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**INFORMATION**

**Keywords:**
Power Shanker distribution
bounded data modeling
simulation study
real data analysis
parameter estimation methods
Tsallis entropy

**DOI:** 10.23967/j.rimni.2026.10.83635

Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería

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## ABSTRACT

It is necessary to construct constrained families of probability distributions when dealing with data that is unit-limited by definition, such as rates, proportions, and probabilities. These models are widely used in fields such as ecology, epidemiology, reliability, and economics. To make sense of the complex real-world events, robust and adaptable statistical methods are required across all these fields. Unit power Shanker distribution (UPSD) is a novel bounded probability distribution that we introduce for this purpose. Key statistical properties, including the quantile function, reliability measures, moments, cumulative distribution function, and probability density function, are derived and studied in detail. Furthermore, the analytical links between the Tsallis entropy and other entropy measures, such as extropy, Shannon, Rényi, and Arimoto, are clarified. Maximum likelihood, Anderson-Darling, Cramér-von Mises, maximum product of spacings, least squares, and variants thereof are among the parameter estimation approaches that we examine. We assess the efficacy of several goodness-of-fit estimation methodologies for bias and mean squared error using a thorough Monte Carlo simulation study. The findings demonstrate that, in every case, the maximum likelihood estimation method outperforms the maximum product of spacings method. In various technical and scientific settings, the UPSD offers a robust and adaptable alternative for analyzing limited data.

**OPEN ACCESS**

**Received:** 07/04/2026

**Accepted:** 09/05/2026

**Published:** 21/07/2026

**DOI**
10.23967/j.rimni.2026.10.83635

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## 1 Introduction

Because they reveal the behavior of random variables (RVs), probability distributions play a crucial role in statistical and data processing. They provide the mathematical groundwork for simulating real-world phenomena, such as the durability of mechanical parts, the average wait time for a service, the average income of a population, the average amount of precipitation, and many other societal and environmental processes. With knowledge of distribution features, researchers and statisticians may make informed decisions, forecast outcomes, and assess risk. In engineering, distributions are used to evaluate the reliability of a system and the durability of a product. They help doctors predict how long someone will live. They help financiers assess market risk. Researchers in environmental science use them to calculate the probabilities of natural disasters such as floods and droughts. These applications highlight why probability distributions are crucial for transforming raw data into insights and solutions applicable across numerous scientific and industrial domains. Shanker distribution (SD) presented by Shanker [1] as a two-component blend of an exponential distribution and a gamma distribution. This mixture structure distribution has several advantageous characteristics, including form, moments, and hazard rate functions, and is used to represent lifetime data. A convex combination of the exponential distribution (with scale parameter $\eta$) and the gamma distribution (with shape parameter two and scale parameter $\eta$) makes up its probability density function (PDF). This mixed structure is thus reflected in the cumulative distribution function (CDF). Shanker [1] demonstrates, using empirical data fitting, that the SD frequently provides a superior fit relative to the exponential and Lindley distributions in particular datasets. The SD is employed to simulate datasets exhibiting non-constant, rising, or bathtub-shaped hazard rates, as observed in numerous real-data lifetime sets where simpler models prove inadequate. A discrete mixture model known as the Poisson SD, which was studied in [2], wherein the SD functions as the mixing distribution for the Poisson parameter. This facilitates modeling overdispersed count data, prevalent in ecology, genetics, and thunderstorms. In the reliability domain, researchers have used truncated-Shanker and weighted-Shanker variants to model lifetimes subject to censoring or weighting (see [3]). TSD is popular because it outperforms the exponential and Lindley distributions. This success led to extensions like the power SD (PSD). It has been utilized in various scientific and technical domains where lifetime or waiting-time statistics are present. Following this, several authors extended and applied the model in various studies (see [4–8], and [9]). The PSD, a two-parameter extension, was introduced by Shanker and Shukla [10] to improve flexibility and accommodate more intricate lifetime data. Let $X$ be a continuous RV from PSD, then the CDF is given by

$$F(x;\eta,\gamma) = 1 - \left(1 + \frac{\eta x^{\gamma}}{\eta^2 + 1}\right) e^{-\eta x^{\gamma}}, \qquad x, \gamma, \eta > 0, \tag{1}$$

where $\eta$ and $\gamma$ are the scale and shape parameters, respectively. Also, the corresponding PDF is given by

$$f(x;\eta,\gamma) = \gamma x^{\gamma-1} \frac{\eta^2}{\eta^2 + 1} (\eta + x^{\gamma}) e^{-\eta x^{\gamma}}. \tag{2}$$

Recently, statisticians have demonstrated increasing interest in the development and analysis of unit distributions characterized by a single parameter, due to their simplicity, interpretability, and extensive use in modeling bounded data, including proportions, rates, and probabilities. Within the unit interval (0, 1), these distributions offer concise and flexible models that can capture different skewness patterns. According to [11–13], and [14], a plethora of recent studies have shed light on this class of models and their extensions. We propose a novel constrained model, the unit PSD (UPSD), that is derived from this approach. A unit-interval transformation is used to obtain it from the PSD. We investigate the main properties, entropy measures, and different parameter estimation methods with full Monte Carlo simulations.

**Motivations**

The motivation for proposing the UPSD arises from both theoretical and practical considerations. The main reasons are outlined as follows:

- It has been demonstrated that the conventional Shanker and PSDs, which are defined on the positive real line $(0, \infty)$, are superior at representing dependability and longevity data. The unit interval $(0, 1)$ is the only valid range for many real-world datasets, including rates, proportions, probabilities, and normalized scores. As a result, a stripped-down version of the PSD is created.
- Traditional bounded models, such as the Beta, Topp-Leone, or Kumaraswamy distributions, may fail to capture skewed or heavy-tailed behavior in some cases. As an alternative model, the UPSD can display a variety of density and hazard rate function shapes due to its shape and scale characteristics.
- By employing a monotonic transformation $X = e^{-T}$ on the power Shanker RV $T$, the resultant model preserves the analytical tractability of the original distribution while conforming to the unit interval. This method guarantees mathematical sophistication and clarity of interpretation.
- Reliability, PDF, and CDF are all defined explicitly in the UPSD. It applies to theoretical and practical studies alike due to the analytical calculability of its moments, quantile function, and entropy measures.
- The information and uncertainty in the suggested model can be better understood by combining Tsallis entropy with related measures like Shannon, Rényi, Arimoto, and extropy. This approach establishes a connection to statistical physics and complex systems.
- Multiple estimation techniques, including maximum likelihood, Anderson-Darling, Cramér-von Mises, maximum product of spacings, and least squares, are analyzed and compared to provide comprehensive inferential methodologies.
- The practical value and durability of the suggested distribution are demonstrated through extensive Monte Carlo simulation experiments that evaluate the finite-sample efficacy of different estimators.
- For data that frequently displays skewness or unusual tail behavior, the UPSD can be used to depict constrained phenomena in reliability engineering, hydrology, economics, biology, and the social sciences.

The remainder of this paper is organized as follows. Section 2 introduces the formulation of the new UPSD and derives its cumulative and probability density functions. Section 3 presents the main statistical properties of the proposed model, including its moments, variance, skewness, kurtosis, and quantile function. In Section 4, several reliability measures, including the survival function, hazard rate, and cumulative hazard function are derived and analyzed. Section 5 is devoted to the study of information-theoretic measures, where the Tsallis entropy and related entropies, such as Shannon, Rényi, Arimoto, and extropy, are obtained and discussed. Section 6 develops a comprehensive set of parameter estimation methods encompassing classical and modern approaches such as maximum Likelihood, Anderson–Darling, Cramér–von Mises, maximum product of spacings, and least squares estimators. Section 7 presents an extensive Monte Carlo simulation study that evaluates the performance of these estimators in terms of bias, mean squared error, and goodness-of-fit statistics. Finally, Section 8 concludes the paper with remarks on the model's flexibility, theoretical importance, and potential applications of the proposed UPSD model in various fields. Concluding remarks are provided in Section 9.

## 2 Formulation of the New Model

In many applications, the observed data are constrained to the unit interval (0, 1), including proportions, rates, normalized scores, and probabilities. A prevalent method for obtaining flexible unit-interval models involves transforming a well-known positive support lifetime distribution via a monotonic mapping, such as $x = e^{-t}$. Inspired by the versatility of the PSD on the interval $(0, \infty)$, which integrates exponential and gamma-like elements and accommodates a diverse array of shapes, we introduce the UPSD by employing the transformation $X = e^{-T}$, where $T$ is a RV that follows the PSD. This transformation converts long-tailed or skewed lifespan distributions into bounded unit-interval behavior while maintaining the shape-controlling parameters of the original model. The resulting two-parameter family is appropriate for modeling bounded data that may display skewness and non-standard tail behavior. Subsequently, we calculate the CDF and the PDF of the UPSD.

**Theorem 1:** *Let $X$ be a continuous RV from $USPD(\eta, \gamma)$. Then the CDF of $X$ is defined as*

$$F(x; \eta, \gamma) = \frac{e^{-\eta(-\log x)^{\gamma}}\,(1 + \eta\,(\eta + (-\log x)^{\gamma}))}{1 + \eta^2}, \eta, \gamma > 0, 0 < x < 1. \tag{3}$$

**Proof:** We have

$$F_X(x) = P(X \le x) = P(e^{-T} \le x) = P(T \ge -\log x) = 1 - P(T < -\log x) = 1 - F_T(-\log x).$$

Using the CDF of the PSD (1) gives

$$F_X(x) = 1 - \left[1 - \left(1 + \frac{\eta(-\log x)^{\gamma}}{1 + \eta^2}\right) e^{-\eta(-\log x)^{\gamma}}\right] = \frac{\left(1 + \eta^2 + \eta(-\log x)^{\gamma}\right) e^{-\eta(-\log x)^{\gamma}}}{1 + \eta^2}$$
$$= \frac{[1 + \eta(\eta + (-\log x)^{\gamma})]\, e^{-\eta(-\log x)^{\gamma}}}{1 + \eta^2}.$$

This completes the proof. □

**Theorem 2:** *The corresponding PDF of the UPSD is obtained as*

$$f(x; \eta, \gamma) = \frac{e^{-\eta(-\log x)^{\gamma}}\ \gamma\eta^2\ (\eta + (-\log x)^{\gamma})\,(-\log x)^{-1+\gamma}}{x\,(1 + \eta^2)} \tag{4}$$

**Proof:** Using the transformation $X = e^{-T}$ (so $t = -\log x$ and $|dt/dx| = 1/x$), the PDF of $X$ is

$$f_X(x) = f_T(-\log x)\left|\frac{dt}{dx}\right| = \frac{\gamma\,\eta^2\left(\eta + (-\log x)^{\gamma}\right)(-\log x)^{\gamma-1}\, e^{-\eta(-\log x)^{\gamma}}}{(1 + \eta^2)\,x}, \qquad 0 < x < 1. \tag{5}$$

This completes the proof. □

**Proposition 1** (Monotonicity and PDF positivity): *The PDF $f(x; \eta, \gamma)$ of UPSD, which is represented in (4), is strictly positive for every $x \in (0, 1)$. Consequently, $F_X$ is strictly increasing on* $(0, 1)$.

**Proof:** For $x \in (0,1)$, it follows that $-\log x > 0$, where $\eta > 0$ and $\gamma > 0$. All components in the numerator of $f_X(x)$ are positive, and the denominator $(1+\eta^2)x$ is likewise positive; hence, $f_X(x) > 0$ for all $x \in (0,1)$. The positivity of the PDF signifies that $F_X'(x) = f_X(x) > 0$, thereby confirming that $F_X$ is strictly increasing on the interval $(0, 1)$. □

**Proposition 2** (Boundary behaviour of the PDF): *The one-sided limits of the PDF at the boundaries depend on $\gamma$ as follows:*

$$\lim_{x\to 0^+} f_X(x) = 0,$$

$$\lim_{x\to 1^-} f_X(x) = \begin{cases} 0, & \text{if } \gamma > 1, \\ \dfrac{\gamma\eta^3}{1+\eta^2}, & \text{if } \gamma = 1, \\ +\infty, & \text{if } 0 < \gamma < 1. \end{cases}$$

*Hence the PDF vanishes at the left endpoint 0, while at the right endpoint 1 the PDF is finite only when $\gamma \geq 1$.*

**Proof:** Write $t = -\log x > 0$. Then $x \to 0^+$ corresponds to $t \to \infty$. Using the form

$$f_X(x) = \frac{\gamma\eta^2(\eta + t^\gamma)t^{\gamma-1}e^{-\eta t^\gamma}}{(1+\eta^2)\,e^{-t}}.$$

If $\gamma > 1$ then $t^{\gamma-1} \to 0$ so $f_X(1^-) = 0$. If $\gamma = 1$ the factor becomes constant and $\lim_{x\to 1^-} f_X(x) = \gamma\eta^3/(1+\eta^2)$. If $0 < \gamma < 1$ then $t^{\gamma-1} \to +\infty$, so $f_X(x) \to +\infty$ as $x \to 1^-$. This proves the claim. □

Fig. 1 depicts the graphical characteristics of the PDF and CDF of the proposed UPSD for different combinations of the shape parameter $\gamma$ and scale parameter $\eta$. Fig. 1 illustrates the PDF curves of the UPSD for various parameter values. The distribution displays a diverse array of shapes, including right-, left-, and almost symmetric forms, depending on the parameter combinations. As $\gamma$ increases, the apex of the PDF shifts to the right, resulting in a distribution that gets more concentrated around elevated values of $x$. More scattered and right-skewed distributions are the result of lower $\gamma$ values. Distribution steepness and tail characteristics are controlled by the scale parameter $\eta$. The UPSD's flexibility demonstrates its ability to suitably characterize constrained data with varying degrees of skewness and kurtosis. As can be seen, the matching CDF curves for similar parameter configurations in Fig. 1. The concept is backed by the fact that all curves on $(0, 1)$ are consistently monotone. Near the origin, lower $\gamma$ values cause a gradual growth, whereas near the upper end of the unit interval, higher values hasten the accumulation of probability mass. For datasets with varying growth patterns and rates of cumulative probability concentration, this demonstrates that the UPSD can replicate them.

## 3 Statistical Properties

### *3.1 Moment*

Let $X$ be a RV following the UPSD. Then, The $r$-th moment of $X$ about the origin is defined by

$$E[X^r] = \int_0^1 x^r f_X(x)\,dx.$$

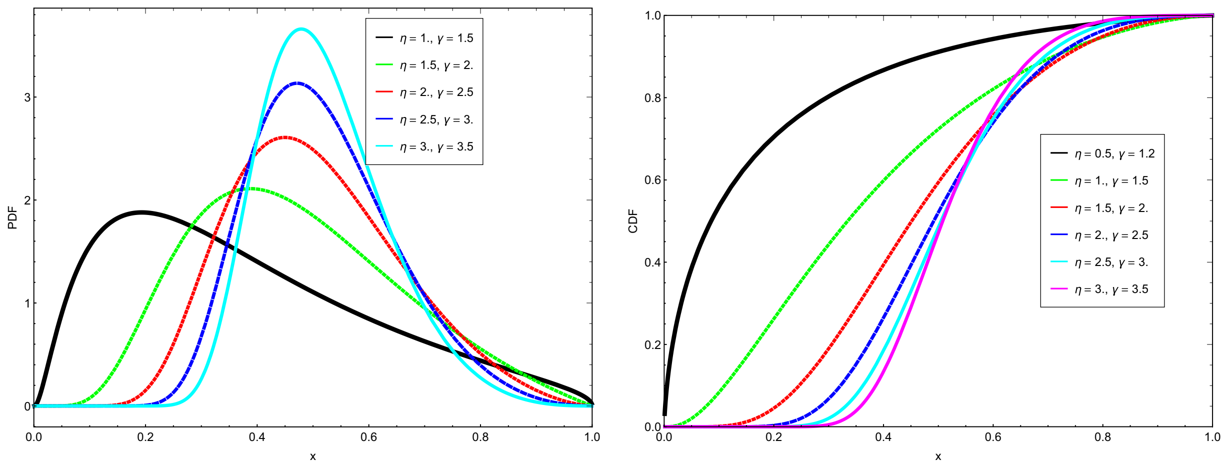


**Figure 1:** The PDF and CDF curves of the UPSD

Applying the transformation $x = e^{-t}$, we obtain

$$E[X^r] = \int_0^1 x^r \frac{f_T(-\log x)}{x}\, dx = \int_\infty^0 e^{-rt} f_T(t)\,(-dt) = \int_0^\infty e^{-rt} f_T(t)\, dt. \tag{6}$$

From (2) and (6), we have

$$E[X^r] = \frac{\gamma\eta^2}{1+\eta^2}\left\{\eta\int_0^\infty t^{\gamma-1}e^{-\eta t^\gamma - rt}\, dt + \int_0^\infty t^{2\gamma-1}e^{-\eta t^\gamma - rt}\, dt\right\}.$$

Taking $u = t^\gamma$. Then

$$\int_0^\infty t^{k\gamma-1}e^{-\eta t^\gamma - rt}\, dt = \frac{1}{\gamma}\int_0^\infty u^{k-1}e^{-\eta u - ru^{1/\gamma}}\, du, \qquad k = 1, 2.$$

Expanding $e^{-ru^{1/\gamma}}$ via its Maclaurin series (which converges absolutely for all $u > 0$), we obtain

$$\int_0^\infty t^{k\gamma-1}e^{-\eta t^\gamma - rt}\, dt = \frac{1}{\gamma}\sum_{n=0}^\infty \frac{(-r)^n}{n!}\int_0^\infty u^{k-1+\frac{n}{\gamma}}e^{-\eta u}\, du = \frac{1}{\gamma}\sum_{n=0}^\infty \frac{(-r)^n}{n!}\,\eta^{-\left(k+\frac{n}{\gamma}\right)}\Gamma\left(k+\frac{n}{\gamma}\right).$$

The series converges absolutely for every $u > 0$, and the partial sums are bounded by an integrable dominating function. Specifically, $\left|e^{-ru^{1/\gamma}}\right| \le e^{|r|u^{1/\gamma}}$, and for any fixed $r$, there exists $N$ such that the tail of the series is dominated by a convergent geometric series. Then

$$E[X^r] = \mu_r = \frac{\eta^2}{1+\eta^2}\sum_{n=0}^\infty \frac{(-r)^n}{n!}\,\eta^{-\frac{n}{\gamma}}\left\{\Gamma\left(1+\frac{n}{\gamma}\right) + \eta^{-2}\Gamma\left(2+\frac{n}{\gamma}\right)\right\}. \tag{7}$$

Expression (7) denotes the $r$-th raw moment of the UPSD as a convergent infinite series, which can be numerically evaluated for any fixed values of $(\eta, \gamma, r)$. For the first moment ($r = 1$), the mean of the UPSD is given by

$$E[X] = \mu_1 = \frac{\eta^2}{1+\eta^2}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\,\eta^{-\frac{n}{\gamma}}\left\{\Gamma\left(1+\frac{n}{\gamma}\right) + \eta^{-2}\Gamma\left(2+\frac{n}{\gamma}\right)\right\}.$$

The variance and coefficient of variation (CV) are defined as

$$\mathrm{Var}(X) = \sigma^2 = \mu_2 - \mu_1^2, \quad CV = \frac{\sigma}{\mu}.$$

The Pearson coefficient of skewness, represented by $\kappa_1$, is articulated in relation to the initial three moments as

$$\kappa_1 = \frac{E[(X-\mu_1)^3]}{\mathrm{Var}(X)^{3/2}} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1^2)^{3/2}}.$$

The excess kurtosis (Fisher's definition), denoted by $\kappa_2$, is

$$\kappa_2 = \frac{E[(X-\mu_1)^4]}{\mathrm{Var}(X)^2} - 3 = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2} - 3. \quad (8)$$

In Table 1, the UPSD's initial four moments, variance, skewness, kurtosis, and coefficient of variation are shown for different parameter combinations of $\eta$ and $\gamma$. These findings provide a complete picture of how distribution parameters affect form and dispersion. Both mean and variance are affected by $\eta$ and $\gamma$. As $\gamma$ rises and $\eta$ stays constant, the mean shifts upward, indicating a bigger probability mass concentration at the top of the unit interval. Conversely, increasing $\gamma$ decreases variance, indicating less dispersion and higher density. Changing $\eta$ affects the distribution's magnitude and spread, enabling better modeling of bounded datasets. Skewness results show that the UPSD can handle positively and negatively skewed distributions depending on parameter settings. Lower $\gamma$ values result in strong right-skewness, while higher values result in more symmetric or slightly left-skewed distributions. Kurtosis results indicate that the distribution may be leptokurtic (heavy-tailed) or platykurtic (light-tailed), proving its adaptability. The coefficient of variation decreases as $\gamma$ increases, showing greater concentration around the mean.

