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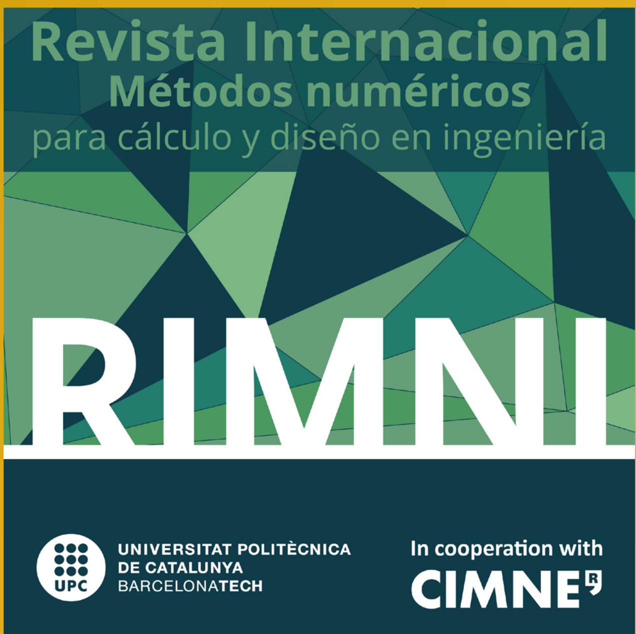

INFORMATION

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ABSTRACT

In this work, we investigate the existence and uniqueness results (EUR) for a class of neutral fractional differential equations (FDEs) with time advance, incorporating the Caputo derivative concerning F. By employing the fixed-point theory, we establish rigorous criteria ensuring the well-posedness of the problem. Additionally, we explore the Ulam-Hyers stability properties of the proposed model, providing a comprehensive analysis of its dynamic behavior. To further support our theoretical findings, we present two examples that illustrate the applicability and effectiveness of the obtained results. These findings contribute to the growing body of research on FDEs and their applications in various scientific and engineering fields.

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1 Introduction

In recent decates, fractional differential equations (FDEs) have attracted considerable interest for their unique ability to model memory and hereditary effects in complex systems across science and engineering. This growing recognition has spurred extensive research into analytical and numerical techniques for solving FDEs [1–3], as well as their applications in diverse areas such as viscoelasticity [4], anomalous diffusion [5], and control theory [6]. Recent advances have further enriched the theoretical framework of fractional calculus, introducing generalized operators and more efficient solution methods for nonlinear FDEs. These innovations highlight the superior capability of fractional models over classical integer-order approaches in accurately representing real-world dynamical processes. Among the key aspects of research in this field is the investigation of stability properties, which play a crucial role in understanding the long-term behavior of solutions. For instance, finite-time stability has been extensively studied, as seen in [7,8], while Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR) stability have been analyzed in depth in works such as [9–12]. In addition to stability

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considerations, problems related to control theory of FDEs have also been explored by numerous scientists [13–15]. These studies have provided valuable insights into the theoretical and practical applications of fractional-order systems, making them an essential topic in mathematical modeling and applied sciences.

In recent years, various types of fractional derivatives have been proposed by scientists, significantly expanding the scope and applicability of fractional calculus [16–18]. These diverse formulations, including the Caputo, Riemann-Liouville, and more generalized fractional operators, have played a fundamental role in advancing various scientific and engineering disciplines. The flexibility and non-local properties of fractional derivatives have enriched mathematical modeling in fields such as physics, medicine, engineering, and stochastic processes, among others. For instance, fractional calculus has been widely employed in epidemiological modeling to describe the complex dynamics of infectious diseases more accurately. In [19], the authors developed a fractional-order model to analyze the spread and control of COVID-19, highlighting the impact of memory effects in disease progression. Similarly, in [20], a fractional-order eco-epidemiological system with infected prey was studied, demonstrating the utility of fractional derivatives in capturing real-world ecological interactions. Furthermore, in [21], a fractional differential system was applied to model the transmission dynamics of hepatitis B, offering deeper insights into disease behavior and control strategies. These examples illustrate the growing importance of FDEs in scientific research, providing more precise and flexible mathematical frameworks for describing complex dynamical systems across various disciplines.

