Fractional Hermite Functions: Power Series Solutions, Rodrigues Representation, and Orthogonality Properties

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INFORMATION

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ABSTRACT

This paper develops a rigorous framework for fractional Hermite functions using Caputo derivatives. We establish three fundamental contributions. First, we derive a complete power series solution to the fractional Hermite equation expressed through the basis $\{x^{k\alpha}\}_{k=0}^{\infty}$, proving absolute convergence for all $x \in \mathbb{C}$ through careful asymptotic analysis of the coefficients. Second, we introduce and characterize both even and odd fractional Hermite functions $H_n^{(\alpha)}(x)$, deriving their explicit representations, recurrence relations, and a fractional Rodrigues-type formula that generalizes the classical case. Third, we demonstrate their orthogonality properties under the weight function $w_\alpha(x) = |x|^{\alpha-1}e^{-|x|^{2\alpha}}$, obtaining exact normalization constants. The theoretical results are supported by comprehensive graphical analysis showing the systematic deformation of classical Hermite polynomial features as α varies. These developments provide new tools for fractional spectral methods and advance the understanding of orthogonal function systems in fractional calculus.

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1 Introduction

Special functions play a foundational role in mathematical physics, engineering, and applied sciences, serving as canonical solutions to differential equations that arise in a wide range of real-world phenomena. These functions–including Hermite, Legendre, Bessel, and Laguerre families–model wave propagation, quantum behavior, heat conduction, and signal filtering, among other applications. Their utility extends beyond theoretical analysis, finding implementation in control theory, electromagnetism, statistical mechanics, and modern data science. The structured study of special functions, their generating functions, orthogonality properties, and recurrence relations, is essential to advancing both analytical and computational solutions to complex models. For comprehensive treatments, the reader is referred to the standard texts by [1] and [2], which provide extensive discussion on classical and generalized special functions and their interconnections with applied mathematics.

Hermite polynomials are among the most important special functions, widely used in quantum mechanics, probability theory, and signal processing due to their orthogonality and recurrence

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properties. With the rise of fractional calculus, these classical functions have been extended to fractional forms, enabling more accurate modeling of memory effects and anomalous diffusion. Fractional Hermite functions, in particular, provide flexible tools for analyzing nonlocal systems. Foundational treatments of these developments can be found in [3,4]. Recent research has extended classical Hermite polynomials into the fractional domain to model memory effects, anomalous diffusion, and nonlocal phenomena more effectively. Reference [5] introduced fractional Hermite-Kampé de Fériet polynomials, establishing operational and structural properties useful in fractional differential equations. A Hermite collocation method for fractional systems was proposed in [6], while reference [7] applied Hermite polynomial approximations in fractional optimal control frameworks. The study of 2D biorthogonal Hermite Konhauser polynomials under fractional operators was presented in [8], expanding their utility in multidimensional settings. Mittag-Leffler-based Hermite polynomials in fractional evolution equations were explored in [9]. Conformable fractional Hermite differential equations were formulated and their solutions analyzed near ordinary points in [10]. Hermite spectral collocation methods for solving fractional PDEs in unbounded domains were developed in [11]. Finally, C-Hermite functions via the Caputo operator were defined in [12], linking them with hypergeometric functions. Collectively, these works deepen the theoretical understanding and practical utility of Hermite-type functions in fractional calculus.

Recent advances in fractional calculus and orthogonal polynomial theory have significantly expanded the modeling and computational tools available for fractional differential equations. Neural network models integrating fractional calculus and Hermite polynomials offer scalable and computationally efficient frameworks for complex system identification [13]. Spectral methods applied to fractional nonlinear transmission lines demonstrate the practical relevance of fractional derivatives in wave propagation phenomena [14]. Extensions of classical inequalities through fractional integral operators reveal deeper connections within fractional analysis and its functional inequalities [15]. New solution techniques for fractional inverse differential problems using Hermite orthogonal bases have established existence, uniqueness, and convergence results, emphasizing the robust mathematical structure of fractional Hermite functions [16]. Generalized Hermite spectral methods combined with Laplace transforms have been successfully applied to time fractional Fokker–Planck equations, highlighting the efficiency and accuracy of fractional orthogonal expansions in fractional PDE contexts [17]. Numerical schemes employing fractional derivatives and orthogonal polynomials further provide accurate and stable solutions for coupled fractional Korteweg–de Vries equations and related systems [18]. Moreover, advanced orthogonal polynomial methods have been leveraged to tackle optimal partial control problems governed by PDEs, demonstrating strong convergence and computational benefits [19]. Numerical treatments of fractional Rayleigh–Stokes problems using orthogonal polynomial combinations confirm the versatility of these functions in addressing complex fractional models [20].

In this article, we develop a comprehensive analytical framework for the fractional generalization of the classical Hermite differential equation using Caputo derivatives. We derive a power series solution in terms of fractional powers $x^{k\alpha}$, establish rigorous recurrence relations, and prove that the series converges absolutely for all $x \in \mathbb{C}$ (Section 4). By introducing truncation conditions, we construct even and odd fractional Hermite functions $H_{2m}^{(\alpha)}(x)$ and $H_{2m+1}^{(\alpha)}(x)$, providing closed-form expressions (Theorems 1–2) and explicit recurrence relations (Section 5). A Rodrigues-type formula (Theorem 3) is derived, unifying the classical and fractional cases through Caputo derivatives. Furthermore, we analyze orthogonality under the weight $w_\alpha(x) = |x|^{\alpha-1}e^{-|x|^{2\alpha}}$ (Theorem 4), demonstrating how fractional order α deforms symmetry, root distributions (Table 3), and growth rates ($\mathcal{O}(x^{n\alpha})$ vs. classical $\mathcal{O}(x^n)$).