**Table 1:** Numerical values of moments, variance, skewness ($\kappa_1$), kurtosis ($\kappa_2$), and coefficient of variation (CV) for the UPSD with different parameter values

| $\gamma$ | $\eta$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma^2$ | $\kappa_1$ | $\kappa_2$ | CV |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.5 | 0.134135 | 0.0889184 | 0.0698279 | 0.0588569 | 0.0709262 | 2.05799 | 5.96744 | 1.98546 |
| 0.5 | 0.8 | 0.287917 | 0.207102 | 0.1695 | 0.146709 | 0.124206 | 0.876084 | 2.197 | 1.22406 |
| 0.5 | 1. | 0.38641 | 0.289318 | 0.24162 | 0.211841 | 0.140005 | 0.412797 | 1.56593 | 0.968331 |
| 0.5 | 1.2 | 0.473266 | 0.366116 | 0.310881 | 0.27548 | 0.142135 | 0.0573945 | 1.40964 | 0.79661 |
| 0.5 | 1.5 | 0.579424 | 0.466146 | 0.403955 | 0.36269 | 0.130413 | −0.366699 | 1.60498 | 0.623252 |
| 0.5 | 2. | 0.703149 | 0.593408 | 0.527713 | 0.481949 | 0.0989894 | −0.923086 | 2.52132 | 0.447453 |
| 0.5 | 2.5 | 0.7823 | 0.682999 | 0.619268 | 0.573039 | 0.0710051 | −1.3814 | 3.8802 | 0.340621 |
| 0.5 | 3. | 0.834747 | 0.747263 | 0.687816 | 0.64319 | 0.0504611 | −1.78241 | 5.55716 | 0.269106 |
| 0.8 | 0.5 | 0.144313 | 0.0762645 | 0.0520587 | 0.0397318 | 0.0554381 | 1.91923 | 5.82724 | 1.63154 |
| 0.8 | 0.8 | 0.288912 | 0.174737 | 0.127302 | 0.101096 | 0.091267 | 0.873422 | 2.47183 | 1.04566 |
| 0.8 | 1. | 0.376917 | 0.242824 | 0.182429 | 0.147605 | 0.100758 | 0.467425 | 1.87134 | 0.842157 |
| 0.8 | 1.2 | 0.452734 | 0.306483 | 0.236006 | 0.193865 | 0.101516 | 0.164825 | 1.68395 | 0.703758 |
| 0.8 | 1.5 | 0.544107 | 0.389945 | 0.309229 | 0.258707 | 0.0938921 | −0.177883 | 1.7487 | 0.563158 |
| 0.8 | 2. | 0.650647 | 0.498206 | 0.409627 | 0.350755 | 0.0748648 | −0.583426 | 2.22673 | 0.420527 |
| 0.8 | 2.5 | 0.720578 | 0.577193 | 0.487257 | 0.424646 | 0.0579604 | −0.873281 | 2.85659 | 0.334106 |

(Continued)

**Table 1 (continued)**

| $\gamma$ | $\eta$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma^2$ | $\kappa_1$ | $\kappa_2$ | CV |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 3. | 0.768959 | 0.636403 | 0.548249 | 0.484552 | 0.0451055 | −1.09468 | 3.51434 | 0.276192 |
| 1. | 0.5 | 0.155556 | 0.072 | 0.044898 | 0.0320988 | 0.0478025 | 1.80129 | 5.6274 | 1.40553 |
| 1. | 0.8 | 0.293887 | 0.161274 | 0.109182 | 0.0819783 | 0.0749045 | 0.866247 | 2.64243 | 0.931264 |
| 1. | 1. | 0.375 | 0.222222 | 0.15625 | 0.12 | 0.0815972 | 0.502769 | 2.07249 | 0.761739 |
| 1. | 1.2 | 0.443842 | 0.278945 | 0.202074 | 0.158017 | 0.0819487 | 0.235398 | 1.86796 | 0.644974 |
| 1. | 1.5 | 0.526154 | 0.353218 | 0.264957 | 0.211697 | 0.0763803 | −0.0599489 | 1.86009 | 0.525264 |
| 1. | 2. | 0.622222 | 0.45 | 0.352 | 0.288889 | 0.0628395 | −0.393606 | 2.1408 | 0.402876 |
| 1. | 2.5 | 0.686137 | 0.521499 | 0.420348 | 0.351969 | 0.0507152 | −0.618731 | 2.51607 | 0.328215 |
| 1. | 3. | 0.73125 | 0.576 | 0.475 | 0.404082 | 0.0412734 | −0.782631 | 2.89061 | 0.277824 |
| 1.2 | 0.5 | 0.168678 | 0.0698251 | 0.0399338 | 0.0267657 | 0.0413728 | 1.68721 | 5.44099 | 1.20586 |
| 1.2 | 0.8 | 0.3 | 0.151849 | 0.0959665 | 0.0682279 | 0.0618491 | 0.864831 | 2.81456 | 0.828984 |
| 1.2 | 1. | 0.374475 | 0.206931 | 0.136782 | 0.0998713 | 0.0666991 | 0.54203 | 2.27045 | 0.689663 |
| 1.2 | 1.2 | 0.436945 | 0.25786 | 0.176471 | 0.13158 | 0.0669397 | 0.306159 | 2.04886 | 0.592128 |
| 1.2 | 1.5 | 0.511242 | 0.324335 | 0.230982 | 0.17652 | 0.0629663 | 0.0497598 | 1.98095 | 0.490826 |
| 1.2 | 2. | 0.598155 | 0.411079 | 0.306795 | 0.241633 | 0.0532894 | −0.231201 | 2.12237 | 0.385928 |
| 1.2 | 2.5 | 0.656643 | 0.475669 | 0.366887 | 0.295445 | 0.044489 | −0.413986 | 2.34243 | 0.321216 |
| 1.2 | 3. | 0.698564 | 0.525469 | 0.415492 | 0.340451 | 0.0374773 | −0.543366 | 2.56266 | 0.277127 |
| 1.5 | 0.5 | 0.189183 | 0.0691443 | 0.0351206 | 0.0214232 | 0.033354 | 1.54637 | 5.25995 | 0.965364 |
| 1.5 | 0.8 | 0.309076 | 0.142606 | 0.0821372 | 0.0540029 | 0.0470776 | 0.877166 | 3.07567 | 0.702007 |
| 1.5 | 1. | 0.374525 | 0.190595 | 0.115928 | 0.0787663 | 0.0503266 | 0.606523 | 2.55724 | 0.598988 |
| 1.5 | 1.2 | 0.428788 | 0.234487 | 0.148619 | 0.103567 | 0.0506283 | 0.408715 | 2.31206 | 0.524752 |
| 1.5 | 1.5 | 0.493068 | 0.291428 | 0.193419 | 0.138766 | 0.0483114 | 0.19619 | 2.17645 | 0.445777 |
| 1.5 | 2. | 0.568607 | 0.365682 | 0.255843 | 0.190029 | 0.0423678 | −0.0306351 | 2.18179 | 0.361998 |
| 1.5 | 2.5 | 0.620145 | 0.421358 | 0.305694 | 0.232796 | 0.0367779 | −0.173581 | 2.26745 | 0.309244 |
| 1.5 | 3. | 0.65771 | 0.464763 | 0.346438 | 0.268965 | 0.0321812 | −0.272448 | 2.36498 | 0.272751 |
| 2. | 0.5 | 0.219777 | 0.0717846 | 0.0311455 | 0.0164457 | 0.0234828 | 1.40252 | 5.20487 | 0.697257 |
| 2. | 0.8 | 0.321314 | 0.134501 | 0.0684071 | 0.0399964 | 0.0312583 | 0.923301 | 3.49621 | 0.550241 |
| 2. | 1. | 0.374743 | 0.173724 | 0.0943917 | 0.0575992 | 0.0332913 | 0.714247 | 2.99867 | 0.486891 |
| 2. | 1.2 | 0.418637 | 0.209003 | 0.119218 | 0.075108 | 0.0337462 | 0.559149 | 2.72473 | 0.438808 |
| 2. | 1.5 | 0.470564 | 0.254316 | 0.152965 | 0.0998593 | 0.0328851 | 0.393028 | 2.52029 | 0.385372 |
| 2. | 2. | 0.532045 | 0.313185 | 0.199816 | 0.135892 | 0.0301128 | 0.219212 | 2.40456 | 0.326157 |
| 2. | 2.5 | 0.574664 | 0.357542 | 0.237354 | 0.166101 | 0.0273031 | 0.112558 | 2.38304 | 0.287536 |
| 2. | 3. | 0.606286 | 0.392466 | 0.268276 | 0.191857 | 0.0248826 | 0.0401204 | 2.38942 | 0.260178 |
| 2.5 | 0.5 | 0.243556 | 0.0762713 | 0.0297824 | 0.0139508 | 0.0169519 | 1.33592 | 5.31009 | 0.534578 |
| 2.5 | 0.8 | 0.329955 | 0.130803 | 0.0608118 | 0.0322397 | 0.0219324 | 0.978848 | 3.87505 | 0.448837 |
| 2.5 | 1. | 0.374614 | 0.163737 | 0.0817668 | 0.045593 | 0.0234015 | 0.808803 | 3.38884 | 0.408355 |
| 2.5 | 1.2 | 0.411178 | 0.192972 | 0.101517 | 0.0587371 | 0.0239043 | 0.6801 | 3.09844 | 0.376017 |
| 2.5 | 1.5 | 0.454468 | 0.230209 | 0.128107 | 0.077176 | 0.0236674 | 0.541772 | 2.8582 | 0.33851 |
| 2.5 | 2. | 0.506023 | 0.278369 | 0.164768 | 0.103865 | 0.0223095 | 0.39859 | 2.6826 | 0.295172 |
| 2.5 | 2.5 | 0.542128 | 0.314696 | 0.194086 | 0.126204 | 0.0207928 | 0.312052 | 2.61206 | 0.265984 |
| 2.5 | 3. | 0.569207 | 0.343425 | 0.218286 | 0.145282 | 0.0194291 | 0.253937 | 2.57829 | 0.244881 |

(Continued)

**Table 1** (continued)

| $\gamma$ | $\eta$ | $\mu_1$ | $\mu_2$ | $\mu_3$ | $\mu_4$ | $\sigma^2$ | $\kappa_1$ | $\kappa_2$ | CV |
|---|---|---|---|---|---|---|---|---|---|
| 3. | 0.5 | 0.261548 | 0.081038 | 0.0296595 | 0.0126786 | 0.0126309 | 1.30819 | 5.4655 | 0.429701 |
| 3. | 0.8 | 0.336097 | 0.129081 | 0.0563296 | 0.0276153 | 0.0161199 | 1.03106 | 4.20474 | 0.37776 |
| 3. | 1. | 0.374275 | 0.157337 | 0.073813 | 0.0382442 | 0.0172545 | 0.886804 | 3.72794 | 0.350961 |
| 3. | 1.2 | 0.405494 | 0.182163 | 0.0900836 | 0.0485859 | 0.017738 | 0.77553 | 3.42939 | 0.328449 |
| 3. | 1.5 | 0.442498 | 0.213573 | 0.111783 | 0.0629632 | 0.0177686 | 0.655538 | 3.169 | 0.301242 |
| 3. | 2. | 0.486756 | 0.25402 | 0.14147 | 0.0836091 | 0.0170884 | 0.532187 | 2.96307 | 0.268559 |
| 3. | 2.5 | 0.517963 | 0.284509 | 0.165119 | 0.10081 | 0.0162233 | 0.45861 | 2.86768 | 0.245907 |
| 3. | 3. | 0.541531 | 0.308667 | 0.184628 | 0.115482 | 0.0154113 | 0.40958 | 2.81403 | 0.229243 |

### 3.2 *Quantile Function*

Let $t = -\log x > 0$ and $s = t^\gamma$ in the CDF (3). For $u \in (0,1)$, the quantile function $Q(u) = F_X^{-1}(u)$ is obtained by solving

$$\frac{\left(1+\eta^2+\eta s\right)e^{-\eta s}}{1+\eta^2} = u.$$

By putting $y = \eta s$ yields

$$\left(1+\eta^2+y\right)e^{-y} = (1+\eta^2)\,u.$$

Let

$$z = y + (1+\eta^2) \quad \Longrightarrow \quad ze^{-z} = (1+\eta^2)\,u\,e^{-(1+\eta^2)}.$$

Hence

$$z = -W\Big(-(1+\eta^2)\,u\,e^{-(1+\eta^2)}\Big),$$

where $W$ denotes the Lambert $W$ function. Since $z = y + (1+\eta^2) \geq 1+\eta^2 > 1$, the appropriate branch is $W_{-1}$ (the lower real branch), and therefore

$$z = -W_{-1}\Big(-(1+\eta^2)\,u\,e^{-(1+\eta^2)}\Big).$$

Therefore, the quantile function is

$$Q(u) = \exp\left\{-\left(\frac{-W_{-1}\big(-(1+\eta^2)\,u\,e^{-(1+\eta^2)}\big)-(1+\eta^2)}{\eta}\right)^{1/\gamma}\right\}, \qquad 0 < u < 1.$$

For $u \to 0^+$, $W_{-1}(\zeta) \to -\infty$, so the inner bracket $\to \infty$, giving $Q(u) \to 0$. For $u \to 1^-$, $\zeta \to -(1+\eta^2)e^{-(1+\eta^2)}$, and one can check that $Q(u) \to 1$. Monotonicity follows from the strict increase of $W_{-1}$ on $(-1/e, 0)$. Hence, $Q(u)$ is a proper quantile function.

## 4 Reliability Measure

Reliability analysis is an essential application of lifespan and bounded–support models. For the UPSD, numerous reliability measures may be calculated in closed form from its probability and cumulative distribution functions. Let $X \sim \text{UPSD}(\eta,\gamma)$. Then the survival (reliability) function

$\bar{F}_X(x) = 1 - F_X(x)$ is

$$\bar{F}(x;\eta,\gamma) = 1 - \frac{\left(1+\eta(\eta+(-\log x)^{\gamma})\right) e^{-\eta(-\log x)^{\gamma}}}{1+\eta^2} = \frac{1}{1+\eta^2}\Big[(1+\eta^2) - (1+\eta(\eta+(-\log x)^{\gamma}))e^{-\eta(-\log x)^{\gamma}}\Big]. \tag{9}$$

The hazard rate (or failure rate) of $X$ is defined as

$$h(x;\eta,\gamma) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\gamma\eta^2\,(\eta+(-\log x)^{\gamma})\,(-\log x)^{\gamma-1}\,e^{-\eta(-\log x)^{\gamma}}}{x\left[(1+\eta^2) - (1+\eta(\eta+(-\log x)^{\gamma}))e^{-\eta(-\log x)^{\gamma}}\right]}. \tag{10}$$

The cumulative hazard function is defined by

$$H_X(x) = -\log\left(\bar{F}_X(x)\right) = -\log\left[\frac{(1+\eta^2) - (1+\eta(\eta+(-\log x)^{\gamma}))e^{-\eta(-\log x)^{\gamma}}}{1+\eta^2}\right]. \tag{11}$$

As $x \to 0^+$, $-\log x \to \infty$ and $e^{-\eta(-\log x)^{\gamma}} \to 0$, so $\bar{F}_X(x) \to 1$ and $h_X(x) \to 0$, indicating high reliability near the lower bound. As $x \to 1^-$, $-\log x \to 0^+$ and the survival function tends to 0, implying $h_X(x) \to \infty$, corresponding to rapid failure near the upper limit of support.

Fig. 2 depicts the reliability function of the UPSD. As anticipated, all curves exhibit a monotonically decreasing trend from around 1 to 0 as $x$ increases, validating that the model satisfies the essential reliability feature. The rate of decrease fluctuates based on the parameters $\eta$ and $\gamma$. Increased values of $\eta$ and $\gamma$ result in a more rapid decline, signifying a greater likelihood of failure in the upper range of $x$, while diminished parameter values are associated with slower decay, hence indicating more dependable systems or processes. The UPSD may replicate components or materials with different aging or degradation characteristics within prescribed lifespans due to its adaptability. Fig. 2 shows the same hazard rate function h(x) for identical parameter values. Hazard rate curves can take on several shapes, such as increasing, decreasing, or bathtub-like, depending on the interplay between $\eta$ and $\gamma$. For small $\gamma$, the hazard function starts low and rises, implying early reliability followed by rapid failure. High $\gamma$ values result in a significant increase in hazard rate toward the upper limit, indicating a rapid decrease as the interval ends. The UPSD can represent a range of real-world reliability phenomena, including early-life stability and wear-out systems, as it captures distinct hazard-rate behaviors.

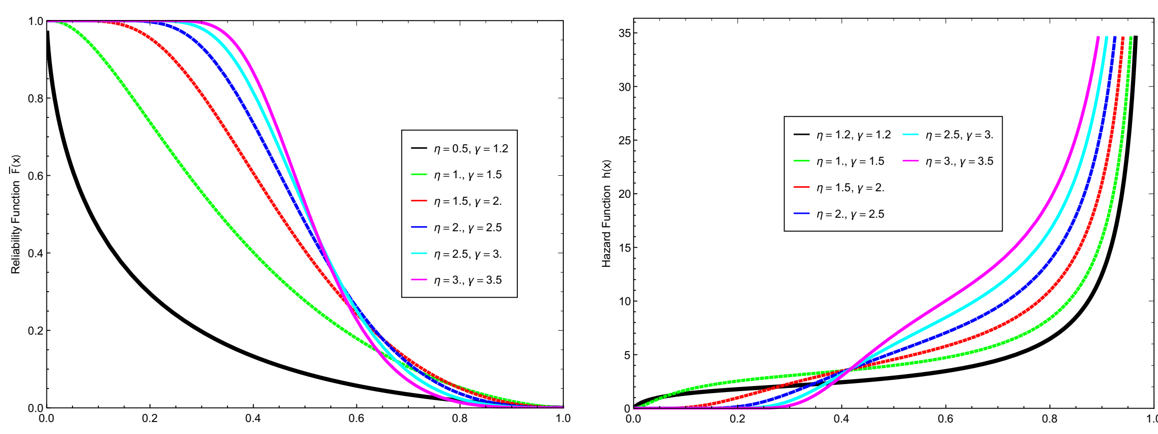


**Figure 2:** The reliability curves of the UPSD

## 5 Tsallis Entropy and Related Entropies

Entropy is fundamental in information theory, physics, and statistical modeling, serving as a metric for the uncertainty or information content of a random variable. The classical measure, introduced by Shannon [15], is extensively utilized owing to its simplicity and robust theoretical underpinnings. Nonetheless, for intricate or non-extensive systems, Shannon entropy may be insufficient to characterize the fundamental uncertainty framework. To address this constraint, Tsallis [16] proposed a generalized entropy that includes a non-additivity parameter, thus offering a more adaptable framework for delineating uncertainty in nonlinear and coupled systems. Subsequently, various alternative entropy measures have been introduced, such as Rényi entropy (cf. [17]), Arimoto entropy (cf. [18]), and extropy (cf. [19]), each providing unique insights into information diversity and the quantification of uncertainty. These entropy metrics have extensive applications in physics, engineering, biology, and statistical inference. Examining the entropic properties of probability distributions yields profound insights into the model's informational content and structural intricacy. Consequently, in this section, we formulate the equations for the Tsallis entropy and the associated entropies for the proposed UPSD to enhance our understanding of its information-theoretic characteristics.

