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REDUCED BASIS METHOD FOR FREQUENCY SWEEPS WITH INTEGRAL EQUATIONS USING LOCALLY ADAPTIVE KERNEL INTERPOLATION – ADMOS 2021

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Abstract. Many electromagnetic and acoustic applications require the ability to explore all solutions in a given frequency window. When the problem is large scale, strategies based on computing a large number of solutions from successive solver calls usually lead to prohibitive computational costs. This is especially the case when the solver relies on an integral equation discretized using the boundary element method (BEM), as this amounts to solving numerous complex, unsymmetrical and fully populated linear systems. The reduced basis method (RBM) is an efficient approach to rapidly and accurately approximate any solution within a given frequency window [1, 2]. In the context of frequency sweeps with the BEM, the success of the RBM essentially depends on the ability to decouple the frequency from the kernel of the underlying integral equation. In this work, we present a new methodology based on the ideas of local adaptivity [3]. The main benefit is that the overall number of operators to be assembled throughout the RBM (or to be computed using adequate low-rank approximation methods) is significantly reduced compared to previous approaches [1, 4]. The proposed methodology is illustrated with the double-layer operator on an academic problem.

1 INTRODUCTION

In this work, we consider a parameterized integral equation discretized using the Boundary Element Method (BEM). The parameter $\mu \in \mathcal{D}$ represents the wavenumber evolving in a given window $\mathcal{D} = [\mu_{\min}, \mu_{\max}] \subset \mathbb{R}$. Let V, W be two $\mathcal{N}$-dimensional approximation spaces, such that the integral equation discretized using the BEM consists in finding $u(\mu) \in V$ satisfying

$$A(\mu)u(\mu) = f(\mu) \quad \text{in } W' \tag{1}$$

where $A(\mu) \in \mathcal{L}(V, W')$ is the μ-dependent discretized integral operator and $f(\mu) \in W'$ a given μ-dependent right-hand side (RHS). Problem (1) is assumed to be well-posed for all $\mu \in \mathcal{D}$. Thus, the solution manifold $\{u(\mu), \mu \in \mathcal{D}\}$ exists and is unique.

The Reduced Basis Method (RBM) is a method of choice for efficiently approximating the solution manifold with rigorous bounds on the approximation error [1, 2]. The efficiency of the RBM is well-known to critically depend on the *affine* nature of the operator and the RHS. In the affine context, there exist two integers M^a, M^f such that

$$A(\mu) = \sum_{m=1}^{M^a} \theta_m^a(\mu) A_m, \quad f(\mu) = \sum_{n=1}^{M^f} \theta_n^f(\mu) f_n, \tag{2}$$

with μ-dependent functions $\theta_m^a, \theta_n^f : \mathcal{D} \to \mathbb{C}$ and μ-independent operator and RHS parts $A_m \in \mathcal{L}(V, W')$, $f_n \in W'$, $1 \leqslant m \leqslant M^a$, $1 \leqslant n \leqslant M^f$. When the operator (or RHS) is not affine, one builds an affine approximation. In the case of a discretized integral operator, the affine approximation typically has a large number of terms M^a because of the non-trivial μ-dependency in the underlying integral equation. Yet M^a being a key contributor to the online complexity of the RBM, a large M^a may ruin the overall performance of the RBM, as has been observed in [1]. To circumvent this issue, the authors in [1] define a maximum number of terms Q and build a domain decomposition of $\mathcal{D}$ in K subdomains. Instead of one *global* affine approximation valid over $\mathcal{D}$, one now constructs K *local* affine approximations, each valid over a given subdomain. The number of subdomains K and the domain decomposition itself are tuned in such a way that the number of terms never exceeds Q in each local affine approximation.

In this work, we revisit the overall RBM for integral equations with three new ingredients: *(i)* a locally adaptive strategy for building local affine approximations not exceeding Q terms, *(ii)* a reduced basis approximation based on nested approximation spaces, generated in a greedy manner and *(iii)* a non-intrusive framework. We show the virtues of each component of the proposed methodology: *(i)* significantly reduces the overall number of distinct μ-independent operator parts to be assembled during the offline stage compared to the *hp* approach proposed in [1, 4], *(ii)* ensures that the number of μ-independent operator parts simultaneously required in memory never exceeds Q during the greedy algorithm used to generate the reduced basis and finally *(iii)* simplifies the implementation, as no other assembly routine than that for assembling $A(\mu)$ at given μ needs to be im-

plemented. Note that we always speak of "assembling" the operator, but in practice this could stand for "computing a representation using low-rank approximation methods".