The existence of solutions and UH stability have been extensively investigated in recent years by numerous scientists [9,10,22]. In particular, the stability analysis of FDEs involving delays has been investigated for various types of fractional derivatives, as seen in [23–25]. For instance, the authors in [23] investigated the UH stability of fractional neutral integrodifferential equations with delay. Makhlouf et al. in [25] demonstrated the UH stability of backward FDEs with time advance.

However, to the best of our knowledge, the stability analysis of backward neutral FDEs with time advance, specifically in the framework of the Caputo derivative concerning $\mathcal{F}$ ($\mathcal{F}$-CD), has not yet been explored in the existing literature. Motivated by this gap, we direct our research toward this novel class of problems.

Based on the work in [25], we consider a backward neutral FDEs with time advance involving the $\mathcal{F}$-CD in this study. First, we establish the EUR employing the fixed-point approach, ensuring the well-posedness of the problem. Next, we present the UH stability results, providing rigorous conditions under which the considered equation exhibits stability properties. Finally, to validate and illustrate our theoretical findings, we present two examples that demonstrate the effectiveness of our findings.

The key contributions and innovations of this study are summarized as follows:

◊ Investigation of the EUR for backward neutral FDEs with time advance via the $\mathcal{F}$-CD employing the fixed point method.

◊ Analysis of the UH stability for the Backward neutral FDEs.

◊ Two examples are provided to demonstrate the significance of the theoretical findings.

2 Preliminaries

In this work, for $t_0,\ T \in \mathbb{R}$ with $t_0 < T$ and $r > 0$ we adopt the following notations:

- $\mathcal{AC}([t_0, T]; \mathbb{R})$ is the space of all Continuous Functions (CF) from $[t_0, T]$ to $\mathbb{R}$.
- $\mathcal{C}([t_0, T]; \mathbb{R})$ is the space of all CF from $[t_0, T]$ to $\mathbb{R}$.

- $\mathcal{D}_l([-r,0];\mathbb{R})$ is the space of all CF from $[-r,0]$ to $\mathbb{R}$.

For $\mathfrak{E} = \mathcal{C}([t_0, T+r];\mathbb{R})$, we define the metric $\mathbf{S}$ on $\mathfrak{E}$ by:

$$\mathbf{S}(u,v) = \sup_{s\in[t_0,T+r]} \left\{ \frac{|u(s)-v(s)|}{\sigma(s)} \right\},$$

with

$$\sigma(s) = \begin{cases} 1, & \forall \quad x \in [T, T+r], \\ e^{\lambda(F(T)-F(s))}, & \forall \quad s \in [t_0, T], \end{cases}$$

where F is a continuous and increasing.

Definition 1. *([26]) For $F \in \mathcal{C}^1([c_1,c_2])$ with $F'(s) > 0$, for every $s \in [c_1,c_2]$ and $0 < \alpha < 1$. The F-CD of $v(s)$ is given by:*

$${}^{C}D^{\alpha,F}_{c_2} v(s) = \frac{1}{F(1-\alpha)} \Big(-\frac{1}{F'(s)} \frac{d}{ds} \Big) \int_s^{c_2} F'(l)(F(l)-F(s))^{-\alpha}(v(l)-v(c_2)\, dl.$$

Lemma 1. *For $v \in \mathcal{AC}([c_1,c_2];\mathbb{R})$, the F-CD of $v(s)$ is defined by:*

$${}^{C}D^{\alpha,F}_{c_2} v(s) = -\frac{1}{F(1-\alpha)} \int_s^{c_2} (F(l)-F(s))^{-\alpha} v'(l)\, dl,$$

with $0 < \alpha < 1$ and $F \in \mathcal{C}^1([c_1,c_2])$ such that $F'(s) > 0$, for every $s \in [c_1,c_2]$.