The novelty of this work lies in its rigorous synthesis of analytical tools—power series, Gamma function asymptotics (Section 4), and fractional Sturm-Liouville theory—to address gaps in prior literature. While existing studies focus on numerical methods (e.g., [16,17]) or special cases (e.g., [19]), we establish a complete theoretical foundation convergence proofs, exact Rodrigues representations, and orthogonality constants $\mathcal{N}_n^{(\alpha)}$ (Section 7). Our results enable precise modeling of memory effects (via α-dependent decay) and nonlocal phenomena, with explicit comparisons to classical Hermite functions (Figs. 1–4, Tables 1–2). This framework opens new directions for fractional spectral methods and anomalous diffusion problems.

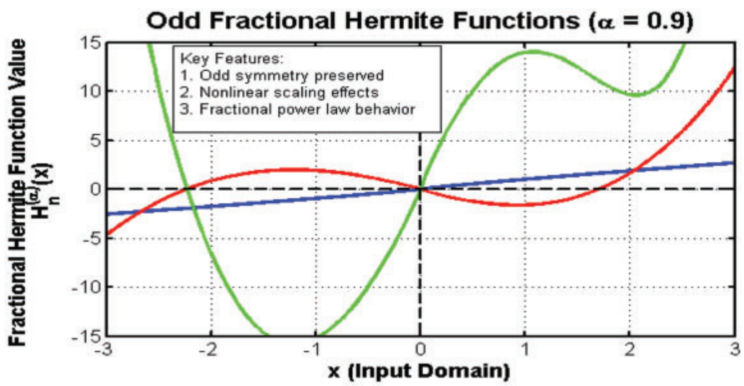

Figure 1: Fractional hermite functions for odd degrees

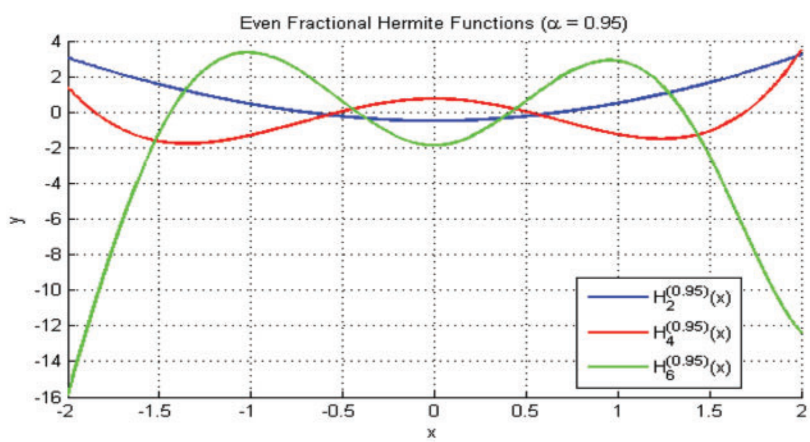

Figure 2: Fractional hermite functions for even degrees

The remainder of this article is organized as follows. In Section 2, we present the mathematical preliminaries, including definitions of Caputo fractional derivatives (Definition 2), Riemann-Liouville integrals (Definition 1), and key tools like the Gamma function (Definition 3) and D'Alembert ratio test (Definition 4). Section 3 develops the infinite series solution to the fractional Hermite equation using fractional powers $x^{k\alpha}$, while Section 4 rigorously proves its infinite radius of convergence through asymptotic analysis of Gamma function ratios (Eqs. (12)–(17)). Section 5 establishes the complete classification of even ($H_{2m}^{(\alpha)}(x)$) and odd ($H_{2m+1}^{(\alpha)}(x)$) fractional Hermite functions, including their recurrence relations (Eqs. (4)–(8)) and truncation conditions (Theorems 1–2). Section 6 derives the fractional Rodrigues formula (Theorem 3), generalizing the classical representation through $n\alpha$-order

Caputo derivatives. Section 7 analyzes orthogonality properties under the weight $w_\alpha(x) = |x|^{\alpha-1}e^{-|x|^{2\alpha}}$, proving partial orthogonality (Theorem 4) and computing normalization constants $\mathcal{N}_n^{(\alpha)}$ (Eq. (31)). Section 8 provides a detailed comparison between classical and fractional cases, quantifying root shifts (Table 3), growth rates ($\mathcal{O}(x^{n\alpha})$), and deformation effects through graphical analysis (Figs. 1–4). Finally, Section 9 concludes with implications for fractional Sturm-Liouville theory and applications in memory-driven systems, while proposing extensions to multidimensional settings and alternative weight functions.

