Optimum Analysis and Parameter Inference of a New Improved Adaptive Progressive Unit-Weibull Model Censoring with Two Physical Applications

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Optimum Analysis and Parameter Inference of a New Improved Adaptive Progressive Unit-Weibull Model Censoring with Two Physical Applications

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ABSTRACT

Recently, a new bounded version of the Weibull lifetime has been introduced, retaining the flexibility of the standard Weibull model while restricting it to the unit interval, making it particularly useful for beta-like modeling in various fields. This paper investigates the improved adaptive Type-II progressively censored plan by applying it to the bounded unit Weibull distribution. From a frequentist perspective, both maximum likelihood and maximum product of spacings methods are used, along with appropriate approximate confidence intervals. Bayesian point estimates, along with Bayesian credible intervals, are obtained using classical procedures that involve Markov-Chain Monte Carlo sampling, with two separate posterior distributions based on the squared error loss. The proposed estimation frameworks encompass both distribution parameters and key reliability metrics, notably the reliability and failure rate functions. Extensive simulations are conducted to evaluate the accuracy and efficiency of the estimators under various censoring levels, sample sizes, and thresholds. To determine the optimal removal strategy under both frequentist estimation paradigms, several metrics are proposed. Additionally, the model is applied to two real-world datasets from engineering reliability; one contains core samples from petroleum reservoirs collected across four cross-sections, and the other pertains to the tensile strength of polyester fibers, illustrating its practical utility and flexibility in modeling. The novelty of this study lies in integrating the bounded support of the unit-Weibull model with an improved adaptive censoring scheme, providing enhanced robustness and adaptability in reliability analysis across different fields.

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1 Introduction

A novel probability model proposed by Mazucheli et al. [1], called the unit-Weibull (Uni-W), is designed for bounded lifetime data within the interval [0, 1]. This distribution extends the traditional Weibull model while ensuring compatibility with proportional and fractional data, making it especially useful in reliability analysis, survival studies, and environmental modeling. They highlight several advantages, including its flexibility to model increasing, decreasing, and constant hazard rates, as well as its ability to approximate other well-known distributions like the beta and Kumaraswamy distributions. Suppose that the lifetime variable, denoted as the random variable X, of an individual testing item follows a Uni-W ($\boldsymbol{\alpha}$) distribution, where $\boldsymbol{\alpha} = (\beta, \delta)^{\top}$ are the shape parameters; then, its probability density function (PDF) (denoted as $f(\cdot)$) and cumulative distribution function (CDF) (denoted as $F(\cdot)$) are given by

$$f(x; \boldsymbol{\alpha}) = \delta\beta x^{-1}(\xi(x))^{\beta-1}e^{-\delta(\xi(x))^{\beta}},\ 0 < x < 1, \tag{1}$$

and

$$F(x; \boldsymbol{\alpha}) = e^{-\delta(\xi(x))^{\beta}}, \tag{2}$$

respectively, where $\xi(x) = -\ln(x)$.

On the other hand, for the lifetime variable Y, the reliability function (RF) (say, $R(\cdot)$), and hazard rate function (HRF) (say, $h(\cdot)$), at a distinct time $t > 0$, are given by

$$R(t; \boldsymbol{\alpha}) = 1 - e^{-\delta(\xi(t))^{\beta}}, \tag{3}$$

and

$$h(t; \boldsymbol{\alpha}) = \frac{\delta\beta t^{-1}(\xi(t))^{\beta-1}e^{-\delta(\xi(t))^{\beta}}}{1 - e^{-\delta(\xi(t))^{\beta}}}, \tag{4}$$

respectively, where $\xi(t) = -\ln(t)$. In recent years, the Uni-W distribution has gained attention due to its bounded support and flexibility in modeling fractional and reliability data. Guerra et al. [2] introduced extended families of Uni-W distributions; Alotaibi et al. [3,4] discussed the estimation issue of Uni-W multicomponent reliability under progressively Type-II censored and stress-strength settings, respectively; and Almetwally et al. [5] proposed optimal censoring schemes of the Uni-W distribution within progressive censoring designs, among others.

Censoring and life testing are fundamental techniques in reliability analysis and survival studies, enabling the estimation of lifetime distributions when complete failure data are unavailable. These methods improve statistical inference in scenarios with time or observation constraints. The progressive Type-II censoring (T2-PC) scheme allows researchers and reliability experts to remove some units from testing for additional evaluation or to reduce experimentation time; see Balakrishnan and Cramer [6].

Kundu and Joarder [7] noted that the T2-PC plan can incur higher time and cost, particularly when one or more extreme lifetime units are under observation. To address this, they proposed the progressive Type-I hybrid censoring (T1-PHC) scheme, which follows the same inspection structure but terminates the test at $T^* = \min(T, X_{m:m:n})$, where T is a pre-specified time. However, a key limitation of this approach is that the required number of failures m may turn out to be relatively small, rendering statistical inference less reliable.

To overcome this limitation, Ng et al. [8] introduced the adaptive progressive Type-II censoring (AT2-PC) scheme, which allows for more flexible and informative data collection when test duration is not a strict constraint.

Building on this, Yan et al. [9] proposed a more flexible approach, termed the improved adaptive Type-II progressive censoring (T2IA-PC) scheme, which enhances testing efficiency by balancing information retention with reduced experimental cost. They demonstrated that T2IA-PC outperforms traditional censoring schemes, particularly in settings with limited test duration or restricted sample sizes.

The T2IA-PC scheme operates as follows: Consider a life-testing experiment with n independent and identically distributed (i.i.d.) units, where the experimenter specifies an integer $m < n$, representing the number of failures before the test ends, and two threshold values T_1 and T_2, with $T_i \in (0, \infty)$ for $i = 1, 2$. A predetermined T2-PC removal scheme is defined by $\mathbf{R} = (R_1, R_2, \ldots, R_m)$, which governs the number of surviving units removed at each failure time during the experiment. Specifically:

(a) At the time of the first failure $X_{1:m:n}$, a subset of R_1 operational units is randomly withdrawn from the study.

(b) Upon the second failure at $X_{2:m:n}$, another set of R_2 remaining units is randomly eliminated.

(c) This process continues sequentially, where at each failure $X_{i:m:n}$, a designated number R_i of units is removed.

(d) This process continues sequentially until the practitioner records one of the following cases:
   - If $X_{m:m:n} < T_1$, (leading to the T2-PC sample) stop the test at $X_{m:m:n}$ (Case-1);
   - If $T_1 < X_{m:m:n} < T_2$, (leading to the AT2-PC sample) stop the test at $X_{m:m:n}$ (Case-2);
   - If $T_2 < X_{m:m:n}$, (leading to the T2IA-PC sample) stop the test at T_2.

(e) All remaining units (say, R^*) are removed from the experiment, ensuring that:

- $R^* = n - m - \sum_{i=1}^{m-1} R_i$ for Case-1;
- $R^* = n - d_2 - \sum_{i=1}^{d_1} R_i$ for Case-2 and -3; where $d_i,\ i = 1, 2$, denote the number of failures that occurred up to $T_i,\ i = 1, 2$.

Now, suppose $x_i = x_{i:m:n}$, for $i = 1, \ldots, m$, represents a T2IA-PC sample collected from the Uni-Weibull distribution with parameter vector $\boldsymbol{\alpha}$. Then, the likelihood function (LF), denoted by $\mathcal{L}(\cdot)$, is given by

$$\mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) = \prod_{i=1}^{\mathcal{A}_2} f(x_i; \boldsymbol{\alpha}) \prod_{i=1}^{\mathcal{A}_1} [R(x_i; \boldsymbol{\alpha})]^{R_i} [R(\tau^*; \boldsymbol{\alpha})]^{R^*}, \tag{5}$$

where $\mathcal{A}_i,\ i = 1, 2$, and τ^* are specified as follows:

$$\mathcal{A}_1 = \begin{cases} m, & \text{for Case-1} \\ d_1, & \text{for Case-2} \\ d_1, & \text{for Case-3} \end{cases}, \mathcal{A}_2 = \begin{cases} m, & \text{for Case-1} \\ m, & \text{for Case-2} \\ d_2, & \text{for Case-3} \end{cases} \text{ and } \tau^* = \begin{cases} 0, & \text{for Case-1} \\ x_m, & \text{for Case-2} \\ T_2, & \text{for Case-3.} \end{cases}$$

Cheng and Amin [10] created the highly profitable estimating methodology known as the MPS approach as an alternative to the traditional ML method. They claimed that the MPS and ML estimators enjoyed the same asymptotic consistency, sufficiency, and efficiency features. According to Anatolyev and Kosenok [11], the invariance property of MPSEs is the same as that of MLEs. They said that because the MPSEs perform better than the MLEs for small samples, the MPS technique is particularly helpful for reliability assessments.

Taking a T2IA-PC sample $\mathbf{x}$ from the Uni-W ($\boldsymbol{\alpha}$) model, the joint spacing function (SF), denoted by $\mathcal{P}(\cdot)$, can be expressed up to proportionality as

$$\mathcal{P}(\boldsymbol{\alpha}|\mathbf{x}) = \prod_{i=1}^{\mathcal{A}_2+1} [F(x_i;\boldsymbol{\alpha}) - F(x_{i-1};\boldsymbol{\alpha})] \prod_{i=1}^{\mathcal{A}_1} [R(x_i;\boldsymbol{\alpha})]^{R_i} [R(\tau^*;\boldsymbol{\alpha})]^{R^*}, \tag{6}$$

where $F(0;\boldsymbol{\alpha}) \equiv 0$ and $F(x_{\mathcal{A}_2+1};\boldsymbol{\alpha}) \equiv 1$; for additional details, see Nassar and Elshahhat [12].

Several recent studies have contributed to the development of flexible lifetime models with samples collected under the improved adaptive progressive Type-II censoring (T2AI-PC) strategy in reliability analysis. This approach reduces test duration and resource consumption while still enabling efficient statistical inference on the underlying lifetime distribution. For example, Nassar and Elshahhat [12] focused on the standard Weibull distribution; Alotaibi et al. [13] considered the Weibull-exponential and XLindley distributions, respectively; and Alqasem and Elshahhat [14] investigated the power half-normal model. In the presence of a T2AI-PC with a competing risks framework, see Dutta and Kayal [15], and Elshahhat and Nassar [16], among others. This work features the integration of the Uni-W distribution into the T2IA-PC framework, an approach not previously addressed in the literature. This integration provides improved estimation accuracy, adaptive test planning, and improved applicability to real-world bounded reliability data.

With the advancement of technology and digital systems, the need for flexible yet efficient censoring schemes is therefore critical, especially when dealing with modern reliability data that exhibit bounded behavior, such as proportions or failure percentages. Motivated by this, given the importance of the Uni-W distribution and the efficiency of T2IA-PC in conducting experiments within a feasible time frame, especially when the test duration is long, this study focuses on developing and evaluating estimation methodologies for the Uni-W distribution under this censoring mechanism. This approach enhances the precision of parameter estimation and broadens the applicability of life-testing models to a wider range of experimental and industrial contexts. The primary goal is to compare classical and Bayesian estimation techniques in extracting meaningful statistical insights from T2IA-PC data. Our key contributions are summarized as follows:

- Developing two inferential frameworks, namely maximum likelihood (ML) and maximum product of spacing (MPS), for the parameter vector $\boldsymbol{\alpha}$, RF $R(t)$, and HRF $h(t)$ of the Uni-W distribution when the T2IA-PC dataset is available.
- Exploring the associated approximate confidence intervals (ACIs) for the acquired estimators developed by ML and MPS approaches for β, δ, $R(t)$, and $h(t)$ are analyzed.
- Implementing Bayesian estimation via the Monte Carlo Markov Chain (MCMC) method using the squared error loss and independent gamma density prior functions. We introduce Bayesian point estimates and their associated Bayesian credible intervals (BCIs) via both ML-based and MPS-based methods.
- Identifying the most efficient estimation method for Uni-W parameters by extensively comparing classical and Bayesian estimators through extensive simulation studies involving diverse experimental designs and precision metrics.
- Determining the optimal removal design that maximizes information retention about unknown parameters. Several optimality criteria are proposed and evaluated under classical estimation frameworks.

- Demonstrating the practical applicability of the proposed estimation techniques in real-world scenarios. To this end, two empirical case studies are conducted using chemical and engineering datasets to validate the effectiveness of the methodologies.

The remainder of the research is classified as Section 2 presents the ML and MPS estimators as well as associated ACIs for the Uni-W distribution parameters, including β, δ, $R(t)$, and $h(t)$. Section 3 deals with the Bayesian point and interval inferences on the same unknown subjects. Monte Carlo comparisons are made in Section 4. Section 5 introduces the ideal removal pattern. Section 6 supplies two physically related applications. Lastly, in Section 7, different conclusions and recommendations are listed.

2 Classical Inference

This part introduces point and asymptotic interval estimators of β, δ, $R(t)$, and $h(t)$ through ML and MPS approaches.

2.1 ML Approach

Let $\mathbf{x} = \{(x_1, R_1), (x_2, R_2), ..., (x_{d_1}, R_{d_1}), (x_{d_1+1}, 0), \ldots, (x_{d_2}, R^*)\}$ be a T2IA-PC sample taken from the Uni-W distribution, consequently, the joint LF can be written up to proportional as

$$\mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) \propto \prod_{i=1}^{\mathcal{A}_2} \delta\beta(\xi(x_i))^{\beta} e^{-\delta(\xi(x_i))^{\beta}} \prod_{i=1}^{\mathcal{A}_1} \left[1 - e^{-\delta(\xi(x_i))^{\beta}}\right]^{R_i} \left[1 - e^{-\delta(\xi(\tau^*))^{\beta}}\right]^{R^*}, \tag{7}$$

where its natural logarithm becomes

$$\begin{aligned}\ln \mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) = {} & \mathcal{A}_2 \ln(\delta\beta) + \sum\nolimits_{i=1}^{\mathcal{A}_2} \beta \ln(\xi(x_i)) - \delta \sum\nolimits_{i=1}^{\mathcal{A}_2} (\xi(x_i))^{\beta} \\ & + \sum_{i=1}^{\mathcal{A}_1} R_i \ln\left(1 - e^{-\delta(\xi(x_i))^{\beta}}\right) + R^* \ln\left(1 - e^{-\delta(\xi(\tau^*))^{\beta}}\right).\end{aligned} \tag{8}$$

The normal equations of β and δ can be derived from (8), respectively, as

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{\mathcal{A}_2}{\beta} + \sum_{i=1}^{\mathcal{A}_2} \ln(\xi(x_i)) \left[1 - \delta(\xi(x_i))^{\beta}\right] - \sum_{i=1}^{\mathcal{A}_1} \frac{\delta R_i \Psi_{1i} \ln(\xi(x_i))}{(1 - e^{-\delta(\xi(x_i))^{\beta}})} - \frac{\delta R^* \Psi_{2i} \ln(\xi(\tau^*))}{(1 - e^{-\delta(\xi(\tau^*))^{\beta}})}, \tag{9}$$

and

$$\frac{\partial \ln \mathcal{L}}{\partial \delta} = \frac{\mathcal{A}_2}{\delta} - \sum_{i=1}^{\mathcal{A}_2} (\xi(x_i))^{\beta} - \sum_{i=1}^{\mathcal{A}_1} \frac{R_i \Psi_{1i}}{(1 - e^{-\delta(\xi(x_i))^{\beta}})} - \frac{R^* \Psi_{2i}}{(1 - e^{-\delta(\xi(\tau^*))^{\beta}})}, \tag{10}$$

where $\Psi_{1i} = \Psi_{1i}(x_i; \beta, \delta) = (\xi(x_i))^{\beta} e^{-\delta(\xi(x_i))^{\beta}}$ and $\Psi_{2i} = \Psi_{2i}(\tau^*; \beta, \delta) = (\xi(\tau^*))^{\beta} e^{-\delta(\xi(\tau^*))^{\beta}}$.

To acquire the MLEs of β and δ, represented by $\hat{\beta}$ and $\hat{\delta}$, it is noted that Eqs. (9) and (10) require computer facilities and numerical techniques due to their development in nonlinear forms. To address this issue, some numerical techniques, such as the Newton-Raphson (NR) method, are advised to produce the required estimates $\hat{\beta}$ and $\hat{\delta}$ as soon as a T2IA-PC dataset is available.

Assume $\hat{R}(t)$ and $\hat{h}(t)$ denote the MLEs of the reliability factors $R(t)$ and $h(t)$ of the Uni-W distribution, respectively. Leveraging the invariance property of the MLEs $\hat{\beta}$ and $\hat{\delta}$ obtained from (9) and (10), respectively, the estimators $\hat{R}(t)$ and $\hat{h}(t)$ can be obtained from (3) and (4), for $t > 0$, as

$$\hat{R}(t) = 1 - e^{-\hat{\delta}(\xi(t))^{\hat{\beta}}},$$

and

$$\hat{h}(t) = \frac{\hat{\delta}\hat{\beta}t^{-1}(\xi(t))^{\hat{\beta}-1}e^{-\hat{\delta}(\xi(t))^{\hat{\beta}}}}{1 - e^{-\hat{\delta}(\xi(t))^{\hat{\beta}}}},$$

respectively, where $\xi(t) = -\ln(t)$.

To find the associated ACI via LF-based (ACI-LF) of β, δ, $R(t)$, or $h(t)$, at a significance level $\varrho\%$, we first acquire the variance-covariance matrix (VCM). To accomplish this, we obtain the inverse of the observed 2×2 Fisher information matrix (FIM), which requires obtaining the second partial derivatives (from (8)) to Uni-W parameters β and δ as follows:

$$\mathcal{L}_{11} = \sum_{i=1}^{\mathcal{A}_2}\left(\frac{1}{\beta} + \ln^2\xi(x_i) - \delta\xi(x_i)^{\beta}(\ln\xi(x_i))^2\right) - \sum_{i=1}^{\mathcal{A}_1} R_i \frac{\delta\Psi_{1i}\ln\xi(x_i)}{(1 - e^{-\delta\xi(x_i)^{\beta}})} - R^* \frac{\delta\Psi_{2i}\ln\xi(\tau^*)}{(1 - e^{-\delta\xi(\tau^*)^{\beta}})}, \tag{11}$$

$$\mathcal{L}_{22} = -\sum_{i=1}^{\mathcal{A}_2}\xi(x_i)^{\beta} + \sum_{i=1}^{\mathcal{A}_1} R_i \frac{\Psi_{1i}\xi(x_i)^{\beta}}{\left(1 - e^{-\delta\xi(x_i)^{\beta}}\right)^2} + R^* \frac{\Psi_{2i}\xi(\tau^*)^{\beta}}{\left(1 - e^{-\delta\xi(\tau^*)^{\beta}}\right)^2}, \tag{12}$$

and

$$\mathcal{L}_{12} = -\sum_{i=1}^{\mathcal{A}_2}\xi(x_i)^{\beta}\ln\xi(x_i) + \sum_{i=1}^{\mathcal{A}_1} R_i \frac{\Psi_{1i}\ln\xi(x_i)}{(1 - e^{-\delta\xi(x_i)^{\beta}})} + R^* \frac{\Psi_{2i}\ln\xi(\tau^*)}{(1 - e^{-\delta\xi(\tau^*)^{\beta}})}, \tag{13}$$

next, the estimated VCM can be estimated (at $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}$) as

$$\widehat{\Sigma}_{\mathcal{L}}(\hat{\boldsymbol{\alpha}}) = \begin{bmatrix} -\mathcal{L}_{11} & -\mathcal{L}_{12} \\ -\mathcal{L}_{21} & -\mathcal{L}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix}. \tag{14}$$

Subsequently, the $100(1-\varrho)\%$ ACI-LF estimators of β and δ can be formulated as

$\hat{\beta} \pm \mathcal{Z}^{\star}\sqrt{\hat{\sigma}_{11}}$ and $\hat{\delta} \pm \mathcal{Z}^{\star}\sqrt{\hat{\sigma}_{22}}$,

respectively, where $\mathcal{Z}^{\star} \equiv z_{\frac{\varrho}{2}}$ represents the upper 0.5 ϱth percentile point of the standard Gaussian distribution.

On the other hand, to create the $100(1-\varrho)\%$ ACI-LF estimators of $R(t)$ and $h(t)$ (at $t > 0$), their estimated variances (denoted by $\hat{\sigma}_R$ and $\hat{\sigma}_h$, respectively) should be first obtained. For this objective, the delta method (discussed by Greene [17]) is utilized to carry out $\hat{\sigma}_R$ and $\hat{\sigma}_h$, respectively, as:

$$\hat{\sigma}_R \cong \begin{bmatrix} \aleph_R^{\circ} & \aleph_R^{\bullet} \end{bmatrix} \Sigma_{\mathcal{L}} \begin{bmatrix} \aleph_R^{\circ} \\ \aleph_R^{\bullet} \end{bmatrix}\Bigg|_{(\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}})} \quad \text{and} \quad \hat{\sigma}_h \cong \begin{bmatrix} \aleph_h^{\circ} & \aleph_h^{\bullet} \end{bmatrix} \Sigma_{\mathcal{L}} \begin{bmatrix} \aleph_h^{\circ} \\ \aleph_h^{\bullet} \end{bmatrix}\Bigg|_{(\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}})}, \tag{15}$$

where $\Psi_t = \Psi_t(t;\beta,\delta) = (\xi(t))^{\beta}e^{-\delta(\xi(t))^{\beta}}$, $\aleph_R^{\circ} = \delta\ln\xi(t)\Psi_t$, $\aleph_R^{\bullet} = -\Psi_t$,

$$\aleph_h^{\circ} = \frac{\delta t^{-1}\xi^{-1}(t)\Psi_t\left(1 - e^{-\delta(\xi(t))^{\beta}}\right)[1 + \beta\ln\xi(t)(1 - \delta(\xi(t))^{\beta})] - \delta^2\beta t^{-1}\Psi_t^2\xi^{-1}(t)\ln(\xi(t))}{\left(1 - e^{-\delta(\xi(t))^{\beta}}\right)^2},$$

and

$$\aleph_h^{\bullet} = \frac{\beta\xi^{-1}(t)\Psi_t\left(1-e^{-\delta(\xi(t))^\beta}\right)[t^{-1}-\delta(\xi(t))^\beta]+\delta\beta t^{-1}\Psi_t^2\xi^{-1}(t)}{\left(1-e^{-\delta(\xi(t))^\beta}\right)^2}.$$

The respective $100(1-\varrho)\%$ ACI-LF estimators of $R(t)$ and $h(t)$ can be created as $\hat{R}(t) \pm \mathcal{Z}^\star\sqrt{\hat{\sigma}_R}$ and $\hat{h}(t) \pm \mathcal{Z}^\star\sqrt{\hat{\sigma}_h}$.