3 Main Results

Consider the following equation:

$${}^{C}D^{\alpha,F}_{T} \Big(\mathbf{e}(s) - G\big(\mathbf{e}(s)\big) \Big) = f(s,\mathbf{e}^s), \quad \forall s \in [t_0, T], \tag{1}$$

$$\mathbf{e}(s) = \varpi(T-s), \quad \forall s \in [T, T+r], \tag{2}$$

where $\varpi \in \mathcal{D}_r = \mathcal{C}([-r,0];\mathbb{R})$, $f \in \mathcal{C}\big([t_0,T] \times \mathcal{D}_r;\mathbb{R})$ and $G \in \mathcal{C}\big(\mathbb{R};\mathbb{R})$.

The function $\mathbf{e}^s$ is given by $\mathbf{e}^s(l) = e(s-l)$ for $l \in [-r,0]$.

The Eq. (1) is equivalent to the following equation (see [26]):

$$\mathbf{e}(s) = G(\mathbf{e}(s)) + \mathbf{e}(T) - G(\mathbf{e}(T)) + \frac{1}{F(\alpha)} \int_s^T F'(l) \Big(F(l) - F(s) \Big)^{\alpha-1} f(l,\mathbf{e}^l) dl.$$

We will now outline the following hypotheses:

$(\mathcal{H}_1): \quad |G(\mathbf{x}_1) - G(\mathbf{x}_2)| \le \mathcal{K}_1 |\mathbf{x}_1 - \mathbf{x}_2|, \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R},$

where $0 < \mathcal{K}_1 < 1$.

$(\mathcal{H}_2): \quad |f(s,\mathbf{y}_1) - f(s,\mathbf{y}_2)| \le \mathcal{K}_2 \|\mathbf{y}_1 - \mathbf{y}_2\|, \quad \forall \mathbf{y}_1, \mathbf{y}_2 \in \mathcal{D}_r \quad$ and $\quad \forall s \in [t_0, T],$

where $\mathcal{K}_2 > 0$.

Definition 2. *The problem (1)–(2) is UH stable if there is $C > 0$ so that for all $\theta > 0$ and for all $\tilde{\mathbf{e}} \in \mathcal{AC}([t_0, T+r];\mathbb{R})$ a solution of:*

$$\left| {}^{C}D^{\alpha,F}_{T} \Big(\tilde{\mathbf{e}}(s) - G\big(\tilde{\mathbf{e}}(s)\big) \Big) - f(s,\tilde{\mathbf{e}}^s) \right| \le \theta,$$

there is $\mathbf{e} \in \mathcal{C}([t_0, T]; \mathbb{R})$ *satisfying the problem (1)–(2) with*

$|\tilde{\mathbf{e}}(s) - \mathbf{e}(s)| \leq C\theta, \quad s \in [t_0, T].$

We define the following constant:

$$M_T^{\alpha,F} = \frac{\lambda^\alpha (F(T) - F(t_0))^\alpha}{[(1-\mathcal{K}_1)\lambda^\alpha - \mathcal{K}_2] F(\alpha+1)} e^{\lambda(F(T)-F(t_0))}, \tag{3}$$

with $\lambda \geq 0$ such that: $\frac{\mathcal{K}_2}{\lambda^\alpha} + \mathcal{K}_1 < 1$.

The following presents the UHR stability for the Eq. (1).

Theorem 1. *Assume that the hypotheses* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are satisfied. If the following inequality:*

$$|{}^C D_T^{\alpha,F}\Big(y(s) - G\big(y(s)\big)\Big) - f(s, y^s)| \leq \theta, \quad \text{for all} \quad s \in [t_0, T], \tag{4}$$

holds, where $y \in \mathcal{AC}([t_0, T+r]; \mathbb{R})$ *and* $\theta > 0$. *Therefore, there is a unique solution* $\mathbf{e}^*$ *of (1) where*

$\mathbf{e}^*(s) = y(s), \quad \forall s \in [T, T+r],$

and

$|\mathbf{e}^*(s) - y(s)| \leq \theta M_T^{\alpha,F}, \quad \forall s \in [t_0, T].$