The paper is structured as follows. In section 2 we introduce the *non-intrusive local affine approximations*. In section 3 we detail the reduced basis approximation. Finally, section 4 is devoted to numerical results on a Helmholtz second-kind integral equation.

2 NON-INTRUSIVE LOCAL AFFINE APPROXIMATIONS

2.1 Defintion

In this approach, Q is decided *a priori* by the user. We introduce $J \geqslant Q$ parameters points $\mathcal{C}^J = \{\mu_j\}_{1\leqslant j\leqslant J} \subset \mathcal{D}$, indexed in inceasing order: $\mu_1 \leqslant \cdots \leqslant \mu_J$. Following a locally adaptive strategy [3], we define for all $\mu \in \mathcal{D}$ the set $\mathcal{T}(\mu)$ comprising of the Q points in $\mathcal{C}^J$ closest to μ in the euclidean distance (thus $\mathcal{T}(\mu)$ is subset of $\mathcal{C}^J$ whose cardinality is exactly Q). Notice that there always exists an index k, $1 \leqslant k \leqslant K$, with $K = J - Q + 1$, such that $\mathcal{T}(\mu) = \{\mu_k, \ldots, \mu_{k+Q-1}\} \equiv \mathcal{T}_k$. Using this observation, we define the k^{th} subdomain as $\mathcal{D}_k = \{\mu \in \mathcal{D},\, \mathcal{T}(\mu) = \mathcal{T}_k\}$.

Now that our domain decomposition of $\mathcal{D}$ in K subdomains is defined, we are ready to introduce the *non-intrusive local affine approximations*, for $k = 1, \ldots, K$,

$$\forall \mu \in \mathcal{D}_k, \quad A_k(\mu) = \sum_{q=1}^{Q} \theta_q^k(\mu) A(\mu_{k+q-1}). \tag{3}$$

Here, the μ-dependent functions $\theta_q^k : \mathcal{D}_k \to \mathbb{C}$, are constructed in such a way that the approximation error $A(\mu) - A_k(\mu)$ is "adequately small" over $\mathcal{D}_k$. Typically, one sets a prescribed tolerance $\eta^{\text{tol}} > 0$ and looks forward to obtaining

$$\sup_{\mu\in\mathcal{D}_k} \sup_{v\in V} \frac{\|A(\mu)v - A_k(\mu)v\|_{W'}}{\|v\|_V} < \eta^{\text{tol}}. \tag{4}$$

2.2 Properties

Assume that, at some point during the offline stage of the RBM, N reduced basis functions $\{\xi_n,\ n = 1, \ldots, N\} \subset V$ (to be defined in section 3) are available and that the Q assembled operators $A(\mu_k), \ldots, A(\mu_{k+Q-1})$ are available in memory. It is then possible to compute all the quantities $\langle A(\mu_i)^* R_W^{-1} A(\mu_j)\xi_n, \xi_m\rangle$, for $i, j \in \{k, \ldots, k+Q-1\}$ and $n, m \in \{1, \ldots, N\}$ (here, $R_W \in \mathcal{L}(W, W')$ denotes the Riesz map on W). These pre-computed quantities will be useful in the online stage of the RBM, for computing $\langle A_k(\mu)^* R_W^{-1} A_k(\mu)\xi_n, \xi_m\rangle$ for any $n, m \in \{1, \ldots, N\}$ and any $\mu \in \mathcal{D}_k$ in $\mathcal{O}(Q^2)$ complexity using the formula (3).