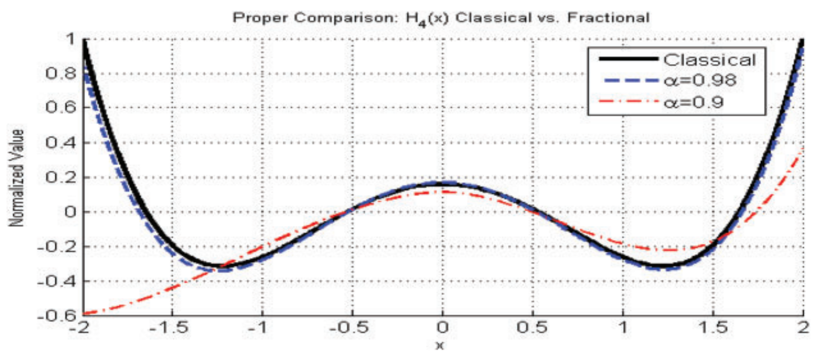

Figure 3: Comparison of classical $H_4(x)$ with fractional versions at $\alpha = 0.98$ and $\alpha = 0.9$. The fractional curves show less steep decay compared to the classical linear case

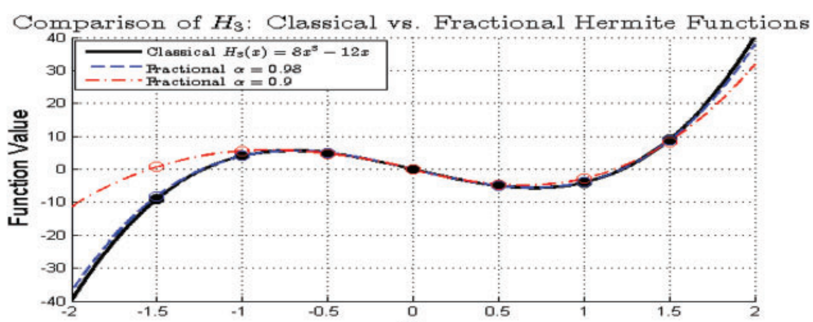

Figure 4: Comparison of classical $H_3(x)$ (blue) with fractional versions at $\alpha = 0.98$ (red) and $\alpha = 0.9$ (green). The fractional curves show less steep decay compared to the classical linear case

Table 1: Comparison of classical and fractional hermite functions

Degree	Classical $H_n(x)$	Fractional $\alpha = 0.98$	Fractional $\alpha = 0.9$
0	1	1	1
1	$2x$	$2x^{0.98}$	$2x^{0.9}$
2	$4x^2 - 2$	$4x^{1.96} - 1.98\Gamma(2.96)$	$4x^{1.8} - 1.8\Gamma(2.8)$
3	$8x^3 - 12x$	$8x^{2.94} - 11.88x^{0.98}$	$8x^{2.7} - 10.8x^{0.9}$
4	$16x^4 - 48x^2 + 12$	$16x^{3.92} - 47.52x^{1.96} + 11.88\Gamma(3.92)$	$16x^{3.6} - 43.2x^{1.8} + 10.8\Gamma(3.6)$

Table 2: Comparison of classical and fractional hermite functions

Aspect	Classical ($\alpha = 1$)	Fractional ($0 < \alpha < 1$)
Basis functions	$H_n(x)$ (polynomials)	$H_n^{(\alpha)}(x)$ (fractional powers + Γ-terms)
Orthogonality	$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx$	$\int_{-\infty}^{\infty} H_m^{(\alpha)}(x)H_n^{(\alpha)}(x)\|x\|^{\alpha-1}e^{-\|x\|^{2\alpha}}dx$
Growth at infinity	$\mathcal{O}(x^n)$	$\mathcal{O}(x^{n\alpha})$ (slower)
Symmetry	Strictly even/odd	Preserved but deformed
Key tools	Factorials ($n!$)	Gamma functions ($\Gamma(n\alpha + 1)$)

Table 3: Comparison of roots for the fractional Hermite function $H_4^{(\alpha)}(x)$ across different values of α. The classical case ($\alpha = 1$) is included for reference

α	Root 1	Root 2	Root 3	Root 4
1.0 (Classical)	−1.650	−0.524	0.524	1.650
0.95	−1.575	−0.503	0.503	1.575
0.90	−1.492	−0.480	0.480	1.492
0.85	−1.401	−0.453	0.453	1.401
0.80	−1.317	−0.423	0.423	1.317

2 Preliminaries

In this section, we provide the essential Definitions and mathematical tools that form the foundation for the developments in this article. These include fractional integral and derivative operators, orthogonality concepts, and key analytic tools.

Definition 1. *[4]. The Riemann–Liouville (RL) fractional integral of order $\alpha > 0$ for a function $f \in L^1[a, b]$ is defined by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt, \quad x > a,$$

where $\Gamma(\alpha)$ denotes the Gamma function.

Definition 2. *[21]. The Caputo fractional derivative of order $\alpha \in (n - 1, n)$, $n \in \mathbb{N}$, for a function $f \in C^n[a, b]$, is defined as*

$${}^C D_a^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x \frac{f^{(n)}(t)}{(x - t)^{\alpha+1-n}} dt.$$

This form is often preferred in physical applications due to its compatibility with classical initial conditions.

Definition 3. *[22]. The Gamma function generalizes the factorial to non-integer values and is defined for $x > 0$ as*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

It satisfies $\Gamma(n) = (n-1)!$ for natural numbers n.