***Tsallis Entropy***

The Tsallis entropy of order $q \in \mathbb{R} \setminus \{1\}$ for a continuous RV with density $f(x)$ is defined as

$$S_q(f) = \frac{1 - \displaystyle\int [f(x)]^q \, dx}{q-1} = \frac{1 - I_q}{q-1}.$$

According to the PDF (4) of UPSD. Using the transformation $x = e^{-t}$ ($t = -\log x$, $dx = -e^{-t}dt$). Then

$$I_q = \int_0^\infty f_T(t)^q e^{-t(1-q)} \, dt \gamma^q \left(\frac{\eta^2}{1+\eta^2}\right)^q \int_0^\infty t^{q(\gamma-1)} (\eta + t^\gamma)^q e^{-q\eta t^\gamma} e^{-t(1-q)} \, dt.$$

Expanding $(\eta+t^\gamma)^q$ via the generalized binomial series and $e^{-t(1-q)}$ by its Maclaurin expansion, then integrating term-by-term using the Gamma function, yields a convergent double series representation:

$$I_q = \gamma^{q-1} \left(\frac{\eta^2}{1+\eta^2}\right)^q \sum_{k=0}^\infty \binom{q}{k} \eta^{q-k} \sum_{n=0}^\infty \frac{(-1)^n (1-q)^n}{n!} \frac{\Gamma\left(q + k - \frac{q-1}{\gamma} + \frac{n}{\gamma}\right)}{(q\eta)^{q+k-\frac{q-1}{\gamma}+\frac{n}{\gamma}}}.$$

Finally, the Tsallis entropy of the UPSD is given by

$$S_q(f) = \frac{1 - \gamma^{q-1} \left(\frac{\eta^2}{1+\eta^2}\right)^q \sum_{k=0}^\infty \binom{q}{k} \eta^{q-k} \sum_{n=0}^\infty \frac{(-1)^n (1-q)^n}{n!} \frac{\Gamma\left(q + k - \frac{q-1}{\gamma} + \frac{n}{\gamma}\right)}{(q\eta)^{q+k-\frac{q-1}{\gamma}+\frac{n}{\gamma}}}}{q-1}$$

For integer $q$, the outer sum truncates, simplifying computation. As $q \to 1$, $S_q$ converges to the Shannon entropy

$$S_1 = -\int_0^1 f_X(x) \log f_X(x) \, dx.$$

**Remark 1:** *(Relations between Tsallis Entropy and Other Information Measures) As a generalization of the traditional entropy measures, the Tsallis entropy is crucial. The majority of well-known entropies can be acquired as limiting or altered situations from its unified framework. The non-extensive parameter q, which regulates the degree of non-additivity and sensitivity to the probability of extreme events, is what determines this measure.*

*1. When $q \to 1$, the Tsallis entropy reduces to the classical Shannon entropy,*

$$\lim_{q\to 1} S_q(X) = -\int f(x)\,\ln f(x)\,dx = H(X).$$

*Hence, the Shannon entropy represents the extensive (additive) limit of Tsallis entropy.*

*2. The Rényi entropy of order q is defined as*

$$R_q(X) = \frac{1}{1-q}\,\ln\left(\int [f(x)]^q\,dx\right).$$

*Comparing with Tsallis entropy, we have the functional relationship*

$$R_q(X) = \frac{1}{1-q}\,\ln\left[\,1-(q-1)S_q(X)\,\right],$$

*and conversely,*

$$S_q(X) = \frac{e^{(1-q)R_q(X)}-1}{1-q}.$$

*Therefore, Renyi and Tsallis entropies are monotonic transformations of each other and contain equivalent information.*

*3. The Arimoto entropy of order q is given by*

$$A_q(X) = \frac{1}{1-q}\left[\left(\int [f(x)]^q\,dx\right)^{\frac{1-q}{q}} - 1\right].$$

*It can be expressed in terms of Tsallis entropy as*

$$A_q(X) = \frac{\left[\,1-(q-1)S_q(X)\,\right]^{\frac{1-q}{q}} - 1}{1-q}.$$

*Both Arimoto and Tsallis entropies share the same non-extensive parameter q and converge to Shannon entropy as $q \to 1$.*

*4. Extropy is considered a complementary dual of entropy, defined as*

$$J(X) = -\frac{1}{2}\int [f(x)]^2\,dx.$$

*It is directly related to Tsallis entropy evaluated at $q = 2$, namely*

$$S_2(X) = 1 - \int [f(x)]^q\,dx \quad \Rightarrow \quad J(X) = \frac{1}{2}\big(S_2(X) - 1\big).$$

*Hence, the extropy corresponds to a specific (quadratic) case of the Tsallis entropy family, emphasizing the concentration rather than the spread of probability mass.*

## 6 Estimation Methods

### 6.1 Method of Maximum Likelihood

The MLE approach [20] usually provides a reliable framework for estimating the parameters for a given distribution based on observed data. Let $x_1, x_2, \ldots, x_n$ be a random sample from the UPSD with parameters $\eta > 0$ and $\gamma > 0$. The corresponding PDF of the UPSD is given by Eq. (4). The likelihood function, $L(\eta, \gamma)$, for the observed sample is expressed as

$$L(\eta, \gamma) = \prod_{i=1}^{n} \frac{e^{-\eta(-\log x_i)^{\gamma}}\, \gamma \eta^2\, \left(\eta + (-\log x_i)^{\gamma}\right) (-\log x_i)^{-1+\gamma}}{x_i\,(1+\eta^2)}. \tag{12}$$

Taking the natural logarithm of Eq. (12), we obtain the log-likelihood function

$$\ell(\eta, \gamma) = n \log \gamma + 2n \log \eta - n \log(1+\eta^2) - \eta \sum_{i=1}^{n} (-\log x_i)^{\gamma} - \sum_{i=1}^{n} \log x_i$$
$$+ \sum_{i=1}^{n} \log\left(\eta + (-\log x_i)^{\gamma}\right) + (\gamma - 1) \sum_{i=1}^{n} \log(-\log x_i). \tag{13}$$

The MLEs $\hat{\eta}$ and $\hat{\gamma}$ are obtained by solving the nonlinear equations

$$\frac{\partial \ell}{\partial \eta} = 0 \quad \text{and} \quad \frac{\partial \ell}{\partial \gamma} = 0.$$

These equations do not yield closed-form solutions and must therefore be solved numerically using iterative optimization techniques such as the NewtonRaphson, BFGS, or NelderMead methods. The approximate standard errors of MLE can be obtained from the inverse of the observed Fisher information matrix, given by

$$\mathbf{I}(\hat{\eta}, \hat{\gamma})^{-1} = -\left[ \left. \frac{\partial^2 \ell(\eta, \gamma)}{\partial (\eta, \gamma)^2} \right|_{\eta=\hat{\eta}, \gamma=\hat{\gamma}} \right]^{-1}.$$

### 6.2 Method of Anderson Darling

The method of Anderson Darling (AD) [21] provides an alternative approach for estimating the unknown parameters for a distribution. This method is based on minimizing the Anderson Darling statistic, which measures the discrepancy between the empirical and theoretical CDFs. The AD statistic gives greater weight to the tails of the distribution, thereby improving the estimation performance when tail behavior is of particular interest.

For a random sample $x_1, x_2, \ldots, x_n$ arranged in ascending order as $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$, the AndersonDarling statistic is defined as

$$A(x_{i:n}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[ \log F(x_{i:n}) + \log S(x_{n-i+1:n}) \right],$$

where $F(x)$ denotes the CDF of the UPSD distribution given by Eq. (3), and $S(x) = 1 - F(x)$ is the corresponding survival function. The estimates of the unknown parameters $(\eta, \gamma)$ are obtained by minimizing $A(x_{i:n})$ with respect to these parameters, that is,

$$(\hat{\eta}, \hat{\gamma}) = \arg\min_{\eta,\gamma>0} A(x_{i:n}).$$

Since a closed-form solution is generally intractable, numerical optimization techniques are employed to obtain the parameter estimates.

### *6.3 Method of Cramer von Mises*

The method of Cramaèr von Mises (CvM) [22] is a powerful technique for parameter estimation that is based on minimizing the integrated squared distance between the empirical and theoretical CDFs. For a random sample $x_1, x_2, \ldots, x_n$ from the UPSD with CDF $F(x)$ given in Eq. (3), the CvM statistic is given as

$$C(\eta, \gamma) = \frac{1}{12n} + \sum_{i=1}^{n} \left[ F(x_{i:n}; \eta, \gamma) - \frac{2i-1}{2n} \right]^2, \tag{14}$$

where $x_{i:n}$ denotes the $i$th order statistic of the sample. The parameters $\eta$ and $\gamma$ of the UPSD distribution are then estimated by minimizing the statistic $C(\eta, \gamma)$ with respect to these parameters.

### *6.4 Method of Maximum Product of Spacings*

The method of maximum product of spacings (MPS), introduced by [23], is another alternative method for estimating model parameters. This method is based on the concept of maximizing the geometric mean of the spacings between consecutive cumulative probabilities derived from the fitted model and the ordered sample observations.

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the ordered sample drawn from the UPSD with the CDF $F(x)$ given in Eq. (3). The spacings between successive cumulative probabilities are defined as

$$I_i(x_{i:n}) = F(x_{i:n}) - F(x_{i-1:n}), \quad i = 1, 2, \ldots, n+1, \tag{15}$$

where $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$. The MPS estimation method seeks to maximize the function

$$\delta(\eta, \gamma) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i(x_{i:n}), \tag{16}$$

with respect to the unknown parameters $\eta$ and $\gamma$.

Equivalently, the MPS estimators $(\hat{\eta}, \hat{\gamma})$ are obtained by solving

$$(\hat{\eta}, \hat{\gamma}) = \arg\max_{\eta,\gamma>0} \delta(\eta, \gamma), \tag{17}$$

where $\delta(\eta, \gamma)$ represents the average log-spacing between consecutive transformed order statistics through the fitted CDF. Under regularity conditions, the MPS estimators are consistent, asymptotically normal, and efficient, sharing the same asymptotic properties as the MLE. Moreover, the MPS method exhibits better small-sample performance and robustness, especially in situations where the likelihood surface is irregular or flat near the optimum.

### *6.5 Methods of Least Squares*

The method of least squares (LS) [24] is one of the most widely used techniques for parameter estimation, particularly when the CDF of the proposed model is available in closed form. This method is based on minimizing the squared differences between the empirical distribution function (EDF) and the theoretical CDF of the model. For a random sample $x_1, x_2, \ldots, x_n$ arranged in ascending order, let $x_{i:n}$ denote the $i$th order statistic. The empirical estimate of the CDF at $x_{i:n}$ is approximated by $\frac{i}{n+1}$, which provides an unbiased estimator of $F(x_{i:n})$.

For the proposed UPSD with CDF defined in Eq. (3), the LS estimators of the parameters $\eta$ and $\gamma$ can be obtained by minimizing the following objective function:

$$V(\eta, \gamma) = \sum_{i=1}^{n} \left[ F(x_{i:n}) - \frac{i}{n+1} \right]^2, \tag{18}$$

where $F(x_{i:n})$ is given in (3).

### *6.6 Methods of Right Tail Anderson Darling*

The right-tail Anderson Darling (RTAD) estimation method [21] is a modification of the classical AD approach, designed to assign greater weight to discrepancies in the right tail of the distribution. This feature makes it particularly suitable for modeling datasets where large values (upper-tail observations) play a crucial role.

Let $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ denote the ordered sample observations from the UPSD with CDF $F(x)$ given in (3) and survival function $S(x) = 1 - F(x)$. The RTAD estimation method minimizes the statistic

$$R(x_{i:n}) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{i:n}) \tag{19}$$

### *6.7 Methods of Weighted Least Squares*

The method of weighted least squares (WLS) introduced by [24] is a simple and efficient estimation approach that minimizes the weighted squared differences between the theoretical CDF values and their corresponding empirical counterparts. For the UPSD with CDF defined in Eq. (3), the WLS estimator is obtained by minimizing the following objective function:

$$W(\eta, \gamma) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ F(x_{i:n}; \eta, \gamma) - \frac{i}{n+1} \right]^2, \tag{20}$$

where $F(x_{i:n}; \eta, \gamma)$ represents the theoretical CDF of the UPSD evaluated at the $i$th ordered sample observation $x_{i:n}$, and the weights $\frac{(n+1)^2(n+2)}{i(n-i+1)}$ are chosen to account for the unequal variances of the empirical distribution function at different order statistics.

### *6.8 Methods of Left Tail Anderson Darling*

The method of left-tail Anderson Darling (LTAD) estimator [25] is a modified version of the classical AD technique that gives greater emphasis to the lower (left) tail of the distribution. This adjustment is particularly suitable for datasets where the probability mass or variation is concentrated near smaller values of the RV, as it allows better estimation performance in the left tail region.

Let $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ denote the ordered observations from the UPSD with CDF $F(x)$ defined in Eq. (3). The LTAD method determines the parameter estimates by minimizing the following objective function:

$$L(\eta, \gamma) = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n}),$$

where $F(x_{i:n})$ is the theoretical CDF of the UPSD evaluated at the $i$th order statistic. The LTAD estimates $(\hat{\eta}, \hat{\gamma})$ are obtained by solving

$$(\hat{\eta}, \hat{\gamma}) = \arg\min_{\eta>0,\,\gamma>0} L(\eta, \gamma).$$

Since the objective function is nonlinear and does not admit a closed-form solution, numerical optimization techniques such as the NewtonRaphson, BFGS, or NelderMead algorithms can be used to obtain the estimates.

### *6.9 Minimum Spacing Absolute Distance*

The method of minimum spacing absolute distance (MSAD) [26] is an alternative estimation procedure based on the concept of spacing between consecutive cumulative probabilities. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ denote the ordered sample observations from the UPSD distribution, whose CDF is given by Eq. (3). Define the spacings as

$$I_i = F(x_{(i)}) - F(x_{(i-1)}), \qquad i = 1, 2, \ldots, n+1,$$

where $F(x_{(0)}) = 0$ and $F(x_{(n+1)}) = 1$. The spacings $I_i$ represent the successive differences between theoretical probabilities under the proposed model.

The MSAD estimator minimizes the total absolute deviation of the spacings from their expected uniform value $1/(n+1)$, and is defined through the following objective function:

$$\zeta(\eta, \gamma) = \sum_{i=1}^{n+1}\left| I_i - \frac{1}{n+1}\right|. \tag{21}$$

The estimates of the parameters $\eta$ and $\gamma$ are obtained by minimizing $\zeta(\eta, \gamma)$ with respect to these parameters, i.e.,

$$(\hat{\eta}_{MSAD}, \hat{\gamma}_{MSAD}) = \arg\min_{\eta>0,\,\gamma>0} \zeta(\eta, \gamma).$$

### *6.10 Minimum Spacing Absolute Log Distance*

The method of minimum spacing absolute log distance (MSALD) which was introduced by [26] is another spacing based estimation procedure used to estimate unknown parameter of a given distribution. It is based on the absolute logarithmic differences between consecutive cumulative probabilities evaluated at the ordered sample observations. Let $x_1 < x_2 < \ldots < x_n$ denote the ordered random sample from the newly proposed UPSD distribution with CDF defined in Eq. (3). Define the spacing quantities as

$$I_i = F(x_{i:n}) - F(x_{i-1:n}), \quad i = 1, 2, \ldots, n+1,$$

where $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$. The corresponding MSALD objective function is given by

$$\Upsilon(\eta, \gamma) = \sum_{i=1}^{n+1} \left| \log I_i - \log \frac{1}{n+1} \right|. \tag{22}$$

The MSALD estimates $(\hat{\eta}, \hat{\gamma})$ of the parameters are obtained by minimizing the above objective function with respect to $\eta$ and $\gamma$. That is,

$$(\hat{\eta}, \hat{\gamma})_{\text{MSALD}} = \arg\min_{\eta,\gamma>0} \{\Upsilon(\eta, \gamma)\}.$$

### *6.11 Anderson Darling Left Tail Second Order*

The Anderson Darling left-tail second-Order (ADLTS) estimation method of [27] provides an alternative to the other estimation method, especially for modeling datasets where the lower tail behavior plays a crucial role. Unlike the conventional AD statistic, which assigns more weight to the tails in general, the ADLTS criterion focuses particularly on the lower tail by introducing a second-order adjustment term. For a random sample $x_1, x_2, \ldots, x_n$ drawn from the new UPSD distribution with $F(x)$ given in Eq. (3), the ADLTS statistic is defined as

$$LTS = 2\sum_{i=1}^{n} \log F(x_{i:n}) + \frac{1}{n}\sum_{i=1}^{n} \frac{(2i-1)}{F(x_{i:n})}, \tag{23}$$

where $F(x_{i:n})$ is the theoretical CDF value of the $i$th ordered observation. The estimation procedure involves determining the parameter vector $\boldsymbol{\theta} = (\eta, \gamma)$ that minimizes the above $LTS$ statistic. That is,

$$\hat{\boldsymbol{\theta}}_{ADLTS} = \arg\min_{\eta,\gamma>0} LTS(\eta, \gamma), \tag{24}$$

where the minimization is typically carried out numerically using iterative algorithms such as the Nelder-Mead or BFGS optimization methods.

### *6.12 Kolmogorov Method*

The Kolmogorov method (KM), which was studied by [27], is a widely used goodness-of-fit technique used to find the suitability of a proposed distribution in modeling a given data set. Here, the test statistic measures the maximum absolute deviation between the EDF and the theoretical CDF of the fitted distribution. Mathematically, it is defined as

$$KM = \max_{1\le i\le n} \left[ \frac{i}{n} - F(x_{i:n}),\ F(x_{i:n}) - \frac{i-1}{n} \right], \tag{25}$$

where $F(x_{i:n})$ denotes the theoretical CDF of the UPSD in our study, given by Eq. (3), and $x_{i:n}$ represents the ordered sample observations. The KM value thus quantifies the largest discrepancy between the observed and expected cumulative proportions. A smaller value of the Kolmogorov statistic indicates a better fit of the proposed model to the empirical data, signifying that the proposed distribution adequately represents the underlying data behavior. Therefore, the Kolmogorov method serves as an effective non-parametric criterion for validating the goodness-of-fit of the new Distribution.

### 6.13 Minimum Spacing Square Distance

The minimum spacing square distance (MSSD) method, which was studied by [26] is an important measure for assessing the goodness-of-fit and uniformity of transformed data points obtained from a proposed probability distribution. For a random sample $x_1, x_2, \ldots, x_n$ drawn from the UPSD distribution in our study, with CDF given in Eq. (3), the corresponding probability integral transform values $I_i = F(x_{(i)})$, where $x_{(i)}$ denotes the $i^{th}$ order statistic, are expected to be approximately uniformly distributed over (0, 1). The MSSD statistic is then defined as

$$\phi_{(x_{i:n})} = \sum_{i=1}^{n+1} \left( I_i - \frac{1}{n+1} \right)^2. \tag{26}$$

A smaller value of $\phi_{(x_{i:n})}$ indicates that the observed spacings between successive $I_i$ are close to those expected under the uniform distribution, implying that the UPSD model provides an adequate fit to the data. Conversely, larger values of $\phi_{(x_{i:n})}$ suggest greater deviation from uniformity and a poorer fit. Hence, the minimum spacing square distance serves as an important tool to evaluate the suitability of a new model for modeling bounded data in (0, 1).

### 6.14 Minimum Spacing Square Log Distance

The minimum spacing square-log distance (MSSLD) estimation method [26] is based on minimizing the squared logarithmic deviation between the observed and theoretical cumulative probabilities. For the UPSD, let $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ denote the ordered sample from the distribution having CDF $F(x)$ defined in (3). The minimum spacing square-log distance (MSSLD) statistic is then expressed as

$$\Psi_{(x_{i:n})} = \sum_{i=1}^{n+1} \left( \log I_i - \log \frac{1}{n+1} \right)^2. \tag{27}$$

The MSSLD estimates of the parameters $\eta$ and $\gamma$ of the UPSD are obtained by minimizing the objective function $\Psi_{(x_{i:n})}$ with respect to these parameters. That is,

$$(\hat{\eta}, \hat{\gamma}) = \arg\min_{\eta,\gamma>0} \Psi_{(x_{i:n})}.$$

### 6.15 Minimum Spacing Linex Distance

The minimum spacing Linex distance (MSLD) [26] is a useful measure to assess the goodness of fit of a proposed statistical model to an observed dataset. It is based on the concept of minimum spacing estimation and utilizes the Linex (linearexponential) loss function, which provides asymmetric penalization for overestimation and underestimation. For our UPSD distribution, with PDF $f(x)$ given in Eq. (4), the MSLD statistic is defined as

$$\Delta_{(x_{i:n})} = \sum_{i=1}^{n+1} \left[ e^{I_i - \frac{1}{n+1}} - \left( I_i - \frac{1}{n+1} \right) - 1 \right],$$

where $I_i = F(x_i) - F(x_{i-1})$ denotes the $i$th spacing computed from the ordered random sample $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$, and $F(x)$ is the CDF of the UPSD given in Eq. (3). The measure $\Delta_{(x_{i:n})}$ quantifies the divergence between the empirical spacings and their expected uniform counterparts $\frac{1}{n+1}$ under the fitted model.

