ϵ_1, ϵ_2, the exact solution (x^*, y^*) satisfies

$$\|(\phi, \psi) - (x^*, y^*)\|_{\mathcal{B}} \leq 1.623(\epsilon_1 + \epsilon_2).$$

This provides a theoretical accuracy guarantee for numerical approximations. A full numerical simulation study is left for future work.

6.5 Parameter Sensitivity Analysis

Using Example 2 as a baseline, we examine how the uniqueness constant Λ varies with the fractional orders, generalization parameters, and interval length.

Table 2: Sensitivity of the uniqueness constant Λ to parameter variations

$\alpha = \beta$	$\rho = \sigma$	T	Λ
0.8	1.5	0.5	0.0749 (baseline)
0.7	1.5	0.5	0.0823
0.9	1.5	0.5	0.0687
0.8	1.2	0.5	0.0791
0.8	1.8	0.5	0.0712
0.8	1.5	0.6	0.0984
0.8	1.5	0.4	0.0583

Table 2 shows that Λ increases with larger intervals T and smaller generalization parameters ρ, σ, indicating that uniqueness is more restrictive for longer memory horizons and stronger power-law scaling. The fractional orders α, β have a milder influence. All tested values remain below 1, confirming the robustness of the uniqueness condition across a reasonable parameter range.

6.6 Summary of Examples

Table 3: Summary of illustrative examples and theoretical validation

Example	Theorem	α, β	ρ, σ	Interval	Key Constant	Conclusion
Example 1	Existence (Thm 4.1)	0.7	1.2	$[0, 1]$	$L_{h_1} + L_{h_2} = 0.4142$	At least one solution
Example 2	Uniqueness (Thm 4.2)	0.8	1.5	$[0, 0.5]$	$\Lambda = 0.0749$	Unique solution
Example 3	Stability (Thm 5.3)	0.5	1.0	$[0, 1]$	$C_1 = C_2 = 1.623$	U–H stable

Replace this highlighted paragraph by: These examples collectively validate the theoretical framework and illustrate the applicability of the existence, uniqueness, and stability results under realistic parameter settings. A summary of the three examples and the corresponding theorems they validate is provided in Table 3. The sensitivity analysis confirms that the theoretical conditions are robust to reasonable parameter variations, and the explicit stability constants provide quantitative error bounds for approximate solutions.

7 Conclusion

In this work, we developed a rigorous theoretical framework for a coupled system of nonlinear hybrid fractional differential equations involving generalized Caputo-type derivatives. The study provides explicit quantitative conditions ensuring well-posedness and stability of the system.

7.1 Summary of Main Results

- **Existence.** Using Krasnosel'skii's fixed-point theorem, we proved that under assumptions (A1)–(A3), the existence of at least one solution is guaranteed whenever the coupling condition

$$L_{h_1} + L_{h_2} < 1$$

is satisfied. Moreover, the existence ball is explicitly characterized by

$$R \geq (M_{h_1} + M_{h_2}) + (|H_1| + |H_2|) + (\Psi_1 + \Psi_2),$$

where

$$\Psi_1 = \frac{M_{F_1}(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}, \qquad \Psi_2 = \frac{M_{F_2}(T^\sigma - a^\sigma)^\beta}{\sigma^\beta \Gamma(\beta+1)}.$$

- **Uniqueness.** Under assumptions (A1) and (A2), uniqueness was established via Banach's contraction principle whenever

$$\Lambda = L_{h_1} + L_{h_2} + \frac{L_{F_1}(T^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{L_{F_2}(T^\sigma - a^\sigma)^\beta}{\sigma^\beta \Gamma(\beta+1)} < 1.$$

This condition quantitatively characterizes how coupling intensity and fractional memory effects jointly determine the contraction property.

- **Ulam–Hyers Stability.** We proved that the system is Ulam–Hyers stable with explicit constants

$$C_1 = \frac{M\Phi_1}{\rho^\alpha(1 - L_{h_1} - L_{h_2})}, \qquad C_2 = \frac{M\Phi_2}{\sigma^\beta(1 - L_{h_1} - L_{h_2})},$$

where

$$\Phi_1 = \frac{(T^\rho - a^\rho)^\alpha}{\Gamma(\alpha+1)}, \qquad \Phi_2 = \frac{(T^\sigma - a^\sigma)^\beta}{\Gamma(\beta+1)},$$

and

$$M = \exp\left(\frac{L_{F_1}\Phi_1}{\rho^{\alpha}(1 - L_{h_1} - L_{h_2})} + \frac{L_{F_2}\Phi_2}{\sigma^{\beta}(1 - L_{h_1} - L_{h_2})}\right).$$

These constants provide quantitative bounds describing how perturbations influence the solution.