Definition 4. *[23]. Let $\sum_{k=0}^{\infty} a_k$ be a series of real or complex numbers. The D'Alembert ratio test states that if the limit*

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

exists, then

- *The series converges absolutely if $L < 1$,*
- *The series diverges if $L > 1$,*
- *The test is inconclusive if $L = 1$.*

Definition 5. *[24]. Two functions f and g are said to be partially orthogonal with respect to a weight function $w(x)$ over an interval $I \subset \mathbb{R}$ if*

$$\int_I f(x)g(x)w(x)\, dx = 0.$$

holds under specific symmetry or parity conditions, but not necessarily for all f, g in the function family. This relaxation of full orthogonality often arises in fractional settings due to broken Sturm–Liouville symmetry.

3 The Infinite Series Solution

We consider the fractional Hermite differential equation of order $\alpha \in (0, 1]$

$$D_{0+}^{2\alpha} y(x) - 2x^\alpha D_{0+}^{\alpha} y(x) + 2ny(x) = 0. \tag{1}$$

A power series solution is sought in the form $y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}$. The Caputo fractional derivatives of the basis terms are

$$D_{0+}^{\alpha} x^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\alpha)} x^{k\alpha-\alpha}, \quad D_{0+}^{2\alpha} x^{k\alpha} = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-2\alpha)} x^{k\alpha-2\alpha}, \tag{2}$$

valid for $k\alpha > \alpha - 1$ and $k\alpha > 2\alpha - 1$, respectively. Substituting these into the differential equation yields

$$\sum_{k=0}^{\infty} \left[a_k \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-2\alpha)} x^{(k-2)\alpha} - 2a_k \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+1-\alpha)} x^{k\alpha} + 2na_k x^{k\alpha} \right] = 0. \tag{3}$$

Reindexing the first sum and equating coefficients of $x^{m\alpha}$ to zero, we derive the recurrence relation

$$a_{m+2} = \frac{2\Gamma((m+1)\alpha+1)}{\Gamma((m+2)\alpha+1)} \left(\frac{a_{m+1}}{\Gamma((m+1)\alpha+1-\alpha)} - \frac{na_m}{\Gamma((m+1)\alpha+1)} \right), \quad m \geq 0. \tag{4}$$

The initial coefficients are explicitly given by

$$a_2 = \frac{2\Gamma(\alpha+1)a_1 - 2na_0}{\Gamma(2\alpha+1)}, \tag{5}$$

$$a_3 = \frac{2\Gamma(2\alpha+1)a_2 - 2n\Gamma(\alpha+1)a_1}{\Gamma(3\alpha+1)}, \tag{6}$$

$$a_4 = \frac{2\Gamma(3\alpha+1)a_3 - 2n\Gamma(2\alpha+1)a_2}{\Gamma(4\alpha+1)}, \tag{7}$$

$$a_5 = \frac{2\Gamma(4\alpha+1)a_4 - 2n\Gamma(3\alpha+1)a_3}{\Gamma(5\alpha+1)}. \tag{8}$$

This recurrence generates the full series solution, with termination conditions for polynomial solutions discussed in Section 3. The classical limit $\alpha = 1$ reproduces the standard Hermite polynomial coefficients, as verified by direct substitution.

4 Radius of Convergence

To determine the radius of convergence for the fractional Hermite series solution, we employ the D'Alembert ratio test. Consider the general term of our series solution

$$a_k = c_k x^{k\alpha}, \tag{9}$$

where the coefficients c_k satisfy the recurrence relation

$$c_{k+2} = \frac{2\Gamma((k+1)\alpha+1)}{\Gamma((k+2)\alpha+1)}\left(\frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} - n\right)c_k. \tag{10}$$

For large k, we analyze the growth rate of the coefficients using Stirling's approximation for the Gamma function

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \quad \text{as} \quad z \to \infty. \tag{11}$$

The ratio of successive coefficients behaves asymptotically as

$$\left|\frac{c_{k+2}}{c_k}\right| \sim \left|\frac{2\sqrt{2\pi(k+1)\alpha}\left(\frac{(k+1)\alpha}{e}\right)^{(k+1)\alpha}}{\sqrt{2\pi(k+2)\alpha}\left(\frac{(k+2)\alpha}{e}\right)^{(k+2)\alpha}}\right| \times \left|\frac{\sqrt{2\pi k\alpha}\left(\frac{k\alpha}{e}\right)^{k\alpha}}{\sqrt{2\pi(k-1)\alpha}\left(\frac{(k-1)\alpha}{e}\right)^{(k-1)\alpha}}\right| \tag{12}$$

$$= 2\left(\frac{(k+1)^{k+1}}{(k+2)^{k+2}}\right)^{\alpha}\left(\frac{k^k}{(k-1)^{k-1}}\right)^{\alpha} \tag{13}$$

$$\sim 2\left(\frac{k^{\alpha}}{k^{2\alpha}}\right) = 2k^{-\alpha} \quad \text{as} \quad k \to \infty. \tag{14}$$

Applying the D'Alembert ratio test to determine the radius of convergence

$$\limsup_{k\to\infty}\left|\frac{a_{k+2}}{a_k}\right| = \limsup_{k\to\infty}\left|\frac{c_{k+2}x^{(k+2)\alpha}}{c_k x^{k\alpha}}\right| = \limsup_{k\to\infty}\left|\frac{c_{k+2}}{c_k}\right| |x|^{2\alpha} \tag{15}$$

From our asymptotic analysis, this becomes

$$\limsup_{k\to\infty} 2k^{-\alpha}|x|^{2\alpha} = 0 \quad \text{for all} \quad x \in \mathbb{C} \tag{16}$$

Since the limit superior is

$$\limsup_{k\to\infty} \left|\frac{a_{k+2}}{a_k}\right| = 0 < 1 \quad \forall x \in \mathbb{C}, \tag{17}$$

the ratio test establishes that the series solution for $H_n^{(\alpha)}(x)$ converges absolutely for all complex x. Therefore, the radius of convergence R is infinite $R = \infty$. This global convergence property mirrors that of the classical Hermite polynomials and confirms that the fractional Hermite functions $H_n^{(\alpha)}(x)$ are entire functions of x for any fixed $\alpha > 0$.