## 7 Numerical Simulation

Simulation is a common method used in statistics. It is used to give data that shows the traits of a certain probability distribution. Simulation studies are especially useful for distributions that are one of a kind or very complicated because they help statisticians figure out how well the distribution works and what the real outcomes are. This section uses carefully planned experiments to see how well the parameter estimating methods described in the estimates section work. We created datasets that matched the theoretical framework from the last section in order to test different strategies. This allowed us to evaluate the estimators according to the previously defined parameter conditions. We used Monte Carlo simulations to create several datasets that follow the structural and distributional assumptions of our model. We used a thorough assessment method based on six basic performance criteria to compare the different estimation methods. The selected metrics seek to capture the fundamental attributes utilized to assess the quality of an estimator. These characteristics include

- Average bias (BIAS), $|Bias(\widehat{\boldsymbol{I}})| = \frac{1}{L}\sum_{i=1}^{L}|\widehat{\boldsymbol{I}} - \boldsymbol{I}|$.
- Mean squared errors (MSE), $MSE = \frac{1}{L}\sum_{i=1}^{L}(\widehat{\boldsymbol{I}} - \boldsymbol{I})^2$.
- Mean relative errors (MRE), $MRE = \frac{1}{L}\sum_{i=1}^{L}|\widehat{\boldsymbol{I}} - \boldsymbol{I}|/\boldsymbol{I}$.
- Average absolute difference ($D_{abs}$), $D_{abs} = \frac{1}{nH}\sum_{i=1}^{H}\sum_{j=1}^{n}|F(x_{ij};\boldsymbol{I}) - F(x_{ij};\widehat{\boldsymbol{I}})|$.
- Maximum absolute difference ($D_{max}$), $D_{max} = \frac{1}{H}\sum_{i=1}^{H}\max_{j=1,\ldots,n}|F(x_{ij};\boldsymbol{I}) - F(x_{ij};\widehat{\boldsymbol{I}})|$.
- Average squared absolute error, $ASAE = \dfrac{1}{n}\sum_{i=1}^{n}\frac{|x_i - \hat{x}_i|}{x_n - x_1}$.

The goal of this work was to determine the optimal method for estimating parameters for our theoretical model through extensive testing using a Monte Carlo simulation. This simulation required the generation of numerous random samples, the collection of extensive data, and the application of analytical techniques to analyze the data. To guarantee statistical reliability and strong conclusions regarding the estimator's performance, it was essential to perform numerous simulated replications to analyze the data. We have carefully recorded the results of our extensive simulation experiments in a set of tables called Tables 2–6. We made detailed visual representations of the numbers in Table 2 to make them easier to understand and help people find patterns. Table 7 presents comprehensive rankings covering all assessment criteria. Furthermore, it provides partial ranks based on specific performance metrics. Table 7 demonstrates that the most efficacious method for determining the estimators of our proposed model is the MLE, followed by the MPSE.

**Table 2:** Numerical values of simulation measures for $\eta = 0.8, \gamma = 0.5$

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | BIAS($\hat\eta$) | 0.11782{3} | 0.112{1} | 0.12811{10} | 0.12148{6} | 0.11943{5} | 0.1182{4} | 0.11645{2} | 0.13377{11} | 0.12704{9} | 0.13582{13} | 0.14105{15} | 0.12697{8} | 0.13515{12} | 0.12519{7} | 0.1367{14} |
| 20 | BIAS($\hat\gamma$) | 0.06813{2} | 0.06871{3} | 0.08365{11} | 0.06981{4} | 0.08003{9} | 0.0852{12} | 0.07556{6} | 0.07762{7} | 0.07948{8} | 0.07472{5} | 0.08931{13} | 0.08354{10} | 0.09561{14} | 0.06767{1} | 0.09649{15} |
| 20 | MSE($\hat\eta$) | 0.02243{2} | 0.02033{1} | 0.02626{9} | 0.0237{6} | 0.02277{4} | 0.02322{5} | 0.02245{3} | 0.02848{11} | 0.02751{10} | 0.03013{14} | 0.03175{15} | 0.02584{7} | 0.03009{13} | 0.02594{8} | 0.02909{12} |
| 20 | MSE($\hat\gamma$) | 0.00816{4} | 0.0081{3} | 0.01313{12} | 0.00741{1} | 0.01052{8} | 0.01291{11} | 0.00939{6} | 0.01069{9} | 0.01001{7} | 0.00835{5} | 0.01402{14} | 0.01188{10} | 0.01398{13} | 0.0075{2} | 0.01441{15} |
| 20 | MRE($\hat\eta$) | 0.14728{3} | 0.14{1} | 0.16014{10} | 0.15185{6} | 0.14929{5} | 0.14775{4} | 0.14556{2} | 0.16721{11} | 0.1588{9} | 0.16977{13} | 0.17631{15} | 0.15872{8} | 0.16893{12} | 0.15648{7} | 0.17087{14} |
| 20 | MRE($\hat\gamma$) | 0.13626{2} | 0.13741{3} | 0.1673{11} | 0.13962{4} | 0.16006{9} | 0.17041{12} | 0.15112{6} | 0.15524{7} | 0.15897{8} | 0.14944{5} | 0.17862{13} | 0.16707{10} | 0.19121{14} | 0.13533{1} | 0.19298{15} |
| 20 | $D_{abs}$ | 0.05575{1} | 0.05613{2} | 0.06076{5} | 0.06084{6} | 0.06149{8} | 0.05999{4} | 0.05848{3} | 0.06177{9} | 0.06322{12} | 0.06742{13} | 0.06242{10} | 0.06271{11} | 0.07231{14} | 0.06101{7} | 0.07315{15} |
| 20 | $D_{max}$ | 0.09131{2} | 0.09083{1} | 0.10103{9} | 0.09498{4} | 0.09981{6} | 0.10027{7} | 0.0948{3} | 0.10033{8} | 0.10206{10} | 0.1061{13} | 0.1036{11} | 0.10469{12} | 0.11663{14} | 0.09665{5} | 0.1178{15} |
| 20 | ASAE | 0.04472{8} | 0.04092{5} | 0.04036{4} | 0.04453{7} | 0.03708{1} | 0.03758{2} | 0.03848{3} | 0.04753{9} | 0.06238{14} | 0.05979{13} | 0.06879{15} | 0.0427{6} | 0.05962{12} | 0.05225{10} | 0.05908{11} |
| 20 | $\sum Ranks$ | 27{2} | 20{1} | 81{8} | 44{4} | 55{6} | 61{7} | 34{3} | 82 {9.5} | 87{11} | 94{12} | 121{14} | 82 {9.5} | 118{13} | 48{5} | 126{15} |
| 55 | BIAS($\hat\eta$) | 0.07004{3} | 0.07097{4} | 0.07379{8} | 0.07308{6} | 0.07299{5} | 0.06922{1} | 0.06965{2} | 0.07834{10} | 0.08412{12} | 0.0765{9} | 0.09192{15} | 0.07329{7} | 0.08801{13} | 0.07911{11} | 0.09015{14} |
| 55 | BIAS($\hat\gamma$) | 0.03885{1} | 0.03941{2} | 0.04743{8} | 0.04104{3} | 0.04839{12} | 0.04825{11} | 0.04305{6} | 0.04365{7} | 0.04774{10} | 0.04201{5} | 0.05381{13} | 0.04764{9} | 0.05781{14} | 0.04125{4} | 0.05875{15} |
| 55 | MSE($\hat\eta$) | 0.00795{4} | 0.00794{3} | 0.0088{8} | 0.0084{5} | 0.00846{6} | 0.00755{2} | 0.00748{1} | 0.00966{10} | 0.01184{12} | 0.0096{9} | 0.0133{15} | 0.00868{7} | 0.01247{13} | 0.00978{11} | 0.01314{14} |
| 55 | MSE($\hat\gamma$) | 0.00253{2} | 0.0026{3} | 0.0037{9} | 0.00252{1} | 0.00372{10} | 0.00385{12} | 0.00297{6} | 0.00309{7} | 0.00359{8} | 0.00277{5} | 0.00475{13} | 0.00373{11} | 0.00529{14} | 0.00265{4} | 0.00547{15} |
| 55 | MRE($\hat\eta$) | 0.08755{3} | 0.08872{4} | 0.09224{8} | 0.09135{6} | 0.09124{5} | 0.08653{1} | 0.08706{2} | 0.09792{10} | 0.10515{12} | 0.09563{9} | 0.1149{15} | 0.09161{7} | 0.11002{13} | 0.09889{11} | 0.11269{14} |
| 55 | MRE($\hat\gamma$) | 0.0777{1} | 0.07883{2} | 0.09486{8} | 0.08208{3} | 0.09678{12} | 0.09651{11} | 0.0861{6} | 0.08731{7} | 0.09549{10} | 0.08402{5} | 0.10762{13} | 0.09529{9} | 0.11563{14} | 0.08251{4} | 0.1175{15} |
| 55 | $D_{abs}$ | 0.03365{1} | 0.03511{3} | 0.0365{7} | 0.03595{4} | 0.0362{5} | 0.03659{8} | 0.03433{2} | 0.03743{9} | 0.04158{13} | 0.03802{10} | 0.04034{12} | 0.03647{6} | 0.04482{14} | 0.03815{11} | 0.04516{15} |
| 55 | $D_{max}$ | 0.05481{1} | 0.05664{3} | 0.0604{6} | 0.05758{4} | 0.05975{5} | 0.06126{11} | 0.05632{2} | 0.06053{7} | 0.06711{13} | 0.06108{10} | 0.06597{12} | 0.0606{8} | 0.0736{14} | 0.06084{9} | 0.07388{15} |
| 55 | ASAE | 0.02635{7} | 0.02354{4} | 0.02267{3} | 0.02683{8} | 0.02246{2} | 0.02193{1} | 0.02359{5} | 0.02762{9} | 0.03818{12} | 0.03555{11} | 0.04384{15} | 0.02477{6} | 0.03901{14} | 0.03305{10} | 0.03874{13} |
| 55 | $\sum Ranks$ | 23{1} | 28{2} | 65{7} | 40{4} | 62{6} | 58{5} | 32{3} | 76{11} | 102{12} | 73{9} | 123 {13.5} | 70{8} | 123 {13.5} | 75{10} | 130{15} |
| 110 | BIAS($\hat\eta$) | 0.05087{7} | 0.05009{3} | 0.05068{5} | 0.04929{2} | 0.05053{4} | 0.04676{1} | 0.05083{6} | 0.05485{9} | 0.05862{12} | 0.05548{11} | 0.0684{14} | 0.05355{8} | 0.06901{15} | 0.05488{10} | 0.06598{13} |
| 110 | BIAS($\hat\gamma$) | 0.027{1} | 0.02803{3} | 0.03241{9} | 0.02793{2} | 0.03142{7} | 0.03352{11} | 0.02889{4} | 0.03135{6} | 0.0334{10} | 0.03174{8} | 0.04011{13} | 0.03368{12} | 0.04197{15} | 0.02927{5} | 0.04108{14} |
| 110 | MSE($\hat\eta$) | 0.00401{4} | 0.00391 {2.5} | 0.00407{6} | 0.00391 {2.5} | 0.00405{5} | 0.00345{1} | 0.00422{7} | 0.00464{9} | 0.0057{12} | 0.00513{11} | 0.00755{14} | 0.0045{8} | 0.00766{15} | 0.00472{10} | 0.00684{13} |
| 110 | MSE($\hat\gamma$) | 0.00115{1} | 0.00125{3} | 0.00174{9} | 0.0012{2} | 0.00157{8} | 0.00179 {10.5} | 0.00135{5} | 0.00154 {6.5} | 0.00179 {10.5} | 0.00154 {6.5} | 0.00261{13} | 0.00181{12} | 0.00276{15} | 0.00132{4} | 0.00267{14} |
| 110 | MRE($\hat\eta$) | 0.06358{7} | 0.06262{3} | 0.06335{5} | 0.06161{2} | 0.06317{4} | 0.05846{1} | 0.06354{6} | 0.06857{9} | 0.07327{12} | 0.06934{11} | 0.0855{14} | 0.06694{8} | 0.08627{15} | 0.0686{10} | 0.08247{13} |
| 110 | MRE($\hat\gamma$) | 0.05401{1} | 0.05607{3} | 0.06483{9} | 0.05585{2} | 0.06284{7} | 0.06703{11} | 0.05778{4} | 0.0627{6} | 0.06681{10} | 0.06348{8} | 0.08021{13} | 0.06736{12} | 0.08395{15} | 0.05854{5} | 0.08217{14} |
| 110 | $D_{abs}$ | 0.02445{2} | 0.02481{3} | 0.0252{6} | 0.02423{1} | 0.02543{7} | 0.0251{5} | 0.02488{4} | 0.02556{8} | 0.02974{13} | 0.02769{11} | 0.02973{12} | 0.02673{10} | 0.03383{15} | 0.02647{9} | 0.03244{14} |
| 110 | $D_{max}$ | 0.0395{2} | 0.0402{3} | 0.04176{7} | 0.03913{1} | 0.04151{5} | 0.04194{8} | 0.04051{4} | 0.04168{6} | 0.04815{12} | 0.04471{11} | 0.04887{13} | 0.04414{10} | 0.05516{15} | 0.04252{9} | 0.05326{14} |
| 110 | ASAE | 0.01861{7} | 0.01674{5} | 0.01602{3} | 0.01876{8} | 0.01596{2} | 0.01552{1} | 0.0166{4} | 0.01899{9} | 0.02727{12} | 0.02475{11} | 0.03283{15} | 0.01726{6} | 0.02957{14} | 0.02351{10} | 0.02933{13} |
| 110 | $\sum Ranks$ | 32{3} | 28.5{2} | 59{7} | 22.5{1} | 49{5} | 49.5{6} | 44{4} | 68.5{8} | 103.5{12} | 88.5{11} | 121{13} | 86{10} | 134{15} | 72{9} | 122{14} |
| 170 | BIAS($\hat\eta$) | 0.03992{2} | 0.04157{5} | 0.04166{6} | 0.03946{1} | 0.04281{8} | 0.04111{4} | 0.04038{3} | 0.04331{9} | 0.04754{12} | 0.04709{11} | 0.05591{15} | 0.0426{7} | 0.05351{14} | 0.04355{10} | 0.05285{13} |
| 170 | BIAS($\hat\gamma$) | 0.02206{1} | 0.02304{4} | 0.02535{8} | 0.02301{3} | 0.02575{9} | 0.02729{12} | 0.02327{5} | 0.02407{6} | 0.02705{11} | 0.02526{7} | 0.03188{14} | 0.02635{10} | 0.03203{15} | 0.02262{2} | 0.03133{13} |
| 170 | MSE($\hat\eta$) | 0.0025{2} | 0.00276{5} | 0.00273{4} | 0.00243{1} | 0.00283{7} | 0.0028{6} | 0.00252{3} | 0.00292{9} | 0.00367{12} | 0.00351{11} | 0.00495{15} | 0.00288{8} | 0.0045{13} | 0.00309{10} | 0.00454{14} |
| 170 | MSE($\hat\gamma$) | 0.00077{1} | 0.00086{5} | 0.00102{9} | 0.00079{2} | 0.00101{8} | 0.00122{12} | 0.00085{4} | 0.00088{6} | 0.00113{11} | 0.00099{7} | 0.00164{15} | 0.00112{10} | 0.00157{14} | 0.00082{3} | 0.00151{13} |
| 170 | MRE($\hat\eta$) | 0.04989{2} | 0.05196{5} | 0.05207{6} | 0.04933{1} | 0.05352{8} | 0.05139{4} | 0.05047{3} | 0.05414{9} | 0.05942{12} | 0.05886{11} | 0.06989{15} | 0.05325{7} | 0.06689{14} | 0.05444{10} | 0.06606{13} |
| 170 | MRE($\hat\gamma$) | 0.04413{1} | 0.04607{4} | 0.0507{8} | 0.04603{3} | 0.0515{9} | 0.05459{12} | 0.04655{5} | 0.04815{6} | 0.05409{11} | 0.05053{7} | 0.06376{14} | 0.0527{10} | 0.06407{15} | 0.04524{2} | 0.06267{13} |
| 170 | $D_{abs}$ | 0.01951{2} | 0.02052{5} | 0.02046{4} | 0.01929{1} | 0.02124 {8.5} | 0.02123{7} | 0.02002{3} | 0.02076{6} | 0.02399{12} | 0.02288{11} | 0.02456{13} | 0.02158{10} | 0.02628{15} | 0.02124 {8.5} | 0.02606{14} |
| 170 | $D_{max}$ | 0.03161{2} | 0.03327{4} | 0.03356{5} | 0.03121{1} | 0.03468{8} | 0.0353{9} | 0.03258{3} | 0.03371{6} | 0.03895{12} | 0.03684{11} | 0.04022{13} | 0.03552{10} | 0.04302{15} | 0.03416{7} | 0.04254{14} |
| 170 | ASAE | 0.01519{9} | 0.01333{4} | 0.01309{3} | 0.01511{7} | 0.01283{2} | 0.01231{1} | 0.01353{5} | 0.01518{8} | 0.02134{12} | 0.02024{11} | 0.02661{15} | 0.01389{6} | 0.02306{13} | 0.01867{10} | 0.02363{14} |
| 170 | $\sum Ranks$ | 22{2} | 41{4} | 53{5} | 20{1} | 67.5{9} | 67{8} | 34{3} | 65{7} | 105{12} | 87{11} | 129{15} | 78{10} | 128{14} | 62.5{6} | 121{13} |
| 240 | BIAS($\hat\eta$) | 0.0333{2} | 0.03347{3} | 0.03581{8} | 0.03306{1} | 0.03416{4} | 0.03499{5} | 0.03549{7} | 0.03654{9} | 0.04069{12} | 0.03911{11} | 0.04732{15} | 0.03532{6} | 0.0472{14} | 0.03717{10} | 0.0454{13} |
| 240 | BIAS($\hat\gamma$) | 0.01712{1} | 0.01938{3} | 0.02204{10} | 0.01822{2} | 0.02092{7} | 0.02367{12} | 0.01978{4} | 0.01994{5} | 0.02363{11} | 0.02161{9} | 0.02814{15} | 0.02117{8} | 0.02657{14} | 0.02051{6} | 0.02586{13} |
| 240 | MSE($\hat\eta$) | 0.00176{2} | 0.00178{3} | 0.00201{8} | 0.0017{1} | 0.00182{4} | 0.00196{7} | 0.00193 {5.5} | 0.00213{9} | 0.00271{12} | 0.00242{11} | 0.00352{14} | 0.00193 {5.5} | 0.00353{15} | 0.00221{10} | 0.00329{13} |
| 240 | MSE($\hat\gamma$) | 0.00048{1} | 6e-04 {3.5} | 0.00079{10} | 0.00051{2} | 7e-04{7} | 0.00087{11} | 6e-04 {3.5} | 0.00063{5} | 0.00088{12} | 0.00073 {8.5} | 0.00122{15} | 0.00073 {8.5} | 0.0011{14} | 0.00065{6} | 0.00104{13} |
| 240 | MRE($\hat\eta$) | 0.04163{2} | 0.04184{3} | 0.04476{8} | 0.04133{1} | 0.0427{4} | 0.04374{5} | 0.04437{7} | 0.04567{9} | 0.05086{12} | 0.04889{11} | 0.05915{15} | 0.04415{6} | 0.059{14} | 0.04647{10} | 0.05675{13} |
| 240 | MRE($\hat\gamma$) | 0.03424{1} | 0.03876{3} | 0.04407{10} | 0.03644{2} | 0.04183{7} | 0.04734{12} | 0.03957{4} | 0.03988{5} | 0.04727{11} | 0.04322{9} | 0.05628{15} | 0.04234{8} | 0.05315{14} | 0.04102{6} | 0.05173{13} |
| 240 | $D_{abs}$ | 0.01633{2} | 0.01639{3} | 0.01786{8} | 0.01608{1} | 0.01702{4} | 0.01794{9} | 0.01728{5} | 0.01746{6} | 0.02014{12} | 0.01891{11} | 0.02055{13} | 0.01783{7} | 0.02286{15} | 0.01803{10} | 0.02239{14} |
| 240 | $D_{max}$ | 0.0262{2} | 0.02683{3} | 0.02938{9} | 0.02583{1} | 0.02796{4} | 0.02984{10} | 0.02807{5} | 0.02841{6} | 0.0329{12} | 0.0306{11} | 0.03383{13} | 0.02921 {7.5} | 0.03712{15} | 0.02921 {7.5} | 0.03632{14} |