For $y \in \mathcal{AC}([t_0, T+r]; \mathbb{R})$ a solution of (4), we consider the operator $\mathfrak{A} : \mathfrak{E} \longrightarrow \mathfrak{E}$ given by:

$$(\mathfrak{A}\mathbf{e})(s) = \begin{cases} y(s), & \forall s \in [T, T+r], \\ G\big(\mathbf{e}(s)\big) + y(T) - G\big(y(T)\big) + \frac{1}{F(\alpha)} \int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1} f(l, \mathbf{e}^l) dl, & \forall s \in [t_0, T]. \end{cases}$$

We have the following result.

Proposition 1. $\mathfrak{A}$ *is a contractive operator*.

Proof. Let $\mathbf{e}_1, \mathbf{e}_2 \in \mathfrak{E}$. Then, we have:

$(\mathfrak{A}\mathbf{e}_1)(s) - (\mathfrak{A}\mathbf{e}_2)(s) = 0, \quad \forall s \in [T, T+r].$

Let $s \in [t_0, T]$. we obtain:

$$\begin{aligned} \Big|(\mathfrak{A}\mathbf{e}_1)(s) - (\mathfrak{A}\mathbf{e}_2)(s)\Big| &= \Big|G\big(\mathbf{e}_1(s)\big) - G\big(\mathbf{e}_2(s)\big) + \frac{1}{F(\alpha)} \int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1} \Big[f(l, \mathbf{e}_1^l) - f(l, \mathbf{e}_2^l)\Big] dl\Big|, \\ &\leq \frac{\mathcal{K}_2}{F(\alpha)} \int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1} \|\mathbf{e}_1^l - \mathbf{e}_2^l\| dl + \mathcal{K}_1 |\mathbf{e}_1(s) - \mathbf{e}_2(s)|, \end{aligned}$$

where:

$$||\mathbf{e}_1^l - \mathbf{e}_2^l|| = \sup_{m \in [-r,0]} \Big(|\mathbf{e}_1(l-m) - \mathbf{e}_2(l-m)|\Big).$$

For $l \in [s, T]$, there exists $m \in [-r, 0]$ with:

$$\begin{aligned} ||\mathbf{e}_1^l - \mathbf{e}_2^l\| &= |\mathbf{e}_1(l-m) - \mathbf{e}_2(l-m)|, \\ &= \frac{|\mathbf{e}_1(l-m) - \mathbf{e}_2(l-m)|}{\sigma(l-m)}\sigma(l-m), \\ &\leq \mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(l). \end{aligned}$$

We have:

$$|\mathbf{e}_1(s) - \mathbf{e}_2(s)| = \frac{|\mathbf{e}_1(s) - \mathbf{e}_2(s)|}{\sigma(s)}\sigma(s) \leq \mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(s).$$

Therefore:

$$\begin{aligned} \Big|(\mathfrak{A}\mathbf{e}_1)(s) - (\mathfrak{A}\mathbf{e}_2)(s)\Big| &\leq \frac{\mathcal{K}_2\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)}{F(\alpha)}\int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1}\sigma(l)dl + \mathcal{K}_1\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(s), \\ &\leq \frac{\mathcal{K}_2\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)}{F(\alpha)}\int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1}e^{\lambda(F(T)-F(l))}dl + \mathcal{K}_1\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(s). \end{aligned} \tag{5}$$

Analogously to the demonstration of Proposition 1 in [25], we obtain:

$$\begin{aligned} \int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1}e^{\lambda(F(T)-F(l))}dl &= e^{\lambda(F(T)-F(s))}\int_0^{\lambda(F(T)-F(s)}\frac{l^{\alpha-1}}{\lambda^\alpha}e^{-l}dl, \\ &\leq \frac{e^{\lambda(F(T)-F(s))}}{\lambda^\alpha}F(\alpha). \end{aligned} \tag{6}$$