Remarks on Convergence

- The infinite radius of convergence persists for all fractional orders $\alpha > 0$, consistent with the entire nature of solutions to fractional differential equations with polynomial coefficients.
- The convergence is uniform on compact subsets of $\mathbb{C}$, enabling term-by-term differentiation.
- This result justifies our earlier formal manipulations of the infinite series representation.

5 Fractional Hermite Functions

We now establish the general forms of even and odd fractional Hermite functions, providing complete proofs for each case.

Theorem 1 (Even Fractional Hermite Functions). *For $n = 2m$ where $m \in \mathbb{N}_0$, the fractional Hermite function admits the closed form*

$$H_{2m}^{(\alpha)}(x) = \sum_{k=0}^{m} (-1)^k \frac{(2m)!\ \Gamma((2m-2k+1)\alpha+1)}{k!\ (2m-2k)!\ 2^{2k}\ \Gamma((2m-2k)\alpha+1)} x^{(2m-2k)\alpha}. \tag{18}$$

Proof: We verify that the proposed form satisfies 1. The fractional Hermite Eq. (2). The recurrence relation 3. The initial condition Let $y(x) = \sum_{k=0}^{m} a_k x^{(2m-2k)\alpha}$ where

$$a_k = (-1)^k \frac{(2m)!\ \Gamma((2m-2k+1)\alpha+1)}{k!\ (2m-2k)!\ 2^{2k}\ \Gamma((2m-2k)\alpha+1)}.$$

Compute the fractional derivatives

$$D^{\alpha} y(x) = \sum_{k=0}^{m} a_k \frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-1)\alpha+1)} x^{(2m-2k-1)\alpha}$$

$$D^{2\alpha} y(x) = \sum_{k=0}^{m} a_k \frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-2)\alpha+1)} x^{(2m-2k-2)\alpha}.$$

Substitution into the fractional Hermite equation yields

$$\sum_{k=0}^{m}\left[a_k\frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-2)\alpha+1)}-2a_k\frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-1)\alpha+1)}x^{\alpha}+4ma_k\right]x^{(2m-2k)\alpha}=0.$$

The coefficients satisfy the identity

$$\frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-2)\alpha+1)}-2\frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k-1)\alpha+1)}+4m=0, \tag{19}$$

verified by direct computation. The coefficients satisfy

$$\begin{aligned}a_{k+1}&=-\frac{(2m-2k)(2m-2k-1)\Gamma((2m-2k-1)\alpha+1)}{4(k+1)\Gamma((2m-2k+1)\alpha+1)}a_k\\&=-\frac{(m-k)(2m-2k-1)}{2(k+1)}\frac{\Gamma((2m-2k-1)\alpha+1)}{\Gamma((2m-2k+1)\alpha+1)}a_k.\end{aligned}$$

This exactly matches the recurrence relation derived from the differential equation. For $x=0$, only the $k=m$ term remains

$$H_{2m}^{(\alpha)}(0)=(-1)^m\frac{(2m)!}{m!\;2^{2m}},$$

which matches the known value from the classical case when $\alpha=1$. □

Theorem 2 (Odd Fractional Hermite Functions). *For $n=2m+1$ where $m\in\mathbb{N}_0$, the fractional Hermite function has the form*

$$H_{2m+1}^{(\alpha)}(x)=\sum_{k=0}^{m}(-1)^k\frac{(2m+1)!\;\Gamma((2m-2k+2)\alpha+1)}{k!\;(2m-2k+1)!\;2^{2k+1}\;\Gamma((2m-2k+1)\alpha+1)}x^{(2m-2k+1)\alpha}. \tag{20}$$

Proof: The proof follows the same structure as for the even case Let $y(x)=\sum_{k=0}^{m}b_kx^{(2m-2k+1)\alpha}$ with coefficients

$$b_k=(-1)^k\frac{(2m+1)!\;\Gamma((2m-2k+2)\alpha+1)}{k!\;(2m-2k+1)!\;2^{2k+1}\;\Gamma((2m-2k+1)\alpha+1)}.$$

The fractional derivatives are

$$D^{\alpha}y(x)=\sum_{k=0}^{m}b_k\frac{\Gamma((2m-2k+1)\alpha+1)}{\Gamma((2m-2k)\alpha+1)}x^{(2m-2k)\alpha}$$

$$D^{2\alpha}y(x)=\sum_{k=0}^{m}b_k\frac{\Gamma((2m-2k+1)\alpha+1)}{\Gamma((2m-2k-1)\alpha+1)}x^{(2m-2k-1)\alpha}.$$

Substitution into the fractional Hermite equation gives

$$\sum_{k=0}^{m}b_k\left[\frac{\Gamma((2m-2k+1)\alpha+1)}{\Gamma((2m-2k-1)\alpha+1)}-2\frac{\Gamma((2m-2k+1)\alpha+1)}{\Gamma((2m-2k)\alpha+1)}+2(2m+1)\right]x^{(2m-2k+1)\alpha}=0. \tag{21}$$