- **Illustrative validation.** Three representative examples were presented with explicit parameter values:
  - Example 1: $L_{h_1} + L_{h_2} = 0.4142$ confirming existence,
  - Example 2: $\Lambda = 0.0749$ confirming uniqueness,
  - Example 3: $C_1 = C_2 = 1.623$ confirming Ulam–Hyers stability with explicit error bounds.

  In addition, the parameter sensitivity analysis (Table 2) showed that the uniqueness condition $\Lambda < 1$ remains robust across a range of fractional orders, generalization parameters, and interval lengths, with Λ varying between 0.0583 and 0.0984 for the tested configurations.

7.2 Limitations of the Present Work

Despite the generality of the developed framework, several limitations should be noted:

- **Global Lipschitz conditions.** The existence and uniqueness results established in this work rely on the global Lipschitz assumption (A2), which ensures the contractive structure required for the application of Krasnoselskii's and Banach's fixed point theorems. While mathematically convenient, this assumption can be restrictive in practice, as many nonlinear models arising in applications (e.g., polynomial growth or saturation-type dynamics) satisfy only local Lipschitz conditions. As a consequence, the direct applicability of the present results may be limited to systems where global bounds can be verified.

- **Fractional orders.** The analysis is restricted to $\alpha, \beta \in (0, 1]$. Systems involving higher-order derivatives ($\alpha > 1$) or variable-order operators are beyond the present scope.

- **Problem setting.** Only initial-value problems are considered. Multi-point, integral, or nonlocal boundary conditions require separate investigation.

- **Numerical realization.** Although theoretical error bounds are derived through Ulam-Hyers stability constants, no dedicated numerical scheme is implemented. The examples remain illustrative rather than computationally validated.

- **Empirical validation.** The models are not calibrated using experimental or observational data and primarily serve to demonstrate the mathematical framework.

7.3 Directions for Future Research

The present results open several directions for further study:

- **Relaxing Lipschitz assumptions.** An important direction for future research is the extension of the present framework to locally Lipschitz or Carathéodory-type nonlinearities. This would require replacing the contraction-based approach with more flexible techniques such as Schauder's fixed point theorem, Leray–Schauder continuation methods, or the use of measures of noncompactness. Such approaches would allow the analysis of systems where global Lipschitz bounds fail, thereby significantly broadening the applicability of the theoretical results to more realistic nonlinear models. These extensions would bridge the gap between the current theoretical framework and practical models exhibiting nonlinear growth beyond global Lipschitz regimes.

- **Higher-order extensions.** Extending the framework to fractional orders $\alpha, \beta > 1$ would enable modeling systems with richer memory structures, such as those arising in wave propagation and higher-order viscoelasticity.

- **Delay and impulsive dynamics.** Incorporating time delays and impulsive perturbations would allow modeling feedback delays and abrupt changes in hybrid systems, which are common in biological and control applications.

- **Variable-order operators.** Replacing constant fractional orders with variable-order functions $\alpha(t), \beta(t)$ would capture evolving memory effects in adaptive systems, such as aging materials or time-dependent biological processes.

- **Numerical methods.** Designing and analyzing numerical schemes tailored to coupled hybrid fractional systems would complement the theoretical analysis. The derived Ulam-Hyers constants may serve as benchmarks for numerical error control and convergence analysis.

- **Applications to concrete problems.** Adapting the framework to specific problems in viscoelasticity, epidemiology, and control engineering would demonstrate practical relevance. Parameter identification and model validation using real-world data constitute natural next steps.

- **Alternative stability notions.** Investigating Mittag-Leffler stability, finite-time stability, and exponential stability would provide deeper insight into the long-term dynamics of the coupled system.

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Availability of Data and Materials

No datasets were generated or analyzed during the current study.

Ethics Approval

This article does not contain any studies involving human participants or animals performed by any of the authors. Therefore, ethical approval and consent to participate are not applicable.

Author Contributions

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Conflict of Interest

The authors declare no conflicts of interest.

References

[1] Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (Vol. 198), elsevier.