2.2 MPS Approach

From (1) and (3), the joint SF (6) becomes

$$\mathcal{P}(\boldsymbol{\alpha}|\mathbf{x}) = \prod_{i=1}^{A_2+1}\left[e^{-\delta(\xi(x_i))^\beta}-e^{-\delta(\xi(x_{i-1}))^\beta}\right]\prod_{i=1}^{A_1}\left[1-e^{-\delta(\xi(x_i))^\beta}\right]^{R_i}\left[1-e^{-\delta(\xi(\tau^*))^\beta}\right]^{R^*}, \tag{16}$$

equivalently, the logarithm-SF is

$$\ln\mathcal{P}(\boldsymbol{\alpha}|\mathbf{x}) = \sum_{i=1}^{A_2+1}\ln\left(e^{-\delta(\xi(x_i))^\beta}-e^{-\delta(\xi(x_{i-1}))^\beta}\right)+\sum_{i=1}^{A_1}R_i\ln\left(1-e^{-\delta(\xi(x_i))^\beta}\right)$$
$$+R^*\ln\left(1-e^{-\delta(\xi(\tau^*))^\beta}\right). \tag{17}$$

The MPS estimators (MPSEs) of β and δ, symbolized by $\check{\beta}$ and $\check{\delta}$, respectively, can be acquired from (17) as

$$\frac{\partial\ln\mathcal{P}}{\partial\beta} = \sum_{i=1}^{A_2+1}\frac{(\delta\ln\xi(x_{i-1})\Psi_{3i}-\delta\ln\xi(x_i)\Psi_{1i})}{e^{-\delta(\xi(x_i))^\beta}-e^{-\delta(\xi(x_{i-1}))^\beta}}-\sum_{i=1}^{\mathcal{A}_1}R_i\left[\frac{\delta\Psi_{1i}\ln(\xi(x_i))}{1-e^{-\delta(\xi(x_i))^\beta}}\right]$$
$$-R^*\left[\frac{\delta\Psi_{2i}\ln(\xi(\tau^*))}{1-e^{-\delta(\xi(\tau^*))^\beta}}\right] \tag{18}$$

and

$$\frac{\partial\ln\mathcal{P}}{\partial\delta} = \sum_{i=1}^{A_2+1}\frac{(\Psi_{3i}-\Psi_{1i})}{e^{-\delta(\xi(x_i))^\beta}-e^{-\delta(\xi(x_{i-1}))^\beta}}-\sum_{i=1}^{\mathcal{A}_1}\frac{R_i\Psi_{1i}}{1-e^{-\delta(\xi(x_i))^\beta}}-\frac{R^*\Psi_{2i}}{1-e^{-\delta(\xi(\tau^*))^\beta}}, \tag{19}$$

where $\Psi_{3i}=\Psi_{1i}(x_{i-1};\beta,\delta)=(\xi(x_{i-1}))^\beta e^{-\delta(\xi(x_{i-1}))^\beta}$. Since the joint-SF is expressed in nonlinear form, we recommend implementing the NR method again to find $\check{\beta}$ and $\check{\delta}$. Once these estimators are evaluated, utilizing their invariance features, the MPSEs of $R(t)$ and $h(t)$, denoted by $\check{R}(t)$ and $\check{h}(t)$, respectively, can be acquired as

$$\check{R}(t) = 1-e^{-\check{\delta}(\xi(t))^{\check{\beta}}},$$

and

$$\check{h}(t) = \frac{\check{\delta}\check{\beta}t^{-1}(\xi(t))^{\check{\beta}-1}e^{-\check{\delta}(\xi(t))^{\check{\beta}}}}{1-e^{-\check{\delta}(\xi(t))^{\check{\beta}}}}.$$

To determine the related ACI via SF-based (ACI-SF) of β, δ, $R(t)$, or $h(t)$, we first derive the observed 2×2 VCM, which is developed by inverting the FIM. To achieve this, by taking the second partial derivatives (from (17)) to Uni-W parameters β and δ, we first get the next quantities $\mathcal{P}_{i,j},\ i,j=1,2$:

$$\mathcal{P}_{11} = \sum_{i=1}^{\mathcal{A}_2+1} \frac{\left(\delta \ln^2 \xi(x_i)\Psi_{1i} - \delta \ln^2 \xi(x_{i-1})\Psi_{3i}\right)}{e^{-\delta(\xi(x_i))^\beta} - e^{-\delta(\xi(x_{i-1}))^\beta}} - \sum_{i=1}^{\mathcal{A}_1} \frac{\delta R_i \Psi_{1i} \ln \xi(x_i)}{(1 - e^{-\delta\xi(x_i)^\beta})} - \frac{\delta R^* \Psi_{2i} \ln \xi(\tau^*)}{(1 - e^{-\delta\xi(\tau^*)^\beta})}, \tag{20}$$

$$\mathcal{P}_{22} = \sum_{i=1}^{\mathcal{A}_2+1} \frac{(\Psi_{1i} - \Psi_{3i})}{e^{-\delta(\xi(x_i))^\beta} - e^{-\delta(\xi(x_{i-1}))^\beta}} + \sum_{i=1}^{\mathcal{A}_1} \frac{R_i \Psi_{1i} \xi(x_i)^\beta}{\left(1 - e^{-\delta\xi(x_i)^\beta}\right)^2} + \frac{R^* \Psi_{2i} \xi(\tau^*)^\beta}{\left(1 - e^{-\delta\xi(\tau^*)^\beta}\right)^2}, \tag{21}$$

and

$$\mathcal{P}_{12} = \sum_{i=1}^{\mathcal{A}_2+1} \frac{(\ln \xi(x_i)\Psi_{1i} - \ln \xi(x_{i-1})\Psi_{3i})}{e^{-\delta(\xi(x_i))^\beta} - e^{-\delta(\xi(x_{i-1}))^\beta}} + \sum_{i=1}^{\mathcal{A}_1} \frac{R_i \Psi_{1i} \ln \xi(x_i)}{(1 - e^{-\delta\xi(x_i)^\beta})} + \frac{R^* \Psi_{2i} \ln \xi(\tau^*)}{(1 - e^{-\delta\xi(\tau^*)^\beta})}. \tag{22}$$

Now, we carry out the estimated VCM of $\check{\beta}$ and $\check{\delta}$ (at $\boldsymbol{\alpha} = \check{\boldsymbol{\alpha}}$) as

$$\widehat{\Sigma}_{\mathcal{P}}(\check{\boldsymbol{\alpha}}) = \begin{bmatrix} -\mathcal{P}_{11} & -\mathcal{P}_{12} \\ -\mathcal{P}_{21} & -\mathcal{P}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \check{\sigma}_{11} & \check{\sigma}_{12} \\ \check{\sigma}_{21} & \check{\sigma}_{22} \end{bmatrix}. \tag{23}$$

As a result, the $100(1-\varrho)\%$ ACI-SF estimators of β and δ can be formulated as

$\check{\beta} \pm \mathcal{Z}^\star \sqrt{\check{\sigma}_{11}}$ and $\check{\delta} \pm \mathcal{Z}^\star \sqrt{\check{\sigma}_{22}}$,

respectively. From (15), leveraging the invariance properties of $\check{\beta}$ and $\check{\delta}$, we replace β and δ by their $\check{\beta}$ and $\check{\delta}$ in turn to get $\check{\sigma}_R$ and $\check{\sigma}_h$, respectively. Thus, the respective $100(1-\varrho)\%$ ACI-SF estimators of $R(t)$ and $h(t)$ can be created as

$\check{R}(t) \pm \mathcal{Z}^\star \sqrt{\check{\sigma}_R}$ and $\check{h}(t) \pm \mathcal{Z}^\star \sqrt{\check{\sigma}_h}$.

3 Bayes Inference

This section implements the Bayesian framework by leveraging two posterior distributions developed from the two frequentist approaches proposed in earlier sections. A key component of the Bayesian estimating approach is selecting the prior distributions that best fit our knowledge of the unknown values. We suppose that both of the Uni-W distribution's parameters, β and δ, follow the gamma distributions and are independent of each other. The joint prior PDF of β and δ can be expressed, free of constants, as

$$\Pi(\boldsymbol{\alpha}) \propto \beta^{a_1-1}\delta^{a_2-1}e^{-(b_1\beta+b_2\delta)},\ a_i, b_i > 0,\ i = 1, 2. \tag{24}$$

3.1 Bayes LF-Based

Using the LF in (7) and the joint prior PDF in (24), the posterior PDF of β and δ (say, $\Pi_{\mathcal{L}}(\cdot)$) becomes

$$\Pi_{\mathcal{L}}(\boldsymbol{\alpha}|\mathbf{x}) = \mathcal{C}_{\mathcal{L}}^{-1}\beta^{\mathcal{A}_2+a_1-1}\delta^{\mathcal{A}_2+a_2-1}e^{-\beta\left(b_1-\sum_{i=1}^{\mathcal{A}_2}\ln\xi(x_i)\right)}e^{-\delta\left(b_2+\sum_{i=1}^{\mathcal{A}_2}(\xi(x_i))^\beta\right)}$$
$$\times \prod_{i=1}^{\mathcal{A}_1}\left[1 - e^{-\delta(\xi(x_i))^\beta}\right]^{R_i}\left[1 - e^{-\delta(\xi(\tau^*))^\beta}\right]^{R^*}, \tag{25}$$

where $\mathcal{C}_{\mathcal{L}}$ is a normalized constant and is defined as

$$\mathcal{C}_{\mathcal{L}} = \int_0^\infty \int_0^\infty \mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) \cdot \Pi(\boldsymbol{\alpha}) \mathrm{d}\beta \mathrm{d}\delta.$$

Following the Bayes continuous setup, the Bayes' point estimator via the SEL of β (as an example), symbolized as $\tilde{\beta}_{\mathcal{L}}$, can be derived from (25) as follows:

$$\tilde{\beta}_{\mathcal{L}} = \frac{\int_0^\infty \int_0^\infty \beta \cdot \mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) \cdot \Pi(\boldsymbol{\alpha}) d\beta d\delta}{\int_0^\infty \int_0^\infty \mathcal{L}(\boldsymbol{\alpha}|\mathbf{x}) \cdot \Pi(\boldsymbol{\alpha}) d\beta d\delta}. \tag{26}$$

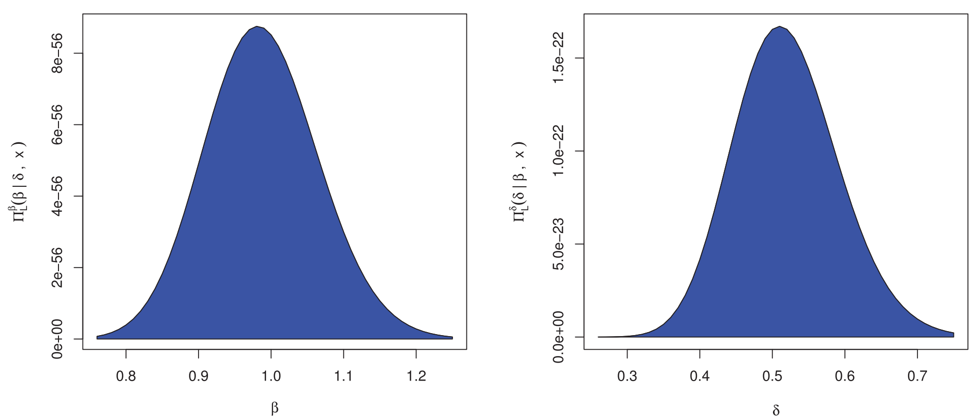
It is clear, from (26), that the Monte Carlo integration approach can be used to get the Bayesian point and credible interval estimates. To perform the MCMC methodology, the full conditional distribution (FCD) for each parameter of β and δ. From (25), the FCDs of β and δ (denoted by $\Pi^{\beta}_{\mathcal{L}}(\cdot)$ and $\Pi^{\delta}_{\mathcal{L}}(\cdot)$) can be represented, respectively, as follows:

$$\Pi^{\beta}_{\mathcal{L}}(\beta|\delta, \mathbf{x}) = \beta^{\mathcal{A}_2+a_1-1} e^{-\beta\left(b_1 - \sum_{i=1}^{\mathcal{A}_2} \ln \xi(x_i)\right) - \delta(\xi(x_i))^{\beta}} \prod_{i=1}^{\mathcal{A}_1} \left[1 - e^{-\delta(\xi(x_i))^{\beta}}\right]^{R_i} \left[1 - e^{-\delta(\xi(\tau^*))^{\beta}}\right]^{R^*}, \tag{27}$$

and

$$\Pi^{\delta}_{\mathcal{L}}(\delta|\beta, \mathbf{x}) = \delta^{\mathcal{A}_2+a_2-1} e^{-\delta\left(b_2 + \sum_{i=1}^{\mathcal{A}_2} (\xi(x_i))^{\beta}\right)} \prod_{i=1}^{\mathcal{A}_1} \left[1 - e^{-\delta(\xi(x_i))^{\beta}}\right]^{R_i} \left[1 - e^{-\delta(\xi(\tau^*))^{\beta}}\right]^{R^*}. \tag{28}$$

The FCDs of β and δ, derived in (27) and (28), cannot all match any known distribution. To overcome this challenge, utilizing a normal proposal as a candidate distribution for β and δ (see Fig. 1), we consider the use of the Metropolis-Hastings (MH) algorithm. Now, to obtain the Bayes-LF and $100(1-\varrho)\%$ BCI-LF estimates of β, δ, $R(t)$, or $h(t)$, do the Markovian steps reported in Algorithm 1.



(a) β (b) δ

Figure 1: The FCD plots of β and δ from the posterior LF-based (**a**,**b**)

Algorithm 1: The MCMC-LF Algorithm
1: **Input:** Start with $\iota = 1$
2: **Input:** Assign initial points of β as $\hat{\beta}$ and of δ as $\hat{\delta}$
3: **Output:** Get β^* from $N(\hat{\beta}, \hat{\sigma}_{11})$
4: **Output:** Calculate $\ell_\beta = \min\left[1, \frac{\Pi^{\beta}_{\mathcal{L}}(\beta^*\|\delta^{(\iota-1)}, \mathbf{x})}{\Pi^{\beta}_{\mathcal{L}}(\beta^{(\iota-1)}\|\delta^{(\iota-1)}, \mathbf{x})}\right]$
5: **Output:** Simulate u from $U(0, 1)$
6: **Output:** Store β^*

(Continued)

Algorithm 1 (continued)

7: As $\beta^{(\iota)}$
8: **if** $u \leq \ell_{\beta}$ **then**
9: Else set $\beta^{(\iota-1)}$
10: **end if**
11: **Input:** Simulate $\delta^{(\iota)}$ from $\Pi^{\delta}_{\mathcal{L}}(\delta|\beta, \mathbf{x})$
12: **Output:** Obtain $R^{(\iota)}$ from (3) and $h^{(\iota)}$ from (4)
13: **Input:** Set $\iota = \iota + 1$
14: **Input:** Redo previous steps $\mathbb{G}$ times, and set $\mathbb{G}^{\star}$ as a burn-in size
15: **Output:** Compute

$\tilde{\beta}_{\mathcal{L}} = \frac{1}{\mathbb{G}^{\star\star}} \sum_{\iota=\mathbb{G}^{\star}+1}^{\mathbb{G}} \beta^{(\iota)}$,

$\tilde{\delta}_{\mathcal{L}} = \frac{1}{\mathbb{G}^{\star\star}} \sum_{\iota=\mathbb{G}^{\star}+1}^{\mathbb{G}} \delta^{(\iota)}$,

$\tilde{R}_{\mathcal{L}}(t) = \frac{1}{\mathbb{G}^{\star\star}} \sum_{\iota=\mathbb{G}^{\star}+1}^{\mathbb{G}} R^{(\iota)}(t)$,

and

$\tilde{h}_{\mathcal{L}}(t) = \frac{1}{\mathbb{G}^{\star\star}} \sum_{\iota=\mathbb{G}^{\star}+1}^{\mathbb{G}} h^{(\iota)}(t)$,

where $\mathbb{G}^{\star\star} = \mathbb{G} - \mathbb{G}^{\star}$

16: **Input:** Sort $\beta^{(\iota)}$, $\delta^{(\iota)}$, $R^{(\iota)}(t)$, and $h^{(\iota)}(t)$ (for $\iota = \mathbb{G}^{\star}+1, \mathbb{G}^{\star}+2, \ldots, \mathbb{G}$)
17: **Output:** Compute the $100(1-\varrho)\%$ BCI-LF estimators of β, δ, $R(t)$, and $h(t)$ as

$\left\{\beta^{\left[\left(\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}, \beta^{\left[\left(1-\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}\right\}$,

$\left\{\delta^{\left[\left(\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}, \delta^{\left[\left(1-\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}\right\}$,

$\left\{R^{\left[\left(\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}(t), R^{\left[\left(1-\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}(t)\right\}$,

and

$\left\{h^{\left[\left(\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}(t), h^{\left[\left(1-\frac{\varrho}{2}\right)\mathbb{G}^{\star\star}\right]}(t)\right\}$

3.2 Bayes SF-Based

Using the SF in (16) and joint prior PDF in (24), the posterior PDF of $\boldsymbol{\alpha}$ (say, $\Pi_{\mathcal{P}}(\cdot)$) becomes

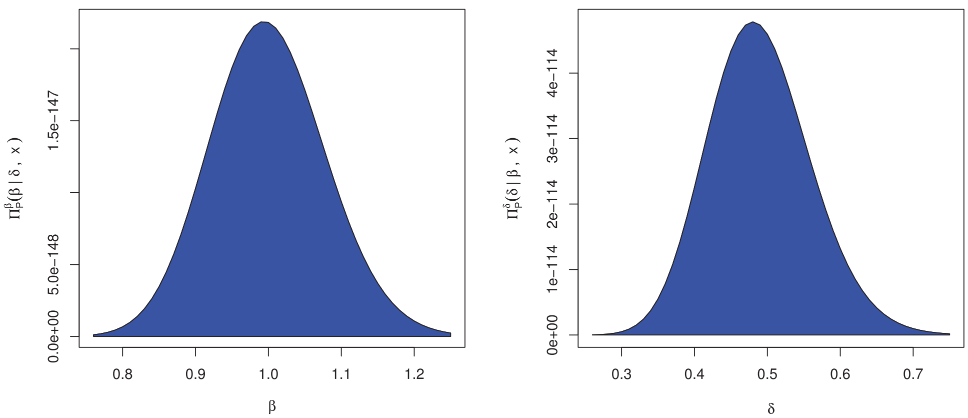
$$\Pi_{\mathcal{P}}(\boldsymbol{\alpha}|\mathbf{x}) = \mathcal{C}_{\mathcal{P}}^{-1} \beta^{a_1-1} \delta^{a_2-1} e^{-(\beta b_1 + \delta b_2)}$$

$$\times \prod_{i=1}^{A_2+1} \left[e^{-\delta(\xi(x_i))^{\beta}} - e^{-\delta(\xi(x_{i-1}))^{\beta}} \right] \prod_{i=1}^{A_1} \left[1 - e^{-\delta(\xi(x_i))^{\beta}} \right]^{R_i} \left[1 - e^{-\delta(\xi(\tau^*))^{\beta}} \right]^{R^*}, \tag{29}$$

where $\mathcal{C}_{\mathcal{P}}$ is a normalized constant and is defined as

$$\mathcal{C}_{\mathcal{P}} = \int_0^\infty \int_0^\infty \mathcal{P}(\boldsymbol{\alpha}|\mathbf{x}) \cdot \Pi(\boldsymbol{\alpha}) \mathrm{d}\beta \mathrm{d}\delta.$$

As we anticipated, from (29), the Bayes estimates from SF-based (Bayes-SF) as well as associated BCI estimates from SF-based (BCI-SF) cannot be derived theoretically. This finding is due to the nonlinear expressions of the posterior PDF (29) of β and δ. Again, we recommend the MCMC methodology for this issue. From (29), the FCDs of β and δ (denoted by $\Pi_{\mathcal{P}}^{\beta}(\cdot)$ and $\Pi_{\mathcal{P}}^{\delta}(\cdot)$) cannot follow any known statistical model. Fig. 2 states that the FCDs of β and δ are fairly symmetrical. For this reason, the candidate distribution of β or δ is taken as utilizing a normal proposal. After that, by setting $\beta \equiv \check{\beta}$ and $\delta \equiv \check{\delta}$ in Algorithm 1, we follow the same Markovian steps to evaluate the Bayes-SF and $100(1-\varrho)\%$ BCI-SF estimators of β, δ, $R(t)$, and $h(t)$.



(a) β (b) δ

Figure 2: The FCD plots of β and δ from the posterior SF-based (**a**,**b**)

4 Numerical Comparisons

In this section, we conduct a series of Monte Carlo simulations to evaluate the accuracy of the Uni-W parameters (including β and δ) and reliability features (including $R(t)$ and $h(t)$) discussed earlier. By generating 1000 T2IA-PC samples from two Uni-W populations, namely Set-1:Uni-W$(1.0, 0.5)$ and Set-2:Uni-W$(2.0, 1.5)$, several simulations are implemented in turn to compare the performance of the acquired estimators for β, δ, $R(t)$, and $h(t)$. At times $t(= 0.1, 0.2)$, the true values of $R(t)$ are taken as $(0.99907, 0.98315)$ and of $h(t)$ are taken as $(0.04205, 0.32608)$ from Set-1 and -2, respectively. All point and interval estimation methodologies are analyzed under various scenarios of n, m, and (T_1, T_2). In Table 1, different scenarios of progressive censoring designs R_i, $i = 1, \ldots, m$, are presented. In this table, for simplicity, censoring patterns like $\mathbf{R} : (5, 5, 5, 0, 0, 0, 0, 0, 0)$ are denoted as $(5^3, 0^6)$. To gather a T2IA-PC sample from the proposed Uni-W model, do the simulating steps discussed in Algorithm 2.