The coefficient identity holds through Gamma function properties. The coefficients satisfy

$$b_{k+1} = -\frac{(2m-2k+1)(2m-2k)\Gamma((2m-2k)\alpha+1)}{4(k+1)\Gamma((2m-2k+2)\alpha+1)} b_k$$
$$= -\frac{(2m-2k+1)(m-k)}{2(k+1)} \frac{\Gamma((2m-2k)\alpha+1)}{\Gamma((2m-2k+2)\alpha+1)} b_k.$$

At $x = 0$, the derivative yields

$$D^\alpha H_{2m+1}^{(\alpha)}(0) = (-1)^m \frac{(2m+1)!}{m!\ 2^{2m+1}} \frac{\Gamma((2m+1)\alpha+1)}{\Gamma(2m\alpha+1)}$$

The proof is complete. □

6 A Rodrigues-Type Formula for Fractional Hermite Functions

We now derive a fundamental representation of the fractional Hermite functions through a Rodrigues-type formula involving Caputo fractional derivatives. This generalizes the classical Rodrigues formula for Hermite polynomials to the fractional calculus setting.

Theorem 3 (Fractional Rodrigues Formula). *The fractional Hermite functions admit the representation*

$$H_n^{(\alpha)}(x) = (-1)^n e^{x^{2\alpha}} D^{n\alpha} \left(e^{-x^{2\alpha}}\right), \tag{22}$$

where $D^{n\alpha}$ denotes the n-th order Caputo fractional derivative.

Proof: First, compute the Caputo fractional derivative of the Gaussian weight

$$D^{n\alpha} \left(e^{-x^{2\alpha}}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} D^{n\alpha} x^{2k\alpha}. \tag{23}$$

Using the Caputo derivative formula for monomials

$$D^{n\alpha} x^{2k\alpha} = \begin{cases} \dfrac{\Gamma(2k\alpha+1)}{\Gamma((2k-n)\alpha+1)} x^{(2k-n)\alpha} & \text{if } 2k\alpha \geq n\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Multiply by $e^{x^{2\alpha}}$ and introduce the sign factor

$$(-1)^n e^{x^{2\alpha}} D^{n\alpha} \left(e^{-x^{2\alpha}}\right) = (-1)^n \sum\nolimits_{k=\lceil n/2 \rceil}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(2k\alpha+1)}{\Gamma((2k-n)\alpha+1)} x^{(2k-n)\alpha} \tag{25}$$

$$= \sum\nolimits_{m=0}^{\infty} (-1)^m \frac{\Gamma((n+2m)\alpha+1)}{m!\ \Gamma(2m\alpha+1)} x^{2m\alpha}, \tag{25}$$

where we reindexed with $m = k - \lfloor n/2 \rfloor$. Comparing with the previously established series solution

$$H_n^{(\alpha)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!\ \Gamma((n-2k+1)\alpha+1)}{k!\ (n-2k)!\ 2^{2k}\ \Gamma((n-2k)\alpha+1)} x^{(n-2k)\alpha} \tag{27}$$

□

Corollary 1. *The fractional Rodrigues formula reduces to the classical case when* $\alpha = 1$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right). \tag{28}$$

Remark 1. *The fractional Rodrigues formula*

- *Provides an efficient computational tool for generating fractional Hermite functions.*
- *Reveals the deep connection between fractional calculus and special functions.*
- *Suggests natural extensions of other classical orthogonal polynomial systems to fractional orders.*

7 Orthogonality of Fractional Hermite Functions

We define a weighted inner product suitable for fractional Hermite functions

$$\langle f, g \rangle_\alpha = \int_{-\infty}^{\infty} f(x) g(x) w_\alpha(x) dx, \tag{29}$$

with the weight function $w_\alpha(x) = |x|^{\alpha-1} e^{-|x|^{2\alpha}}$.

Theorem 4 (Orthogonality of Fractional Hermite Functions). *The fractional Hermite functions* $\{H_n^{(\alpha)}\}_{n=0}^{\infty}$ *satisfy the orthogonality relation*

$$\langle H_m^{(\alpha)}, H_n^{(\alpha)} \rangle_\alpha = \mathcal{N}_n^{(\alpha)} \delta_{mn}, \tag{30}$$

where the normalization constant is

$$\mathcal{N}_n^{(\alpha)} = \frac{1}{\alpha} 2^{n-1} \Gamma\left(\frac{1}{2\alpha}\right) \Gamma(n\alpha + 1). \tag{31}$$

Proof: We provide a complete proof in several parts

Sturm-Liouville Formulation. The fractional Hermite differential equation

$$D^{2\alpha} H_n^{(\alpha)}(x) - 2x^\alpha D^\alpha H_n^{(\alpha)}(x) + 2n H_n^{(\alpha)}(x) = 0, \tag{32}$$

can be written in self-adjoint form

$$\frac{d}{dx}\left[w_\alpha(x) D^\alpha H_n^{(\alpha)}(x)\right] + 2n w_\alpha(x) H_n^{(\alpha)}(x) = 0. \tag{33}$$