(Continued)

**Table 2 (continued)**

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 240 | ASAE | 0.01258$^{\{7\}}$ | 0.01125$^{\{4\}}$ | 0.0109$^{\{3\}}$ | 0.01269$^{\{8\}}$ | 0.01063$^{\{2\}}$ | 0.01057$^{\{1\}}$ | 0.01128$^{\{5\}}$ | 0.01327$^{\{9\}}$ | 0.01824$^{\{12\}}$ | 0.01685$^{\{11\}}$ | 0.02394$^{\{15\}}$ | 0.01185$^{\{6\}}$ | 0.02001$^{\{14\}}$ | 0.01572$^{\{10\}}$ | 0.0197$^{\{13\}}$ |
| 240 | $\sum Ranks$ | 20$^{\{2\}}$ | 38.5$^{\{3\}}$ | 71$^{\{9\}}$ | 19$^{\{1\}}$ | 51$^{\{4\}}$ | 69$^{\{8\}}$ | 56$^{\{5\}}$ | 61$^{\{7\}}$ | 103$^{\{12\}}$ | 89.5$^{\{11\}}$ | 127$^{\{15\}}$ | 59.5$^{\{6\}}$ | 126$^{\{14\}}$ | 73.5$^{\{10\}}$ | 116$^{\{13\}}$ |
| 320 | BIAS($\hat{\eta}$) | 0.0289$^{\{2\}}$ | 0.02948$^{\{3\}}$ | 0.03129$^{\{8\}}$ | 0.02958$^{\{4\}}$ | 0.03021$^{\{5\}}$ | 0.03033$^{\{6\}}$ | 0.02878$^{\{1\}}$ | 0.03117$^{\{7\}}$ | 0.03736$^{\{12\}}$ | 0.03472$^{\{11\}}$ | 0.04085$^{\{15\}}$ | 0.03162$^{\{9\}}$ | 0.03929$^{\{13\}}$ | 0.03275$^{\{10\}}$ | 0.0395$^{\{14\}}$ |
| 320 | BIAS($\hat{\gamma}$) | 0.01506$^{\{1\}}$ | 0.01628$^{\{3\}}$ | 0.01925$^{\{9\}}$ | 0.01584$^{\{2\}}$ | 0.01862$^{\{8\}}$ | 0.02082$^{\{12\}}$ | 0.01648$^{\{4\}}$ | 0.01782$^{\{6\}}$ | 0.0207$^{\{11\}}$ | 0.01812$^{\{7\}}$ | 0.02406$^{\{15\}}$ | 0.01999$^{\{10\}}$ | 0.02215$^{\{13\}}$ | 0.0168$^{\{5\}}$ | 0.02219$^{\{14\}}$ |
| 320 | MSE($\hat{\eta}$) | 0.0013$^{\{1\}}$ | 0.00139$^{\{4\}}$ | 0.00153$^{\{7\}}$ | 0.00136$^{\{3\}}$ | 0.00141$^{\{5\}}$ | 0.00146$^{\{6\}}$ | 0.00131$^{\{2\}}$ | 0.00156$^{\{8.5\}}$ | 0.00217$^{\{12\}}$ | 0.0019$^{\{11\}}$ | 0.0027$^{\{15\}}$ | 0.00156$^{\{8.5\}}$ | 0.00245$^{\{13\}}$ | 0.0017$^{\{10\}}$ | 0.00247$^{\{14\}}$ |
| 320 | MSE($\hat{\gamma}$) | 0.00036$^{\{1\}}$ | 0.00042$^{\{3.5\}}$ | 0.00057$^{\{8.5\}}$ | 4e-04$^{\{2\}}$ | 0.00057$^{\{8.5\}}$ | 0.00068$^{\{12\}}$ | 0.00042$^{\{3.5\}}$ | 0.00051$^{\{6.5\}}$ | 0.00066$^{\{11\}}$ | 0.00051$^{\{6.5\}}$ | 0.00092$^{\{15\}}$ | 0.00064$^{\{10\}}$ | 0.00076$^{\{13.5\}}$ | 0.00044$^{\{5\}}$ | 0.00076$^{\{13.5\}}$ |
| 320 | MRE($\hat{\eta}$) | 0.03612$^{\{2\}}$ | 0.03685$^{\{3\}}$ | 0.03911$^{\{8\}}$ | 0.03697$^{\{4\}}$ | 0.03776$^{\{5\}}$ | 0.03791$^{\{6\}}$ | 0.03598$^{\{1\}}$ | 0.03897$^{\{7\}}$ | 0.0467$^{\{12\}}$ | 0.0434$^{\{11\}}$ | 0.05106$^{\{15\}}$ | 0.03952$^{\{9\}}$ | 0.04912$^{\{13\}}$ | 0.04094$^{\{10\}}$ | 0.04938$^{\{14\}}$ |
| 320 | MRE($\hat{\gamma}$) | 0.03011$^{\{1\}}$ | 0.03257$^{\{3\}}$ | 0.03851$^{\{9\}}$ | 0.03169$^{\{2\}}$ | 0.03724$^{\{8\}}$ | 0.04163$^{\{12\}}$ | 0.03297$^{\{4\}}$ | 0.03565$^{\{6\}}$ | 0.04139$^{\{11\}}$ | 0.03624$^{\{7\}}$ | 0.04813$^{\{15\}}$ | 0.03998$^{\{10\}}$ | 0.0443$^{\{13\}}$ | 0.03361$^{\{5\}}$ | 0.04439$^{\{14\}}$ |
| 320 | $D_{abs}$ | 0.01388$^{\{1\}}$ | 0.01437$^{\{3\}}$ | 0.01542$^{\{8\}}$ | 0.01443$^{\{4\}}$ | 0.015$^{\{5\}}$ | 0.01527$^{\{7\}}$ | 0.01421$^{\{2\}}$ | 0.01507$^{\{6\}}$ | 0.01797$^{\{13\}}$ | 0.01665$^{\{11\}}$ | 0.01769$^{\{12\}}$ | 0.01569$^{\{9\}}$ | 0.01938$^{\{15\}}$ | 0.01579$^{\{10\}}$ | 0.01931$^{\{14\}}$ |
| 320 | $D_{max}$ | 0.02237$^{\{1\}}$ | 0.02319$^{\{2\}}$ | 0.02528$^{\{7\}}$ | 0.02325$^{\{4\}}$ | 0.02457$^{\{5.5\}}$ | 0.02544$^{\{9\}}$ | 0.0232$^{\{3\}}$ | 0.02457$^{\{5.5\}}$ | 0.02912$^{\{12\}}$ | 0.02669$^{\{11\}}$ | 0.02919$^{\{13\}}$ | 0.02585$^{\{10\}}$ | 0.03142$^{\{15\}}$ | 0.02535$^{\{8\}}$ | 0.03129$^{\{14\}}$ |
| 320 | ASAE | 0.01086$^{\{7\}}$ | 0.00975$^{\{4\}}$ | 0.00957$^{\{3\}}$ | 0.01094$^{\{8\}}$ | 0.00921$^{\{1\}}$ | 0.00937$^{\{2\}}$ | 0.0098$^{\{5\}}$ | 0.01142$^{\{9\}}$ | 0.0162$^{\{12\}}$ | 0.01473$^{\{11\}}$ | 0.0207$^{\{15\}}$ | 0.01036$^{\{6\}}$ | 0.0178$^{\{13\}}$ | 0.01343$^{\{10\}}$ | 0.01805$^{\{14\}}$ |
| 320 | $\sum Ranks$ | 17$^{\{1\}}$ | 27.5$^{\{3\}}$ | 66.5$^{\{7\}}$ | 46$^{\{4\}}$ | 50$^{\{5\}}$ | 71$^{\{8\}}$ | 24.5$^{\{2\}}$ | 60.5$^{\{6\}}$ | 105$^{\{12\}}$ | 85.5$^{\{11\}}$ | 129$^{\{15\}}$ | 80.5$^{\{10\}}$ | 120.5$^{\{13\}}$ | 72$^{\{9\}}$ | 124.5$^{\{14\}}$ |

**Table 3:** Numerical values of simulation measures for $\eta = 1.5, \gamma = 0.85$

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | BIAS($\hat{\eta}$) | 0.21598[10] | 0.2006[3] | 0.22824[14] | 0.1888[1] | 0.20141[5] | 0.24286[15] | 0.20813[7] | 0.20928[8] | 0.19385[2] | 0.20967[9] | 0.21726[11] | 0.22518[13] | 0.20393[6] | 0.20111[4] | 0.22186[12] |
| 20 | BIAS($\hat{\gamma}$) | 0.13568[4] | 0.13053[2] | 0.15852[11] | 0.12847[1] | 0.15117[9] | 0.15674[10] | 0.14282[5] | 0.14442[6] | 0.1499[8] | 0.14797[7] | 0.17857[14] | 0.16249[12] | 0.17217[13] | 0.13086[3] | 0.18299[15] |
| 20 | MSE($\hat{\eta}$) | 0.082[11] | 0.06905[4] | 0.094[14] | 0.06062[1] | 0.0708[6] | 0.11335[15] | 0.07793[9] | 0.07763[8] | 0.06344[2] | 0.07309[7] | 0.07807[10] | 0.09006[13] | 0.07061[5] | 0.06595[3] | 0.08443[12] |
| 20 | MSE($\hat{\gamma}$) | 0.03324[4] | 0.0289[3] | 0.04491[12] | 0.02498[1] | 0.0374[9] | 0.04296[11] | 0.03386[5] | 0.03599[7] | 0.03631[8] | 0.0344[6] | 0.05659[15] | 0.04521[13] | 0.04243[10] | 0.0266[2] | 0.0487[14] |
| 20 | MRE($\hat{\eta}$) | 0.14399[10] | 0.13373[3] | 0.15216[14] | 0.12586[1] | 0.13428[5] | 0.16191[15] | 0.13875[7] | 0.13952[8] | 0.12923[2] | 0.13978[9] | 0.14484[11] | 0.15012[13] | 0.13596[6] | 0.13407[4] | 0.14791[12] |
| 20 | MRE($\hat{\gamma}$) | 0.15962[4] | 0.15357[2] | 0.18649[11] | 0.15114[1] | 0.17784[9] | 0.1844[10] | 0.16802[5] | 0.16991[6] | 0.17635[8] | 0.17409[7] | 0.21009[14] | 0.19116[12] | 0.20256[13] | 0.15395[3] | 0.21528[15] |
| 20 | $D_{abs}$ | 0.05804[2] | 0.05721[1] | 0.06061[9] | 0.0594[4] | 0.05932[3] | 0.06189[10] | 0.06049[8] | 0.05958[6] | 0.06251[11] | 0.06617[12] | 0.06655[13] | 0.05949[5] | 0.0688[14] | 0.06029[7] | 0.07459[15] |
| 20 | $D_{max}$ | 0.094[3] | 0.09164[1] | 0.1001[10] | 0.09283[2] | 0.09569[5] | 0.10214[11] | 0.09684[7] | 0.09623[6] | 0.09993[9] | 0.10451[12] | 0.10889[13] | 0.09911[8] | 0.11072[14] | 0.0954[4] | 0.11903[15] |
| 20 | ASAE | 0.04521[9] | 0.04049[4] | 0.03972[3] | 0.04453[8] | 0.03762[1] | 0.0425[6] | 0.03825[2] | 0.04168[5] | 0.05925[14] | 0.05909[12] | 0.05183[11] | 0.04284[7] | 0.05919[13] | 0.05149[10] | 0.06119[15] |
| 20 | $\sum Ranks$ | 57[6] | 23[2] | 98[12] | 20[1] | 52[4] | 103[13] | 55[5] | 60[7] | 64[8] | 81[9] | 112[14] | 96[11] | 94[10] | 40[3] | 125[15] |
| 55 | BIAS($\hat{\eta}$) | 0.1259[10] | 0.11661[2] | 0.13046[11] | 0.11397[1] | 0.1256[9] | 0.13381[13] | 0.12051[4] | 0.12064[5] | 0.1254[8] | 0.13217[12] | 0.11785[3] | 0.12431[7] | 0.14558[14] | 0.12419[6] | 0.14632[15] |
| 55 | BIAS($\hat{\gamma}$) | 0.07441[1] | 0.07789[2] | 0.09423[11] | 0.07921[3] | 0.0923[8] | 0.09302[9] | 0.08262[5] | 0.08448[7] | 0.09818[12] | 0.08271[6] | 0.10637[14] | 0.09418[10] | 0.10623[13] | 0.08224[4] | 0.1087[15] |
| 55 | MSE($\hat{\eta}$) | 0.02541[8] | 0.02156[2] | 0.02777[11] | 0.02096[1] | 0.02643[9] | 0.03043[13] | 0.02408[5] | 0.02292[4] | 0.02706[10] | 0.02836[12] | 0.02253[3] | 0.02487[6] | 0.0347[14] | 0.02528[7] | 0.03606[15] |
| 55 | MSE($\hat{\gamma}$) | 0.00898[1] | 0.00984[3] | 0.0146[9] | 0.00936[2] | 0.01406[8] | 0.01472[10] | 0.01115[6] | 0.01234[7] | 0.01487[11] | 0.01045[4] | 0.01759[13] | 0.01517[12] | 0.01799[14] | 0.01052[5] | 0.01875[15] |
| 55 | MRE($\hat{\eta}$) | 0.08393[10] | 0.07774[2] | 0.08697[11] | 0.07598[1] | 0.08374[9] | 0.0892[13] | 0.08034[4] | 0.08043[5] | 0.0836[8] | 0.08812[12] | 0.07857[3] | 0.08287[7] | 0.09706[14] | 0.08279[6] | 0.09755[15] |
| 55 | MRE($\hat{\gamma}$) | 0.08755[1] | 0.09164[2] | 0.11086[11] | 0.09319[3] | 0.10858[8] | 0.10943[9] | 0.0972[5] | 0.09939[7] | 0.1155[12] | 0.09731[6] | 0.12515[14] | 0.11081[10] | 0.12498[13] | 0.09676[4] | 0.12788[15] |
| 55 | $D_{abs}$ | 0.03505[2] | 0.0343[1] | 0.03778[9] | 0.03539[4] | 0.037[7] | 0.03744[8] | 0.03522[3] | 0.03584[5] | 0.04005[12] | 0.04027[13] | 0.03936[11] | 0.03615[6] | 0.04492[14] | 0.03806[10] | 0.04539[15] |
| 55 | $D_{max}$ | 0.0566[3] | 0.05534[1] | 0.06227[10] | 0.05657[2] | 0.0606[8] | 0.062[9] | 0.05717[4] | 0.05816[5] | 0.06489[13] | 0.06387[11] | 0.06452[12] | 0.05984[6] | 0.07284[14] | 0.06052[7] | 0.07359[15] |
| 55 | ASAE | 0.02649[9] | 0.02344[4] | 0.02259[2] | 0.02624[8] | 0.02242[1] | 0.02528[7] | 0.02306[3] | 0.02364[5] | 0.03667[13] | 0.03385[12] | 0.03175[11] | 0.0248[6] | 0.0379[14] | 0.03102[10] | 0.03899[15] |
| 55 | $\sum Ranks$ | 45[4] | 19[1] | 85[10] | 25[2] | 67[7] | 91[12] | 39[3] | 50[5] | 99[13] | 88[11] | 84[9] | 70[8] | 124[14] | 59[6] | 135[15] |
| 110 | BIAS($\hat{\eta}$) | 0.08616[5] | 0.08425[2] | 0.08873[8] | 0.0803[1] | 0.08543[3] | 0.09237[11] | 0.0874[6] | 0.08587[4] | 0.09611[12] | 0.09812[13] | 0.08758[7] | 0.08985[10] | 0.1074[15] | 0.08927[9] | 0.10546[14] |
| 110 | BIAS($\hat{\gamma}$) | 0.05362[1] | 0.05475[3] | 0.06594[12] | 0.05471[2] | 0.06094[7] | 0.06467[10] | 0.05729[5] | 0.05796[6] | 0.06565[11] | 0.06147[8] | 0.07602[13] | 0.06431[9] | 0.07623[14] | 0.05585[4] | 0.07638[15] |
| 110 | MSE($\hat{\eta}$) | 0.01165[5] | 0.01143[2] | 0.01257[8] | 0.00985[1] | 0.01145[3] | 0.01361[11] | 0.01259[9] | 0.01154[4] | 0.01515[13] | 0.01497[12] | 0.01205[6] | 0.01312[10] | 0.02119[15] | 0.01252[7] | 0.01855[14] |
| 110 | MSE($\hat{\gamma}$) | 0.00445[1] | 0.00472[3] | 0.00703[11] | 0.00451[2] | 0.0058[7] | 0.00681[9] | 0.00535[6] | 0.00533[5] | 0.00695[10] | 0.00583[8] | 0.00928[15] | 0.00709[12] | 0.00859[13] | 0.00482[4] | 0.00888[14] |
| 110 | MRE($\hat{\eta}$) | 0.05744[5] | 0.05616[2] | 0.05915[8] | 0.05353[1] | 0.05695[3] | 0.06158[11] | 0.05827[6] | 0.05725[4] | 0.06407[12] | 0.06541[13] | 0.05839[7] | 0.0599[10] | 0.0716[15] | 0.05951[9] | 0.07031[14] |
| 110 | MRE($\hat{\gamma}$) | 0.06309[1] | 0.06441[3] | 0.07758[12] | 0.06437[2] | 0.0717[7] | 0.07608[10] | 0.0674[5] | 0.06819[6] | 0.07723[11] | 0.07232[8] | 0.08943[13] | 0.07565[9] | 0.08969[14] | 0.0657[4] | 0.08986[15] |
| 110 | $D_{abs}$ | 0.02495[3] | 0.02469 [1.5] | 0.02603[8] | 0.02469 [1.5] | 0.02507[4] | 0.02573[7] | 0.02516[5] | 0.0255[6] | 0.02907[12] | 0.02923[13] | 0.02905[11] | 0.02607[9] | 0.03286[15] | 0.02651[10] | 0.03282[14] |
| 110 | $D_{max}$ | 0.04022[3] | 0.03979[2] | 0.04299[10] | 0.03956[1] | 0.04121[5] | 0.04294[8] | 0.04092[4] | 0.0413[6] | 0.04697[12] | 0.04667[11] | 0.04767[13] | 0.04297[9] | 0.05328[15] | 0.04238[7] | 0.05318[14] |
| 110 | ASAE | 0.01839[8] | 0.01629[4] | 0.01595[2] | 0.01849[9] | 0.01554[1] | 0.01742[7] | 0.01623[3] | 0.01641[5] | 0.02598[13] | 0.02435[12] | 0.02389[11] | 0.01733[6] | 0.02832[15] | 0.02196[10] | 0.02782[14] |
| 110 | $\sum Ranks$ | 32[3] | 22.5[2] | 79[8] | 20.5[1] | 40[4] | 84 [9.5] | 49[6] | 46[5] | 106[13] | 98[12] | 96[11] | 84 [9.5] | 131[15] | 64[7] | 128[14] |
| 170 | BIAS($\hat{\eta}$) | 0.06502[2] | 0.06939[6] | 0.07122[8] | 0.06551[3] | 0.06917[5] | 0.07054[7] | 0.06468[1] | 0.0658[4] | 0.08155[13] | 0.07493[12] | 0.07234[10] | 0.07137[9] | 0.08545[15] | 0.0739[11] | 0.08492[14] |
| 170 | BIAS($\hat{\gamma}$) | 0.03992[1] | 0.04322[3] | 0.05079[10] | 0.04173[2] | 0.04735[7] | 0.05325[12] | 0.04503[4] | 0.04714[5] | 0.05174[11] | 0.04764[8] | 0.06239[15] | 0.05076[9] | 0.05656[13] | 0.0472[6] | 0.05919[14] |
| 170 | MSE($\hat{\eta}$) | 0.00686[3] | 0.00753[5] | 0.00811[9] | 0.00685[2] | 0.00755[6] | 0.00802[8] | 0.00674[1] | 0.0069[4] | 0.01035[13] | 0.00908[12] | 0.00795[7] | 0.00823[10] | 0.01177[15] | 0.00886[11] | 0.01145[14] |
| 170 | MSE($\hat{\gamma}$) | 0.00258[1] | 0.00293[3] | 0.00411[10] | 0.00273[2] | 0.00356[7] | 0.00458[12] | 0.00316[4] | 0.00362[8] | 0.0042[11] | 0.00352[6] | 0.00619[15] | 0.00405[9] | 0.00499[13] | 0.00339[5] | 0.00538[14] |
| 170 | MRE($\hat{\eta}$) | 0.04335[2] | 0.04626[6] | 0.04748[8] | 0.04367[3] | 0.04611[5] | 0.04702[7] | 0.04312[1] | 0.04386[4] | 0.05437[13] | 0.04995[12] | 0.04822[10] | 0.04758[9] | 0.05697[15] | 0.04927[11] | 0.05661[14] |
| 170 | MRE($\hat{\gamma}$) | 0.04697[1] | 0.05084[3] | 0.05975[10] | 0.04909[2] | 0.0557[7] | 0.06264[12] | 0.05298[4] | 0.05546[5] | 0.06087[11] | 0.05605[8] | 0.07339[15] | 0.05971[9] | 0.06654[13] | 0.05553[6] | 0.06963[14] |
| 170 | $D_{abs}$ | 0.01874[1] | 0.02004[4] | 0.02069[8] | 0.01986[3] | 0.0204[6] | 0.02052[7] | 0.01927[2] | 0.02006[5] | 0.02407[12] | 0.02239[11] | 0.02426[13] | 0.02083[9] | 0.02566[14] | 0.02236[10] | 0.02603[15] |
| 170 | $D_{max}$ | 0.03018[1] | 0.03237[4] | 0.03402[7] | 0.03171[3] | 0.03325[6] | 0.03405[8] | 0.0314[2] | 0.03258[5] | 0.03873[12] | 0.03595[11] | 0.03965[13] | 0.03431[9] | 0.04154[14] | 0.03561[10] | 0.04208[15] |
| 170 | ASAE | 0.01448[8] | 0.01299[4] | 0.01274[2] | 0.01477[9] | 0.01264[1] | 0.01395[7] | 0.01294[3] | 0.01325[5] | 0.02109[13] | 0.0193[11] | 0.01986[12] | 0.0139[6] | 0.02288[15] | 0.01787[10] | 0.02267[14] |
| 170 | $\sum Ranks$ | 20[1] | 38[4] | 72[7] | 29[3] | 50[6] | 80 [9.5] | 22[2] | 45[5] | 109[12] | 91[11] | 110[13] | 79[8] | 127[14] | 80 [9.5] | 128[15] |
| 240 | BIAS($\hat{\eta}$) | 0.05737[4] | 0.05659[2] | 0.06083[10] | 0.05615[1] | 0.05922[6] | 0.06072[9] | 0.05702[3] | 0.0576[5] | 0.06786[13] | 0.06519[12] | 0.05924[7] | 0.06143[11] | 0.07379[15] | 0.05997[8] | 0.0715[14] |
| 240 | BIAS($\hat{\gamma}$) | 0.03351[1] | 0.03693[3] | 0.04304[9] | 0.03574[2] | 0.04377[10] | 0.04632[12] | 0.03872[5] | 0.04039[6] | 0.04449[11] | 0.0412[7] | 0.05511[15] | 0.04299[8] | 0.04858[14] | 0.03725[4] | 0.04824[13] |
| 240 | MSE($\hat{\eta}$) | 0.00524[4] | 0.00503[2] | 0.00582[9] | 0.0049[1] | 0.00554[6] | 0.00601[11] | 0.00516[3] | 0.00525[5] | 0.00726[13] | 0.00667[12] | 0.00558[7] | 0.00596[10] | 0.00876[15] | 0.00561[8] | 0.00819[14] |
| 240 | MSE($\hat{\gamma}$) | 0.00178[1] | 0.00215[3] | 0.00294[9] | 0.00194[2] | 0.00295[10] | 0.00343[12] | 0.00236[5] | 0.00256 [6.5] | 0.00312[11] | 0.00256 [6.5] | 0.00481[15] | 0.00293[8] | 0.00366[13] | 0.00218[4] | 0.00371[14] |
| 240 | MRE($\hat{\eta}$) | 0.03825[4] | 0.03773[2] | 0.04055[10] | 0.03744[1] | 0.03948[6] | 0.04048[9] | 0.03801[3] | 0.0384[5] | 0.04524[13] | 0.04346[12] | 0.03949[7] | 0.04095[11] | 0.0492[15] | 0.03998[8] | 0.04767[14] |
| 240 | MRE($\hat{\gamma}$) | 0.03942[1] | 0.04345[3] | 0.05064[9] | 0.04205[2] | 0.0515[10] | 0.05449[12] | 0.04555[5] | 0.04752[6] | 0.05234[11] | 0.04848[7] | 0.06483[15] | 0.05058[8] | 0.05715[14] | 0.04383[4] | 0.05675[13] |
| 240 | $D_{abs}$ | 0.01617[1] | 0.01665[3] | 0.018[10] | 0.01661[2] | 0.01767[8] | 0.01786[9] | 0.01683[4] | 0.01723[5] | 0.02047[12] | 0.01938[11] | 0.02049[13] | 0.01765[7] | 0.02211[15] | 0.01755[6] | 0.0219[14] |
| 240 | $D_{max}$ | 0.02601[1] | 0.02692[3] | 0.02941[9] | 0.02669[2] | 0.02907[7] | 0.02968[10] | 0.02735[4] | 0.02795[5] | 0.03287[12] | 0.03106[11] | 0.03361[13] | 0.02914[8] | 0.03566[15] | 0.02825[6] | 0.03521[14] |
| 240 | ASAE | 0.0122[8] | 0.01111[4] | 0.01074[2] | 0.01243[9] | 0.01064[1] | 0.01187[7] | 0.01091[3] | 0.01112[5] | 0.01805[13] | 0.01614[11] | 0.01725[12] | 0.01166[6] | 0.01927[15] | 0.01497[10] | 0.0191[14] |
| 240 | $\sum Ranks$ | 25 [2.5] | 25 [2.5] | 77 [8.5] | 22[1] | 64[7] | 91[11] | 35[4] | 48.5[5] | 109[13] | 89.5[10] | 104[12] | 77 [8.5] | 131[15] | 58[6] | 124[14] |
| 320 | BIAS($\hat{\eta}$) | 0.04748[2] | 0.04666[1] | 0.04863[5] | 0.04807[3] | 0.04977[6] | 0.05356[9] | 0.05094[7] | 0.04816[4] | 0.05832[13] | 0.05416[11] | 0.05399[10] | 0.05167[8] | 0.06273[14] | 0.0554[12] | 0.06501[15] |
| 320 | BIAS($\hat{\gamma}$) | 0.03014[1] | 0.03143[3] | 0.03724[10] | 0.03177[4] | 0.03627[8] | 0.03782[11] | 0.03113[2] | 0.03548[6] | 0.0365[9] | 0.03593[7] | 0.04837[15] | 0.03964[12] | 0.04289[14] | 0.03416[5] | 0.04195[13] |
| 320 | MSE($\hat{\eta}$) | 0.00358[2] | 0.00356[1] | 0.00374[5] | 0.00365[3] | 0.00396[6] | 0.00454[10] | 0.0042[8] | 0.00366[4] | 0.00549[13] | 0.00451[9] | 0.00457[11] | 0.00416[7] | 0.00633[14] | 0.00487[12] | 0.00674[15] |