It follows from (5) and (6) that:

$$\Big|(\mathfrak{A}\mathbf{e}_1)(s) - (\mathfrak{A}\mathbf{e}_2)(s)\Big| \leq \frac{\mathcal{K}_2}{\lambda^\alpha}\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)e^{\lambda(F(T)-F(s))} + \mathcal{K}_1\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(s) = \left(\frac{\mathcal{K}_2}{\lambda^\alpha} + \mathcal{K}_1\right)\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2)\sigma(s).$$

Therefore, we get:

$$\mathbf{S}(\mathfrak{A}\mathbf{e}_1, \mathfrak{A}\mathbf{e}_2) \leq \left(\tfrac{\mathcal{K}_2}{\lambda^\alpha} + \mathcal{K}_1\right)\mathbf{S}(\mathbf{e}_1, \mathbf{e}_2).$$

Since $\left(\frac{\mathcal{K}_2}{\lambda^\alpha} + \mathcal{K}_1\right) < 1$. Then, $\mathfrak{A}$ is contractive.

Remark 1. *In assumption* $(\mathcal{H}_1)$*, the condition* $\mathcal{K}_1 < 1$ *is necessary to ensure that* $\mathfrak{A}$ *is a contraction.*

Now, we present the demonstration of Theorem 1.

Proof. We obtain:

$$(\mathfrak{A}y)(s) - y(s) = 0, \quad \text{for each} \quad s \in [T, T+r].$$

Using (4), we get:

$$\begin{aligned} |y(s) - (\mathfrak{A}y)(s)| &\leq \frac{\theta}{F(\alpha)}\int_s^T F'(l)\Big(F(l) - F(s)\Big)^{\alpha-1}dl \\ &\leq \frac{\theta}{F(\alpha+1)}\Big(F(T) - F(s)\Big)^\alpha. \end{aligned}$$

Therefore:

$$\mathbf{S}(y, \mathfrak{A}y) \leq \frac{\theta}{\digamma(\alpha+1)}\Big(\digamma(T)-\digamma(t_0)\Big)^{\alpha}.$$

It follows from Theorem 2.1 in [11] that there is a unique solution $\mathbf{e}^*$ of (1)–(2), such that $\mathbf{e}^*(s) = y(s)$ for every $s \in [T, T+r]$, with:

$$\mathbf{S}(y, \mathbf{e}^*) \leq \frac{\theta}{\digamma(\alpha+1)}\Big(\digamma(T)-\digamma(t_0)\Big)^{\alpha}\frac{1}{1-\left(\frac{\mathcal{K}_2}{\lambda^{\alpha}}+\mathcal{K}_1\right)},$$

$$\leq \frac{\lambda^{\alpha}\Big(\digamma(T)-\digamma(t_0)\Big)^{\alpha}}{[(1-\mathcal{K}_1)\lambda^{\alpha}-\mathcal{K}_2]\digamma(\alpha+1)}\theta.$$

Thus:

$$|y(s)-\mathbf{e}^*(s)| \leq \theta M_T^{\alpha,\digamma}, \quad \forall s \in [t_0, T].$$

□

Remark 2. *If $G = 0$, we obtain the findings given in [25].*

4 Examples

This section provides two ullistratives examples to demonstrate the practical application of the theoretical findings.

Example 1. *Consider the following Hadamard-fractional equation ($\digamma(t) = \ln(t)$):*

$${}^{CH}D_5^{0.8,\digamma}\Big(\mathbf{e}(s)-G(\mathbf{e}(s))\Big) = f(s, \mathbf{e}^s), \quad \forall s \in [1,5], \tag{7}$$

$$\mathbf{e}(s) = \varpi(5-s), \quad \forall s \in [5, 5.01], \tag{8}$$

where $f(s,y) = \dfrac{1}{25}\sin(s)y(-0.01)$ *and* $G(\mathbf{e}) = 0.5\cos(\mathbf{e})$. The functions f *and* G *satisfy the hypotheses* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$:

$$|f(s,\mathbf{e})-f(s,\mathbf{v})| \leq \frac{1}{25}\|\mathbf{e}-\mathbf{v}\|, \quad \forall s \in [1,5],$$

$$|G(x)-G(y)| \leq \frac{1}{2}|x-y|, \quad \forall x, y \in \mathbb{R}.$$

Now, let's consider this equation:

$${}^{CH}D_5^{0.8,\digamma}\Big(y(s)-G(y(s))\Big) = f(s, y^s) + \frac{1}{50}, \quad \forall s \in [1,5],$$

$$y(s) = \mathbf{e}(s), \quad \forall s \in [5, 5.01],$$

We have $y \in \mathcal{C}([1, 5.01]; \mathbb{R})$ is a solution to the inequality (4), with $\theta = \frac{1}{50}$:

$$\left|{}^{CH}D_5^{0.8,\digamma}\Big(y(s)-G(y(s))\Big) - f(s, y^s)\right| \leq \tfrac{1}{50}, \quad \forall s \in [1,5].$$

Therefore, from Theorem 1, we deduce that the problem (7)–(8) has a unique solution $\mathbf{e}^*$ *such that:*

$$\mathbf{e}^*(s) = y(s), \quad \forall s \in [5, 5.01],$$

and

$$|\mathbf{e}^*(s) - y(s)| \leq \frac{1}{50} M_5^{0.8,\mathcal{F}}, \quad \forall s \in [1, 5].$$

Example 2. *Let* $\mathcal{F}(s) = 5 + s^4$. *Consider the following equation:*

$$^C D_4^{0.5,\mathcal{F}} \Big(\mathbf{e}(s) - G(\mathbf{e}(s))\Big) = f(s, \mathbf{e}^s), \quad \forall s \in [2, 4], \tag{9}$$

$$\mathbf{e}(s) = \varpi(4 - s), \quad \forall s \in [4, 4.1], \tag{10}$$

where $f(s, y) = \cos(3s)y(-0.1)$ *and* $G(\mathbf{e}) = 0.2\sin(\mathbf{e})$. *The functions* f *and* G *satisfy the hypotheses* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$:

$$|f(s, \mathbf{e}) - f(s, \mathbf{v})| \leq \|\mathbf{e} - \mathbf{v}\|, \quad \forall s \in [2, 4],$$

$$|G(x) - G(y)| \leq 0.2|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Let's consider this equation:

$$^C D_4^{0.5,\mathcal{F}} \Big((y(s) - G(y(s))\Big) = f(s, y^s) + 0.1, \quad \forall s \in [2, 4],$$
$$y(s) = \mathbf{e}(s), \quad \forall s \in [4, 4.1].$$

If $y \in \mathcal{C}([2, 4.1]; \mathbb{R})$ *is a solution to the inequality:*

$$|^C D_4^{0.5,\mathcal{F}} \big(y(s) - G(y(s))\big) - f(s, y^s)| \leq 0.1, \quad \forall s \in [2, 4],$$

Therefore, from Theorem 1, we deduce that the problem (9)–(10) has a unique solution $\mathbf{e}^*$ *with:*

$$\mathbf{e}^*(s) = y(s), \quad \forall s \in [4, 4.1],$$

and

$$|\mathbf{e}^*(s) - y(s)| \leq 0.1 M_4^{0.5,\mathcal{F}}, \quad \forall s \in [2, 4].$$

5 Conclusion

In this study, several important objectives have been demonstrated. We have established the existence and uniqueness of solutions for a class of backward neutral differential equations with time advance, incorporating the $\mathcal{F}$-CD. This result ensures the well-posedness of the problem and provides a strong theoretical foundation for further studies in this area.

Additionally, we have investigated the stability properties of the considered system. Specifically, we have proven the UH stability. These findings contribute to a deeper understanding of the long-term behavior of solutions and their sensitivity to perturbations, which is crucial for applications in various scientific and engineering fields.

Further research may investigate stability analysis, numerical approximations, and potential applications of these equations across disciplines such as finance and physics, where fractional modeling is of significant importance.

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