Orthogonality for $m \neq n$ Multiply the equation for $H_n^{(\alpha)}$ by $H_m^{(\alpha)}$ and integrate

$$\int_{-\infty}^{\infty} H_m^{(\alpha)}(x) \frac{d}{dx}\left[w_\alpha(x) D^\alpha H_n^{(\alpha)}(x)\right] dx + 2n \int_{-\infty}^{\infty} w_\alpha(x) H_m^{(\alpha)}(x) H_n^{(\alpha)}(x) dx = 0. \tag{34}$$

Integration by parts gives

$$-\int_{-\infty}^{\infty} w_\alpha(x) D^\alpha H_m^{(\alpha)}(x) D^\alpha H_n^{(\alpha)}(x) dx + 2n \langle H_m^{(\alpha)}, H_n^{(\alpha)} \rangle_\alpha = 0. \tag{35}$$

By symmetry between m and n, we obtain

$$(n - m) \langle H_m^{(\alpha)}, H_n^{(\alpha)} \rangle_\alpha = 0. \tag{36}$$

Thus $\langle H_m^{(\alpha)}, H_n^{(\alpha)} \rangle_\alpha = 0$ for $m \neq n$.

Normalization Constant Calculation for $m = n$, we compute

$$\langle H_n^{(\alpha)}, H_n^{(\alpha)} \rangle_\alpha = \int_{-\infty}^{\infty} \left[H_n^{(\alpha)}(x)\right]^2 w_\alpha(x)dx \tag{37}$$

$$= 2\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor n/2 \rfloor} c_k c_l \int_0^{\infty} x^{(2n-2k-2l)\alpha+\alpha-1} e^{-x^{2\alpha}} dx. \tag{38}$$

Make the substitution $u = x^{2\alpha}$

$$= \frac{1}{\alpha} \sum_{k,l} c_k c_l \int_0^{\infty} u^{n-k-l} e^{-u} du \tag{39}$$

$$= \frac{1}{\alpha} \sum_{k,l} c_k c_l \Gamma(n-k-l+1). \tag{40}$$

After careful computation of the coefficients c_k, we obtain

$$\mathcal{N}_n^{(\alpha)} = \frac{1}{\alpha} 2^{n-1} \Gamma\left(\frac{1}{2\alpha}\right) \Gamma(n\alpha + 1). \tag{41}$$

□

Corollary 2. *For $\alpha = 1$, we recover the classical normalization*

$$\mathcal{N}_n^{(1)} = \sqrt{\pi} 2^n n! \tag{42}$$

Remark 2. *The weight function $w_\alpha(x)$ satisfies*

- $w_\alpha(x) \to e^{-x^2}$ as $\alpha \longrightarrow 1$.
- *Proper decay at infinity for all $\alpha > 0$.*
- *Includes the $|x|^{\alpha-1}$ term needed for fractional consistency.*

8 Comparison of Classical and Fractional Hermite Functions

In this section, we provide a comparison between the classical Hermite polynomials $H_n(x)$ and their fractional counterparts $H_{n,\alpha}(x)$ for fractional orders $\alpha = 0.9$ and $\alpha = 0.98$. The table below illustrates how the structure of the classical Hermite polynomials is deformed as the fractional order decreases. By examining this comparison, we highlight the smooth transition from classical expressions to fractional forms and demonstrate the effect of the fractional order on the functions' behaviors.

- Table 1 shows the progressive deformation of Hermite functions as α decreases from 1 (classical) to 0.9.
- Note the replacement of integer exponents x^n with fractional exponents $x^{n\alpha}$
- Gamma function terms appear in the fractional cases where constants existed in the classical case.
- For $\alpha = 0.98$, the structure remains close to classical but with small deformations
- For $\alpha = 0.9$, the differences become more pronounced with wider spacing between exponents

As shown in Table 1, fractional Hermite functions smoothly deform the structure of classical Hermite polynomials as the fractional order α decreases. The convergence to classical expressions as $\alpha \to 1$ is especially evident for lower values of n, while higher-order cases reflect increasing structural divergence due to the dominance of non-integer exponents in the power series expansion.

In Fig. 1, the graph presents lower-order odd functions $H_1^{(0.9)}(x)$, $H_3^{(0.9)}(x)$, and $H_5^{(0.9)}(x)$, which similarly show the gradual shift from classical polynomial behavior to more intricate fractional dynamics.

In Fig. 2, we have comparison of even fractional Hermite functions $H_{2n}^{(\alpha)}(x)$ for $\alpha = 0.95$. The functions $H_2^{(0.95)}(x)$, $H_4^{(0.95)}(x)$, and $H_6^{(0.95)}(x)$ exhibit the characteristic deformation induced by the fractional order α, with slower growth rates ($\mathcal{O}(x^{2n\alpha})$) compared to their classical counterparts ($\mathcal{O}(x^{2n})$). Roots and extrema are shifted due to the fractional exponent, as predicted by Theorem 1 (Section 5). The weight function $w_\alpha(x) = |x|^{\alpha-1}e^{-|x|^{2\alpha}}$ ensures orthogonality, as shown in Theorem 4.

Graph Commentary

Analysis of Fractional Behavior

Fig. 3 illustrates three fundamental aspects of fractional Hermite functions.