(Continued)

**Table 3 (continued)**

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 320 | MSE($\hat{\gamma}$) | 0.00142[1] | 0.00158[4] | 0.00219[10] | 0.00155[3] | 0.00209[8] | 0.00229[11] | 0.00154[2] | 0.00188[6] | 0.0021[9] | 0.002[7] | 0.00366[15] | 0.00244[12] | 0.00283[14] | 0.00184[5] | 0.00267[13] |
| 320 | MRE($\hat{\eta}$) | 0.03166[2] | 0.03111[1] | 0.03242[5] | 0.03204[3] | 0.03318[6] | 0.0357[9] | 0.03396[7] | 0.03211[4] | 0.03888[13] | 0.03611[11] | 0.03599[10] | 0.03445[8] | 0.04182[14] | 0.03693[12] | 0.04334[15] |
| 320 | MRE($\hat{\gamma}$) | 0.03546[1] | 0.03697[3] | 0.04382[10] | 0.03738[4] | 0.04267[8] | 0.04449[11] | 0.03662[2] | 0.04174[6] | 0.04294[9] | 0.04227[7] | 0.0569[15] | 0.04663[12] | 0.05046[14] | 0.04019[5] | 0.04935[13] |
| 320 | $D_{abs}$ | 0.01388[1] | 0.0139[2] | 0.01448[4] | 0.01441[3] | 0.01471[7] | 0.01528 [8.5] | 0.01462[5] | 0.01463[6] | 0.01725[12] | 0.01611[10] | 0.01838[13] | 0.01528 [8.5] | 0.01913[14] | 0.0164[11] | 0.01936[15] |
| 320 | $D_{max}$ | 0.02243[1] | 0.02255[2] | 0.02407[6] | 0.02317[3] | 0.02426[7] | 0.02539[8] | 0.02367[4] | 0.02384[5] | 0.02762[12] | 0.026[10] | 0.03007[13] | 0.02542[9] | 0.03089[14] | 0.02626[11] | 0.03114[15] |
| 320 | ASAE | 0.01054[8] | 0.00954[3] | 0.00918 [1.5] | 0.0108[9] | 0.00918 [1.5] | 0.0101[6] | 0.00961[4] | 0.00978[5] | 0.0151[12] | 0.01397[11] | 0.01518[13] | 0.0103[7] | 0.01661[14] | 0.01304[10] | 0.01697[15] |
| 320 | $\sum$ *Ranks* | 19[1] | 20[2] | 56.5[6] | 35[3] | 57.5[7] | 83.5 [10.5] | 41[4] | 46[5] | 102[12] | 83 [8.5] | 115[13] | 83.5 [10.5] | 126[14] | 83 [8.5] | 129[15] |