At this point, if $k < K$, we are *almost* able to compute $\langle A_{k+1}(\mu)^* R_W^{-1} A_{k+1}(\mu)\xi_n, \xi_m\rangle$ for any $n, m \in \{1, \ldots, N\}$ and any $\mu \in \mathcal{D}_{k+1}$ with $\mathcal{O}(Q^2)$ complexity. Indeed, the only missing quantities are the $\langle A(\mu_i)^* R_W^{-1} A(\mu_{k+Q})\xi_n, \xi_m\rangle$ and the $\langle A(\mu_{k+Q})^* R_W^{-1} A(\mu_i)\xi_n, \xi_m\rangle$, for $i \in \{k+1, \ldots, k+Q\}$ and $n, m \in \{1, \ldots, N\}$. Computing these missing quantities

offline requires the Q operators $A(\mu_{k+1}), \dots, A(\mu_{k+Q})$ to be available in memory. This shows how, thanks to formula (3), the number of operators simultaneously required in memory will never exceed Q. Overall, the number of operator assemblies will be J.

3 REDUCED BASIS APPROXIMATION

3.1 Nested approximation spaces

Let us introduce K discrete sets $\mathcal{S}_k \subset \mathcal{D}_k$, $k = 1, \dots, K$. We recursively define the K nested approximation spaces,

$$\begin{aligned} V^1 &= \mathrm{Span}\{u(\check{\mu}),\, \check{\mu} \in \mathcal{S}_1\}, \\ V^k &= V^{k-1} \oplus \mathrm{Span}\{u(\check{\mu}),\, \check{\mu} \in \mathcal{S}_k\}, \quad 2 \leqslant k \leqslant K. \end{aligned} \tag{5}$$

Clearly, $V^1 \subset V^2 \subset \cdots \subset V^K \subset V$. Assuming that solutions at different parameter points are linearly independent, the approximation space V^k, $1 \leqslant k \leqslant K$ is of dimension $N(k) = \sum_{\kappa=1}^{k} \mathrm{Card}(\mathcal{S}_\kappa)$, where $\mathrm{Card}(\cdot)$ denotes the cardinality (0 when the set is empty).

Here, we recall that $\mathcal{D}_1$ contains the lowest and $\mathcal{D}_K$ the highest possible values of $\mu \in \mathcal{D}$. Since μ denotes the wavenumber (or frequency), the approximation space V^1 must be thought of as the span of some low-frequency solutions and V^K as the span of solutions ranging from the lowest to the highest possible frequencies.

3.2 Least-squares approximation

For all $k = 1, \dots, K$, define the reduced μ-dependent test spaces $W_\mu^k = R_W^{-1} A_k(\mu) V^k \subset W$. For all $\mu \in \mathcal{D}$, there exists an integer $k = 1, \dots, K$ such that $\mu \in \mathcal{D}_k$. We define the reduced basis approximation at this value of μ as $u_k^{\mathtt{rb}}(\mu) \in V^k$, solution to the Petrov-Galerkin formulation

$$\langle A_k(\mu) u_k^{\mathtt{rb}}(\mu), w_k \rangle = \langle f(\mu), w_k \rangle \quad \forall w_k \in W_\mu^k, \tag{6}$$

with duality brackets between W and W'. Equivalently, $u_k^{\mathtt{rb}}(\mu) \in V^k$ is the solution to the Galerkin formulation

$$\langle A_k(\mu)^* R_W^{-1} A_k(\mu) u_k^{\mathtt{rb}}(\mu), v_k \rangle = \langle A_k(\mu)^* R_W^{-1} f(\mu), v_k \rangle \quad \forall v_k \in V^k, \tag{7}$$

with duality brackets between V and V'. Existence and uniqueness of the reduced basis solution is guaranteed when $A_k(\mu)$ satisfies an inf-sup condition. Given that $A(\mu)$ satisfies an inf-sup condition (from the well-posedness of (1)), an adequately small $\eta^{\mathrm{tol}} > 0$ in (4) will ensure the well-posedness of the reduced basis problem.

3.3 Error estimation

Denoting $\alpha(\mu) = \inf_{v \in V} \|A(\mu)v\|_{W'} / \|v\|_V$ the inf-sup constant of $A(\mu)$, the following *a posteriori* error estimate holds,

$$\|u(\mu) - u_k^{\mathtt{rb}}(\mu)\|_V \leqslant \frac{1}{\alpha(\mu)} \|A_k(\mu) u_k^{\mathtt{rb}}(\mu) - f(\mu)\|_{W'} + \frac{\eta^{\mathrm{tol}}}{\alpha(\mu)} \|u_k^{\mathtt{rb}}(\mu)\|_V. \tag{8}$$

Typically, $\eta^{\text{tol}} > 0$ is chosen sufficiently small so that the second term in the error estimate can be neglected. Thus, the practical error estimate is simply the residual norm divided by the inf-sup constant.