Table 1: Removal designs in Monte Carlo evaluations

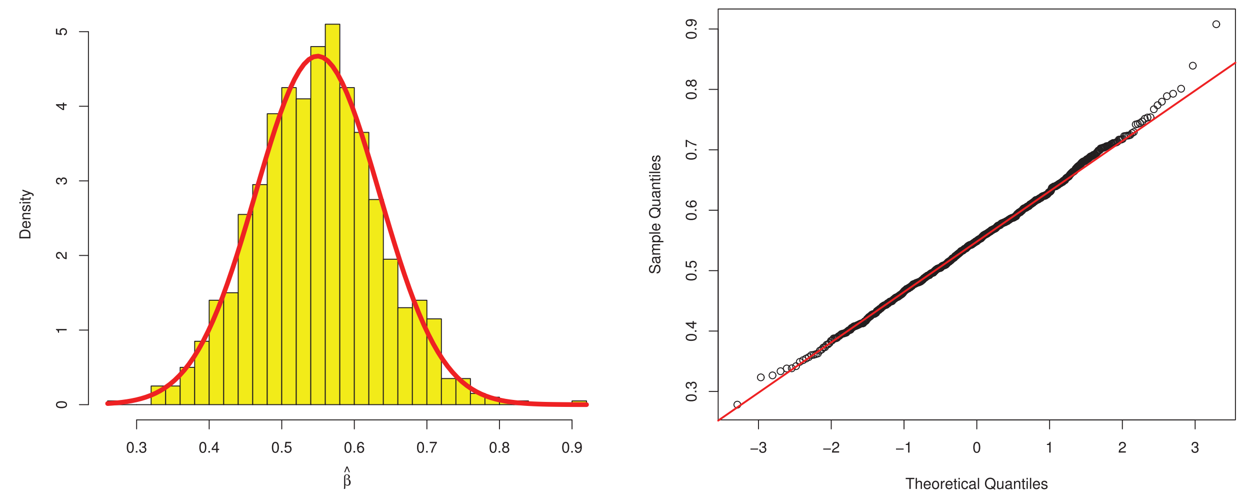
(n, m)	**R**		
PC $\rightarrow$	[1]	[2]	[3]
(30, 15)	$(5^3, 0^{12})$	$(0^6, 5^3, 0^6)$	$(0^{12}, 5^3)$
(30, 25)	$(5^1, 0^{24})$	$(0^{12}, 5^1, 0^{12})$	$(0^{24}, 5^1)$
(60, 30)	$(5^6, 0^{24})$	$(0^{12}, 5^6, 0^{12})$	$(0^{24}, 5^6)$
(60, 50)	$(5^2, 0^{48})$	$(0^{24}, 5^2, 0^{24})$	$(0^{48}, 5^2)$
(90, 45)	$(5^9, 0^{36})$	$(0^{18}, 5^9, 0^{18})$	$(0^{36}, 5^9)$
(90, 70)	$(5^4, 0^{66})$	$(0^{33}, 5^4, 0^{33})$	$(0^{66}, 5^4)$

Algorithm 2: The T2IA-PC generation algorithm

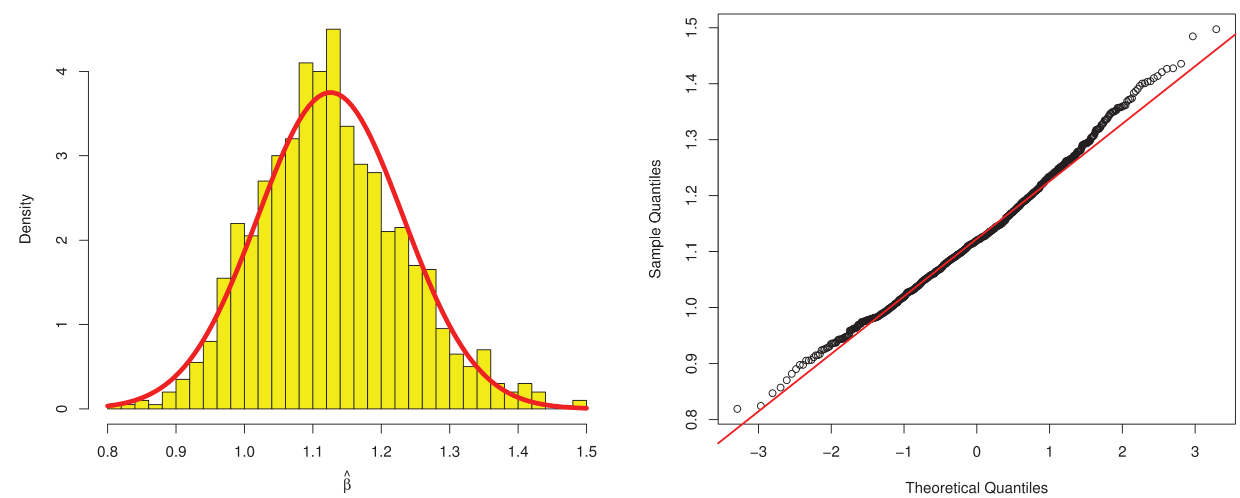
Input: Assign the values of Uni-W ($\boldsymbol{\alpha}$)
Input: Assign the levels of n , m , R_i, $i = 1, 2, \ldots, m$, and T_i, $i = 1, 2$
Input: Create g independent units with size m (say, $\nabla_1, \nabla_2, \ldots, \nabla_m$)
Input: Set $\Delta_i = \nabla_i^{\left(i+\sum_{j=m-i+1}^{m} S_j\right)^{-1}}$, $i = 1, 2, \ldots, m$
Input: Set $U_i = 1 - \Delta_m \Delta_{m-1} \cdots \Delta_{m-i+1}$ for $i = 1, 2, \ldots, m$
Input: Set $X_i = \exp\left(-\left[-\frac{1}{\delta}\log(u_i)\right]^{\frac{1}{\beta}}\right)$, $i = 1, 2, \ldots, m$
Output: Find d_1 at T_1, and discard X_i for $i = d_1 + 2, \ldots, m$
Output: Produce $(X_{d_1+2}, \ldots, X_m)$ data from $f(x)\left[R\left(x_{d_1+1}\right)\right]^{-1}$
Output: Stop the T2IA-PC test:
At X_m (Case-1)
if $X_m < T_1$ **then**
  Set $R^* = n - m - \sum_{i=1}^{m-1} R_i$
end if
At X_m (Case-2)
if $T_1 < X_m < T_2$ **then**
  Set $R^* = n - d_2 - \sum_{i=1}^{d_1} R_i$
end if
At T_2 (Case-3)
if $T_2 < X_m$ **then**
  Set $R^* = n - d_2 - \sum_{i=1}^{d_1} R_i$
end if

To evaluate the asymptotically normal distribution of the MLE of β when its support is $\beta \leq 1$, we plot the histogram (with a Gaussian kernel) and the quantile-quantile (QQ) for 1000 MLEs of β when $(n, m) = (100, 50)$, $(T_1, T_2) = (0.2, 0.4)$, and $\mathbf{R} : (1^{50})$. Fig. 3a shows a slightly skewed histogram with mild deviations from the overlaid normal curve, while the corresponding Q–Q plot indicates that the bulk of the empirical distribution aligns well with the theoretical normal quantiles. These results suggest that, despite the low value of β, the MLE exhibits reasonably well-behaved sampling properties.

In contrast, Fig. 3b shows that the histogram of $\hat{\beta}$ is approximately symmetric, and the Q–Q plot follows a strong linear trend with minimal deviations, indicating a good fit to the normal distribution. These simulation results demonstrate that the specific censoring scheme and estimation framework employed in this study effectively mitigate the known limitations of the MLE for small values of β. Consequently, standard inferential procedures based on asymptotic normality appear to be valid and reliable across the range of β values considered. For additional details on the issue of assessing the asymptotically normal distribution of the MLE, one may refer to Zou and Wen [18].



(a) For Uni-W$(0.5, 1)$



(b) For Uni-W$(1, 1)$

Figure 3: Normality diagrams for 1000 MLEs of Uni-W shape parameter (**a**,**b**)

From each AT2PHC sample, in the frequentist viewpoint, the estimates developed by ML and MPS techniques (in addition to their 95% ACIs) of β, δ, $R(t)$, and $h(t)$ are evaluated. To examine the influence of gamma priors on the parameter vector $\boldsymbol{\alpha}$, two informative sets of hyperparameters $(a_i, b_i),\ i = 1, 2$, are selected using the past-information elicitation algorithm proposed by Nassar and

Elshahhat [12]. Based on a collection of 10,000 complete historical samples, the assigned values of (a_1, b_1, a_2, b_2) are taken as

- From Set-1
- Using log-LF, we get (41.73103, 41.08557, 76.44668, 148.6706);
- Using log-SF, we get (36.88423, 35.86467, 72.50526, 133.0251),
- From Set-2
- Using log-LF, we get (40.50317, 19.55254, 76.44668, 49.5568);
- Using log-SF, we get (34.80288, 15.89363, 72.50022, 44.33855).

For each parameter, following Algorithm 1, we set the total number of MCMC iterations to $\mathbb{G} = 12{,}000$, with a burn-in period of $\mathbb{G}^\star = 2000$. Consequently, from the remaining $\mathbb{G}^{\star\star} = 10{,}000$ iterations, the Bayes estimates and their associated 95% BCIs for β, δ, $R(t)$, and $h(t)$ are obtained under both the likelihood-based and product-of-spacings estimation approaches. We recommend installing two packages in `R` (version 4.2.2) to implement the proposed estimation methods: the `maxLik` package (by Henningsen and Toomet [19]) for maximizing likelihood functions, and the `coda` package (by Plummer et al. [20]) for analyzing MCMC output and computing Bayesian estimates and credible intervals.

Specifically, the average estimates (Av.Es) of β are given by

$$Av.E(\beta) = \frac{1}{1000}\sum_{i=1}^{1000} \breve{\beta}^{(i)},$$

where $\breve{\beta}^{(i)}$ is the estimate of β at ith sample.

Two criteria for the point estimates of β, namely root mean squared errors (RMSEs) and mean relative absolute biases (MABs), are utilized as

$$\text{RMSE}(\breve{\beta}) = \sqrt{\frac{1}{1000}\sum\nolimits_{i=1}^{H} \left(\breve{\beta}^{(i)} - \beta\right)^2},$$

and

$$\text{MAB}(\breve{\beta}) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000} \left|\breve{\beta}^{(i)} - \beta\right|,$$

respectively.

Moreover, to compare the acquired interval estimates of β, we consider two criteria, namely: average interval lengths (AILs) and coverage percentages (CPs) as

$$\text{AIL}_{(1-\epsilon)\%}(\beta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000} \left(\mathcal{U}_{\breve{\beta}^{(i)}} - \mathcal{L}_{\breve{\beta}^{(i)}}\right),$$

and

$$\text{CP}_{(1-\epsilon)\%}(\beta) = \frac{1}{1000}\sum\nolimits_{i=1}^{1000} \aleph^\star_{\left(\mathcal{L}_{\breve{\beta}^{(i)}};\mathcal{U}_{\breve{\beta}^{(i)}}\right)}(\beta),$$

respectively, where $\aleph^{\star}(\cdot)$ is an indicator. Tables 2–9 presented the Av.Es, RMSEs, and MABs in the first, second, and third columns, respectively. On the other hand, Tables 10–17 presented the AILs and CPs in the first and second columns, respectively.

Table 2: The point assessments of β from Set-1

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.2, 0.4)$													
(30, 15)	[1]	2.041	1.063	1.041	1.627	0.653	0.627	1.080	0.302	0.220	1.069	0.275	0.224
	[2]	2.387	1.416	1.387	1.833	0.900	0.876	1.044	0.360	0.273	1.021	0.492	0.470
	[3]	2.057	1.180	1.057	1.630	0.770	0.747	1.041	0.315	0.256	1.032	0.291	0.323
(30, 25)	[1]	1.776	0.815	0.776	1.384	0.413	0.385	1.072	0.262	0.191	1.049	0.230	0.170
	[2]	1.776	0.835	0.776	1.384	0.424	0.390	1.071	0.264	0.192	1.048	0.257	0.192
	[3]	1.851	0.894	0.851	1.400	0.437	0.400	1.070	0.300	0.217	1.040	0.264	0.192
(60, 30)	[1]	1.244	0.237	0.199	1.067	0.198	0.160	1.078	0.174	0.145	1.162	0.185	0.157
	[2]	1.306	0.363	0.321	1.075	0.279	0.176	1.012	0.192	0.160	1.014	0.228	0.169
	[3]	1.130	0.300	0.258	0.941	0.213	0.174	1.034	0.188	0.152	1.035	0.209	0.162
(60, 50)	[1]	0.965	0.205	0.170	0.913	0.169	0.138	1.039	0.154	0.126	1.032	0.164	0.133
	[2]	0.986	0.210	0.169	0.913	0.178	0.144	1.041	0.164	0.135	1.034	0.173	0.137
	[3]	1.012	0.214	0.179	0.964	0.184	0.148	1.031	0.167	0.143	1.023	0.175	0.136
(90, 45)	[1]	1.189	0.161	0.128	1.146	0.146	0.116	1.072	0.136	0.108	1.072	0.139	0.112
	[2]	1.125	0.175	0.135	1.075	0.167	0.127	1.006	0.147	0.123	1.010	0.157	0.124
	[3]	1.084	0.165	0.135	1.036	0.166	0.119	1.037	0.142	0.117	1.039	0.156	0.116
(90, 70)	[1]	1.108	0.143	0.115	1.076	0.127	0.106	1.028	0.114	0.088	1.026	0.119	0.091
	[2]	1.108	0.146	0.118	1.076	0.131	0.107	1.032	0.116	0.092	1.030	0.121	0.095
	[3]	1.152	0.150	0.123	1.123	0.135	0.111	1.015	0.126	0.095	1.013	0.129	0.098
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	2.123	0.939	0.919	1.747	0.563	0.534	0.884	0.247	0.193	0.875	0.281	0.220
	[2]	2.418	1.215	1.418	1.876	0.857	0.833	1.026	0.358	0.322	1.039	0.396	0.355
	[3]	1.919	1.142	1.123	1.534	0.656	0.630	0.999	0.272	0.214	0.996	0.316	0.247
(30, 25)	[1]	1.658	0.652	0.615	1.377	0.368	0.329	1.043	0.218	0.170	1.065	0.232	0.173
	[2]	1.615	0.693	0.658	1.323	0.415	0.379	1.035	0.223	0.183	1.055	0.245	0.189
	[3]	1.779	0.818	0.779	1.459	0.494	0.459	1.037	0.228	0.174	1.064	0.261	0.196
(60, 30)	[1]	1.111	0.229	0.196	0.935	0.203	0.176	0.999	0.173	0.138	0.993	0.186	0.155
	[2]	1.313	0.368	0.324	1.112	0.237	0.216	1.011	0.205	0.155	1.009	0.224	0.184
	[3]	1.111	0.240	0.197	0.935	0.208	0.167	0.999	0.184	0.146	0.993	0.198	0.157
(60, 50)	[1]	1.015	0.198	0.164	0.971	0.165	0.133	1.031	0.152	0.130	1.041	0.155	0.126
	[2]	0.973	0.198	0.168	0.930	0.172	0.148	1.028	0.155	0.123	1.037	0.166	0.136
	[3]	1.032	0.206	0.179	0.980	0.186	0.154	1.022	0.173	0.138	1.033	0.176	0.149
(90, 45)	[1]	0.826	0.150	0.121	0.784	0.142	0.115	1.011	0.133	0.104	1.011	0.135	0.109
	[2]	1.174	0.198	0.168	1.118	0.159	0.126	1.007	0.150	0.125	1.005	0.152	0.120

(Continued)

Table 2 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
	[3]	1.026	0.163	0.133	0.986	0.149	0.117	0.999	0.138	0.113	0.995	0.140	0.110
(90, 70)	[1]	1.160	0.129	0.105	1.137	0.115	0.094	1.027	0.108	0.084	1.034	0.113	0.088
	[2]	1.111	0.140	0.113	1.087	0.129	0.101	1.018	0.121	0.099	1.022	0.124	0.096
	[3]	1.185	0.141	0.115	1.170	0.132	0.105	1.014	0.126	0.100	1.020	0.128	0.102

Table 3: The point assessments of β from Set-2

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	3.462	1.494	1.462	3.063	1.599	1.532	1.553	0.629	0.561	1.690	0.639	0.577
	[2]	3.197	2.150	2.090	3.581	1.783	1.738	2.177	0.683	0.611	2.369	0.702	0.504
	[3]	3.394	2.022	1.968	3.738	1.619	1.581	2.148	0.662	0.603	2.270	0.684	0.590
(30, 25)	[1]	2.988	1.434	1.399	3.482	1.486	1.457	2.166	0.437	0.318	2.308	0.574	0.412
	[2]	3.199	1.447	1.399	3.482	1.543	1.482	2.156	0.450	0.329	2.281	0.605	0.548
	[3]	3.408	1.460	1.408	3.532	1.543	1.482	2.152	0.520	0.371	2.324	0.615	0.438
(60, 30)	[1]	2.136	1.229	1.207	1.928	1.256	1.189	1.524	0.330	0.242	1.581	0.400	0.295
	[2]	3.029	1.243	1.221	2.771	1.386	1.362	2.095	0.436	0.335	2.187	0.534	0.385
	[3]	2.733	1.241	1.216	2.559	1.384	1.361	2.090	0.417	0.304	2.156	0.529	0.408
(60, 50)	[1]	3.147	1.114	1.052	3.129	1.086	1.063	2.119	0.293	0.227	2.196	0.341	0.245
	[2]	3.052	1.208	1.138	3.058	1.114	1.058	2.106	0.294	0.217	2.174	0.354	0.262
	[3]	3.138	1.213	1.147	3.189	1.187	1.129	2.090	0.303	0.225	2.179	0.368	0.270
(90, 45)	[1]	1.718	0.565	0.526	1.740	0.599	0.557	1.561	0.234	0.179	1.598	0.275	0.207
	[2]	2.516	1.073	1.029	2.557	0.836	0.771	2.055	0.280	0.207	2.115	0.331	0.259
	[3]	2.526	0.774	0.733	2.527	0.611	0.559	2.067	0.242	0.189	2.112	0.283	0.217
(90, 70)	[1]	3.221	0.284	0.242	3.361	0.247	0.190	2.102	0.205	0.167	2.157	0.215	0.165
	[2]	3.207	0.330	0.292	3.362	0.323	0.273	2.083	0.228	0.182	2.130	0.254	0.202
	[3]	3.216	0.558	0.516	3.457	0.564	0.527	2.052	0.232	0.176	2.117	0.273	0.209
$(T_1, T_2) = (0.7, 0.9)$													
(30, 15)	[1]	3.193	1.931	1.883	3.619	1.550	1.486	2.113	0.487	0.353	2.303	0.679	0.468
	[2]	3.346	2.098	2.035	3.593	1.660	1.619	2.128	0.521	0.369	2.337	0.717	0.511
	[3]	3.883	1.976	1.926	3.514	1.636	1.593	2.124	0.504	0.368	2.310	0.696	0.481
(30, 25)	[1]	3.077	1.170	1.146	3.216	1.389	1.361	1.917	0.443	0.319	2.133	0.620	0.417
	[2]	3.261	1.308	1.261	3.395	1.448	1.395	2.105	0.450	0.330	2.275	0.643	0.441
	[3]	3.298	1.345	1.298	3.486	1.495	1.514	2.107	0.485	0.372	2.296	0.671	0.456
(60, 30)	[1]	2.859	1.063	1.019	2.710	1.109	1.048	2.041	0.314	0.236	2.134	0.374	0.276
	[2]	3.019	1.119	1.077	2.756	1.262	1.216	2.067	0.362	0.310	2.164	0.405	0.298
	[3]	2.741	1.115	1.047	2.578	1.215	1.194	2.062	0.332	0.247	2.150	0.378	0.279
(60, 50)	[1]	2.929	0.926	0.929	2.930	0.852	0.834	2.063	0.296	0.228	2.149	0.346	0.260

(Continued)

Table 3 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
	[2]	2.929	0.993	0.929	2.930	0.915	0.930	2.063	0.306	0.231	2.149	0.355	0.273
	[3]	3.047	1.033	1.010	3.048	0.985	0.930	2.065	0.312	0.233	2.159	0.367	0.273
(90, 45)	[1]	2.347	0.644	0.614	2.354	0.637	0.595	2.013	0.239	0.183	2.075	0.278	0.210
	[2]	2.586	0.899	0.859	2.595	0.827	0.756	2.036	0.276	0.223	2.099	0.336	0.246
	[3]	2.378	0.781	0.741	2.373	0.771	0.710	2.035	0.250	0.194	2.092	0.290	0.222
(90, 70)	[1]	2.613	0.399	0.356	2.834	0.391	0.354	1.730	0.228	0.174	1.804	0.263	0.198
	[2]	3.010	0.420	0.383	3.194	0.414	0.375	2.035	0.233	0.168	2.096	0.267	0.205
	[3]	3.146	0.630	0.586	3.361	0.629	0.578	2.036	0.236	0.178	2.103	0.271	0.216

Table 4: The point assessments of δ from Set-1

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
							$(T_1, T_2) = (0.2, 0.4)$						
(30, 15)	[1]	0.550	0.462	0.434	0.878	0.504	0.496	0.645	0.214	0.144	0.612	0.221	0.158
	[2]	0.663	0.336	0.326	0.849	0.443	0.435	0.522	0.150	0.109	0.575	0.181	0.120
	[3]	0.658	0.363	0.358	0.843	0.463	0.453	0.545	0.153	0.113	0.599	0.196	0.141
(30, 25)	[1]	0.572	0.303	0.297	0.996	0.404	0.398	0.532	0.135	0.095	0.576	0.167	0.124
	[2]	0.574	0.269	0.269	0.935	0.391	0.384	0.525	0.118	0.088	0.563	0.143	0.108
	[3]	0.574	0.292	0.285	0.935	0.394	0.388	0.526	0.130	0.098	0.564	0.145	0.119
(60, 30)	[1]	0.733	0.241	0.233	0.860	0.385	0.378	0.525	0.115	0.087	0.561	0.128	0.098
	[2]	0.701	0.208	0.201	0.788	0.364	0.359	0.074	0.097	0.075	0.520	0.102	0.079
	[3]	0.733	0.240	0.233	0.837	0.369	0.360	0.513	0.105	0.081	0.543	0.115	0.088
(60, 50)	[1]	0.865	0.170	0.163	0.898	0.355	0.349	0.519	0.086	0.068	0.545	0.098	0.076
	[2]	0.858	0.166	0.158	0.888	0.347	0.337	0.513	0.078	0.064	0.537	0.090	0.070
	[3]	0.858	0.168	0.159	0.888	0.350	0.343	0.512	0.085	0.065	0.536	0.091	0.073
(90, 45)	[1]	0.659	0.161	0.154	0.677	0.332	0.337	0.515	0.077	0.061	0.539	0.085	0.064
	[2]	0.620	0.127	0.120	0.633	0.297	0.288	0.486	0.070	0.054	0.505	0.080	0.062
	[3]	0.654	0.153	0.145	0.670	0.324	0.337	0.499	0.073	0.063	0.519	0.083	0.063
(90, 70)	[1]	0.770	0.101	0.093	0.859	0.197	0.189	0.511	0.060	0.047	0.532	0.068	0.053
	[2]	0.797	0.085	0.075	0.790	0.170	0.161	0.505	0.052	0.046	0.524	0.059	0.047
	[3]	0.749	0.094	0.084	0.815	0.187	0.177	0.504	0.056	0.044	0.521	0.064	0.052
							$(T_1, T_2) = (0.5, 0.7)$						
(30, 15)	[1]	0.641	0.351	0.346	0.860	0.477	0.470	0.538	0.185	0.157	0.597	0.210	0.222
	[2]	0.643	0.328	0.323	0.806	0.432	0.425	0.604	0.133	0.098	0.686	0.184	0.136
	[3]	0.681	0.350	0.345	0.867	0.455	0.448	0.554	0.145	0.105	0.613	0.197	0.145
(30, 25)	[1]	0.584	0.277	0.287	0.970	0.382	0.376	0.522	0.126	0.092	0.560	0.128	0.097
	[2]	0.593	0.255	0.249	0.925	0.374	0.367	0.516	0.101	0.075	0.550	0.122	0.093

(Continued)

Table 4 (continued)