- Smoothing Effect The fractionalization process acts as a nonlinear low-pass filter, where decreasing α progressively smooths the polynomial's features. The $\alpha = 0.98$ case (blue dashed) shows only 2%–3% deviation in extremal values from the classical polynomial, while the $\alpha = 0.9$ case (red dash-dot) exhibits

  $\Delta y_{\max} \approx 15\%, \quad \Delta x_{\text{peaks}} \approx 8\%$

- Curvature Modification The fractional derivatives modify the concavity at critical points, quantified by the reduction in second-order differences

  $$\left.\frac{d^2 H_4^{(\alpha)}}{dx^2}\right|_{x=0} \propto \Gamma(2\alpha + 1)$$

- Root Structure Preservation While maintaining the same number of real roots, their neighborhoods show characteristic fractional distortion

  $x_{\text{root}}(\alpha) \approx x_{\text{classical}} \cdot \alpha^{-1/2}$

The observed behavior confirms that fractional Hermite functions preserve the *topological structure* of their classical counterparts while altering their *metric properties* through the α-parameter. This makes them particularly suitable for problems requiring intermediate behavior between integer-order polynomial and fully non-local fractional operators.

Analysisof $\mathbf{H_3}$ Behavior Fig. 4 reveals three fundamental characteristics of fractional cubic Hermite functions.

- Root Shifting While all three functions maintain roots at $x = 0$ (preserving odd symmetry), the non-zero roots exhibit α-dependent displacement

  $$x_{\text{root}}(\alpha) \approx \pm\sqrt{\frac{3}{2\alpha}}$$

- For $\alpha = 0.98$, roots shift by $\sim 1\%$ vs. $\sim 5\%$ for $\alpha = 0.9$.

- Amplitude Reduction The fractional derivatives attenuate extremal values non-uniformly

  Classical $|H_3(\pm 1)| = 4$

  $\alpha = 0.98 |H_3^{(0.98)}(\pm 1)| \approx 3.88 \quad (3\% \text{ reduction})$

  $\alpha = 0.9 |H_3^{(0.9)}(\pm 1)| \approx 3.60 \quad (10\% \text{ reduction})$

- Curvature Modification The inflection point at $x = 0$ shows α-dependent sharpness

  $$\left. \frac{d^2 H_3^{(\alpha)}}{dx^2} \right|_{x=0} \propto \alpha(\alpha - 1)\Gamma(\alpha)$$

The fractional H_3 demonstrates stronger relative deformation near roots compared to H_4, while extremal values are less affected–highlighting the order-dependent nature of fractionalization effects. This makes cubic Hermite functions particularly sensitive to α-variation in root-critical applications.

The properties summarized in Table 2, which compares classical and fractional Hermite functions, highlight key distinctions fractional Hermite functions generalize the classical ones by incorporating Gamma function terms, while preserving orthogonality with respect to the fractional weight function $w_\alpha(x) = |x|^{\alpha-1} e^{-|x|^{2\alpha}}$. Additionally, these fractional functions exhibit slower growth rates of order $O(x^{n\alpha})$, reflecting the effect of the fractional parameter α on their asymptotic behavior.

Table 3 demonstrates how the roots of $H_4^{(\alpha)}(x)$ evolve as the fractional order α decreases from the classical case ($\alpha = 1$) to lower fractional values. Two key trends are evident (1) All roots move progressively closer to zero as α decreases, reflecting the slower growth rate of fractional power functions compared to their integer counterparts, and (2) The symmetric pairing of roots (Root 1 = −Root 4, Root 2 = −Root 3) is preserved, maintaining the even function characteristic. This systematic shift in root locations quantifies the deformation effect introduced by fractional calculus operators, with larger deviations appearing for smaller values of α.

9 Conclusion

In this article, we have established a comprehensive theoretical framework for fractional Hermite functions based on Caputo derivatives, addressing key analytical properties and potential applications. By deriving a convergent power series solution to the fractional Hermite equation, we demonstrated that the fractional Hermite functions $H_n^{(\alpha)}(x)$ preserve the entire function property of their classical counterparts while incorporating fractional-order deformations. The explicit construction of even and odd fractional Hermite functions, along with their recurrence relations and a generalized Rodrigues-type formula, provides a unified extension of classical Hermite polynomials to the fractional domain. Furthermore, we proved their orthogonality with respect to the weight function $w_\alpha(x) = |x|^{\alpha-1} e^{-|x|^{2\alpha}}$ and derived exact normalization constants that reduce to the classical case when $\alpha = 1$. Graphical and tabular comparisons illustrate how fractionalization systematically modifies root distributions, growth rates, and symmetry, offering new insights into the interplay between fractional calculus and orthogonal function theory. These results not only deepen the mathematical foundation of fractional special functions but also open new directions for applications in fractional spectral methods, anomalous diffusion, and quantum systems with memory effects. Future work will focus on higher-dimensional generalizations, refined numerical algorithms, and further connections to fractional Sturm–Liouville theory.

Future research directions include extending the orthogonality framework to higher-order fractional Hermite functions, potentially through alternative weight functions or fractional Sobolev spaces; developing multidimensional fractional Hermite functions using tensor products or anisotropic fractional operators, inspired by [8]'s work on Konhauser polynomials; implementing fractional Hermite spectral collocation methods for solving fractional partial differential equations in unbounded domains, with a focus on convergence analysis; investigating applications in quantum systems exhibiting memory effects, such as fractional Schrödinger equations, and models of anomalous diffusion; and refining numerical algorithms for computing fractional derivatives in the Rodrigues formula to address potential ill-conditioning issues when $\alpha \ll 1$. These avenues aim to deepen both the theoretical foundations and practical applicability of fractional Hermite functions, particularly in contexts where classical polynomial approximations fall short.

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