**Table 4:** Numerical values of simulation measures for $\eta = 2.5, \gamma = 1.5$

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | BIAS($\hat{\eta}$) | 0.42269[7] | 0.42468[8] | 0.49933[14] | 0.39639[3] | 0.4437[10] | 0.51784[15] | 0.44588[11] | 0.43269[9] | 0.38857[2] | 0.41655[6] | 0.41487[5] | 0.38531[1] | 0.46758[12] | 0.40579[4] | 0.47625[13] |
| 20 | BIAS($\hat{\gamma}$) | 0.24789[4] | 0.21615[1] | 0.27892[12] | 0.22785[2] | 0.26565[10] | 0.2755[11] | 0.25697[6] | 0.25361[5] | 0.26185[9] | 0.26134[8] | 0.3312[15] | 0.25799[7] | 0.31291[14] | 0.24069[3] | 0.31188[13] |
| 20 | MSE($\hat{\eta}$) | 0.33761[9] | 0.32613[6] | 0.46614[14] | 0.25728[1] | 0.34766[10] | 0.50005[15] | 0.35239[12] | 0.36381[13] | 0.2701[2] | 0.29587[5] | 0.33382[7] | 0.2865[4] | 0.33669[8] | 0.28495[3] | 0.35214[11] |
| 20 | MSE($\hat{\gamma}$) | 0.1053[5] | 0.07801[2] | 0.13797[12] | 0.07775[1] | 0.10941[7] | 0.12866[11] | 0.11024[8] | 0.11396[9] | 0.10735[6] | 0.10386[4] | 0.18937[15] | 0.11781[10] | 0.13846[13] | 0.09033[3] | 0.1418[14] |
| 20 | MRE($\hat{\eta}$) | 0.16908[7] | 0.16987[8] | 0.19973[14] | 0.15856[3] | 0.17748[10] | 0.20713[15] | 0.17835[11] | 0.17308[9] | 0.15543[2] | 0.16662[6] | 0.16595[5] | 0.15412[1] | 0.18703[12] | 0.16232[4] | 0.1905[13] |
| 20 | MRE($\hat{\gamma}$) | 0.16526[4] | 0.1441[1] | 0.18595[12] | 0.1519[2] | 0.1771[10] | 0.18366[11] | 0.17131[6] | 0.16908[5] | 0.17456[9] | 0.17423[8] | 0.2208[15] | 0.17199[7] | 0.20861[14] | 0.16046[3] | 0.20792[13] |
| 20 | $D_{abs}$ | 0.05661[2] | 0.0556[1] | 0.05956[8] | 0.05746[3] | 0.05887[5] | 0.0597[9] | 0.05955[7] | 0.05905[6] | 0.06295[12] | 0.06685[13] | 0.06284[11] | 0.05821[4] | 0.07121[15] | 0.06051[10] | 0.07091[14] |
| 20 | $D_{max}$ | 0.09156[3] | 0.08854[1] | 0.09764[9] | 0.09028[2] | 0.09446[5] | 0.09795[10] | 0.09514[7] | 0.09458[6] | 0.10019[11] | 0.10511[13] | 0.10341[12] | 0.09233[4] | 0.11401[15] | 0.0956[8] | 0.11328[14] |
| 20 | ASAE | 0.0455[9] | 0.04107[3] | 0.04121[4] | 0.04259[5] | 0.0402[2] | 0.04445[7] | 0.03933[1] | 0.04269[6] | 0.05617[13] | 0.0538[12] | 0.05196[11] | 0.04511[8] | 0.06051[14] | 0.04894[10] | 0.0612[15] |
| 20 | $\sum$ *Ranks* | 50[5] | 31[2] | 99[12] | 22[1] | 69 [8.5] | 104[13] | 69 [8.5] | 68[7] | 66[6] | 75[10] | 96[11] | 46[3] | 117[14] | 48[4] | 120[15] |
| 55 | BIAS($\hat{\eta}$) | 0.23868[2] | 0.26311[9] | 0.29086[11] | 0.22679[1] | 0.29088[12] | 0.31837[14] | 0.24927[7] | 0.24532[5] | 0.26082[8] | 0.27464[10] | 0.24603[6] | 0.24464[4] | 0.32524[15] | 0.24351[3] | 0.30551[13] |
| 55 | BIAS($\hat{\gamma}$) | 0.12882[1] | 0.1439[4] | 0.15905[10] | 0.13729[2] | 0.1583[9] | 0.16343[11] | 0.14396[5] | 0.15366[7] | 0.16619[12] | 0.146[6] | 0.18787[13] | 0.15714[8] | 0.19662[15] | 0.13916[3] | 0.19171[14] |
| 55 | MSE($\hat{\eta}$) | 0.10068[3] | 0.11841[7] | 0.15115[13] | 0.08056[1] | 0.14466[11] | 0.19434[15] | 0.11187[6] | 0.10613[5] | 0.11964[8] | 0.12452[10] | 0.10564[4] | 0.12188[9] | 0.17073[14] | 0.09954[2] | 0.14858[12] |
| 55 | MSE($\hat{\gamma}$) | 0.02722[1] | 0.03279[4] | 0.04182[10] | 0.02807[2] | 0.04105[9] | 0.04373[12] | 0.03436[6] | 0.03935[7] | 0.04276[11] | 0.03403[5] | 0.05916[15] | 0.04069[8] | 0.05866[14] | 0.03032[3] | 0.05664[13] |
| 55 | MRE($\hat{\eta}$) | 0.09547[2] | 0.10524[9] | 0.11634[11] | 0.09072[1] | 0.11635[12] | 0.12735[14] | 0.09971[7] | 0.09813[5] | 0.10433[8] | 0.10985[10] | 0.09841[6] | 0.09785[4] | 0.1301[15] | 0.0974[3] | 0.1222[13] |
| 55 | MRE($\hat{\gamma}$) | 0.08588[1] | 0.09593[4] | 0.10603[10] | 0.09153[2] | 0.10554[9] | 0.10895[11] | 0.09597[5] | 0.10244[7] | 0.11079[12] | 0.09733[6] | 0.12525[13] | 0.10476[8] | 0.13108[15] | 0.09277[3] | 0.12781[14] |
| 55 | $D_{abs}$ | 0.03351[1] | 0.03445[4] | 0.03599[7] | 0.03376[3] | 0.0375[10] | 0.03741[9] | 0.03361[2] | 0.03576[6] | 0.0414[13] | 0.04087[12] | 0.0407[11] | 0.0354[5] | 0.04551[15] | 0.03639[8] | 0.045[14] |
| 55 | $D_{max}$ | 0.05378[1] | 0.05614[4] | 0.05913[8] | 0.05402[2] | 0.06092[9] | 0.06136[10] | 0.05459[3] | 0.058[7] | 0.06616[12] | 0.06438[11] | 0.0663[13] | 0.05742[5] | 0.07374[15] | 0.05797[6] | 0.07279[14] |
| 55 | ASAE | 0.02457[8] | 0.02271[2] | 0.02331[5] | 0.02421[6] | 0.02281[3] | 0.0246[9] | 0.0226[1] | 0.02302[4] | 0.03324[13] | 0.03079[12] | 0.03024[11] | 0.02452[7] | 0.03543[15] | 0.02882[10] | 0.03536[14] |
| 55 | $\sum$ *Ranks* | 20 [1.5] | 47[5] | 85[10] | 20 [1.5] | 84[9] | 105[13] | 42[4] | 53[6] | 97[12] | 82[8] | 92[11] | 58[7] | 133[15] | 41[3] | 121[14] |
| 110 | BIAS($\hat{\eta}$) | 0.17365[5] | 0.1703[2] | 0.198[11] | 0.15849[1] | 0.20026[12] | 0.22486[14] | 0.18438[8] | 0.17092[4] | 0.19123[10] | 0.18552[9] | 0.17086[3] | 0.17467[6] | 0.24389[15] | 0.17859[7] | 0.22345[13] |
| 110 | BIAS($\hat{\gamma}$) | 0.09602[1] | 0.1002[3] | 0.11205[9] | 0.09783[2] | 0.1123[10] | 0.11494[11] | 0.10429[6] | 0.10442[7] | 0.12139[12] | 0.1117[8] | 0.14084[15] | 0.10421[5] | 0.13782[14] | 0.10139[4] | 0.13631[13] |
| 110 | MSE($\hat{\eta}$) | 0.04969[5] | 0.04827[2] | 0.06666[11] | 0.03945[1] | 0.06869[12] | 0.08768[14] | 0.05642[9] | 0.04828[3] | 0.05837[10] | 0.05475[8] | 0.04854[4] | 0.05409[7] | 0.10096[15] | 0.05034[6] | 0.08001[13] |
| 110 | MSE($\hat{\gamma}$) | 0.01437[1] | 0.01608[3] | 0.01991[10] | 0.01464[2] | 0.0199[9] | 0.02146[11] | 0.01723[5] | 0.01793[7] | 0.02286[12] | 0.01916[8] | 0.0311[15] | 0.01771[6] | 0.02946[14] | 0.01637[4] | 0.02893[13] |
| 110 | MRE($\hat{\eta}$) | 0.06946[5] | 0.06812[2] | 0.0792[11] | 0.0634[1] | 0.0801[12] | 0.08995[14] | 0.07375[8] | 0.06837[4] | 0.07649[10] | 0.07421[9] | 0.06834[3] | 0.06987[6] | 0.09755[15] | 0.07144[7] | 0.08938[13] |
| 110 | MRE($\hat{\gamma}$) | 0.06401[1] | 0.0668[3] | 0.0747[9] | 0.06522[2] | 0.07487[10] | 0.07663[11] | 0.06953[6] | 0.06961[7] | 0.08093[12] | 0.07447[8] | 0.09389[15] | 0.06947[5] | 0.09188[14] | 0.06759[4] | 0.09088[13] |
| 110 | $D_{abs}$ | 0.02373[1] | 0.02447[3] | 0.02485[5] | 0.02422[2] | 0.02611[8] | 0.02622[9] | 0.02467[4] | 0.02529[6] | 0.02942[12] | 0.02854[11] | 0.02978[13] | 0.02532[7] | 0.03317[15] | 0.02684[10] | 0.03294[14] |
| 110 | $D_{max}$ | 0.03845[1] | 0.03965[3] | 0.041[7] | 0.03899[2] | 0.04255[8] | 0.04336[10] | 0.04021[4] | 0.04077[5] | 0.04758[12] | 0.04545[11] | 0.04837[13] | 0.04086[6] | 0.05384[15] | 0.04287[9] | 0.05298[14] |
| 110 | ASAE | 0.01649[6] | 0.01529[2] | 0.01549[4] | 0.01651[7] | 0.01545[3] | 0.01679[8] | 0.01521[1] | 0.01579[5] | 0.02285[13] | 0.02137[11] | 0.02162[12] | 0.01686[9] | 0.02472[15] | 0.01991[10] | 0.02471[14] |
| 110 | $\sum$ *Ranks* | 26[3] | 23[2] | 77[8] | 20[1] | 84[10] | 102[12] | 51[5] | 48[4] | 103[13] | 83[9] | 93[11] | 57[6] | 132[15] | 61[7] | 120[14] |
| 170 | BIAS($\hat{\eta}$) | 0.13352[2] | 0.141[6] | 0.1548[11] | 0.13343[1] | 0.1539[10] | 0.17258[13] | 0.13911[5] | 0.13695[4] | 0.16137[12] | 0.15152[9] | 0.13417[3] | 0.14746[8] | 0.17516[14] | 0.14287[7] | 0.17865[15] |
| 170 | BIAS($\hat{\gamma}$) | 0.07246[1] | 0.0827[4] | 0.09058[10] | 0.0764[2] | 0.08991[9] | 0.09452[12] | 0.07878[3] | 0.08676[7] | 0.09281[11] | 0.08599[6] | 0.11803[15] | 0.08965[8] | 0.10974[14] | 0.08415[5] | 0.10695[13] |
| 170 | MSE($\hat{\eta}$) | 0.0287[2] | 0.03201[6] | 0.03943[11] | 0.02781[1] | 0.03748[10] | 0.04768[13] | 0.03105[5] | 0.02947[3] | 0.04183[12] | 0.03642[8] | 0.0304[4] | 0.03702[9] | 0.04818[14] | 0.03246[7] | 0.05278[15] |
| 170 | MSE($\hat{\gamma}$) | 0.00837[1] | 0.01093[4] | 0.01345[11] | 0.00894[2] | 0.01224[8] | 0.01411[12] | 0.01019[3] | 0.01196[7] | 0.01343[10] | 0.01135[6] | 0.02207[15] | 0.01288[9] | 0.01828[14] | 0.01112[5] | 0.01757[13] |
| 170 | MRE($\hat{\eta}$) | 0.05341[2] | 0.0564[6] | 0.06192[11] | 0.05337[1] | 0.06156[10] | 0.06903[13] | 0.05565[5] | 0.05478[4] | 0.06455[12] | 0.06061[9] | 0.05367[3] | 0.05898[8] | 0.07006[14] | 0.05715[7] | 0.07146[15] |
| 170 | MRE($\hat{\gamma}$) | 0.0483[1] | 0.05513[4] | 0.06039[10] | 0.05094[2] | 0.05994[9] | 0.06302[12] | 0.05252[3] | 0.05784[7] | 0.06187[11] | 0.05732[6] | 0.07868[15] | 0.05977[8] | 0.07316[14] | 0.0561[5] | 0.0713[13] |
| 170 | $D_{abs}$ | 0.01884[1] | 0.02047[6] | 0.02072[7] | 0.01982[2] | 0.02042[5] | 0.021 [8.5] | 0.01989[3] | 0.02038[4] | 0.0245[12] | 0.02289[11] | 0.02462[13] | 0.021 [8.5] | 0.02635[15] | 0.02175[10] | 0.02628[14] |
| 170 | $D_{max}$ | 0.03041[1] | 0.03305[5] | 0.03385[7] | 0.03172[2] | 0.03338[6] | 0.03486[9] | 0.03199[3] | 0.03293[4] | 0.03924[12] | 0.03657[11] | 0.04011[13] | 0.03408[8] | 0.04278[15] | 0.03487[10] | 0.04257[14] |
| 170 | ASAE | 0.01306[8] | 0.01193[2] | 0.01199[3] | 0.01302[7] | 0.01216[4] | 0.013[6] | 0.01182[1] | 0.0123[5] | 0.01796[13] | 0.0165[11] | 0.01785[12] | 0.01325[9] | 0.01986[15] | 0.01536[10] | 0.01938[14] |
| 170 | $\sum$ *Ranks* | 19[1] | 43[4] | 81[10] | 20[2] | 71[7] | 98.5[12] | 31[3] | 45[5] | 105[13] | 77[9] | 93[11] | 75.5[8] | 129[15] | 66[6] | 126[14] |
| 240 | BIAS($\hat{\eta}$) | 0.11224[1] | 0.11643[3] | 0.14049[12] | 0.11649[4] | 0.12994[9] | 0.14897[13] | 0.11759[6] | 0.11994[7] | 0.13471[11] | 0.13031[10] | 0.11605[2] | 0.11757[5] | 0.15153[14] | 0.12194[8] | 0.15409[15] |
| 240 | BIAS($\hat{\gamma}$) | 0.06441[1] | 0.06564[2] | 0.07572[9] | 0.06717[3] | 0.07858[10] | 0.07919[11] | 0.06735[4] | 0.07002[6] | 0.08085[12] | 0.07446[8] | 0.10289[15] | 0.07394[7] | 0.08845[13] | 0.06979[5] | 0.08899[14] |
| 240 | MSE($\hat{\eta}$) | 0.02013[1] | 0.02187[4] | 0.03223[12] | 0.0206[2] | 0.02633[9] | 0.03587[13] | 0.02255[5] | 0.023[6] | 0.03061[11] | 0.02707[10] | 0.02065[3] | 0.02405[8] | 0.03713[14] | 0.02364[7] | 0.03815[15] |
| 240 | MSE($\hat{\gamma}$) | 0.00664[1] | 0.0067[2] | 0.00933[9] | 0.00696[3] | 0.00963[10] | 0.01012[11] | 0.00727[4] | 0.00775[6] | 0.01047[12] | 0.00866[7] | 0.01656[15] | 0.00877[8] | 0.01199[13] | 0.00747[5] | 0.01257[14] |
| 240 | MRE($\hat{\eta}$) | 0.0449[1] | 0.04657[3] | 0.05619[12] | 0.0466[4] | 0.05197[9] | 0.05959[13] | 0.04704[6] | 0.04797[7] | 0.05388[11] | 0.05212[10] | 0.04642[2] | 0.04703[5] | 0.06061[14] | 0.04878[8] | 0.06164[15] |
| 240 | MRE($\hat{\gamma}$) | 0.04294[1] | 0.04376[2] | 0.05048[9] | 0.04478[3] | 0.05239[10] | 0.05279[11] | 0.0449[4] | 0.04668[6] | 0.0539[12] | 0.04964[8] | 0.0686[15] | 0.04929[7] | 0.05897[13] | 0.04653[5] | 0.05933[14] |
| 240 | $D_{abs}$ | 0.01674[3] | 0.01627[1] | 0.01772[7] | 0.01702[4] | 0.01749[6] | 0.0184[10] | 0.0164[2] | 0.01747[5] | 0.02079[12] | 0.01929[11] | 0.02158[13] | 0.01774[8] | 0.02208[14] | 0.01823[9] | 0.02225[15] |
| 240 | $D_{max}$ | 0.02687[3] | 0.02631[1] | 0.02908[8] | 0.02732[4] | 0.02871 [6.5] | 0.03041[10] | 0.02669[2] | 0.02818[5] | 0.03355[12] | 0.03096[11] | 0.03503[13] | 0.02871 [6.5] | 0.03554[14] | 0.02925[9] | 0.03597[15] |
| 240 | ASAE | 0.01065[6] | 0.00996[2] | 0.01014[4] | 0.01093[9] | 0.01006[3] | 0.01082[8] | 0.00985[1] | 0.01032[5] | 0.01494[12] | 0.01378[11] | 0.01498[13] | 0.01077[7] | 0.01618[14] | 0.01278[10] | 0.01665[15] |
| 240 | $\sum$ *Ranks* | 18[1] | 20[2] | 82[9] | 36[4] | 72.5[8] | 100[12] | 34[3] | 53[5] | 105[13] | 86[10] | 91[11] | 61.5[6] | 123[14] | 66[7] | 132[15] |
| 320 | BIAS($\hat{\eta}$) | 0.09386[1] | 0.10109[6] | 0.10798[10] | 0.09772[4] | 0.11878[12] | 0.12405[13] | 0.1033[8] | 0.09709[2] | 0.11734[11] | 0.1078[9] | 0.09729[3] | 0.10078[5] | 0.12837[14] | 0.10265[7] | 0.12989[15] |
| 320 | BIAS($\hat{\gamma}$) | 0.05273[1] | 0.05695[3] | 0.06678[10] | 0.05448[2] | 0.06731[11] | 0.06785[12] | 0.05854[4] | 0.06298[7] | 0.06614[9] | 0.0631[8] | 0.08636[15] | 0.06211[6] | 0.07413[13] | 0.05892[5] | 0.07939[14] |
| 320 | MSE($\hat{\eta}$) | 0.01375[1] | 0.01603[5] | 0.01935[10] | 0.01473[3] | 0.02277[12] | 0.02538[13] | 0.01743[8] | 0.0144[2] | 0.02229[11] | 0.01822[9] | 0.01484[4] | 0.01723[7] | 0.0265[14] | 0.01647[6] | 0.0269[15] |

(Continued)

**Table 4 (continued)**

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 320 | MSE($\hat{\gamma}$) | $0.00437^{[1]}$ | $0.00504^{[3]}$ | $0.00707^{[10]}$ | $0.00464^{[2]}$ | $0.00726^{[11]}$ | $0.00731^{[12]}$ | $0.00543^{[4]}$ | $0.00624^{[7]}$ | $0.00685^{[9]}$ | $0.00605^{[6]}$ | $0.01225^{[15]}$ | $0.00631^{[8]}$ | $0.00848^{[13]}$ | $0.00545^{[5]}$ | $0.00973^{[14]}$ |
| 320 | MRE($\hat{\eta}$) | $0.03755^{[1]}$ | $0.04044^{[6]}$ | $0.04319^{[10]}$ | $0.03909^{[4]}$ | $0.04751^{[12]}$ | $0.04962^{[13]}$ | $0.04132^{[8]}$ | $0.03884^{[2]}$ | $0.04694^{[11]}$ | $0.04312^{[9]}$ | $0.03892^{[3]}$ | $0.04031^{[5]}$ | $0.05135^{[14]}$ | $0.04106^{[7]}$ | $0.05196^{[15]}$ |
| 320 | MRE($\hat{\gamma}$) | $0.03516^{[1]}$ | $0.03797^{[3]}$ | $0.04452^{[10]}$ | $0.03632^{[2]}$ | $0.04487^{[11]}$ | $0.04523^{[12]}$ | $0.03903^{[4]}$ | $0.04199^{[7]}$ | $0.04409^{[9]}$ | $0.04207^{[8]}$ | $0.05758^{[15]}$ | $0.04141^{[6]}$ | $0.04942^{[13]}$ | $0.03928^{[5]}$ | $0.05292^{[14]}$ |
| 320 | $D_{abs}$ | $0.0138^{[1]}$ | $0.0143^{[3]}$ | $0.01467^{[6]}$ | $0.01425^{[2]}$ | $0.01557^{[8]}$ | $0.01576^{[10]}$ | $0.0144^{[4]}$ | $0.01493^{[7]}$ | $0.01739^{[12]}$ | $0.01642^{[11]}$ | $0.01825^{[13]}$ | $0.01461^{[5]}$ | $0.01912^{[14]}$ | $0.01565^{[9]}$ | $0.0195^{[15]}$ |
| 320 | $D_{max}$ | $0.02211^{[1]}$ | $0.02315^{[3]}$ | $0.0242^{[7]}$ | $0.02289^{[2]}$ | $0.02561^{[9]}$ | $0.0259^{[10]}$ | $0.02335^{[4]}$ | $0.02414^{[6]}$ | $0.02794^{[12]}$ | $0.0262^{[11]}$ | $0.02974^{[13]}$ | $0.02375^{[5]}$ | $0.03078^{[14]}$ | $0.02498^{[8]}$ | $0.03153^{[15]}$ |
| 320 | ASAE | $0.00912^{[6]}$ | $0.00846^{[2]}$ | $0.00869^{[4]}$ | $0.00926^{[8]}$ | $0.00867^{[3]}$ | $0.00916^{[7]}$ | $0.00844^{[1]}$ | $0.00882^{[5]}$ | $0.0128^{[13]}$ | $0.0118^{[11]}$ | $0.01267^{[12]}$ | $0.00954^{[9]}$ | $0.0143^{[15]}$ | $0.0109^{[10]}$ | $0.01406^{[14]}$ |
| 320 | $\sum Ranks$ | $14^{[1]}$ | $34^{[3]}$ | $77^{[8]}$ | $29^{[2]}$ | $89^{[10]}$ | $102^{[13]}$ | $45^{[4.5]}$ | $45^{[4.5]}$ | $97^{[12]}$ | $82^{[9]}$ | $93^{[11]}$ | $56^{[6]}$ | $124^{[14]}$ | $62^{[7]}$ | $131^{[15]}$ |