(n, m)	PC	MLE Av.E	MLE RMSE	MLE MAB	MPSE Av.E	MPSE RMSE	MPSE MAB	Bayes-LF Av.E	Bayes-LF RMSE	Bayes-LF MAB	Bayes-SF Av.E	Bayes-SF RMSE	Bayes-SF MAB
	[3]	0.599	0.276	0.270	0.948	0.377	0.371	0.522	0.112	0.078	0.557	0.127	0.095
(60, 30)	[1]	0.723	0.249	0.244	0.835	0.367	0.360	0.521	0.098	0.073	0.554	0.118	0.090
	[2]	0.738	0.243	0.238	0.838	0.357	0.340	0.527	0.090	0.069	0.561	0.110	0.085
	[3]	0.738	0.245	0.240	0.838	0.361	0.346	0.527	0.096	0.071	0.561	0.114	0.087
(60, 50)	[1]	0.846	0.230	0.223	0.876	0.347	0.338	0.513	0.088	0.065	0.536	0.102	0.078
	[2]	0.823	0.179	0.173	0.846	0.324	0.319	0.507	0.071	0.055	0.527	0.082	0.065
	[3]	0.845	0.187	0.181	0.871	0.341	0.335	0.508	0.073	0.061	0.529	0.085	0.069
(90, 45)	[1]	0.644	0.151	0.144	0.661	0.320	0.315	0.512	0.069	0.054	0.534	0.081	0.063
	[2]	0.738	0.149	0.141	0.762	0.295	0.290	0.501	0.062	0.048	0.503	0.070	0.055
	[3]	0.673	0.150	0.143	0.689	0.313	0.306	0.516	0.068	0.050	0.538	0.078	0.061
(90, 70)	[1]	0.770	0.107	0.099	0.819	0.268	0.262	0.506	0.058	0.046	0.524	0.066	0.052
	[2]	0.785	0.084	0.074	0.837	0.142	0.133	0.499	0.048	0.039	0.515	0.053	0.042
	[3]	0.785	0.085	0.075	0.837	0.179	0.170	0.502	0.059	0.046	0.520	0.061	0.051

Table 5: The point assessments of δ from Set-2

(n, m)	PC	MLE Av.E	MLE RMSE	MLE MAB	MPSE Av.E	MPSE RMSE	MPSE MAB	Bayes-LF Av.E	Bayes-LF RMSE	Bayes-LF MAB	Bayes-SF Av.E	Bayes-SF RMSE	Bayes-SF MAB
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	2.479	1.649	1.635	3.551	2.236	2.207	1.518	0.817	0.628	1.667	1.249	1.003
	[2]	2.542	1.549	1.527	3.707	2.082	2.051	2.012	0.533	0.476	2.141	0.649	0.580
	[3]	2.547	1.589	1.573	2.994	2.182	2.152	1.409	0.610	0.517	1.534	0.815	0.694
(30, 25)	[1]	2.086	1.528	1.506	3.497	2.028	1.997	1.504	0.397	0.323	1.603	0.477	0.373
	[2]	2.022	1.313	1.283	3.195	1.672	1.695	1.421	0.279	0.216	1.505	0.322	0.267
	[3]	2.022	1.365	1.331	3.195	1.720	1.770	1.379	0.305	0.253	1.448	0.362	0.278
(60, 30)	[1]	2.783	1.278	1.258	3.131	1.652	1.631	1.507	0.272	0.224	1.592	0.271	0.222
	[2]	3.171	1.183	1.169	3.652	1.520	1.494	1.831	0.230	0.191	2.027	0.227	0.180
	[3]	2.831	1.261	1.241	3.130	1.581	1.630	1.385	0.236	0.215	1.454	0.258	0.202
(60, 50)	[1]	3.006	1.122	1.103	2.892	1.443	1.439	1.503	0.215	0.171	1.564	0.221	0.177
	[2]	3.027	1.062	1.042	2.909	1.429	1.410	1.418	0.193	0.165	1.469	0.206	0.160
	[3]	3.027	1.068	1.047	2.909	1.431	1.409	1.355	0.200	0.163	1.396	0.213	0.190
(90, 45)	[1]	2.312	1.060	1.044	2.224	1.415	1.392	1.500	0.192	0.154	1.561	0.198	0.152
	[2]	2.613	0.830	0.812	2.507	1.302	1.287	1.590	0.160	0.136	1.654	0.176	0.137
	[3]	2.544	1.001	0.979	2.470	1.367	1.350	1.364	0.164	0.129	1.410	0.185	0.142
(90, 70)	[1]	2.615	0.615	0.595	2.787	1.023	1.007	1.505	0.156	0.123	1.556	0.171	0.134
	[2]	2.289	0.534	0.512	2.384	0.770	0.752	1.903	0.133	0.106	2.013	0.153	0.118
	[3]	2.673	0.603	0.586	2.850	0.988	0.970	1.505	0.144	0.114	1.558	0.165	0.126

(Continued)

Table 5 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.7, 0.9)$													
(30, 15)	[1]	2.588	1.715	1.671	3.317	2.107	2.077	1.559	0.514	0.398	1.774	0.725	0.593
	[2]	2.758	1.536	1.514	3.214	1.987	1.963	1.734	0.431	0.312	1.896	0.570	0.431
	[3]	2.387	1.605	1.582	3.344	2.031	2.001	1.585	0.462	0.406	1.828	0.574	0.417
(30, 25)	[1]	2.066	1.393	1.369	3.501	1.870	1.844	1.525	0.385	0.285	1.654	0.569	0.514
	[2]	2.095	1.265	1.235	3.577	1.796	1.771	1.856	0.273	0.205	2.078	0.342	0.262
	[3]	2.005	1.280	1.247	3.271	1.846	1.817	1.530	0.338	0.249	1.660	0.502	0.360
(60, 30)	[1]	2.735	1.231	1.204	3.059	1.738	1.714	1.528	0.253	0.196	1.636	0.339	0.252
	[2]	2.704	1.129	1.115	2.910	1.540	1.521	1.637	0.237	0.183	1.729	0.319	0.241
	[3]	2.747	1.187	1.173	3.021	1.579	1.559	1.539	0.247	0.193	1.658	0.321	0.236
(60, 50)	[1]	2.818	1.110	1.088	2.914	1.437	1.414	1.510	0.217	0.169	1.578	0.279	0.211
	[2]	2.741	0.909	0.887	2.595	1.353	1.331	1.513	0.188	0.147	1.582	0.224	0.174
	[3]	3.014	0.983	0.965	2.831	1.360	1.342	1.513	0.202	0.155	1.582	0.245	0.191
(90, 45)	[1]	2.465	0.873	0.858	2.342	1.255	1.231	1.510	0.169	0.131	1.580	0.220	0.165
	[2]	2.869	0.803	0.789	2.731	0.973	0.957	1.710	0.157	0.124	1.830	0.169	0.140
	[3]	2.358	0.866	0.852	2.252	1.115	1.095	1.513	0.167	0.131	1.591	0.210	0.159
(90, 70)	[1]	2.669	0.584	0.566	2.842	0.899	0.884	1.498	0.152	0.121	1.547	0.140	0.112
	[2]	2.347	0.526	0.505	3.463	0.743	0.724	1.373	0.126	0.101	1.410	0.133	0.109
	[3]	2.352	0.541	0.522	2.457	0.862	0.842	1.292	0.149	0.123	1.321	0.135	0.110

Table 6: The point assessments of $R(t)$ from Set-1

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.2, 0.4)$													
(30, 15)	[1]	0.982	0.202	0.201	0.976	0.196	0.195	0.783	0.102	0.669	0.799	0.099	0.070
	[2]	0.964	0.191	0.191	0.961	0.181	0.183	0.786	0.087	0.060	0.798	0.088	0.071
	[3]	0.969	0.179	0.189	0.961	0.180	0.180	0.791	0.077	0.062	0.803	0.083	0.067
(30, 25)	[1]	0.939	0.166	0.158	0.948	0.168	0.167	0.792	0.064	0.051	0.807	0.068	0.054
	[2]	0.945	0.172	0.164	0.957	0.177	0.177	0.789	0.075	0.059	0.806	0.078	0.058
	[3]	0.939	0.160	0.158	0.948	0.158	0.167	0.793	0.061	0.047	0.807	0.067	0.054
(60, 30)	[1]	0.905	0.128	0.124	0.884	0.118	0.116	0.783	0.060	0.047	0.790	0.058	0.046
	[2]	0.870	0.120	0.110	0.844	0.108	0.103	0.788	0.054	0.043	0.794	0.055	0.045
	[3]	0.888	0.111	0.107	0.868	0.104	0.102	0.720	0.051	0.041	0.805	0.051	0.039
(60, 50)	[1]	0.856	0.100	0.098	0.849	0.092	0.088	0.790	0.043	0.035	0.798	0.047	0.038
	[2]	0.871	0.104	0.098	0.866	0.101	0.102	0.787	0.046	0.038	0.796	0.050	0.041
	[3]	0.856	0.096	0.090	0.849	0.090	0.085	0.790	0.043	0.035	0.798	0.047	0.038

(Continued)

Table 6 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
(90, 45)	[1]	0.852	0.095	0.090	0.845	0.077	0.073	0.782	0.043	0.035	0.786	0.046	0.037
	[2]	0.842	0.085	0.080	0.834	0.074	0.068	0.788	0.041	0.033	0.792	0.043	0.036
	[3]	0.860	0.082	0.076	0.854	0.073	0.068	0.797	0.038	0.030	0.802	0.040	0.032
(90, 70)	[1]	0.879	0.080	0.073	0.883	0.071	0.065	0.784	0.035	0.028	0.790	0.037	0.030
	[2]	0.890	0.082	0.076	0.897	0.072	0.065	0.784	0.036	0.029	0.790	0.038	0.031
	[3]	0.879	0.069	0.062	0.883	0.060	0.054	0.787	0.033	0.026	0.793	0.035	0.028
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	0.981	0.201	0.201	0.977	0.197	0.196	0.783	0.108	0.089	0.797	0.111	0.091
	[2]	0.970	0.189	0.189	0.955	0.185	0.185	0.781	0.076	0.060	0.792	0.085	0.068
	[3]	0.972	0.184	0.183	0.965	0.180	0.179	0.738	0.070	0.054	0.749	0.078	0.061
(30, 25)	[1]	0.926	0.151	0.149	0.943	0.167	0.166	0.788	0.063	0.050	0.801	0.066	0.053
	[2]	0.941	0.161	0.160	0.960	0.175	0.175	0.788	0.065	0.048	0.803	0.070	0.055
	[3]	0.930	0.147	0.145	0.946	0.163	0.162	0.790	0.060	0.048	0.804	0.065	0.049
(60, 30)	[1]	0.904	0.127	0.123	0.887	0.120	0.118	0.782	0.058	0.046	0.790	0.064	0.051
	[2]	0.867	0.114	0.112	0.843	0.111	0.107	0.781	0.052	0.041	0.787	0.055	0.044
	[3]	0.867	0.105	0.103	0.843	0.108	0.106	0.781	0.050	0.038	0.787	0.054	0.042
(60, 50)	[1]	0.856	0.096	0.091	0.850	0.090	0.085	0.787	0.045	0.036	0.795	0.048	0.039
	[2]	0.872	0.098	0.096	0.866	0.102	0.100	0.786	0.048	0.038	0.795	0.051	0.040
	[3]	0.863	0.093	0.087	0.857	0.081	0.076	0.788	0.044	0.035	0.797	0.047	0.038
(90, 45)	[1]	0.861	0.091	0.085	0.853	0.078	0.072	0.781	0.042	0.033	0.786	0.045	0.036
	[2]	0.831	0.088	0.082	0.824	0.075	0.070	0.781	0.041	0.031	0.785	0.043	0.035
	[3]	0.779	0.087	0.081	0.769	0.070	0.063	0.773	0.037	0.030	0.773	0.039	0.032
(90, 70)	[1]	0.877	0.059	0.052	0.881	0.051	0.045	0.788	0.034	0.027	0.794	0.036	0.029
	[2]	0.892	0.082	0.076	0.899	0.070	0.063	0.784	0.036	0.028	0.790	0.038	0.030
	[3]	0.883	0.037	0.030	0.887	0.036	0.028	0.787	0.032	0.026	0.793	0.033	0.028

Table 7: The point assessments of $R(t)$ from Set-2

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	0.981	0.207	0.207	0.988	0.200	0.198	0.979	0.111	0.091	0.986	0.113	0.093
	[2]	0.978	0.197	0.196	0.985	0.185	0.188	0.976	0.077	0.061	0.989	0.087	0.069
	[3]	0.964	0.184	0.194	0.980	0.184	0.184	0.968	0.071	0.056	0.978	0.080	0.062
(30, 25)	[1]	0.978	0.171	0.163	0.984	0.172	0.171	0.976	0.064	0.051	0.982	0.068	0.054
	[2]	0.981	0.176	0.169	0.987	0.181	0.180	0.979	0.066	0.050	0.985	0.071	0.057
	[3]	0.980	0.164	0.163	0.985	0.162	0.171	0.977	0.062	0.049	0.983	0.066	0.050
(60, 30)	[1]	0.982	0.131	0.127	0.986	0.121	0.119	0.980	0.059	0.047	0.984	0.066	0.052

(Continued)

Table 7 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
	[2]	0.979	0.123	0.113	0.983	0.110	0.106	0.977	0.053	0.042	0.981	0.057	0.045
	[3]	0.962	0.114	0.110	0.973	0.106	0.104	0.960	0.051	0.039	0.975	0.055	0.043
(60, 50)	[1]	0.979	0.103	0.101	0.982	0.094	0.090	0.988	0.046	0.037	0.980	0.049	0.040
	[2]	0.982	0.107	0.101	0.986	0.104	0.104	0.983	0.049	0.039	0.984	0.052	0.041
	[3]	0.981	0.098	0.092	0.985	0.092	0.087	0.979	0.045	0.036	0.983	0.048	0.039
(90, 45)	[1]	0.982	0.098	0.093	0.985	0.079	0.075	0.980	0.043	0.034	0.980	0.046	0.037
	[2]	0.979	0.087	0.082	0.982	0.076	0.070	0.976	0.042	0.032	0.979	0.044	0.036
	[3]	0.962	0.084	0.078	0.970	0.075	0.070	0.960	0.038	0.030	0.968	0.040	0.033
(90, 70)	[1]	0.977	0.082	0.075	0.979	0.073	0.066	0.972	0.034	0.027	0.988	0.037	0.030
	[2]	0.982	0.084	0.078	0.985	0.073	0.066	0.980	0.037	0.029	0.983	0.038	0.031
	[3]	0.980	0.071	0.064	0.983	0.061	0.055	0.978	0.032	0.027	0.981	0.033	0.028
$(T_1, T_2) = (0.7, 0.9)$													
(30, 15)	[1]	0.979	0.207	0.207	0.988	0.199	0.197	0.981	0.110	0.091	0.986	0.113	0.093
	[2]	0.980	0.197	0.196	0.989	0.186	0.189	0.978	0.077	0.061	0.983	0.087	0.069
	[3]	0.984	0.186	0.194	0.990	0.184	0.183	0.982	0.071	0.055	0.990	0.079	0.062
(30, 25)	[1]	0.980	0.171	0.166	0.987	0.171	0.174	0.973	0.064	0.051	0.980	0.068	0.054
	[2]	0.973	0.176	0.169	0.987	0.180	0.180	0.988	0.066	0.049	0.985	0.071	0.056
	[3]	0.981	0.164	0.163	0.989	0.161	0.170	0.978	0.061	0.049	0.987	0.066	0.050
(60, 30)	[1]	0.981	0.131	0.127	0.986	0.120	0.118	0.980	0.059	0.047	0.984	0.064	0.052
	[2]	0.982	0.123	0.113	0.987	0.112	0.108	0.979	0.053	0.042	0.978	0.056	0.045
	[3]	0.984	0.114	0.110	0.988	0.106	0.104	0.982	0.050	0.038	0.986	0.055	0.043
(60, 50)	[1]	0.988	0.103	0.101	0.986	0.094	0.089	0.981	0.046	0.037	0.984	0.049	0.039
	[2]	0.982	0.107	0.107	0.986	0.103	0.102	0.986	0.048	0.038	0.981	0.052	0.041
	[3]	0.982	0.098	0.092	0.986	0.091	0.086	0.980	0.044	0.036	0.978	0.047	0.038
(90, 45)	[1]	0.986	0.098	0.093	0.985	0.079	0.075	0.979	0.043	0.034	0.983	0.046	0.037
	[2]	0.987	0.087	0.082	0.985	0.075	0.070	0.980	0.042	0.032	0.983	0.044	0.036
	[3]	0.983	0.085	0.080	0.987	0.074	0.070	0.981	0.038	0.030	0.982	0.040	0.033
(90, 70)	[1]	0.985	0.078	0.072	0.986	0.072	0.062	0.987	0.034	0.027	0.978	0.037	0.029
	[2]	0.982	0.084	0.078	0.985	0.073	0.066	0.980	0.037	0.029	0.983	0.038	0.031
	[3]	0.980	0.071	0.064	0.987	0.061	0.055	0.983	0.032	0.027	0.980	0.034	0.028

Table 8: The point assessments of $h(t)$ from Set-1

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.2, 0.4)$													
(30, 15)	[1]	1.146	0.619	0.611	1.148	0.463	0.456	0.926	0.396	0.295	0.985	0.439	0.289
	[2]	1.293	0.857	0.809	0.834	0.792	0.758	1.058	0.649	0.644	0.210	0.722	0.716
	[3]	0.891	0.624	0.616	1.280	0.493	0.528	0.987	0.428	0.246	1.044	0.489	0.352
(30, 25)	[1]	0.921	0.445	0.428	0.952	0.359	0.332	0.926	0.268	0.208	0.942	0.300	0.227
	[2]	0.892	0.494	0.477	0.977	0.407	0.371	0.929	0.311	0.235	0.946	0.316	0.237
	[3]	0.904	0.528	0.510	1.027	0.427	0.397	0.951	0.383	0.293	0.968	0.339	0.264
(60, 30)	[1]	0.836	0.354	0.280	1.052	0.316	0.248	0.929	0.255	0.202	0.885	0.242	0.192
	[2]	1.184	0.414	0.380	1.053	0.407	0.330	0.964	0.267	0.202	1.007	0.290	0.225
	[3]	0.967	0.376	0.300	1.247	0.364	0.297	0.930	0.255	0.199	0.967	0.285	0.221
(60, 50)	[1]	1.239	0.247	0.190	1.294	0.224	0.175	0.920	0.208	0.151	0.938	0.216	0.155
	[2]	1.209	0.286	0.214	1.294	0.249	0.179	0.917	0.218	0.166	0.935	0.223	0.176
	[3]	1.137	0.324	0.263	1.211	0.252	0.221	0.938	0.221	0.175	0.959	0.224	0.181
(90, 45)	[1]	0.858	0.183	0.147	0.904	0.180	0.143	0.852	0.169	0.136	0.871	0.172	0.133
	[2]	0.946	0.213	0.166	1.007	0.200	0.159	0.950	0.179	0.146	0.982	0.191	0.152
	[3]	0.981	0.204	0.157	1.042	0.188	0.146	0.906	0.176	0.141	0.938	0.179	0.143
(90, 70)	[1]	0.959	0.162	0.134	0.913	0.143	0.120	0.912	0.123	0.104	0.930	0.130	0.101
	[2]	0.991	0.175	0.139	1.031	0.167	0.130	0.905	0.135	0.107	0.922	0.137	0.107
	[3]	0.940	0.179	0.142	0.969	0.175	0.137	0.934	0.137	0.110	0.954	0.143	0.114
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	1.286	0.573	0.564	1.082	0.520	0.509	1.258	0.354	0.264	1.397	0.449	0.325
	[2]	1.201	0.815	0.724	0.992	0.729	0.664	1.037	0.607	0.599	0.978	0.659	0.526
	[3]	1.189	0.645	0.640	1.207	0.567	0.530	1.014	0.359	0.259	1.087	0.461	0.325
(30, 25)	[1]	0.980	0.436	0.415	1.326	0.407	0.357	0.917	0.263	0.202	0.932	0.279	0.225
	[2]	0.954	0.486	0.444	0.975	0.408	0.371	0.936	0.286	0.200	0.955	0.296	0.224
	[3]	0.978	0.496	0.478	1.226	0.428	0.385	0.935	0.349	0.257	0.951	0.353	0.258
(60, 30)	[1]	0.999	0.327	0.289	1.292	0.288	0.232	0.971	0.229	0.180	1.014	0.262	0.204
	[2]	1.176	0.414	0.389	1.167	0.397	0.339	0.955	0.243	0.183	0.993	0.275	0.210
	[3]	0.989	0.396	0.336	0.948	0.349	0.286	0.971	0.233	0.183	1.014	0.270	0.210
(60, 50)	[1]	1.125	0.284	0.257	1.005	0.234	0.176	0.910	0.182	0.143	0.922	0.201	0.156
	[2]	1.193	0.314	0.265	0.928	0.250	0.220	0.919	0.188	0.149	0.933	0.215	0.161
	[3]	1.105	0.326	0.270	0.845	0.284	0.222	0.929	0.221	0.169	0.943	0.245	0.187
(90, 45)	[1]	1.370	0.183	0.149	0.938	0.175	0.146	0.959	0.150	0.120	0.962	0.158	0.127
	[2]	0.887	0.253	0.201	1.114	0.224	0.176	0.944	0.176	0.147	0.970	0.188	0.157
	[3]	1.055	0.251	0.200	1.078	0.213	0.170	0.952	0.175	0.141	0.982	0.187	0.150
(90, 70)	[1]	0.922	0.140	0.114	0.971	0.133	0.103	0.903	0.116	0.093	0.914	0.119	0.095

(Continued)

Table 8 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
	[2]	0.983	0.156	0.130	0.980	0.154	0.121	0.921	0.142	0.112	0.936	0.149	0.117
	[3]	0.898	0.163	0.131	1.075	0.171	0.131	0.925	0.148	0.119	0.939	0.155	0.125