4 NUMERICAL RESULTS

4.1 Helmholtz formulation using the double layer potential

Consider the unit sphere $S = \{y \in \mathbb{R}^3, \|y\| = 1\}$ and the wavenumber interval $\mathcal{D} = [0.92, 6.0]$ (corresponding to the frequency interval [50, 325]Hz taking into account the speed of sound $c = 340\text{m/s}$). Given the Dirichlet data $g_d(\mu) \in H^{1/2}(S)$, we are interested by the following second-kind integral equation: *find* $\phi(\mu) \in H^{1/2}(S)$*, such that*

$$\left(M_\mu + \frac{1}{2}I\right)\phi(\mu) = g_d(\mu), \tag{9}$$

where $M_\mu : H^{1/2}(S) \to H^{1/2}(S)$ denotes the double layer integral operator at wavenumber μ. This formulation can be used to solve the Helmholtz exterior problem, where S is the boundary of the scattering object.

In our numerical experiments, we consider a triangulation of S with $\mathcal{N} = 642$ vertices. This triangulation defines a polyhedral approximation to the unit sphere, which we denote Γ. Let V be the Lagrange P^1 finite element approximation space constructed upon the triangulation (note that $\dim(V) = \mathcal{N}$). We set $W = V$ and consider the operator $A(\mu) \in \mathcal{L}(V, W')$ defined for all $(v, w) \in V \times W$ by

$$\langle A(\mu)v, w\rangle = \int_\Gamma \overline{w(x)} \int_\Gamma \partial_{n(y)}\mathcal{G}(x, y; \mu)v(y)\mathrm{d}\gamma_y\mathrm{d}\gamma_x + \frac{1}{2}\int_\Gamma \overline{w(x)}v(x)\mathrm{d}\gamma_x \tag{10}$$

where $\mathcal{G}(x, y; \mu) = \frac{e^{i\mu\|x-y\|}}{4\pi\|x-y\|}$ is the Helmholtz fundamental solution and $\partial_{n(y)}$ the derivative with respect to the normal $n(y)$, outgoing Γ. We also define the RHS $f(\mu) \in W'$ as $\langle f(\mu), w\rangle = \int_\Gamma g_d(x; \mu)\overline{w(x)}\mathrm{d}\gamma_x$, where the Dirichlet data is the plane wave $g_d(\cdot; \mu) : x \in \Gamma \mapsto e^{i\mu(x-x_0)\cdot d}$, with direction $d = (1, 0, 0)^T$ and origin $x_0 = (-5, 0, 0)^T$.

4.2 Local affine approximations

Given the problem at hand, we can be satisfied by a non-intrusive local affine approximation just for the fully-populated part of our operator, corresponding to the discretization of M_μ. Indeed, the $\frac{1}{2}I$ part is independent from μ and yields a sparse matrix. To start with, we observe that the fully-populated part of our operator writes as a convolution with a kernel taking the specific form

$$\partial_{n(y)}\mathcal{G}(x, y; \mu) = \psi(\|x - y\|; \mu)\mathcal{H}(x, y), \tag{11}$$

where $\mathcal{H} : \Gamma \times \Gamma \to \mathbb{C}$ does not depend on μ and where we can give an explicit analytical expression for $\psi(\cdot; \mu) : \mathbb{R}_+ \to \mathbb{C}$. Thus, the work to be done is brought to finding J

wavenumbers $\{\mu_j\}_{1\leqslant j\leqslant J} \subset \mathcal{D}$ and a domain-decomposition of $\mathcal{D} = \cup_{k=1}^{K} \mathcal{D}_k$, such that for all $k = 1, \ldots, K$, we can build a local interpolant under the form

$$\mathcal{I}_k \psi(\cdot;\mu) = \sum_{q=1}^{Q} \theta_q^k(\mu) \psi(\cdot;\mu_{k+q-1}) \tag{12}$$