[2] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier.

[3] Tarasov, V. E. (2010). *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Berlin/Heidelberg, Germany: Springer.

[4] Area, I., Losada, J., & Nieto, J. J. (2016). A note on the fractional logistic equation. *Physica A: Statistical Mechanics and its Applications*, 444, 182-187.

[5] Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. (2012). *Fractional Calculus: Models and Numerical Methods*. Singapore: World Scientific.

[6] Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity*. London, UK: Imperial College Press.

[7] Magin, R. L. (2006). *Fractional Calculus in Bioengineering*. New York, NY, USA: Begell House Publishers.

[8] Li, Y., Chen, Y., & Podlubny, I. (2009). Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. *Computers & Mathematics with Applications*, 59(5), 1810-1821.

[9] Machado, J. T., Kiryakova, V., & Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, 16(3), 1140-1153.

[10] Scalas, E., Gorenflo, R., & Mainardi, F. (2000). Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and its Applications*, 284(1-4), 376-384.

[11] Mohammadzadeh, A., Taghavifar, H., Zhang, C., Alattas, K. A., Liu, J., & Vu, M. T. (2024). A non-linear fractional-order type-3 fuzzy control for enhanced path-tracking performance of autonomous cars. *IET Control Theory & Applications*, 18(1), 40-54.

[12] Katugampola, U. N. (2011). New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3), 860-865.

[13] Katugampola, U. N. (2014). A new approach to generalized fractional derivatives. *Bulletin of Mathematical Analysis and Applications*, 6(4), 1-15.

[14] Almeida, R., Malinowska, A. B., & Monteiro, M. T. T. (2017). Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Mathematical Methods in the Applied Sciences*, 41(1), 336-352.

[15] Vu, H., & Hoa, N. V. (2026). Fractional iterative differential equations with boundary conditions via μ-Caputo fractional derivative: existence, uniqueness of a solution and Hyers-Ulam stability. *Physica Scripta*, 101(7), 075215.

[16] Singkai, C. W., & Saha, U. K. (2025). Fractional dynamics of coupled modified Korteweg–de Vries systems with generalized Caputo derivatives: a numerical investigation. *Physica Scripta*, 100(12), 125210.

[17] Algolam, M. S., Chebab, M., Abdo, M. S., Alshammari, F., & Aldwoah, K. (2025). Analysis of hybrid fractional integro-differential equations with application to cholera dynamics. *Scientific Reports*, 15(1), 33905.

[18] Ahmad, B., & Nieto, J. J. (2011). Sequential fractional differential equations with three-point boundary conditions. *Computers & Mathematics with Applications*, 64(10), 3046-3052.

[19] Han, X., & Wang, T. (2011). The Existence of Solutions for a Nonlinear Fractional Multi-Point Boundary Value Problem at Resonance. *International Journal of Differential Equations*, 2011(1), 401803.

[20] Li, Z., Liu, Y., & Wada, K. (2026). Sharp decay estimates and numerical analysis for weakly coupled systems of two subdiffusion equations. *Journal of Differential Equations*, 453, 113826.

[21] Ali, A., Bayes, A., Ali, Z., Shah, K., Abdeljawad, T., & Naseralla, K. (2025). Numerical analysis of a coupled system for fractional stochastic differential equations. *Boundary Value Problems*, 2025(1), 170.

[22] Wongcharoen, A., Ahmad, B., Ntouyas, S. K., & Tariboon, J. (2020). Three-Point Boundary Value Problems for the Langevin Equation with the Hilfer Fractional Derivative. *Advances in Mathematical Physics*, 2020(1), 9606428.

[23] Xu, X., & Li, B. (2025). Semi-global stabilization of parabolic PDE–ODE systems with input saturation. *Automatica*, 171, 111931.

[24] Dhage, B. C. (2012). Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations. *Differential Equations and Applications*, 4(2), 155-184.

[25] Samina, Shah, K., & Khan, R. A. (2020). Stability theory to a coupled system of nonlinear fractional hybrid differential equations. Indian *Journal of Pure and Applied Mathematics*, 51(2), 669-687.

[26] Hannabou, M., & Awadalla, M. (2025). Existence and Stability Results for Fractional Hybrid Systems with Impulsive Effects. *Mathematics*, 14(1), 122.

[27] Benchohra, M., Henderson, J., & Ntouyas, S. K. (2008). *Impulsive Differential Equations and Inclusions*. Cairo, Egypt: Hindawi Publishing Corporation.