Table 9: The point assessments of $h(t)$ from Set-2

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
$(T_1, T_2) = (0.5, 0.7)$													
(30, 15)	[1]	0.623	0.574	0.539	0.424	0.567	0.532	0.413	0.396	0.289	0.606	0.463	0.354
	[2]	0.313	0.632	0.597	0.229	0.624	0.590	0.223	0.415	0.313	0.304	0.511	0.396
	[3]	0.326	0.601	0.584	0.257	0.593	0.576	0.250	0.406	0.296	0.318	0.488	0.375
(30, 25)	[1]	0.322	0.485	0.443	0.260	0.479	0.438	0.357	0.199	0.163	0.242	0.201	0.171
	[2]	0.330	0.521	0.481	0.274	0.514	0.475	0.321	0.204	0.160	0.245	0.205	0.175
	[3]	0.315	0.539	0.526	0.243	0.532	0.519	0.325	0.207	0.169	0.243	0.219	0.188
(60, 30)	[1]	0.673	0.347	0.357	0.541	0.342	0.352	0.304	0.162	0.131	0.247	0.172	0.143
	[2]	0.318	0.429	0.418	0.264	0.424	0.412	0.320	0.193	0.159	0.260	0.185	0.149
	[3]	0.335	0.387	0.378	0.291	0.382	0.373	0.315	0.168	0.130	0.256	0.173	0.145
(60, 50)	[1]	0.320	0.324	0.324	0.279	0.320	0.320	0.319	0.145	0.118	0.270	0.152	0.125
	[2]	0.339	0.325	0.326	0.302	0.321	0.322	0.319	0.149	0.121	0.270	0.155	0.128
	[3]	0.318	0.331	0.329	0.269	0.327	0.325	0.321	0.156	0.126	0.269	0.169	0.141
(90, 45)	[1]	0.673	0.320	0.319	0.358	0.314	0.313	0.315	0.131	0.106	0.272	0.138	0.116
	[2]	0.326	0.323	0.323	0.286	0.319	0.319	0.326	0.140	0.113	0.283	0.152	0.125
	[3]	0.344	0.321	0.320	0.313	0.315	0.315	0.323	0.133	0.108	0.280	0.140	0.117
(90, 70)	[1]	0.331	0.247	0.241	0.299	0.229	0.222	0.393	0.124	0.099	0.316	0.129	0.106
	[2]	0.355	0.270	0.279	0.328	0.290	0.289	0.325	0.127	0.103	0.286	0.134	0.110
	[3]	0.327	0.299	0.298	0.288	0.302	0.301	0.326	0.129	0.103	0.285	0.135	0.112
$(T_1, T_2) = (0.7, 0.9)$													
(30, 15)	[1]	0.312	0.532	0.353	0.239	0.516	0.527	0.294	0.215	0.174	0.210	0.223	0.194
	[2]	0.347	0.593	0.290	0.241	0.584	0.584	0.320	0.264	0.215	0.224	0.251	0.216
	[3]	0.324	0.572	0.376	0.232	0.554	0.567	0.312	0.216	0.178	0.218	0.225	0.192
(30, 25)	[1]	0.361	0.412	0.377	0.480	0.392	0.434	0.314	0.181	0.143	0.253	0.185	0.156
	[2]	0.330	0.465	0.275	0.503	0.447	0.471	0.322	0.187	0.155	0.267	0.197	0.170
	[3]	0.248	0.502	0.352	0.474	0.493	0.511	0.306	0.200	0.164	0.236	0.212	0.182
(60, 30)	[1]	0.309	0.332	0.352	0.378	0.327	0.349	0.296	0.154	0.124	0.353	0.165	0.138
	[2]	0.247	0.372	0.413	0.282	0.362	0.409	0.310	0.177	0.145	0.257	0.184	0.152
	[3]	0.352	0.349	0.373	0.375	0.332	0.370	0.326	0.168	0.137	0.283	0.174	0.144

(Continued)

Table 9 (continued)

(n, m)	PC	MLE			MPSE			Bayes-LF			Bayes-SF		
		Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB	Av.E	RMSE	MAB
(60, 50)	[1]	0.230	0.318	0.320	0.371	0.315	0.317	0.312	0.129	0.105	0.272	0.136	0.110
	[2]	0.276	0.319	0.322	0.424	0.318	0.319	0.330	0.132	0.102	0.294	0.138	0.114
	[3]	0.212	0.324	0.325	0.396	0.323	0.322	0.310	0.142	0.116	0.262	0.152	0.126
(90, 45)	[1]	0.325	0.309	0.315	0.305	0.303	0.309	0.655	0.123	0.099	0.569	0.127	0.103
	[2]	0.342	0.314	0.319	0.346	0.312	0.315	0.318	0.127	0.103	0.278	0.135	0.112
	[3]	0.383	0.310	0.316	0.320	0.308	0.312	0.335	0.124	0.100	0.304	0.129	0.107
(90, 70)	[1]	0.249	0.296	0.238	0.292	0.292	0.225	0.291	0.099	0.080	0.322	0.101	0.080
	[2]	0.328	0.299	0.291	0.344	0.295	0.285	0.345	0.105	0.082	0.319	0.106	0.085
	[3]	0.299	0.302	0.294	0.314	0.296	0.298	0.319	0.108	0.086	0.280	0.113	0.092

Table 10: The interval assessments of β from Set-1

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(T_1, T_2) = (0.2, 0.4)$									
(30, 15)	[1]	1.017	0.933	1.072	0.921	0.903	0.935	0.710	0.941
	[2]	1.255	0.932	1.374	0.920	1.106	0.934	0.874	0.938
	[3]	1.029	0.933	1.192	0.921	0.911	0.935	0.814	0.939
(30, 25)	[1]	0.904	0.937	0.929	0.925	0.812	0.939	0.679	0.943
	[2]	0.944	0.935	0.990	0.923	0.821	0.937	0.698	0.943
	[3]	0.980	0.934	1.020	0.922	0.890	0.936	0.703	0.941
(60, 30)	[1]	0.702	0.940	0.723	0.928	0.638	0.942	0.533	0.946
	[2]	0.880	0.939	0.914	0.927	0.729	0.941	0.666	0.945
	[3]	0.718	0.940	0.735	0.928	0.661	0.942	0.552	0.946
(60, 50)	[1]	0.656	0.942	0.671	0.930	0.546	0.944	0.462	0.948
	[2]	0.674	0.942	0.684	0.930	0.548	0.944	0.497	0.948
	[3]	0.704	0.941	0.713	0.929	0.609	0.943	0.528	0.947
(90, 45)	[1]	0.546	0.945	0.597	0.933	0.506	0.947	0.432	0.951
	[2]	0.597	0.943	0.669	0.931	0.523	0.945	0.456	0.949
	[3]	0.588	0.944	0.607	0.932	0.516	0.946	0.449	0.950
(90, 70)	[1]	0.501	0.946	0.529	0.934	0.440	0.949	0.403	0.952
	[2]	0.513	0.946	0.542	0.934	0.470	0.948	0.413	0.952
	[3]	0.535	0.945	0.552	0.933	0.485	0.947	0.422	0.951
$(T_1, T_2) = (0.5, 0.7)$									
(30, 15)	[1]	0.946	0.935	0.980	0.924	0.840	0.938	0.720	0.942
	[2]	1.107	0.934	1.170	0.923	1.087	0.937	0.736	0.941

(Continued)

Table 10 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[3]	0.963	0.935	1.004	0.924	0.865	0.938	0.713	0.942
(30, 25)	[1]	0.835	0.939	0.865	0.928	0.790	0.942	0.678	0.945
	[2]	0.845	0.937	0.875	0.926	0.803	0.940	0.689	0.943
	[3]	0.907	0.936	0.948	0.925	0.833	0.939	0.694	0.943
(60, 30)	[1]	0.643	0.942	0.648	0.931	0.600	0.945	0.513	0.949
	[2]	0.773	0.941	0.791	0.930	0.713	0.943	0.660	0.945
	[3]	0.668	0.942	0.678	0.931	0.632	0.945	0.529	0.948
(60, 50)	[1]	0.593	0.944	0.603	0.933	0.525	0.947	0.435	0.951
	[2]	0.611	0.944	0.641	0.933	0.545	0.947	0.458	0.951
	[3]	0.633	0.943	0.645	0.932	0.581	0.946	0.501	0.949
(90, 45)	[1]	0.530	0.947	0.537	0.936	0.473	0.950	0.426	0.952
	[2]	0.585	0.945	0.594	0.934	0.519	0.948	0.431	0.951
	[3]	0.546	0.946	0.551	0.935	0.510	0.949	0.429	0.952
(90, 70)	[1]	0.476	0.948	0.481	0.937	0.406	0.952	0.364	0.953
	[2]	0.487	0.948	0.491	0.937	0.436	0.951	0.408	0.953
	[3]	0.515	0.947	0.516	0.936	0.443	0.950	0.417	0.952

Table 11: The interval assessments of β from Set-2

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
					$(T_1, T_2) = (0.5, 0.7)$				
(30, 15)	[1]	1.715	0.903	2.010	0.891	1.341	0.906	1.500	0.909
	[2]	1.960	0.900	2.228	0.889	1.770	0.903	1.592	0.906
	[3]	1.864	0.902	2.188	0.891	1.640	0.905	1.453	0.908
(30, 25)	[1]	1.640	0.906	1.846	0.895	1.257	0.909	1.303	0.911
	[2]	1.657	0.905	1.879	0.893	1.279	0.908	1.320	0.911
	[3]	1.675	0.905	1.880	0.894	1.296	0.908	1.490	0.909
(60, 30)	[1]	1.162	0.909	1.255	0.898	1.126	0.912	1.147	0.916
	[2]	1.245	0.908	1.350	0.897	1.236	0.911	1.229	0.913
	[3]	1.240	0.908	1.310	0.897	1.215	0.911	1.195	0.914
(60, 50)	[1]	1.116	0.910	1.185	0.899	0.937	0.913	0.996	0.916
	[2]	1.126	0.910	1.191	0.899	0.968	0.913	1.002	0.916
	[3]	1.121	0.910	1.194	0.899	1.024	0.913	1.122	0.917
(90, 45)	[1]	0.938	0.914	0.981	0.903	0.872	0.917	0.808	0.920
	[2]	0.992	0.912	1.045	0.901	0.917	0.915	0.908	0.919
	[3]	0.960	0.913	0.995	0.902	0.891	0.916	0.816	0.920
(90, 70)	[1]	0.898	0.915	0.911	0.904	0.675	0.922	0.703	0.923

(Continued)

Table 11 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[2]	0.902	0.915	0.917	0.904	0.820	0.918	0.715	0.922
	[3]	0.913	0.914	0.924	0.903	0.843	0.917	0.722	0.922
$(T_1, T_2) = (0.7, 0.9)$									
(30, 15)	[1]	1.653	0.905	1.877	0.894	1.526	0.908	1.375	0.911
	[2]	1.836	0.902	2.192	0.891	1.757	0.905	1.535	0.908
	[3]	1.681	0.904	1.971	0.893	1.580	0.907	1.388	0.910
(30, 25)	[1]	1.524	0.908	1.687	0.897	1.179	0.911	1.217	0.913
	[2]	1.558	0.907	1.736	0.896	1.189	0.910	1.226	0.913
	[3]	1.601	0.907	1.830	0.896	1.266	0.910	1.374	0.911
(60, 30)	[1]	1.131	0.912	1.202	0.900	1.007	0.915	1.129	0.918
	[2]	1.229	0.911	1.335	0.899	1.163	0.914	1.206	0.915
	[3]	1.145	0.910	1.237	0.899	1.024	0.913	1.178	0.916
(60, 50)	[1]	1.022	0.912	1.125	0.901	0.922	0.915	0.902	0.918
	[2]	1.089	0.912	1.164	0.901	0.939	0.915	1.059	0.918
	[3]	1.114	0.913	1.190	0.901	0.977	0.916	1.069	0.919
(90, 45)	[1]	0.929	0.917	0.969	0.905	0.855	0.920	0.827	0.923
	[2]	0.998	0.915	1.036	0.903	0.879	0.918	0.894	0.921
	[3]	0.987	0.916	0.992	0.904	0.870	0.919	0.880	0.922
(90, 70)	[1]	0.896	0.917	0.903	0.906	0.734	0.924	0.655	0.925
	[2]	0.870	0.918	0.911	0.906	0.744	0.920	0.667	0.924
	[3]	0.885	0.917	0.943	0.905	0.780	0.920	0.691	0.924

Table 12: The interval assessments of δ from Set-1

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(T_1, T_2) = (0.2, 0.4)$									
(30, 15)	[1]	0.614	0.938	0.766	0.925	0.287	0.952	0.381	0.939
	[2]	0.516	0.942	0.604	0.929	0.270	0.956	0.365	0.943
	[3]	0.541	0.941	0.652	0.928	0.276	0.955	0.377	0.942
(30, 25)	[1]	0.473	0.944	0.588	0.931	0.262	0.958	0.344	0.945
	[2]	0.461	0.945	0.542	0.932	0.255	0.959	0.310	0.946
	[3]	0.470	0.944	0.565	0.931	0.259	0.958	0.333	0.945
(60, 30)	[1]	0.412	0.947	0.465	0.934	0.243	0.961	0.303	0.948
	[2]	0.344	0.950	0.379	0.937	0.215	0.964	0.292	0.951
	[3]	0.358	0.949	0.398	0.936	0.233	0.963	0.295	0.950

(Continued)

Table 12 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
(60, 50)	[1]	0.330	0.950	0.359	0.937	0.210	0.964	0.284	0.951
	[2]	0.321	0.951	0.350	0.938	0.190	0.965	0.270	0.952
	[3]	0.322	0.951	0.345	0.938	0.196	0.965	0.276	0.952
(90, 45)	[1]	0.285	0.952	0.339	0.939	0.188	0.966	0.241	0.953
	[2]	0.266	0.953	0.304	0.940	0.171	0.967	0.228	0.954
	[3]	0.282	0.952	0.317	0.939	0.185	0.966	0.239	0.953
(90, 70)	[1]	0.257	0.953	0.287	0.940	0.165	0.967	0.223	0.954
	[2]	0.236	0.954	0.264	0.941	0.147	0.968	0.187	0.955
	[3]	0.248	0.954	0.274	0.941	0.154	0.968	0.218	0.955
				$(T_1, T_2) = (0.5, 0.7)$					
(30, 15)	[1]	0.512	0.944	0.615	0.931	0.255	0.958	0.361	0.945
	[2]	0.411	0.948	0.471	0.935	0.247	0.962	0.339	0.949
	[3]	0.502	0.947	0.599	0.934	0.249	0.961	0.356	0.948
(30, 25)	[1]	0.398	0.950	0.449	0.937	0.237	0.964	0.325	0.951
	[2]	0.368	0.951	0.426	0.938	0.232	0.965	0.303	0.952
	[3]	0.386	0.950	0.437	0.937	0.232	0.964	0.321	0.951
(60, 30)	[1]	0.347	0.953	0.384	0.940	0.230	0.967	0.295	0.954
	[2]	0.283	0.956	0.326	0.943	0.207	0.970	0.268	0.957
	[3]	0.336	0.955	0.372	0.942	0.216	0.969	0.287	0.956
(60, 50)	[1]	0.278	0.956	0.299	0.943	0.189	0.970	0.258	0.957
	[2]	0.269	0.957	0.289	0.944	0.188	0.971	0.245	0.958
	[3]	0.273	0.957	0.292	0.944	0.188	0.971	0.256	0.958
(90, 45)	[1]	0.265	0.958	0.281	0.945	0.185	0.972	0.224	0.959
	[2]	0.233	0.959	0.248	0.946	0.180	0.973	0.219	0.960
	[3]	0.263	0.958	0.276	0.945	0.183	0.972	0.220	0.959
(90, 70)	[1]	0.224	0.959	0.237	0.946	0.163	0.973	0.213	0.960
	[2]	0.210	0.960	0.215	0.947	0.149	0.974	0.206	0.961
	[3]	0.214	0.960	0.221	0.947	0.156	0.974	0.211	0.961

Table 13: The interval assessments of δ from Set-2

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
				$(T_1, T_2) = (0.5, 0.7)$					
(30, 15)	[1]	1.347	0.905	1.574	0.893	1.305	0.918	1.424	0.906
	[2]	1.090	0.909	1.354	0.897	1.043	0.922	1.313	0.910

(Continued)

Table 13 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[3]	1.243	0.908	1.437	0.896	1.044	0.921	1.363	0.909
(30, 25)	[1]	1.019	0.910	1.261	0.898	0.992	0.923	1.223	0.911
	[2]	0.937	0.912	1.108	0.900	0.915	0.925	1.023	0.913
	[3]	0.972	0.911	1.183	0.899	0.966	0.924	1.089	0.912
(60, 30)	[1]	0.920	0.912	1.066	0.900	0.866	0.925	1.002	0.913
	[2]	0.850	0.917	0.957	0.904	0.766	0.930	0.884	0.917
	[3]	0.860	0.916	0.984	0.903	0.854	0.929	0.963	0.917
(60, 50)	[1]	0.767	0.917	0.941	0.904	0.746	0.930	0.820	0.917
	[2]	0.707	0.918	0.893	0.905	0.689	0.931	0.766	0.918
	[3]	0.747	0.918	0.903	0.905	0.693	0.931	0.810	0.918
(90, 45)	[1]	0.678	0.919	0.754	0.906	0.647	0.932	0.718	0.919
	[2]	0.612	0.920	0.713	0.907	0.602	0.933	0.645	0.920
	[3]	0.637	0.919	0.734	0.906	0.619	0.932	0.672	0.919
(90, 70)	[1]	0.552	0.922	0.667	0.910	0.542	0.935	0.597	0.923
	[2]	0.515	0.923	0.541	0.911	0.504	0.936	0.526	0.925
	[3]	0.536	0.923	0.660	0.911	0.514	0.936	0.564	0.924
$(T_1, T_2) = (0.7, 0.9)$									
(30, 15)	[1]	1.235	0.911	1.449	0.899	1.215	0.924	1.348	0.912
	[2]	1.125	0.915	1.275	0.903	1.006	0.927	1.206	0.916
	[3]	1.183	0.914	1.384	0.902	1.014	0.927	1.302	0.915
(30, 25)	[1]	0.990	0.916	1.212	0.904	0.956	0.929	1.065	0.917
	[2]	0.909	0.918	1.170	0.905	0.862	0.931	0.948	0.919
	[3]	0.942	0.917	1.196	0.905	0.945	0.929	1.058	0.918
(60, 30)	[1]	0.882	0.918	1.115	0.906	0.829	0.932	0.906	0.919
	[2]	0.806	0.923	0.936	0.910	0.695	0.933	0.790	0.923
	[3]	0.818	0.922	1.108	0.909	0.785	0.933	0.892	0.922
(60, 50)	[1]	0.796	0.923	0.927	0.910	0.672	0.935	0.774	0.923
	[2]	0.732	0.924	0.911	0.911	0.663	0.935	0.728	0.924
	[3]	0.744	0.924	0.923	0.911	0.670	0.934	0.733	0.924
(90, 45)	[1]	0.684	0.925	0.881	0.912	0.657	0.935	0.713	0.925
	[2]	0.644	0.926	0.828	0.913	0.560	0.936	0.589	0.926
	[3]	0.672	0.925	0.854	0.912	0.652	0.935	0.707	0.925
(90, 70)	[1]	0.590	0.928	0.662	0.915	0.545	0.937	0.580	0.926
	[2]	0.542	0.929	0.596	0.916	0.468	0.938	0.514	0.927
	[3]	0.544	0.929	0.647	0.916	0.514	0.937	0.559	0.927

Table 14: The interval assessments of $R(t)$ from Set-1

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
				$(T_1, T_2) = (0.2, 0.4)$					
(30, 15)	[1]	0.310	0.962	0.306	0.963	0.138	0.971	0.133	0.970
	[2]	0.279	0.963	0.266	0.964	0.130	0.972	0.125	0.971
	[3]	0.260	0.963	0.253	0.964	0.127	0.972	0.122	0.971
(30, 25)	[1]	0.240	0.964	0.238	0.965	0.122	0.973	0.116	0.972
	[2]	0.254	0.964	0.249	0.965	0.124	0.973	0.118	0.972
	[3]	0.233	0.965	0.231	0.966	0.119	0.974	0.113	0.973
(60, 30)	[1]	0.218	0.966	0.220	0.967	0.112	0.975	0.108	0.974
	[2]	0.189	0.967	0.186	0.968	0.111	0.976	0.105	0.975
	[3]	0.183	0.967	0.182	0.968	0.105	0.976	0.096	0.975
(60, 50)	[1]	0.179	0.968	0.172	0.969	0.083	0.977	0.085	0.976
	[2]	0.181	0.967	0.180	0.968	0.085	0.976	0.088	0.976
	[3]	0.171	0.968	0.167	0.969	0.081	0.977	0.082	0.976
(90, 45)	[1]	0.168	0.968	0.157	0.969	0.078	0.977	0.069	0.976
	[2]	0.157	0.969	0.149	0.970	0.074	0.978	0.063	0.977
	[3]	0.148	0.970	0.146	0.971	0.073	0.979	0.058	0.978
(90, 70)	[1]	0.141	0.970	0.139	0.971	0.048	0.979	0.049	0.978
	[2]	0.145	0.970	0.142	0.971	0.050	0.979	0.053	0.978
	[3]	0.138	0.971	0.133	0.972	0.037	0.980	0.045	0.979
				$(T_1, T_2) = (0.5, 0.7)$					
(30, 15)	[1]	0.287	0.965	0.290	0.966	0.139	0.975	0.131	0.974
	[2]	0.254	0.966	0.255	0.967	0.128	0.976	0.123	0.975
	[3]	0.250	0.966	0.251	0.967	0.123	0.976	0.121	0.975
(30, 25)	[1]	0.234	0.967	0.233	0.968	0.120	0.977	0.113	0.976
	[2]	0.236	0.967	0.236	0.968	0.121	0.977	0.114	0.976
	[3]	0.219	0.968	0.218	0.969	0.118	0.978	0.111	0.977
(60, 30)	[1]	0.206	0.969	0.207	0.970	0.117	0.979	0.109	0.978
	[2]	0.181	0.970	0.180	0.971	0.110	0.980	0.107	0.979
	[3]	0.179	0.970	0.179	0.971	0.109	0.980	0.103	0.979
(60, 50)	[1]	0.168	0.971	0.168	0.971	0.087	0.981	0.082	0.980
	[2]	0.169	0.970	0.170	0.971	0.090	0.980	0.087	0.979
	[3]	0.167	0.971	0.166	0.972	0.084	0.981	0.079	0.980
(90, 45)	[1]	0.163	0.971	0.152	0.972	0.080	0.981	0.078	0.980
	[2]	0.150	0.972	0.150	0.973	0.077	0.982	0.071	0.981
	[3]	0.148	0.972	0.141	0.973	0.075	0.983	0.059	0.982
(90, 70)	[1]	0.137	0.973	0.136	0.974	0.043	0.983	0.049	0.982

(Continued)

Table 14 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[2]	0.139	0.973	0.138	0.974	0.065	0.983	0.058	0.982
	[3]	0.135	0.973	0.131	0.975	0.036	0.984	0.035	0.983