that is a "good" approximation to $\psi(\cdot;\mu)$ for $\mu \in \mathcal{D}_k$. A tool for achieving this is for instance the locally adaptive method developped in [3]. Note that there is a singularity $\lim_{r\to 0} \psi(r;\mu) = \infty$. Fortunately, we can decompose $\psi(\cdot;\mu) = \psi^{\mathrm{ns}}(\cdot;\mu) + \psi^{\mathrm{s}}(\cdot)$, where $\psi^{\mathrm{ns}}(\cdot;\mu)$ is non-singular and contains all the dependency in μ, while $\psi^{\mathrm{s}}(\cdot)$ is singular and independent from μ. Thus, to obtain (12), one applies the locally adaptive tools not directly to $\psi(\cdot;\mu)$ but to the non-singular part $\psi^{\mathrm{ns}}(\cdot;\mu)$.

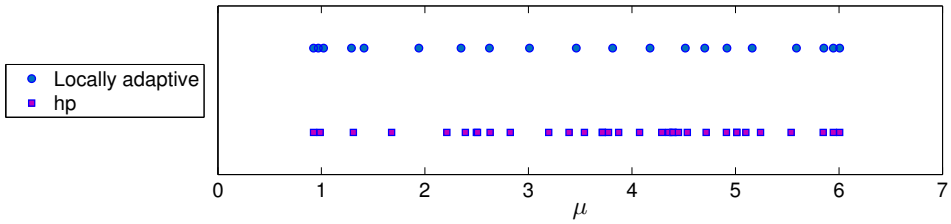

Figure 1: The $J = 20$ parameter points selected by the locally adaptive algorithm [3] and the 34 parameter points selected by the *hp* algorithm [4]. Both algorithms applied to $\psi^{\mathrm{ns}}(\cdot;\mu)$ with $Q = 7$ and prescribed tolerance of 10^{-7} on the approximation error in $L^\infty([0, r_{\max}] \times \mathcal{D})$ norm (with $r_{\max} = 2$, the diameter of the unit sphere).

We show on fig. 1 the $J = 20$ wavenumbers obtained when applying the locally adaptive algorithm on $\psi^{\mathrm{ns}}(\cdot;\mu)$, with $Q = 7$ and a prescribed tolerance of 10^{-7}. These wavenumbers correspond to the wavenumbers at which the double-layer integral operator will have to be assembled at some point during the offline phase of the RBM. For comparison, the *hp* method (or elementwise method) of [1, 4] applied with the same Q and the same prescribed tolerance yields 34 wavenumbers, also shown on fig. 1. Thus, the gain from using the locally adaptive strategy rather than the *hp* method is 14 offline assemblies, which is quite significant given that assembly costs are dominant in the BEM.

4.3 Reduced basis generation

The locally adaptive method gives us a domain-decomposition $\mathcal{D} = \cup_{k=1}^{K} \mathcal{D}_k$, with $K = J - Q + 1 (= 14)$ subdomains. To generate the reduced basis, we proceed subdomain after subdomain starting at $k = 1$. To start the process, the Q double-layer operators at wavenumbers μ_j, $j = 1, \ldots, Q$ must be assembled, then we proceed as follows:

1. classical greedy iterations: the set $\mathcal{S}_k$ is enriched until a prescribed tolerance $\epsilon^{\mathrm{rb}} > 0$ is reached on $\max_{\mu\in\Xi_k} \Delta_k(\mu)$, where $\Delta_k(\mu)$ is an estimator for the error $\|u(\mu) - u_k^{\mathrm{rb}}(\mu)\|_V$

and $\Xi_k \subset \mathcal{D}_k$ a discrete surrogate set of parameters. The enrichment strategy is based on enriching $\mathcal{S}_k$ by appending the maximizer of the error estimator over Ξ_k.

2. when the prescribed tolerance is reached, the process terminates if $k = K$, else the double-layer operator at wavenumber $\mu = \mu_{k+Q}$ is assembled, overwriting the memory that was allocated for the double-layer operator at wavenumber $\mu = \mu_k$ (accordingly to section 2.2) and the process is repeated with $k \leftarrow k + 1$.

Recalling that $\mathcal{D}_1$ contains the lowest and $\mathcal{D}_K$ the highest possible frequencies, the process is really that of a *frequency-sweep*, from the low to the high frequencies.