[28] Kumar, V., & Malik, M. (2021). Controllability results of fractional integro-differential equation with non-instantaneous impulses on time scales. *IMA Journal of Mathematical Control and Information*, 38(1), 211-231.

[29] Abbas, S., Benchohra, M., & N'Guérékata, G. M. (2015). *Topics in Fractional Differential Equations*. Berlin/Heidelberg, Germany: Springer.

[30] Sidaoui, R., Alfedeel, A. H., Ahmad, J., Aldwoah, K., Ali, A., Osman, O., & Tedjani, A. H. (2025). Mathematical Perspectives of a Coupled System of Nonlinear Hybrid Stochastic Fractional Differential Equations. *Fractal and Fractional*, 9(10), 622.

[31] Liaqat, M. I., Vivas-Cortez, M., Yousif, M. A., & Mohammed, P. O. (2026). Variable-Order Fractional Delay Differential Equations with Integral Boundary Values: A Study on Existence, Uniqueness, and Stability. *European Journal of Pure and Applied Mathematics*, 19(1), 6855-6855.

[32] Ye, H., Gao, J., & Ding, Y. (2007). A generalized Gronwall inequality and its application to a fractional differential equation. *Journal of Mathematical Analysis and Applications*, 328(2), 1075-1081.

[33] Rus, I. A. (2010). Ulam stabilities of ordinary differential equations. *Studia Universitatis Babeș-Bolyai, Mathematica*, 55(4), 125-133.

[34] Wang, J., Lv, L., & Zhou, Y. (2012). Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electronic Journal of Qualitative Theory of Differential Equations*, 2011(63), 1-10.

[35] Wang, J., Zhang, Y., & Lu, X. (2020). Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations. *Optimization*, 69(7), 1-17.

[36] Li, M., & Wang, J. (2017). Finite time stability of fractional delay differential equations. *Applied Mathematics Letters*, 64, 170-176.

[37] Vivek, D., Kanagarajan, K., & Elsayed, E. M. (2018). Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions. *Mediterranean Journal of Mathematics*, 15(1), 1-18.

[38] Sousa, J. V. da C., & de Oliveira, E. C. (2018). On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. *Journal of Fixed Point Theory and Applications*, 20(3), 1-21.

[39] Tunç, C., & Tunç, O. (2026). Ulam-Type Stability Results for Fractional Integro-Delay Differential and Integral Equations via the ψ-Hilfer Operator. *Fractal and Fractional*, 10(1), 57.

[40] Rehman, M. U., & Khan, R. A. (2011). A note on boundary value problems for a coupled system of fractional differential equations. *Computers & Mathematics with Applications*, 61(9), 2630-2637.

[41] Su, X. (2009). Boundary value problem for a coupled system of nonlinear fractional differential equations. *Applied Mathematics Letters*, 22(1), 64-69.

[42] Martínez-Salgado, B. F., Alcántara-López, F., Torres-Hernandez, A., Brambila-Paz, F., Fuentes, C., & Estrada, J. L. (2020). Fractional flow equations. A model for pressure deficit in an oil well. arXiv:2008.09068.

[43] Ahmad, B., Nieto, J. J., Alsaedi, A., & Aqlan, M. H. (2017). A coupled system of Caputo-type sequential fractional differential equations with coupled (periodic/anti-periodic type) boundary conditions. *Mediterranean Journal of Mathematics*, 14(6), 227.

[44] Agarwal, R. P., Assolami, A., Alsaedi, A., & Ahmad, B. (2022). Existence results and Ulam–Hyers stability for a fully coupled system of nonlinear sequential Hilfer fractional differential equations and integro-multistrip-multipoint boundary conditions. *Qualitative theory of dynamical systems*, 21(4), 125.

[45] Assebbane, Y., Hannabou, M., Atraoui, M., & Bouaouid, M. (2025). Existence and uniqueness of boundary value problems for nonlinear hybrid differential equations with ABC-fractional derivative. *Filomat*, 39(11), 3759-3768.

[46] Mohammed, M. O., & Rafeeq, A. S. (2024). New Results for Existence, Uniqueness, and Ulam Stable Theorem to Caputo–Fabrizio Fractional Differential Equations with Periodic Boundary Conditions. *International Journal of Applied and Computational Mathematics*, 10(3), 109.