Table 15: The interval assessments of $R(t)$ from Set-2

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(T_1, T_2) = (0.5, 0.7)$									
(30, 15)	[1]	0.302	0.963	0.222	0.964	0.101	0.973	0.091	0.972
	[2]	0.278	0.964	0.211	0.965	0.074	0.974	0.064	0.973
	[3]	0.251	0.964	0.177	0.965	0.071	0.974	0.062	0.973
(30, 25)	[1]	0.152	0.965	0.146	0.966	0.066	0.975	0.055	0.974
	[2]	0.165	0.965	0.169	0.966	0.067	0.975	0.059	0.974
	[3]	0.147	0.966	0.129	0.968	0.065	0.976	0.053	0.975
(60, 30)	[1]	0.147	0.967	0.117	0.968	0.060	0.977	0.052	0.976
	[2]	0.139	0.968	0.105	0.969	0.054	0.978	0.051	0.977
	[3]	0.131	0.968	0.101	0.969	0.053	0.978	0.048	0.977
(60, 50)	[1]	0.125	0.969	0.082	0.971	0.048	0.979	0.044	0.978
	[2]	0.126	0.968	0.084	0.970	0.051	0.978	0.048	0.977
	[3]	0.116	0.969	0.068	0.972	0.047	0.980	0.042	0.978
(90, 45)	[1]	0.111	0.969	0.058	0.970	0.046	0.978	0.042	0.978
	[2]	0.106	0.970	0.053	0.972	0.046	0.980	0.041	0.979
	[3]	0.087	0.972	0.048	0.973	0.045	0.981	0.040	0.980
(90, 70)	[1]	0.076	0.973	0.036	0.974	0.040	0.982	0.026	0.981
	[2]	0.086	0.972	0.037	0.973	0.043	0.981	0.029	0.980
	[3]	0.068	0.974	0.035	0.974	0.039	0.982	0.023	0.982
$(T_1, T_2) = (0.7, 0.9)$									
(30, 15)	[1]	0.279	0.967	0.269	0.968	0.079	0.976	0.078	0.975
	[2]	0.232	0.968	0.182	0.969	0.072	0.977	0.072	0.976
	[3]	0.214	0.968	0.171	0.969	0.070	0.977	0.064	0.976
(30, 25)	[1]	0.166	0.969	0.142	0.970	0.066	0.978	0.056	0.977
	[2]	0.181	0.969	0.144	0.970	0.067	0.978	0.061	0.977
	[3]	0.145	0.970	0.113	0.971	0.065	0.980	0.053	0.978
(60, 30)	[1]	0.144	0.971	0.107	0.972	0.057	0.980	0.052	0.979
	[2]	0.137	0.972	0.107	0.973	0.048	0.981	0.049	0.980
	[3]	0.133	0.972	0.104	0.973	0.047	0.981	0.047	0.980

(Continued)

Table 15 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
(60, 50)	[1]	0.120	0.973	0.053	0.974	0.045	0.983	0.039	0.981
	[2]	0.123	0.972	0.055	0.973	0.046	0.982	0.040	0.980
	[3]	0.097	0.973	0.052	0.975	0.045	0.984	0.038	0.981
(90, 45)	[1]	0.096	0.973	0.047	0.974	0.040	0.982	0.035	0.981
	[2]	0.096	0.974	0.047	0.975	0.039	0.984	0.034	0.982
	[3]	0.084	0.975	0.036	0.976	0.038	0.985	0.032	0.984
(90, 70)	[1]	0.077	0.976	0.028	0.977	0.037	0.986	0.020	0.985
	[2]	0.082	0.975	0.032	0.976	0.038	0.985	0.025	0.984
	[3]	0.073	0.977	0.027	0.977	0.035	0.986	0.019	0.986

Table 16: The interval assessments of $h(t)$ from Set-1

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
$(T_1, T_2) = (0.2, 0.4)$									
(30, 15)	[1]	1.369	0.937	1.281	0.939	0.840	0.944	0.835	0.946
	[2]	1.889	0.929	1.670	0.932	1.006	0.935	0.870	0.937
	[3]	1.554	0.933	1.385	0.936	0.937	0.940	0.850	0.942
(30, 25)	[1]	1.216	0.943	1.117	0.945	0.771	0.950	0.789	0.952
	[2]	1.227	0.942	1.150	0.944	0.795	0.949	0.804	0.951
	[3]	1.248	0.940	1.157	0.942	0.821	0.947	0.810	0.949
(60, 30)	[1]	0.989	0.946	0.924	0.948	0.658	0.953	0.625	0.955
	[2]	1.149	0.944	1.070	0.946	0.706	0.951	0.694	0.953
	[3]	0.991	0.946	0.931	0.948	0.681	0.953	0.637	0.955
(60, 50)	[1]	0.867	0.952	0.807	0.954	0.609	0.959	0.557	0.961
	[2]	0.884	0.951	0.827	0.953	0.613	0.958	0.587	0.960
	[3]	0.939	0.948	0.895	0.950	0.622	0.955	0.619	0.957
(90, 45)	[1]	0.775	0.956	0.684	0.958	0.528	0.963	0.433	0.965
	[2]	0.793	0.954	0.754	0.956	0.576	0.961	0.474	0.963
	[3]	0.786	0.955	0.739	0.957	0.576	0.962	0.453	0.964
(90, 70)	[1]	0.638	0.960	0.574	0.962	0.391	0.967	0.343	0.969
	[2]	0.677	0.958	0.615	0.960	0.487	0.965	0.404	0.967
	[3]	0.678	0.958	0.635	0.960	0.505	0.965	0.419	0.967
$(T_1, T_2) = (0.5, 0.7)$									
(30, 15)	[1]	1.125	0.942	1.069	0.945	0.788	0.947	0.754	0.950
	[2]	1.574	0.935	1.413	0.937	0.924	0.939	0.844	0.941

(Continued)

Table 16 (continued)

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[3]	1.469	0.938	1.312	0.941	0.898	0.943	0.827	0.946
(30, 25)	[1]	1.002	0.949	0.959	0.951	0.734	0.952	0.727	0.953
	[2]	1.042	0.947	0.977	0.950	0.757	0.951	0.747	0.952
	[3]	1.096	0.945	1.009	0.948	0.785	0.950	0.751	0.951
(60, 30)	[1]	0.829	0.952	0.792	0.954	0.675	0.956	0.632	0.959
	[2]	0.952	0.950	0.891	0.952	0.720	0.954	0.682	0.957
	[3]	0.889	0.952	0.852	0.954	0.700	0.956	0.652	0.959
(60, 50)	[1]	0.721	0.958	0.699	0.960	0.613	0.962	0.534	0.965
	[2]	0.749	0.956	0.715	0.958	0.622	0.961	0.574	0.964
	[3]	0.774	0.954	0.749	0.955	0.630	0.958	0.606	0.961
(90, 45)	[1]	0.646	0.962	0.613	0.963	0.524	0.966	0.453	0.969
	[2]	0.692	0.960	0.673	0.962	0.566	0.964	0.522	0.967
	[3]	0.648	0.961	0.633	0.963	0.558	0.965	0.496	0.968
(90, 70)	[1]	0.527	0.966	0.516	0.968	0.373	0.970	0.324	0.974
	[2]	0.557	0.964	0.544	0.966	0.447	0.968	0.355	0.973
	[3]	0.598	0.964	0.582	0.966	0.513	0.968	0.432	0.971

Table 17: The interval assessments of $h(t)$ from Set-2

(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
				$(T_1, T_2) = (0.5, 0.7)$					
(30, 15)	[1]	0.750	0.950	0.717	0.955	0.100	0.957	0.067	0.959
	[2]	0.791	0.941	0.724	0.954	0.124	0.955	0.081	0.958
	[3]	0.772	0.946	0.722	0.955	0.108	0.957	0.076	0.959
(30, 25)	[1]	0.609	0.956	0.604	0.959	0.055	0.963	0.046	0.965
	[2]	0.727	0.955	0.686	0.957	0.058	0.962	0.051	0.963
	[3]	0.749	0.953	0.701	0.956	0.076	0.960	0.057	0.963
(60, 30)	[1]	0.559	0.959	0.545	0.961	0.051	0.966	0.036	0.967
	[2]	0.594	0.957	0.568	0.961	0.053	0.964	0.044	0.966
	[3]	0.577	0.959	0.553	0.961	0.052	0.966	0.038	0.967
(60, 50)	[1]	0.502	0.965	0.488	0.967	0.047	0.968	0.027	0.971
	[2]	0.543	0.964	0.525	0.966	0.048	0.977	0.031	0.969
	[3]	0.553	0.961	0.534	0.963	0.050	0.968	0.032	0.969
(90, 45)	[1]	0.466	0.969	0.462	0.971	0.034	0.970	0.020	0.972
	[2]	0.491	0.967	0.476	0.969	0.042	0.969	0.023	0.972
	[3]	0.477	0.968	0.469	0.970	0.040	0.969	0.022	0.972
(90, 70)	[1]	0.424	0.973	0.414	0.975	0.025	0.971	0.009	0.975

(Continued)

Table 17 (continued)

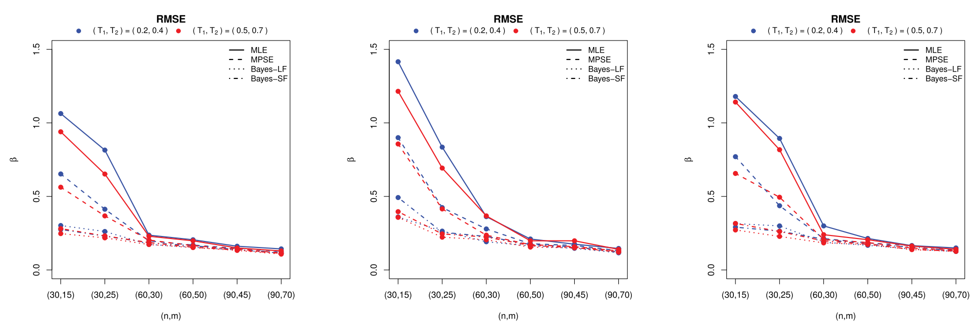
(n, m)	PC	ACI-LF		ACI-SF		BCI-LF		BCI-SF	
		AIL	CP	AIL	CP	AIL	CP	AIL	CP
	[2]	0.449	0.971	0.440	0.973	0.027	0.971	0.010	0.974
	[3]	0.457	0.971	0.455	0.973	0.030	0.971	0.012	0.974
$(T_1, T_2) = (0.7, 0.9)$									
(30, 15)	[1]	0.769	0.956	0.723	0.960	0.097	0.963	0.070	0.965
	[2]	0.795	0.947	0.751	0.959	0.115	0.960	0.077	0.964
	[3]	0.775	0.952	0.735	0.960	0.101	0.961	0.072	0.965
(30, 25)	[1]	0.717	0.962	0.685	0.964	0.059	0.966	0.051	0.969
	[2]	0.748	0.961	0.703	0.963	0.062	0.965	0.052	0.968
	[3]	0.754	0.959	0.712	0.961	0.093	0.963	0.065	0.966
(60, 30)	[1]	0.567	0.965	0.543	0.967	0.052	0.969	0.037	0.972
	[2]	0.624	0.963	0.562	0.966	0.056	0.967	0.042	0.970
	[3]	0.588	0.965	0.555	0.967	0.053	0.969	0.039	0.972
(60, 50)	[1]	0.537	0.971	0.526	0.973	0.039	0.971	0.025	0.974
	[2]	0.552	0.970	0.528	0.972	0.049	0.980	0.028	0.983
	[3]	0.559	0.967	0.530	0.969	0.052	0.971	0.032	0.974
(90, 45)	[1]	0.473	0.975	0.455	0.977	0.035	0.973	0.021	0.976
	[2]	0.501	0.973	0.483	0.975	0.038	0.972	0.022	0.975
	[3]	0.488	0.974	0.469	0.976	0.037	0.972	0.022	0.975
(90, 70)	[1]	0.466	0.979	0.406	0.981	0.026	0.974	0.010	0.977
	[2]	0.468	0.977	0.435	0.979	0.027	0.974	0.016	0.977
	[3]	0.471	0.977	0.435	0.979	0.029	0.974	0.020	0.977

From the results presented in Tables 2–17, based on the lowest RMSE, MAB, and AIL values and the highest CP values, the following key observations can be made:

- All proposed estimators for β, δ, $R(t)$, and $h(t)$ perform well under the evaluated scenarios.
- Increasing n (or m) improves the accuracy of all estimates, while reducing $\sum_{j=1}^{m} R_j$ results in similar improvements.
- Bayesian point estimates derived from the Bayes-LF or Bayes-SF methods consistently outperform those from other approaches.
- As T_i, $i = 1, 2$ increase, the RMSE, MAB, and AIL values developed from the classical (or Bayes') approaches of β, δ, $R(t)$, or $h(t)$ decrease. In other terms, longer test durations (T_i $i = 1, 2,$) lead to more efficient results. The opposite finding is also reached when comparing the same parameters regarding their simulated CP values.
- As β and δ grow, it can be noted that:
  - The RMSE and MAB values of β, δ, and $R(t)$ increased while those of $h(t)$ decreased;
  - The AIL values of β and δ increased while those of $R(t)$ and $h(t)$ decreased;
  - The CP values of β and δ decreased while those of $R(t)$ and $h(t)$ increased.

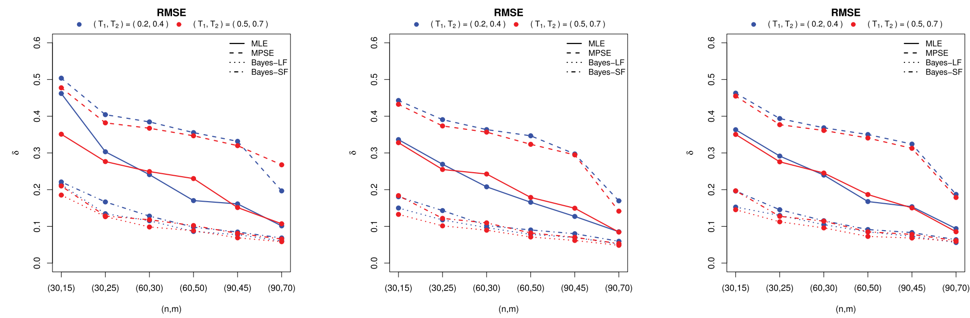
- Comparing the three proposed removal patterns PC[i] for $i = 1, 2, 3$, for each group (n, m), it is observed that:
  – The results of β and $h(t)$ perform satisfactorily using PC [1] (i.e., left-censoring);
  – The results of $R(t)$ perform satisfactorily using PC [3] (i.e., right-censoring);
  – The results of δ perform satisfactorily using PC [2] (i.e., middle-censoring);.
- Comparing point estimation results, we noticed that:
  – In the classical point of view, the SF approach provides the most accurate estimates for β, $R(t)$, and $h(t)$ than others, while the LF approach is better in estimating δ;
  – In Bayes' point of view, the Bayes-LF approach provides the most accurate estimates for all unknown quantities β, δ, $R(t)$, and $h(t)$ compared to others.
- Comparing interval estimation results, we noted that:
  – In the classical point of view, the ACI-LF approach provides more accurate estimates for β and δ than others, while the ACI-SF approach is better at estimating $R(t)$ and $h(t)$;
  – In Bayes' point of view, the BCI-SF approach provides the most accurate estimates for β, $R(t)$, and $h(t)$, while the BCI-LF approach is better at estimating δ.
- In summary, for analyzing data obtained via the T2IA-PC mechanism, we recommended utilizing the Bayesian framework through a product of spacings setup combined with Metropolis-Hastings sampling to evaluate the parameters in addition to the reliability features of the Uni-W model.

Fig. 4 provides graphical representations of the simulation results from Set (as an example), specifically illustrating the behavior of β, δ, $R(t)$, and $h(t)$ across their simulated RMSE results. The visual diagrams in Fig. 4 corroborate and reinforce the patterns and conclusions presented earlier, offering further empirical validation of the estimator performance.

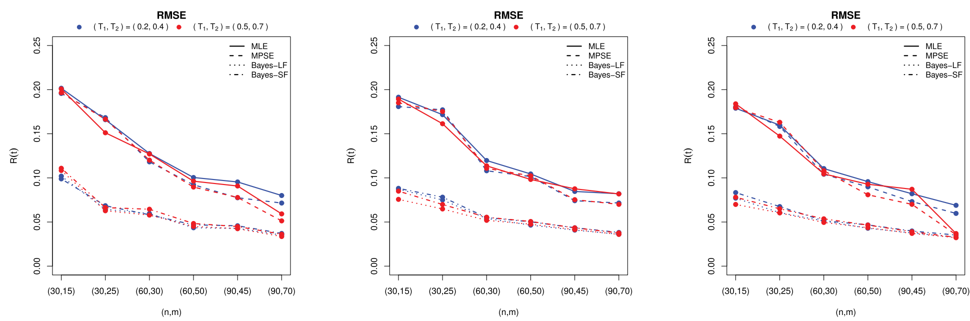


(a) β

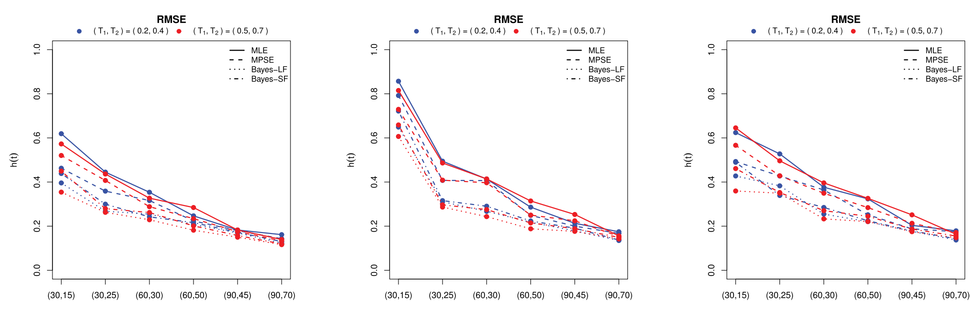
Figure 4: (Continued)



(b) δ



(c) $R(t)$



(d) $h(t)$

(i) PC[1] (ii) PC[2] (iii) PC[3]

Figure 4: The RMSE plots of β, δ, $R(t)$, and $h(t)$ **(a–d)**

5 Optimal T2-PC Plan

Over the last twenty years, extensive research efforts have been dedicated to identifying the most effective strategies for data censoring within statistical analysis. Given specific values of (n, m), the set of feasible censoring configurations comprises all arrangements of $\mathbf{R}$ that satisfy the constraint $m + \sum_{i=1}^{m} R_i = n$. The key challenge in this context lies in selecting a progressive censoring scheme that maximizes the available information about the unknown parameters of interest, ensuring its superiority over alternative censoring methods.

A fundamental concern is the determination of how best to extract meaningful inferences about unknown parameters when dealing with censored datasets. Another crucial aspect is the comparative evaluation of distinct censoring methodologies to assess their efficiency and effectiveness in parameter estimation. For a more comprehensive discussion on this topic, including comparative analyses of censoring techniques, the reader is referred to Ng et al. [21], Pradhan and Kundu [22], Sen et al. [23], Ashour et al. [24], and references therein.

Ultimately, the objective is to identify a censoring design that yields the highest possible information content regarding the parameters under study. Following the LF approach (for instance), Table 18 presents a selection of criteria that serve as guiding principles in the pursuit of an optimal censoring strategy.

Table 18: Four criteria of the ideal T2-PC plan

Criterion	Method
C[1]	Maximize $-\sum_{i=1}^{2}\mathcal{L}_{ii},\ i=1,2$
C[2]	Minimize $\sum_{i=1}^{2}\sigma_{ii},\ i=1,2$
C[3]	Minimize $\sigma_{11}\sigma_{22}-\sigma_{12}^{2}$
C[4]	Minimize $\mathrm{Var}(\log(\Lambda_\nu)),\ 0<\nu<1$

According to criterion C[1], the primary objective is to maximize the estimated trace of the observed FIM. In contrast, for criteria C[i], $i = 2, 3$, the aim is to minimize the estimated determinant and trace of the 2×2 asymptotic VCM, respectively. Additionally, under criterion C[4], the optimal design should minimize the variance associated with the logarithm of the νth quantile of the proposed model. From (2), the logarithm of the quantile function for the Uni-Weibull model, denoted by $\log(\Lambda_\nu)$, is expressed as:

$$\log(\Lambda_\nu) = -\left[-\delta^{-1}\log(\nu)\right]^{\frac{1}{\beta}},\quad 0<\nu<1.$$

To approximate the variance of $\log(\Lambda_\nu)$, the delta method is employed. The optimal censoring scheme is identified by selecting the one that minimizes criteria C[i], $i = 2, 3, 4$, while maximizing C[1]. It is important to note that all proposed criteria in Table 18 are evaluated based on the frequentist estimates of the Uni-Weibull parameters β and δ.

6 Real-World Data Applications

The purpose of this section is to illustrate how the suggested approaches can be adjusted and modified to fit real-world events. To achieve this, two actual physics data applications are provided.

6.1 Petroleum Data

Core samples from petroleum reservoirs provide valuable insights into the geological, petrophysical, and reservoir characteristics, enabling more accurate evaluation of hydrocarbon reserves, improved recovery strategies, and better decision-making in exploration and production.

This application analyzed core samples from petroleum reservoirs sampled by four cross-sections, and 48 observations were made; see Mazucheli et al. [25]. The dataset involves measurements from petroleum core samples, where each sample was analyzed for permeability and cross-sectional pore geometry. Specifically, each cross-section includes three variables: the total area of pores, the total perimeter of pores, and a derived shape-related measure. In this example, we focus on a specific shape variable calculated as the squared total perimeter divided by the total pore area. Table 19 lists the records of petroleum reservoirs.

Table 19: Records from petroleum reservoirs

0.090	0.114	0.117	0.122	0.133	0.145	0.148	0.149	0.151	0.153	0.154	0.162
0.162	0.164	0.164	0.167	0.173	0.177	0.179	0.182	0.183	0.190	0.192	0.198
0.200	0.200	0.201	0.204	0.204	0.225	0.229	0.230	0.232	0.240	0.254	0.263
0.263	0.276	0.276	0.281	0.291	0.312	0.327	0.329	0.341	0.420	0.439	0.464

Before proceeding further, to validate the suitability of the Uni-W distribution, we first compute the MLEs of β and δ, along with their corresponding standard errors (St.Ers) and 95% ACI-LF estimates, including their interval widths (Int.Ws); see Table 20. The results presented in Table 20 indicate that the Uni-W model provides an adequate fit to the petroleum dataset, as confirmed by the Kolmogorov–Smirnov (KS) statistic and its associated P-value. Additionally, Fig. 5 offers four graphical assessments of model fit: (a) data histograms with fitted density curves, (b) plots of the empirical and estimated reliability functions, (c) scaled total time on test (TTT) transforms, and (d) a contour plot of the log-likelihood function. Each of these graphical summaries supports the goodness-of-fit conclusion drawn from Table 20. Specifically, Fig. 5c suggests that the petroleum dataset exhibits an increasing failure rate, consistent with one of the possible failure rate shapes of the Uni-W model. Moreover, Fig. 5d confirms the existence and uniqueness of the MLEs for β and δ. Based on these findings, we propose using the estimates $\hat{\beta} \approx 5.1414$ and $\hat{\delta} \approx 0.0604$ as suitable initial values for all subsequent calculations.

Table 20: Fit results for the Uni-W model from petroleum dataset

Par.	MLE (St.Er)	95% ACI-LF			KS (p-value)
		Low.	Upp.	Int.W	
β	5.1414 (0.5749)	4.0146	6.2682	2.2536	0.0848 (0.8803)
δ	0.0604 (0.0235)	0.0144	0.1065	0.0021	

Now, from the entire petroleum dataset, three various T2IA-PC samples (each with $m = 24$) based on different choices of $\mathbf{R}$ and T_i, $i = 1, 2$, are created; see Table 21. To highlight the practical differences and advantages of various censoring schemes, we include a comparative analysis among the traditional T2-PC and AT2-PC plans, with the proposed T2IA-PC schemes. Specifically, Table 21 presents three censored datasets, labeled as $\mathcal{S}[1]$, $\mathcal{S}[2]$, and $\mathcal{S}[3]$, which correspond to the T2-PC, AT2-PC, and T2IA-PC schemes, respectively. This enables a direct evaluation of the estimation performance under each censoring plan, illustrating the practical improvements offered by the T2IA-PC method.