Consistent with the error estimate (8), we take as error estimator the residual norm divided by an approximation $\tilde{\alpha}(\mu)$ for the inf-sup constant (obtained by interpolation), that is $\Delta_k(\mu) = \|A_k(\mu)u_k^{\mathrm{rb}}(\mu) - f(\mu)\|_{W'}/\tilde{\alpha}(\mu)$. We use surrogate sets Ξ_k based on discretizing $\mathcal{D}_k$ uniformly with 100 points. Setting $\epsilon^{\mathrm{rb}} = 10^{-2}$, we have obtained nested approximation spaces V^k, $1 \leqslant k \leqslant K$ and have consigned their dimension in table 1. We note

Table 1: Dimension of the nested approximation spaces V^k, $1 \leqslant k \leqslant K$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dim(V^k)$	4	4	5	5	6	8	8	9	9	11	12	12	13	16

that the approximation space is sometimes the same between two successive subdomains. This shows that the frequency-sweep procedure is able to detect, when arriving on a new subdomain, that the approximation space from the previous subdomain is good enough to approximate the solution. When this happens, no solution needs to be computed and the frequency sweep may continue with the next subdomain. This is also visible on fig.

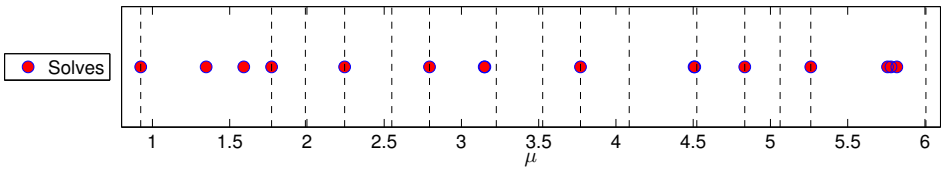

Figure 2: The points $\check{\mu} \in \mathcal{S}_k$, $k = 1, \ldots, K$ for which the solutions $u(\check{\mu})$ have been computed. The dotted vertical lines indicate the boundary of the subdomains $\mathcal{D}_k$, $k = 1, \ldots, K$ (thus each interspace between two dotted vertical lines corresponds to a subdomain).

2, where we have plotted the points $\check{\mu}$ in the sets $\mathcal{S}_k$, $1 \leqslant k \leqslant K$, where the solutions $u(\check{\mu})$ have been computed to serve as basis functions for the approximation spaces V^k, $1 \leqslant k \leqslant K$. We explain the stronger concentration of points in the neighborhood of the wavenumbers 3.1, 4.5 and 5.8 by the presence of interior resonant frequencies, where $\alpha(\mu)$ becomes relatively small ($\approx 10^{-2}$, while $\mathcal{O}(1)$ away from these resonant frequencies).

4.4 Validation

In order to assess the quality of the reduced basis approximation, we pick 80 random parameter points in $\mathcal{D}$. We compute the reduced basis solution $u_k^{\mathrm{rb}}(\mu)$ (whose coordinates in the approximation space V^k can be obtained with $\mathcal{N}$-independent complexity) as well as the truth solution $u(\mu)$ (which requires assembling fully-populated BEM matrix and calling a solver). We then compute the effectivity index $\mathrm{eff}(\mu) = \Delta_k(\mu)/\|u(\mu) - u_k^{\mathrm{rb}}(\mu)\|_V$. We present some statistics in table 2. We find that the over/under-estimation of the error

Table 2: Statistics of the effectivity index (80 random samples of $\mu \in \mathcal{D}$).

Minimum	Mean	Maximum
0.61 (at $\mu \approx 4.5$)	2.8	61.18 (at $\mu \approx 3.1$)

stays within reasonable bounds. Thus, the tolerance ϵ^{rb} prescribed *a priori* by the user indeed reflects the true level of accuracy of the obtained reduced basis solutions.

5 CONCLUSIONS

We have revisited RBM for integral equations parameterized by the frequency, by introducing local affine approximations and nested approximation spaces. The methodology has been illustrated with the double-layer operator on an academic problem. We found the overall number of operator assemblies to be significantly reduced compared to previous approaches. Future work will be applying the methodology to various integral equations.

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