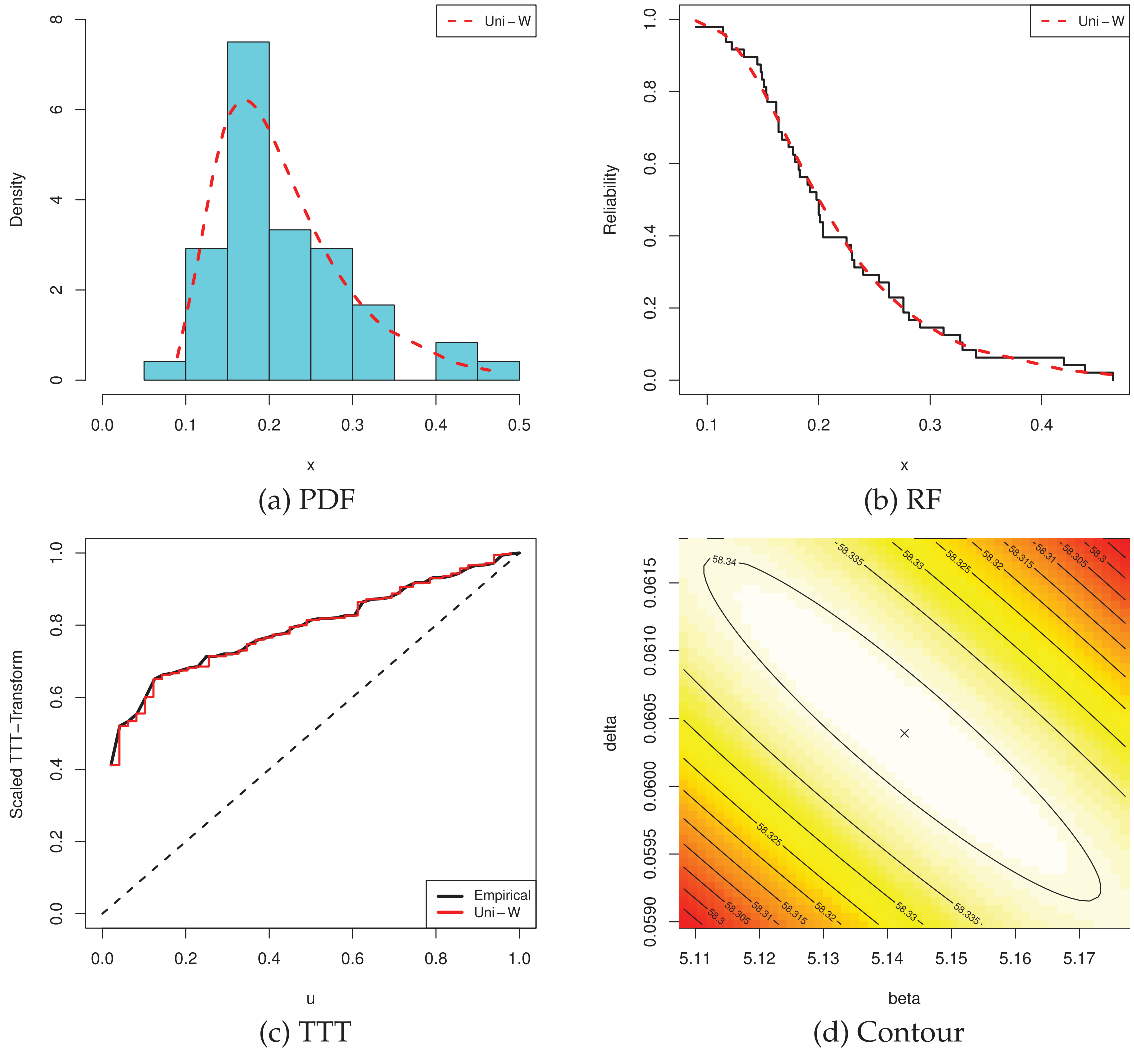


Figure 5: Fit plots for the Uni-W model from petroleum dataset (**a**–**d**)

Table 21: Three T2IA-PC samples from petroleum dataset

Sample	**R**	$T_1(d_1)$	$T_2(d_2)$	R^*	T^*	Censored data
$\mathcal{S}[1]$	$(4^6, 0^{18})$	0.345(24)	0.375(24)	0	0.341	0.090, 0.117, 0.122, 0.133, 0.145, 0.148, 0.154, 0.162, 0.173, 0.177, 0.179, 0.182, 0.192, 0.198, 0.201, 0.225, 0.230, 0.254, 0.263, 0.276, 0.291, 0.312, 0.329, 0.341
$\mathcal{S}[2]$	$(0^9, 4^6, 0^9)$	0.165(13)	0.285(24)	8	0.276	0.090, 0.114, 0.117, 0.122, 0.133, 0.145, 0.148, 0.149, 0.151, 0.153, 0.154, 0.162, 0.164, 0.167, 0.173, 0.177, 0.179, 0.182, 0.192, 0.198, 0.225, 0.229, 0.232, 0.276

(Continued)

Table 21 (continued)

Sample	R	$T_1(d_1)$	$T_2(d_2)$	R^*	T^*	Censored data
$\mathcal{S}[3]$	$(0^{18}, 4^6)$	0.191(20)	0.200(22)	18	0.200	0.090, 0.114, 0.117, 0.122, 0.133, 0.145, 0.148, 0.149, 0.151, 0.153, 0.154, 0.162, 0.162, 0.164, 0.164, 0.167, 0.173, 0.177, 0.179, 0.190, 0.192, 0.198

For each dataset $\mathcal{S}[i]$, $i = 1, 2, 3$, the frequentist estimates—including ML and MPS—as well as the Bayesian estimates—including Bayes-LF and Bayes-SF—are computed for the parameters β, δ, and the reliability and hazard functions, $R(t)$ and $h(t)$, evaluated at $t = 0.15$. In addition, for each $\mathcal{S}[i]$, asymptotic confidence intervals (ACI-LF and ACI-SF) and Bayesian credible intervals (BCI-LF and BCI-SF) for all quantities of interest are also constructed. Due to the lack of prior information about the Uni-Weibull parameters β and δ, we employ improper gamma priors in the Bayesian framework. A total of 50,000 MCMC iterations are generated, discarding the initial 10,000 as burn-in. The resulting point estimates (with their St.Ers) and interval estimates (with their Int.Ws) are reported in Table 22. The findings in Table 22 demonstrate that the Bayesian estimates of β, δ, $R(t)$, and $h(t)$ obtained via the Bayes-LF and Bayes-SF approaches outperform the corresponding frequentist ML and MPS methods in terms of estimation accuracy. Furthermore, the Bayesian credible intervals (BCI-LF and BCI-SF) yield narrower intervals and more stable results compared to their frequentist counterparts (ACI-LF and ACI-SF), confirming the superiority of the Bayesian approach in this setting. As shown in Table 22, the performance of the T2IA-PC scheme outperforms the traditional T2-PC and AT2-PC under the given accuracy criteria.

Table 22: Estimates of β, δ, $R(t)$, and $h(t)$ from petroleum dataset

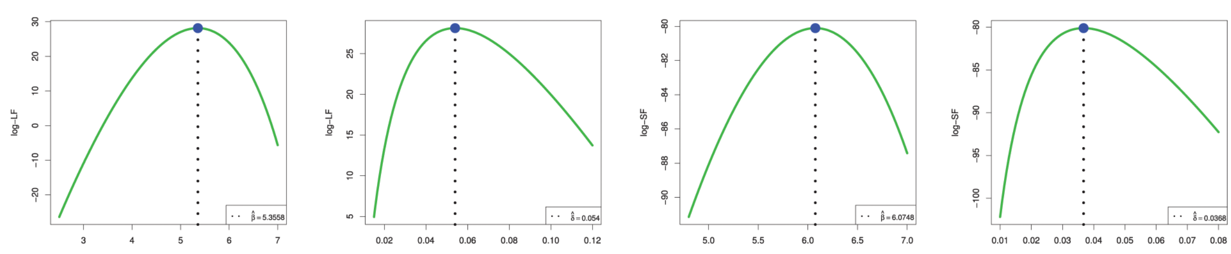
Sample	Par.	MLE		Bayes-LF		95% ACI-LF			95% BCI-LF		
		MPSE		Bayes-SF		95% ACI-SF			95% BCI-SF		
		Est.	St.Er	Est.	St.Er	low.	Upp.	Int.W	low.	Upp.	Int.W
S[1]	β	5.3558	0.7843	5.3558	0.0125	3.8186	6.8930	3.0745	5.3314	5.3804	0.0490
		6.0748	0.9717	6.0748	0.0080	4.1703	7.9794	3.8090	6.0591	6.0907	0.0316
	δ	0.0540	0.0300	0.0539	0.0072	0.0047	0.1127	0.1080	0.0407	0.0687	0.0280
		0.0367	0.0241	0.0366	0.0050	0.0039	0.0840	0.0801	0.0273	0.0467	0.0194
	$R(0.15)$	0.8110	0.0525	0.8056	0.0427	0.7081	0.9139	0.2058	0.7145	0.8797	0.1651
		0.8342	0.0515	0.8280	0.0417	0.7333	0.9352	0.2020	0.7369	0.8981	0.1612
	$h(0.15)$	7.3079	1.8041	7.3837	1.0211	3.7719	10.844	7.0720	5.4482	9.4193	3.9711
		7.6234	2.0072	7.7401	1.1921	3.6893	11.558	7.8682	5.5303	10.179	4.6487
S[2]	β	4.5908	0.7187	4.5908	0.0124	3.1821	5.9994	2.8173	4.5665	4.6153	0.0488
		4.8874	0.8114	4.8874	0.0080	3.2970	6.4778	3.1808	4.8717	4.9033	0.0316
	δ	0.0885	0.0434	0.0880	0.0091	0.0034	0.1736	0.1702	0.0708	0.1062	0.0354
		0.0771	0.0415	0.0768	0.0066	0.0042	0.1585	0.1543	0.0641	0.0900	0.0259
	$R(0.15)$	0.8124	0.0467	0.8080	0.0332	0.7210	0.9039	0.1830	0.7376	0.8660	0.1284

(Continued)

Table 22 (continued)

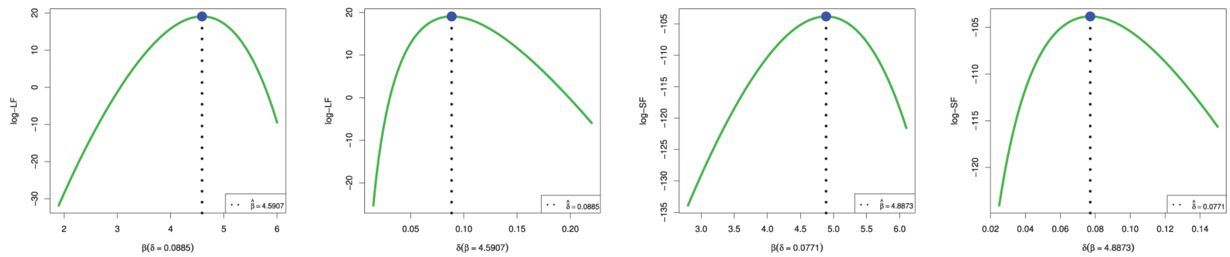
Sample	Par.	MLE		Bayes-LF		95% ACI-LF			95% BCI-LF		
		MPSE		Bayes-SF		95% ACI-SF			95% BCI-SF		
		Est.	St.Er	Est.	St.Er	low.	Upp.	Int.W	low.	Upp.	Int.W
		0.8285	0.0458	0.8253	0.0266	0.7388	0.9182	0.1794	0.7688	0.8723	0.1035
	$h(0.15)$	6.2333	1.4183	6.2990	0.6825	3.4535	9.0130	5.5595	5.0198	7.6741	2.6543
		6.2680	1.4777	6.3206	0.6114	3.3716	9.1643	5.7927	5.1745	7.5600	2.3855
S[3]	β	4.9613	0.7876	4.9613	0.0125	3.4176	6.5050	3.0874	4.9371	4.9859	0.0488
		4.9967	0.9086	4.9967	0.0080	3.2159	6.7774	3.5615	4.9810	5.0125	0.0316
	δ	0.0694	0.0370	0.0690	0.0079	−0.0031	0.1418	0.1449	0.0542	0.0849	0.0307
		0.0738	0.0439	0.0735	0.0065	−0.0123	0.1600	0.1723	0.0610	0.0865	0.0255
	$R(0.15)$	0.8105	0.0462	0.8055	0.0369	0.7198	0.9011	0.1812	0.7271	0.8693	0.1422
		0.8364	0.0451	0.8331	0.0269	0.7480	0.9248	0.1768	0.7758	0.8802	0.1044
	$h(0.15)$	6.7815	1.5414	6.8558	0.8145	3.7603	9.8027	6.0424	5.3339	8.4985	3.1646
		6.2171	1.5459	6.2743	0.6451	3.1873	9.2469	6.0597	5.0715	7.5844	2.5129

To verify the existence and uniqueness of the proposed ML and MPS estimates for the parameters β and δ, Fig. 6 displays the profile plots of the log-LF and log-SF. Based on the samples $\mathcal{S}[i]$, $i = 1, 2, 3$, collected from the petroleum dataset, Fig. 6 confirms that the ML and MPS estimates for both parameters exist and are unique. To assess the convergence behavior of the remaining 40,000 MCMC iterations (after a 10,000 burn-in), Fig. 7 presents the density and trace plots for β, δ, $R(t)$, and $h(t)$. In each plot, the solid line denotes the posterior mean (i.e., Bayes point estimate), while the dashed lines represent the corresponding credible interval bounds. These visualizations confirm that the MCMC algorithm performs effectively, with the selected burn-in length being sufficient. Moreover, the posterior distributions generated by the Bayes-LF and Bayes-SF methods reveal the following characteristics in Fig. 7: (i) the estimates of β are approximately symmetric; (ii) the distributions of δ and $h(t)$ are positively skewed; and (iii) the estimates of $R(t)$ exhibit negative skewness.

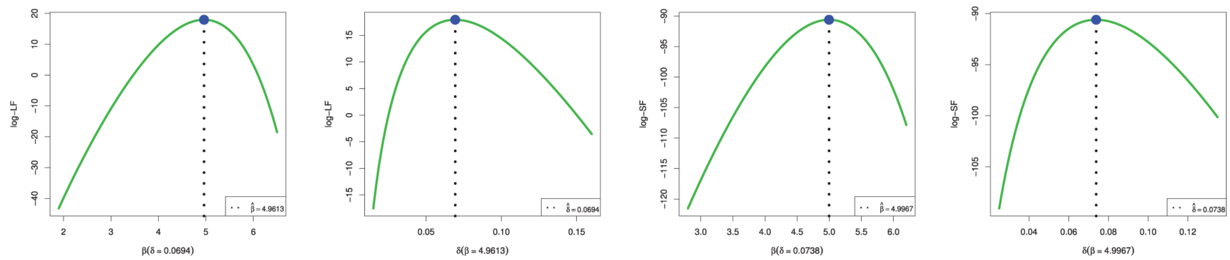


(a) Sample $\mathcal{S}[1]$

Figure 6: (Continued)

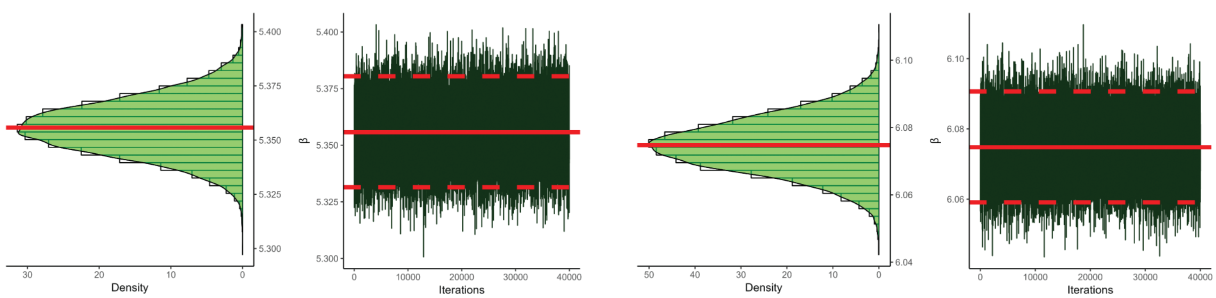


(b) Sample $\mathcal{S}[2]$

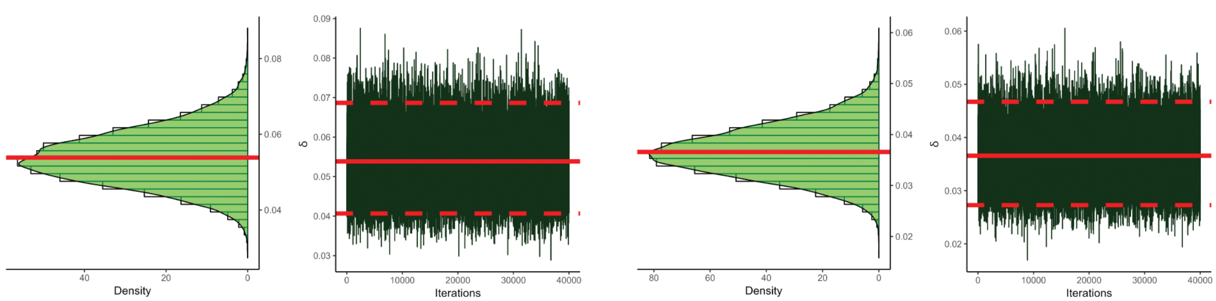


(c) Sample $\mathcal{S}[3]$

Figure 6: The log-LF (left) and log-SF (right) of β and δ from petroleum dataset (**a–c**)

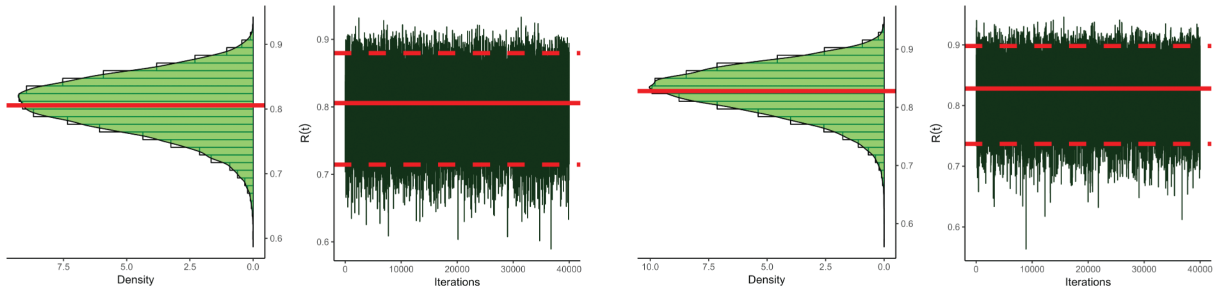


(a) β

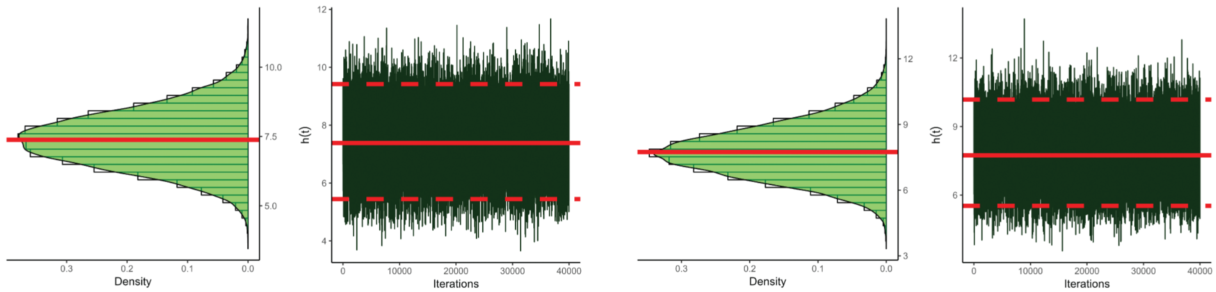


(b) δ

Figure 7: (Continued)



(c) $R(t)$



(d) $h(t)$

Figure 7: The density and trace plots from Bayes-LF (left) and Bayes-SF (right) iterations of β, δ, $R(t)$, and $h(t)$ from petroleum dataset (**a**–**d**)

We now illustrate the process of selecting the optimal censoring scheme using the datasets $\mathcal{S}[i]$, $i = 1, 2, 3$, generated from the petroleum real-world dataset. Based on the ML and MPS estimation methods, Table 23 presents the computed values for all optimality criteria $\mathsf{C}[i]$, $i = 1, 2, 3, 4$. The findings summarized in Table 23 validate the effectiveness of our proposed censoring strategies observed in the Monte Carlo experiments. Specifically, under both ML and MPS approaches:

Table 23: Optimum censoring from petroleum dataset

Sample	C[1]	C[2]	C[3]	C[4]		
$\nu \rightarrow$				0.3	0.6	0.9
			via ML			
			via MPS			
$\mathcal{S}[1]$	12725.55	0.61605	0.000048	0.000092	0.000260	0.000993
	24871.41	0.94481	0.000038	0.000087	0.000220	0.000872
$\mathcal{S}[2]$	5808.076	0.51844	0.000089	0.000110	0.000323	0.001384
	7361.109	0.66015	0.000090	0.000102	0.000305	0.001348
$\mathcal{S}[3]$	9711.805	0.62171	0.000064	0.000102	0.000272	0.001203
	7940.244	0.82742	0.000104	0.000103	0.000334	0.001532

- For criteria $\mathsf{C}[1]$, $\mathsf{C}[3]$, and $\mathsf{C}[4]$, the left-censoring scheme used in $\mathcal{S}[1]$ is identified as the optimal censoring strategy;
- For criterion $\mathsf{C}[2]$, the middle-censoring scheme used in $\mathcal{S}[2]$ emerges as the most optimal among the considered designs.

6.2 Polyester Data

In textiles, industrial materials, and high-performance products requiring strength and longevity, the tensile strength of polyester fibers is a critical property that ensures durability, flexibility, and resistance to stretching, making them ideal. This application examines thirty measurements of tensile strength of polyester fibers; see Mazucheli et al. [26]. This dataset was tested under standard ASTM D2256/D2256M-10 protocols using a universal testing machine. Table 24 lists the strengths of polyester fibers.

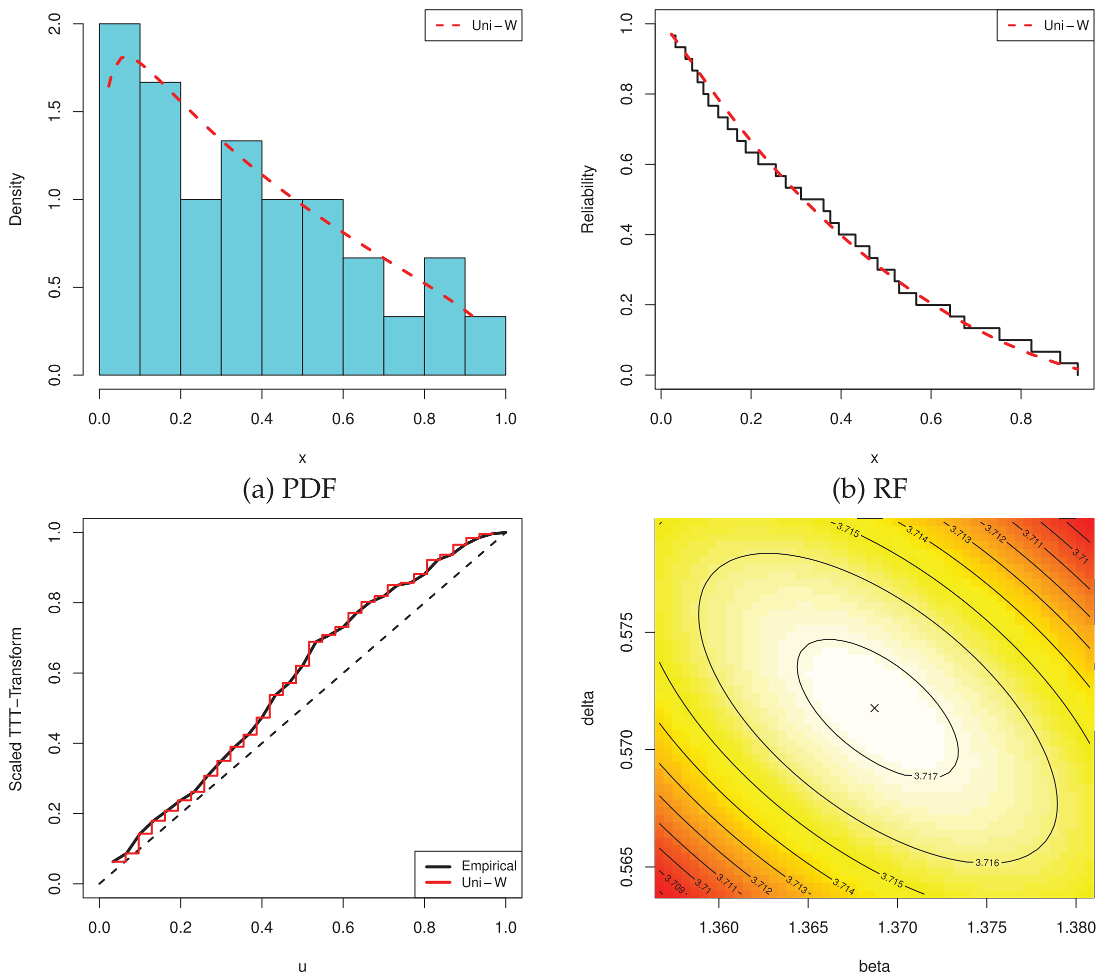
Table 24: Tensile strength of polyester fibers

0.023	0.032	0.054	0.069	0.081	0.094	0.105	0.127	0.148	0.169
0.188	0.216	0.255	0.277	0.311	0.361	0.376	0.395	0.432	0.463
0.481	0.519	0.529	0.567	0.642	0.674	0.752	0.823	0.887	0.926

To ensure the validity of the Uni-W distribution, we begin by calculating the MLEs and 95% ACI-LF estimates of β and δ; see Table 25. Additionally, the *KS* statistic and its associated *p*-value are computed; see Table 25. The results in Table 25 confirm that the Uni-W model provides a good fit for the polyester fibers dataset. The graphical representations shown in Fig. 8 support the fitting results detailed in Table 25. Specifically, Fig. 8c demonstrates that the polyester fibers dataset follows an increasing failure rate, aligning with one of the Uni-W model's failure rate behaviors. Fig. 8d confirms that the MLEs of β and δ are exist and unique. As a sequence, we recommend using the estimates $\hat{\beta} \cong 5.1414$ and $\hat{\delta} \cong 0.5717$ as reliable starting points for all subsequent calculations.

Table 25: Fit results for the Uni-W model from polyester fibers dataset

Par.	MLE (St.Er)	95% ACI-LF			*KS* (*p*-value)
		Low.	Upp.	Int.W	
β	5.1414 (0.5749)	0.9756	1.7621	0.7865	0.0576 (0.9998)
δ	0.5717 (0.1320)	0.3129	0.8305	0.5176	



(a) PDF (b) RF (c) TTT (d) Contour

Figure 8: Fit plots for the Uni-W model from polyester fibers dataset (**a**–**d**)

From Table 24, three distinct T2IA-PC samples, each with $m = 15$, are generated based on different configurations of $\mathbf{R}$ and T_i, $i = 1, 2$; see Table 26. Similar to Table 21, to highlight the practical differences and advantages of the censoring scheme, Table 26 presents results from three distinct censored datasets, labeled as $\mathcal{S}[i]$, $i = 1, 2, 3$, which correspond to the T2-PC, AT2-PC, and T2IA-PC schemes, respectively. This allows a direct evaluation of estimation accuracy under each censoring plan, illustrating the practical efficiency gains offered by the T2IA-PC method.

Table 26: Three T2IA-PC samples from polyester fibers dataset

Sample	$\mathbf{R}$	$T_1(d_1)$	$T_2(d_2)$	R^*	T^*	Censored data
$\mathcal{S}[1]$	$(3^5, 0^{10})$	0.55(15)	0.65(15)	0	0.529	0.023, 0.032, 0.054, 0.081, 0.094, 0.148, 0.169, 0.216, 0.255, 0.277, 0.311, 0.361, 0.395, 0.463, 0.529

(Continued)

Table 26 (continued)

Sample	**R**	$T_1(d_1)$	$T_2(d_2)$	R^*	T^*	Censored data
$\mathcal{S}[2]$	$(0^5, 3^5, 0^5)$	0.15(8)	0.45(15)	6	0.432	0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.127, 0.148, 0.188, 0.216, 0.277, 0.311, 0.376, 0.395, 0.432
$\mathcal{S}[3]$	$(0^{10}, 3^5)$	0.21(11)	0.400(14)	13	0.400	0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.255, 0.361, 0.395

Following the estimation scenario provided in Section 6.1, the proposed point inferential approached (including ML, MPS, Bayes-LF, and Bayes-SF) in addition to the proposed interval inferential approached (including ACI-LF, ACI-SF, BCI-LF, and BCI-SF) are utilized to evaluated β, δ, $R(t)$, and $h(t)$ (evaluated at $t = 0.1$).

Table 27 summarizes the point estimates (along with their St.Ers) and interval estimates (including their Int.Ws). Table 27 shows that the Bayesian estimates of β, δ, $R(t)$, and $h(t)$ produced using the Bayes-LF and Bayes-SF techniques are better than those obtained from the ML and MPS approaches. Table 27 also demonstrates that the credible interval estimates of β, δ, $R(t)$, and $h(t)$ using the BCI-LF and BCI-SF techniques are superior to those provided by the ACI-LF and ACI-SF methods.

Table 27: Estimates of β, δ, $R(t)$, and $h(t)$ from polyester fibers dataset

Sample	Par.	MLE		Bayes-LF		95% ACI-LF			95% BCI-LF		
		MPSE		Bayes-SF		95% ACI-SF			95% BCI-SF		
		Est.	St.Er	Est.	St.Er	Low.	Upp.	Int.W	Low.	Upp.	Int.W
S[1]	β	2.0990	0.3928	1.3292	2.8688	1.5396	2.0984	0.0491	2.0026	2.1956	0.1929
		2.2548	0.4449	1.3828	3.1268	1.7440	2.2539	0.0492	2.1569	2.3509	0.1940
	δ	0.2645	0.1146	0.0398	0.4891	0.4493	0.2622	0.0371	0.1926	0.3374	0.1448
		0.2451	0.1151	0.0196	0.4706	0.4510	0.2428	0.0365	0.1742	0.3162	0.1420
	$R(0.1)$	0.7819	0.0708	0.6431	0.9207	0.2776	0.7739	0.0481	0.6709	0.8568	0.1859
		0.7996	0.0714	0.6596	0.9395	0.2799	0.7904	0.0505	0.6817	0.8751	0.1934
	$h(0.1)$	3.8720	1.1923	1.5351	6.2088	4.6737	3.9299	0.5130	2.9717	4.9656	1.9939
		3.9454	1.2910	1.4151	6.4756	5.0605	4.0199	0.5958	2.9211	5.2309	2.3098
S[2]	β	1.4971	0.2946	0.9196	2.0745	1.1549	1.4956	0.0482	1.4014	1.5910	0.1896
		1.5149	0.3156	0.8963	2.1335	1.2372	1.5136	0.0488	1.4186	1.6097	0.1911
	δ	0.5172	0.1588	0.2061	0.8284	0.6223	0.5139	0.0448	0.4274	0.6028	0.1754
		0.5374	0.1697	0.2048	0.8701	0.6652	0.5348	0.0452	0.4473	0.6244	0.1770
	$R(0.1)$	0.8352	0.0559	0.7255	0.9448	0.2193	0.8307	0.0283	0.7714	0.8801	0.1087
		0.8506	0.0550	0.7428	0.9585	0.2157	0.8468	0.0263	0.7913	0.8930	0.1017
	$h(0.1)$	2.3134	0.6737	0.9930	3.6339	2.6409	2.3384	0.2339	1.9036	2.8131	0.9095
		2.1966	0.6725	0.8786	3.5146	2.6360	2.2183	0.2301	1.7893	2.6892	0.8999
S[3]	β	1.1783	0.2514	0.6855	1.6711	0.9856	1.1764	0.0481	1.0823	1.2713	0.1891

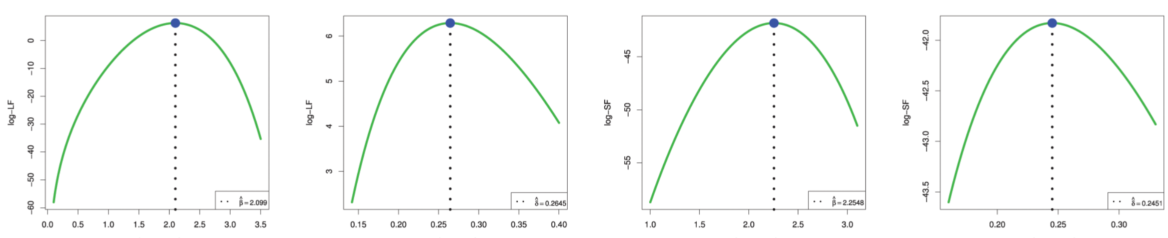
(Continued)

Table 27 (continued)

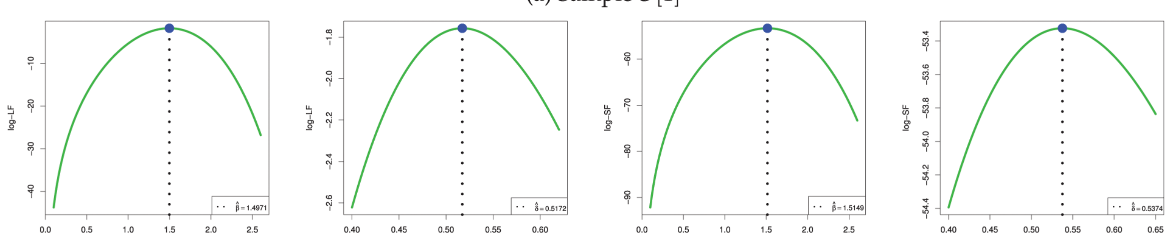
Sample	Par.	MLE		Bayes-LF		95% ACI-LF			95% BCI-LF		
		MPSE		Bayes-SF		95% ACI-SF			95% BCI-SF		
		Est.	St.Er	Est.	St.Er	Low.	Upp.	Int.W	Low.	Upp.	Int.W
		1.1602	0.2622	0.6463	1.6741	1.0278	1.1583	0.0485	1.0633	1.2540	0.1907
	δ	0.6940	0.1840	0.3333	1.0547	0.7213	0.6910	0.0470	0.6001	0.7841	0.1840
		0.7362	0.1970	0.3501	1.1223	0.7722	0.7340	0.0472	0.6422	0.8270	0.1848
	$R(0.1)$	0.8434	0.0540	0.7375	0.9493	0.2118	0.8403	0.0227	0.7930	0.8811	0.0881
		0.8559	0.0535	0.7511	0.9608	0.2096	0.8534	0.0208	0.8102	0.8912	0.0810
	$h(0.1)$	1.7615	0.5140	0.7541	2.7689	2.0148	1.7729	0.1434	1.5038	2.0633	0.5596
		1.6431	0.5006	0.6621	2.6242	1.9621	1.6515	0.1334	1.4003	1.9228	0.5225

Fig. 9 depicts log-LF and log-SF profile plots to confirm the offered ML and MPS estimations for β and δ are present and unique. Fig. 9 verifies the existence and uniqueness of the ML and MPS estimates for β and δ in the polyester fibers dataset, $\mathcal{S}[i]$, $i = 1, 2, 3$. Fig. 10 displays density and trace graphs for β, δ, $R(t)$, and $h(t)$ to demonstrate the convergence rate of the remaining 40,000 MCMC iterations. It shows that the MCMC approach works well and converges superiorly, and the specified burn-in sample size is sufficient.

All the facts derived from Fig. 10 support the same distribution behaviors for β, δ, $R(t)$, and $h(t)$ shown in Fig. 7. To illustrate the process of selecting the optimal censoring technique, we consider $\mathcal{S}[i]$ for $i = 1, 2, 3$, generated from the polyester fibers dataset. Using the ML and MPS approaches, Table 28 presents the fitted values of the proposed optimal criteria $\mathsf{C}[i]$ for $i = 1, 2, 3, 4$.

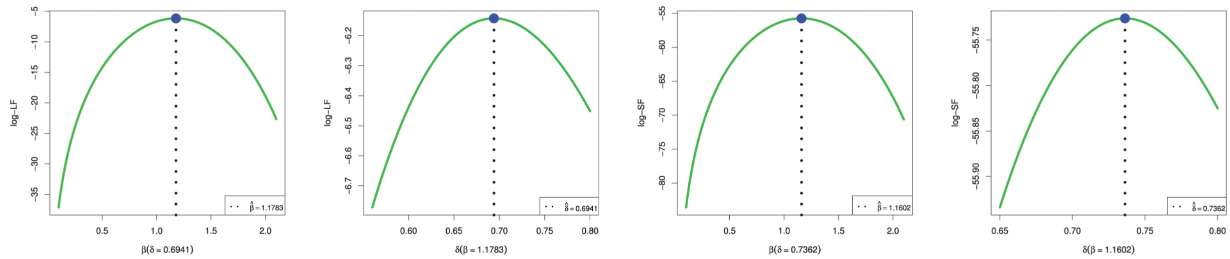


(a) Sample $\mathcal{S}[1]$



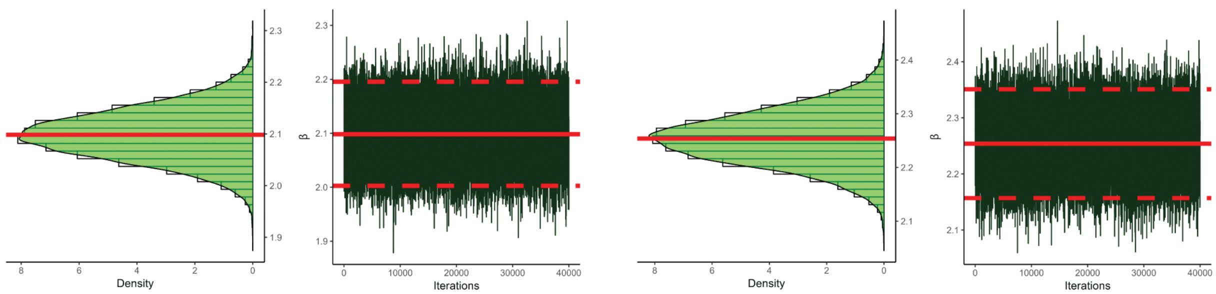
(b) Sample $\mathcal{S}[2]$

Figure 9: (Continued)

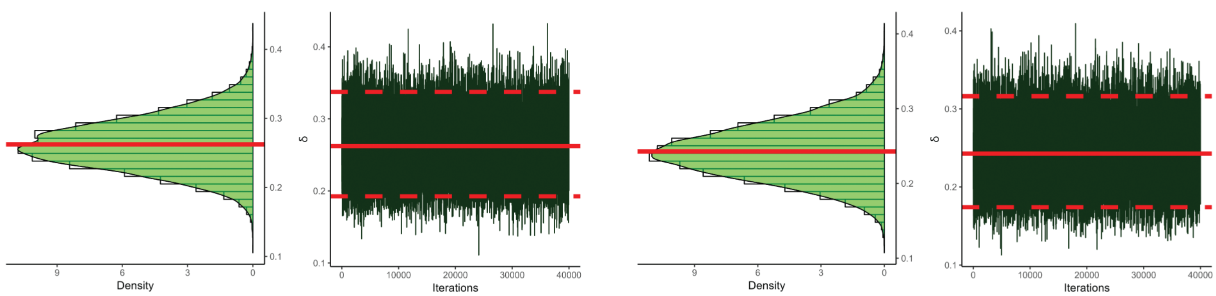


(c) Sample $\mathcal{S}[3]$

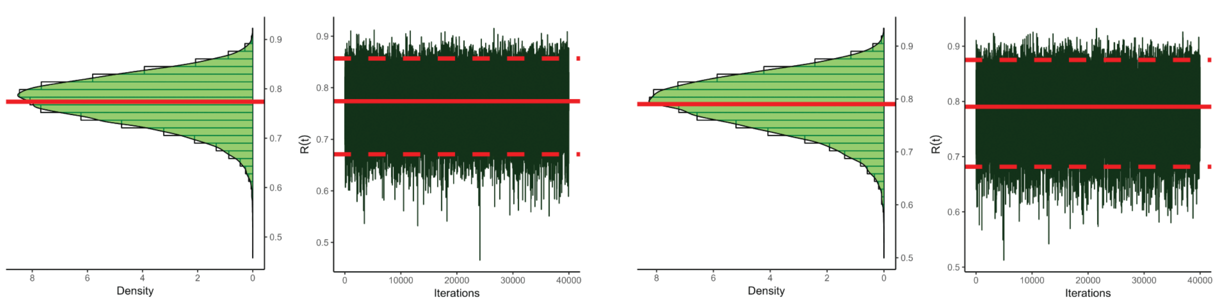
Figure 9: The log-LF (left) and log-SF (right) of β and δ from polyester fibers dataset (**a**–**c**)



(a) β

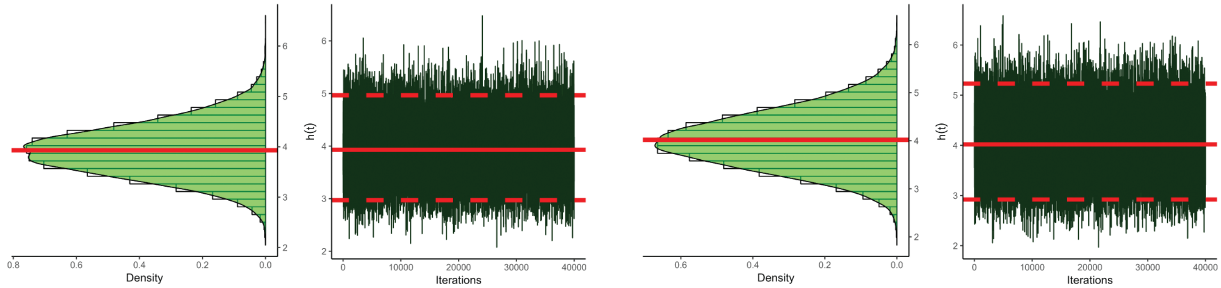


(b) δ



(c) $R(t)$

Figure 10: (Continued)



(d) $h(t)$

Figure 10: The density and trace plots from Bayes-LF (left) and Bayes-SF (right) iterations of β, δ, $R(t)$, and $h(t)$ from polyester fibers dataset (**a**–**d**)

Table 28: Optimum censoring from polyester fibers dataset

Sample	C[1]	C[2]	C[3]	C[4]		
$\nu \rightarrow$				0.3	0.6	0.9
			via ML			
			via MPS			
$\mathcal{S}[1]$	364.162	0.16739	0.00046	0.00082	0.00300	0.00906
	377.128	0.21118	0.00056	0.00081	0.00285	0.00909
$\mathcal{S}[2]$	136.160	0.11201	0.00082	0.00169	0.00576	0.00918
	119.471	0.12842	0.00107	0.00185	0.00619	0.00985
$\mathcal{S}[3]$	93.1308	0.09708	0.00104	0.00298	0.00911	0.00755
	80.5400	0.10755	0.00134	0.00342	0.01008	0.00771

The findings in Table 28 validate our proposed censoring strategies within the Monte Carlo experiments, showing that, for both the ML and MPS approaches:

- Regarding C[i], $i = 1, 3, 4$, the left-censoring (in $\mathcal{S}[1]$) is the optimum censoring than others;
- Regarding C[3], the right-censoring used in $\mathcal{S}[3]$ is the optimum censoring compared to others.

Using the datasets $\mathcal{S}[i]$, $i = 1, 2, 3$, collected from the petroleum dataset (as reported in Table 21), all estimated values of β exceed one, indicating an increasing hazard rate. This behavior aligns with failure mechanisms typically dominated by wear-out or aging processes, suggesting that the proposed Uni-W model effectively captures the progressive deterioration over time observed in mechanical systems.

In contrast, for the datasets $\mathcal{S}[i]$, $i = 1, 2, 3$, derived from the polyester fiber dataset (see Table 26), the data are characterized by shorter lifespans and inherently bounded outcomes. These properties align well with the unit support of the Uni-W model. The bounded structure of the model accommodates the biological and physical constraints of the polyester data and yields interpretable estimates for early-life failure risks.

7 Concluding Remarks

This study investigated the estimation of parameters, reliability, and failure rate functions for the newly proposed Uni-W distribution under an improved adaptive progressively Type-II censored sampling scheme. Two widely adopted estimation techniques, ML and MPS, were employed. Approximate confidence intervals for each parameter were derived using the delta method to estimate the variances necessary for constructing intervals for the reliability and hazard functions. From a Bayesian perspective, the Markov Chain Monte Carlo approach was implemented to obtain Bayesian estimates under the squared error loss function, as well as to construct Bayesian credible intervals for the model parameters. Bayesian inference was carried out using two posterior formulations: one based on the likelihood function and the other on the product of spacings. A comprehensive simulation study assessed the performance and robustness of the proposed estimators. Moreover, multiple optimality criteria were introduced to identify the most efficient censoring scheme among various possible removal patterns. The practical utility of the proposed methodology was demonstrated using two real-life case studies, supported by graphical diagnostics and empirical findings. Overall, the results confirm that the proposed inferential approaches yield robust and reliable parameter estimates and effectively describe the behavior of the Uni-W distribution under real-world censoring conditions. For future work, the methodologies developed in this study can be extended to accelerated life testing frameworks. Additionally, we recommend adopting a Bayesian framework that integrates the product of spacings approach via the Metropolis–Hastings algorithm to achieve efficient parameter estimation and accurate assessment of reliability characteristics for the Uni-W distribution.

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