**Table 5:** Numerical values of simulation measures for $\eta = 0.4, \gamma = 3.0$

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | BIAS($\hat{\eta}$) | 0.07453[2] | 0.08873[3] | 0.09721[11] | 0.0951[9] | 0.09452[8] | 0.09115[6] | 0.08976[4] | 0.10007[12] | 0.08999[5] | 0.09555[10] | 0.1185[15] | 0.06253[1] | 0.11211[14] | 0.09263[7] | 0.11194[13] |
| 20 | BIAS($\hat{\gamma}$) | 0.31561[2] | 0.39105[5] | 0.48542[12] | 0.39621[6] | 0.43515[10] | 0.44948[11] | 0.421[8] | 0.4335[9] | 0.33726[3] | 0.40292[7] | 0.56089[15] | 0.13829[1] | 0.51944[13] | 0.37957[4] | 0.52393[14] |
| 20 | MSE($\hat{\eta}$) | 0.00882[2] | 0.012[3] | 0.01444[11] | 0.01388[9] | 0.01378[8] | 0.0127[5] | 0.01227[4] | 0.015[12] | 0.01323[6] | 0.01439[10] | 0.0211[15] | 0.007[1] | 0.01979[14] | 0.01361[7] | 0.01929[13] |
| 20 | MSE($\hat{\gamma}$) | 0.16886[2] | 0.25735[7] | 0.42936[14] | 0.23313[5] | 0.30709[8] | 0.37161[11] | 0.3126[9] | 0.32536[10] | 0.21605[3] | 0.24764[6] | 0.55778[15] | 0.06273[1] | 0.40866[13] | 0.22486[4] | 0.39896[12] |
| 20 | MRE($\hat{\eta}$) | 0.18633[2] | 0.22183[3] | 0.24302[11] | 0.23776[9] | 0.23629[8] | 0.22787[6] | 0.22441[4] | 0.25018[12] | 0.22498[5] | 0.23888[10] | 0.29624[15] | 0.15632[1] | 0.28027[14] | 0.23157[7] | 0.27986[13] |
| 20 | MRE($\hat{\gamma}$) | 0.1052[2] | 0.13035[5] | 0.16181[12] | 0.13207[6] | 0.14505[10] | 0.14983[11] | 0.14033[8] | 0.1445[9] | 0.11242[3] | 0.13431[7] | 0.18696[15] | 0.0461[1] | 0.17315[13] | 0.12652[4] | 0.17464[14] |
| 20 | $D_{abs}$ | 0.05087[1] | 0.05839[4] | 0.06062[9] | 0.06002[7] | 0.05987[6] | 0.06082[11] | 0.0571[3] | 0.06075[10] | 0.06505[12] | 0.06056[8] | 0.06515[13] | 0.05435[2] | 0.06938[14] | 0.05973[5] | 0.07164[15] |
| 20 | $D_{max}$ | 0.08241[2] | 0.09334[4] | 0.10013[11] | 0.09358[5] | 0.09628[8] | 0.09998[10] | 0.09259[3] | 0.0981[9] | 0.10065[12] | 0.09617[7] | 0.10696[13] | 0.07962[1] | 0.1114[14] | 0.09437[6] | 0.11488[15] |
| 20 | ASAE | 0.04608[7] | 0.04271[4] | 0.04432[6] | 0.044[5] | 0.04241[3] | 0.04186[1] | 0.04208[2] | 0.04859[9] | 0.05801[12] | 0.05195[11] | 0.05961[14] | 0.05002[10] | 0.05858[13] | 0.04823[8] | 0.06071[15] |
| 20 | $\sum$ Ranks | 22[2] | 38[3] | 97[12] | 61[6.5] | 69[8] | 72[9] | 45[4] | 92[11] | 61[6.5] | 76[10] | 130[15] | 19[1] | 122[13] | 52[5] | 124[14] |
| 55 | BIAS($\hat{\eta}$) | 0.04856[2] | 0.05464[5] | 0.05894[10] | 0.05428[4] | 0.05659[7] | 0.0557[6] | 0.05402[3] | 0.05845[9] | 0.05907[11] | 0.06048[12] | 0.07713[15] | 0.03999[1] | 0.06901[13] | 0.05794[8] | 0.07001[14] |
| 55 | BIAS($\hat{\gamma}$) | 0.21618[3] | 0.21971[5] | 0.27848[12] | 0.21884[4] | 0.25132[10] | 0.26187[11] | 0.24139[7] | 0.24834[9] | 0.21219[2] | 0.24552[8] | 0.33378[15] | 0.09906[1] | 0.29789[13] | 0.23583[6] | 0.30095[14] |
| 55 | MSE($\hat{\eta}$) | 0.00361[2] | 0.00463[4] | 0.00542[9] | 0.00466[5] | 0.00503[7] | 0.00472[6] | 0.00457[3] | 0.00543[10] | 0.00572[11] | 0.00583[12] | 0.00936[15] | 0.00273[1] | 0.00747[13] | 0.00534[8] | 0.00779[14] |
| 55 | MSE($\hat{\gamma}$) | 0.07419[2] | 0.07682[4] | 0.12709[12] | 0.07529[3] | 0.10284[9] | 0.10988[10] | 0.09427[8] | 0.10999[11] | 0.08308[5] | 0.09345[7] | 0.18752[15] | 0.02779[1] | 0.13619[14] | 0.0879[6] | 0.13266[13] |
| 55 | MRE($\hat{\eta}$) | 0.12141[2] | 0.13659[5] | 0.14735[10] | 0.1357[4] | 0.14146[7] | 0.13924[6] | 0.13505[3] | 0.14613[9] | 0.14767[11] | 0.1512[12] | 0.19282[15] | 0.09998[1] | 0.17251[13] | 0.14485[8] | 0.17501[14] |
| 55 | MRE($\hat{\gamma}$) | 0.07206[3] | 0.07324[5] | 0.09283[12] | 0.07295[4] | 0.08377[10] | 0.08729[11] | 0.08046[7] | 0.08278[9] | 0.07073[2] | 0.08184[8] | 0.11126[15] | 0.03302[1] | 0.0993[13] | 0.07861[6] | 0.10032[14] |
| 55 | $D_{abs}$ | 0.03163[1] | 0.03574[4] | 0.03589[6] | 0.03577[5] | 0.03723[9] | 0.03708[8] | 0.03526[3] | 0.03606[7] | 0.04197[13] | 0.03979[11] | 0.04156[12] | 0.03357[2] | 0.04459[14] | 0.03737[10] | 0.04497[15] |
| 55 | $D_{max}$ | 0.05157[2] | 0.05702[4] | 0.0595[7] | 0.05659[3] | 0.06007[9] | 0.06081[10] | 0.05749[5] | 0.0584[6] | 0.06533[12] | 0.06323[11] | 0.06863[13] | 0.05031[1] | 0.07174[14] | 0.05977[8] | 0.07233[15] |
| 55 | ASAE | 0.02279[4] | 0.02235[3] | 0.02297[5] | 0.02338[7] | 0.02312[6] | 0.02164[1] | 0.02212[2] | 0.0249[8] | 0.02967[12] | 0.02842[11] | 0.03375[15] | 0.02568[9] | 0.03233[14] | 0.02621[10] | 0.03137[13] |
| 55 | $\sum$ Ranks | 21[2] | 39 [3.5] | 83[11] | 39 [3.5] | 74[8] | 69[6] | 41[5] | 78[9] | 79[10] | 92[12] | 130[15] | 18[1] | 121[13] | 70[7] | 126[14] |
| 110 | BIAS($\hat{\eta}$) | 0.03638[3] | 0.03628[2] | 0.03895[7] | 0.03821[5] | 0.04117[10] | 0.03729[4] | 0.03885[6] | 0.04096[9] | 0.04266[12] | 0.04203[11] | 0.06002[15] | 0.02728[1] | 0.04892[13] | 0.0404[8] | 0.0497[14] |
| 110 | BIAS($\hat{\gamma}$) | 0.14835[2] | 0.15856[3] | 0.18474[11] | 0.16005[4] | 0.18216[10] | 0.18804[12] | 0.16453[6] | 0.17119[9] | 0.16757[7] | 0.16408[5] | 0.23971[15] | 0.07728[1] | 0.21218[13] | 0.17096[8] | 0.21525[14] |
| 110 | MSE($\hat{\eta}$) | 0.00209[2] | 0.00215[3] | 0.00232[6] | 0.00229[5] | 0.00269[10] | 0.00225[4] | 0.00233[7] | 0.00259[8] | 0.00303[12] | 0.0029[11] | 0.00564[15] | 0.00126[1] | 0.00385[13] | 0.00262[9] | 0.00387[14] |
| 110 | MSE($\hat{\gamma}$) | 0.0354[2] | 0.0406[4] | 0.05369[11] | 0.03835[3] | 0.05236[10] | 0.05849[12] | 0.04281[5] | 0.04733[8] | 0.049[9] | 0.04382[6] | 0.09231[15] | 0.01438[1] | 0.06999[14] | 0.04461[7] | 0.06865[13] |
| 110 | MRE($\hat{\eta}$) | 0.09094[3] | 0.09069[2] | 0.09738[7] | 0.09553[5] | 0.10293[10] | 0.09323[4] | 0.09711[6] | 0.10241[9] | 0.10664[12] | 0.10506[11] | 0.15006[15] | 0.06819[1] | 0.1223[13] | 0.10101[8] | 0.12424[14] |
| 110 | MRE($\hat{\gamma}$) | 0.04945[2] | 0.05285[3] | 0.06158[11] | 0.05335[4] | 0.06072[10] | 0.06268[12] | 0.05484[6] | 0.05706[9] | 0.05586[7] | 0.05469[5] | 0.0799[15] | 0.02576[1] | 0.07073[13] | 0.05699[8] | 0.07175[14] |
| 110 | $D_{abs}$ | 0.02327[2] | 0.02412[3] | 0.02538[8] | 0.0243[4] | 0.0262[10] | 0.02561[9] | 0.02477[6] | 0.02449[5] | 0.02963[12] | 0.02751[11] | 0.03166[13] | 0.02244[1] | 0.03179[14] | 0.02529[7] | 0.0321[15] |
| 110 | $D_{max}$ | 0.03768[2] | 0.03898[3] | 0.04191[8] | 0.03911[4] | 0.04268[10] | 0.0426[9] | 0.04001[6] | 0.03975[5] | 0.047[12] | 0.04365[11] | 0.05182[14] | 0.03413[1] | 0.05134[13] | 0.04086[7] | 0.05196[15] |
| 110 | ASAE | 0.01486[6] | 0.01421[3] | 0.0146[4] | 0.01508[7] | 0.0148[5] | 0.01406[1] | 0.01408[2] | 0.01591[8] | 0.01925[12] | 0.01788[11] | 0.02308[15] | 0.0163[9] | 0.02114[13] | 0.0168[10] | 0.02119[14] |
| 110 | $\sum$ Ranks | 24[2] | 26[3] | 73[9] | 41[4] | 85[11] | 67[6] | 50[5] | 70[7] | 95[12] | 82[10] | 132[15] | 17[1] | 119[13] | 72[8] | 127[14] |
| 170 | BIAS($\hat{\eta}$) | 0.02832[2] | 0.02992[3] | 0.03364[11] | 0.03104[5] | 0.03252[9] | 0.03133[6] | 0.03048[4] | 0.03231[7] | 0.03477[12] | 0.0328[10] | 0.04715[15] | 0.02311[1] | 0.03881[13] | 0.03238[8] | 0.04159[14] |
| 170 | BIAS($\hat{\gamma}$) | 0.12128[2] | 0.12812[4] | 0.14402[10] | 0.12748[3] | 0.14516[11] | 0.15248[12] | 0.12848[5] | 0.13787[9] | 0.13152[6] | 0.13762[8] | 0.18687[15] | 0.07011[1] | 0.16428[13] | 0.13526[7] | 0.1796[14] |
| 170 | MSE($\hat{\eta}$) | 0.00125[2] | 0.00144[3] | 0.00175[11] | 0.00153[5] | 0.00162[7] | 0.00154[6] | 0.00148[4] | 0.0017[9] | 0.00197[12] | 0.00172[10] | 0.0035[15] | 0.00094[1] | 0.00232[13] | 0.00164[8] | 0.00272[14] |
| 170 | MSE($\hat{\gamma}$) | 0.02269[2] | 0.02654[5] | 0.03351[11] | 0.02497[3] | 0.03324[10] | 0.03727[12] | 0.02574[4] | 0.03057[9] | 0.02989[8] | 0.0297[7] | 0.05478[15] | 0.01149[1] | 0.04073[13] | 0.02898[6] | 0.04834[14] |
| 170 | MRE($\hat{\eta}$) | 0.07081[2] | 0.07479[3] | 0.08409[11] | 0.07761[5] | 0.0813[9] | 0.07834[6] | 0.0762[4] | 0.08077[7] | 0.08692[12] | 0.08199[10] | 0.11787[15] | 0.05777[1] | 0.09701[13] | 0.08096[8] | 0.10396[14] |
| 170 | MRE($\hat{\gamma}$) | 0.04043[2] | 0.04271[4] | 0.04801[10] | 0.04249[3] | 0.04839[11] | 0.05083[12] | 0.04283[5] | 0.04596[9] | 0.04384[6] | 0.04587[8] | 0.06229[15] | 0.02337[1] | 0.05476[13] | 0.04509[7] | 0.05987[14] |
| 170 | $D_{abs}$ | 0.01769[1] | 0.01941[3] | 0.02151[10] | 0.01982[4] | 0.021[8] | 0.02091[7] | 0.02001[5] | 0.02019[6] | 0.02463[12] | 0.02181[11] | 0.02529[13] | 0.01868[2] | 0.02627[15] | 0.02122[9] | 0.02579[14] |
| 170 | $D_{max}$ | 0.02888[2] | 0.03148[3] | 0.03491[10] | 0.03183[4] | 0.03413[7] | 0.03455[9] | 0.03239[5] | 0.03285[6] | 0.03891[12] | 0.03508[11] | 0.04118[13] | 0.02861[1] | 0.0422[15] | 0.03421[8] | 0.04198[14] |
| 170 | ASAE | 0.011[4] | 0.01082[3] | 0.01118[6] | 0.01126[7] | 0.01111[5] | 0.01066[2] | 0.01063[1] | 0.01211[8] | 0.01481[12] | 0.01349[11] | 0.01769[15] | 0.01217[9] | 0.01588[14] | 0.01299[10] | 0.01587[13] |
| 170 | $\sum$ Ranks | 19[2] | 31[3] | 90[11] | 39[5] | 77[9] | 72[8] | 37[4] | 70[6] | 92[12] | 86[10] | 131[15] | 18[1] | 122[13] | 71[7] | 125[14] |
| 240 | BIAS($\hat{\eta}$) | 0.02538[4] | 0.02502[2] | 0.0274[9] | 0.02529[3] | 0.02766[10] | 0.02671[6] | 0.02629[5] | 0.02694[7] | 0.02902[11] | 0.02926[12] | 0.04084[15] | 0.01953[1] | 0.03389[13] | 0.02709[8] | 0.03425[14] |
| 240 | BIAS($\hat{\gamma}$) | 0.10421[2] | 0.10503[4] | 0.12328[10] | 0.10444[3] | 0.12542[11] | 0.12545[12] | 0.10767[6] | 0.10738[5] | 0.11777[9] | 0.11567[8] | 0.1642[15] | 0.05871[1] | 0.14341[13] | 0.1098[7] | 0.14605[14] |
| 240 | MSE($\hat{\eta}$) | 0.00099[3] | 0.00097[2] | 0.00117[9] | 0.001[4] | 0.0012[10] | 0.0011[6] | 0.00107[5] | 0.00115 [7.5] | 0.00135[11] | 0.00138[12] | 0.00267[15] | 0.00066[1] | 0.00182[14] | 0.00115 [7.5] | 0.00181[13] |
| 240 | MSE($\hat{\gamma}$) | 0.01678[2] | 0.01759[4] | 0.024[10] | 0.0168[3] | 0.02515[12] | 0.02451[11] | 0.0182[5] | 0.01835[6] | 0.02382[9] | 0.02124[8] | 0.04285[15] | 0.00823[1] | 0.03251[13] | 0.01863[7] | 0.03297[14] |
| 240 | MRE($\hat{\eta}$) | 0.06346[4] | 0.06255[2] | 0.06849[9] | 0.06322[3] | 0.06915[10] | 0.06677[6] | 0.06573[5] | 0.06735[7] | 0.07256[11] | 0.07316[12] | 0.10211[15] | 0.04882[1] | 0.08472[13] | 0.06772[8] | 0.08561[14] |
| 240 | MRE($\hat{\gamma}$) | 0.03474[2] | 0.03501[4] | 0.04109[10] | 0.03481[3] | 0.04181[11] | 0.04182[12] | 0.03589[6] | 0.03579[5] | 0.03926[9] | 0.03856[8] | 0.05473[15] | 0.01957[1] | 0.0478[13] | 0.0366[7] | 0.04868[14] |
| 240 | $D_{abs}$ | 0.01582[1] | 0.01634[4] | 0.01759[9] | 0.01633[3] | 0.01733[7] | 0.01742[8] | 0.01715[5] | 0.0173[6] | 0.02019[12] | 0.01935[11] | 0.02083[13] | 0.01603[2] | 0.02195[15] | 0.01839[10] | 0.02181[14] |
| 240 | $D_{max}$ | 0.0256[2] | 0.02639[4] | 0.02883[9] | 0.02624[3] | 0.0284[7] | 0.02871[8] | 0.02756[5] | 0.02771[6] | 0.03226[12] | 0.03088[11] | 0.03425[13] | 0.02456[1] | 0.03537 [14.5] | 0.02933[10] | 0.03537 [14.5] |
| 240 | ASAE | 0.00904[6] | 0.00857[1] | 0.00898[5] | 0.00907[7] | 0.00887[4] | 0.00877[3] | 0.00873[2] | 0.00966[8] | 0.01202[12] | 0.0112[11] | 0.01443[15] | 0.00998[9] | 0.01288[14] | 0.01059[10] | 0.01286[13] |
| 240 | $\sum$ Ranks | 26[2] | 27[3] | 80[9] | 32[4] | 82[10] | 72[7] | 44[5] | 57.5[6] | 96[12] | 93[11] | 131[15] | 18[1] | 122.5[13] | 74.5[8] | 124.5[14] |
| 320 | BIAS($\hat{\eta}$) | 0.02211[5] | 0.02169[4] | 0.02305[7] | 0.02127[3] | 0.02326[8] | 0.02088[2] | 0.02249[6] | 0.02457[10] | 0.02461[11] | 0.02523[12] | 0.03752[15] | 0.01811[1] | 0.0302[14] | 0.02449[9] | 0.02924[13] |
| 320 | BIAS($\hat{\gamma}$) | 0.08943[2] | 0.09091[5] | 0.10248[8] | 0.08958[3] | 0.10638[11] | 0.10893[12] | 0.09008[4] | 0.10377[9] | 0.10052[7] | 0.10468[10] | 0.14988[15] | 0.05572[1] | 0.1241[14] | 0.09648[6] | 0.12177[13] |
| 320 | MSE($\hat{\eta}$) | 0.00079 [5.5] | 0.00074[4] | 0.00082[7] | 0.00071[3] | 0.00087[8] | 0.00069[2] | 0.00079[5.5] | 0.00093[9] | 0.00101[11.5] | 0.00101[11.5] | 0.00227[15] | 0.00058[1] | 0.00143[14] | 0.00097[10] | 0.00133[13] |

(Continued)

**Table 5 (continued)**

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 320 | MSE($\hat{\gamma}$) | $0.01276^{\{3\}}$ | $0.01297^{\{4\}}$ | $0.01693^{\{7\}}$ | $0.01237^{\{2\}}$ | $0.01827^{\{11\}}$ | $0.0183^{\{12\}}$ | $0.01302^{\{5\}}$ | $0.01694^{\{8\}}$ | $0.01709^{\{10\}}$ | $0.01706^{\{9\}}$ | $0.03658^{\{15\}}$ | $0.00793^{\{1\}}$ | $0.02357^{\{14\}}$ | $0.01494^{\{6\}}$ | $0.02287^{\{13\}}$ |
| 320 | MRE($\hat{\eta}$) | $0.05527^{\{5\}}$ | $0.05423^{\{4\}}$ | $0.05761^{\{7\}}$ | $0.05318^{\{3\}}$ | $0.05815^{\{8\}}$ | $0.05219^{\{2\}}$ | $0.05622^{\{6\}}$ | $0.06143^{\{10\}}$ | $0.06152^{\{11\}}$ | $0.06308^{\{12\}}$ | $0.0938^{\{15\}}$ | $0.04528^{\{1\}}$ | $0.07551^{\{14\}}$ | $0.06122^{\{9\}}$ | $0.0731^{\{13\}}$ |
| 320 | MRE($\hat{\gamma}$) | $0.02981^{\{2\}}$ | $0.0303^{\{5\}}$ | $0.03416^{\{8\}}$ | $0.02986^{\{3\}}$ | $0.03546^{\{11\}}$ | $0.03631^{\{12\}}$ | $0.03003^{\{4\}}$ | $0.03459^{\{9\}}$ | $0.03351^{\{7\}}$ | $0.03489^{\{10\}}$ | $0.04996^{\{15\}}$ | $0.01857^{\{1\}}$ | $0.04137^{\{14\}}$ | $0.03216^{\{6\}}$ | $0.04059^{\{13\}}$ |
| 320 | $D_{abs}$ | $0.01388^{\{1\}}$ | $0.0142^{\{4\}}$ | $0.01516^{\{8\}}$ | $0.01406^{\{2\}}$ | $0.01495^{\{6.5\}}$ | $0.01495^{\{6.5\}}$ | $0.01458^{\{5\}}$ | $0.01541^{\{9\}}$ | $0.01694^{\{12\}}$ | $0.01656^{\{11\}}$ | $0.01904^{\{14\}}$ | $0.01419^{\{3\}}$ | $0.01969^{\{15\}}$ | $0.0157^{\{10\}}$ | $0.01891^{\{13\}}$ |
| 320 | $D_{max}$ | $0.02243^{\{2\}}$ | $0.0229^{\{4\}}$ | $0.02469^{\{7\}}$ | $0.02259^{\{3\}}$ | $0.02452^{\{6\}}$ | $0.02488^{\{8\}}$ | $0.02333^{\{5\}}$ | $0.02497^{\{9\}}$ | $0.02714^{\{12\}}$ | $0.02661^{\{11\}}$ | $0.03142^{\{14\}}$ | $0.02182^{\{1\}}$ | $0.03165^{\{15\}}$ | $0.02511^{\{10\}}$ | $0.03052^{\{13\}}$ |
| 320 | ASAE | $0.00755^{\{6\}}$ | $0.00747^{\{3\}}$ | $0.00757^{\{7\}}$ | $0.0075^{\{4.5\}}$ | $0.0075^{\{4.5\}}$ | $0.00727^{\{2\}}$ | $0.00723^{\{1\}}$ | $0.00834^{\{8.5\}}$ | $0.00998^{\{12\}}$ | $0.00939^{\{11\}}$ | $0.01266^{\{15\}}$ | $0.00834^{\{8.5\}}$ | $0.0112^{\{14\}}$ | $0.0088^{\{10\}}$ | $0.01081^{\{13\}}$ |
| 320 | $\sum$ *Ranks* | $31.5^{\{3\}}$ | $37^{\{4\}}$ | $66^{\{7\}}$ | $26.5^{\{2\}}$ | $74^{\{8\}}$ | $58.5^{\{6\}}$ | $41.5^{\{5\}}$ | $81.5^{\{10\}}$ | $93.5^{\{11\}}$ | $97.5^{\{12\}}$ | $133^{\{15\}}$ | $18.5^{\{1\}}$ | $128^{\{14\}}$ | $76^{\{9\}}$ | $117^{\{13\}}$ |

**Table 6:** Numerical values of simulation measures for $\eta = 3.5, \gamma = 0.2$

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | BIAS($\hat{\eta}$) | 0.7538[12] | 0.72316[9] | 0.8165[14] | 0.64915[3] | 0.71366[7] | 0.85923[15] | 0.67884[6] | 0.77581[13] | 0.56839[2] | 0.66168[5] | 0.72149[8] | 0.10483[1] | 0.7418[10] | 0.66132[4] | 0.74588[11] |
| 20 | BIAS($\hat{\gamma}$) | 0.03181[6] | 0.03051[2] | 0.03458[10] | 0.03085[4] | 0.03249[7] | 0.03479[12] | 0.03175[5] | 0.03387[9] | 0.03476[11] | 0.03279[8] | 0.03842[13] | 0.02948[1] | 0.04052[14] | 0.03055[3] | 0.04351[15] |
| 20 | MSE($\hat{\eta}$) | 1.06785[12] | 0.93714[11] | 1.18863[14] | 0.66174[3] | 0.83006[9] | 1.28442[15] | 0.81695[8] | 1.11836[13] | 0.64351[2] | 0.71023[4] | 0.88623[10] | 0.06197[1] | 0.75327[7] | 0.74982[6] | 0.74349[5] |
| 20 | MSE($\hat{\gamma}$) | 0.00174[8] | 0.00153[4] | 0.00201[11] | 0.00145[2] | 0.0017[7] | 0.00203[12] | 0.00155[5] | 0.00192[9] | 0.00193[10] | 0.00158[6] | 0.00249[14] | 0.00145[2] | 0.00239[13] | 0.00145[2] | 0.00261[15] |
| 20 | MRE($\hat{\eta}$) | 0.21537[12] | 0.20662[9] | 0.23329[14] | 0.18547[3] | 0.2039[7] | 0.2455[15] | 0.19395[6] | 0.22166[13] | 0.1624[2] | 0.18905[5] | 0.20614[8] | 0.02995[1] | 0.21194[10] | 0.18895[4] | 0.21311[11] |
| 20 | MRE($\hat{\gamma}$) | 0.15907[6] | 0.15254[2] | 0.17292[10] | 0.15423[4] | 0.16247[7] | 0.17395[12] | 0.15874[5] | 0.16933[9] | 0.17379[11] | 0.16394[8] | 0.19209[13] | 0.14739[1] | 0.2026[14] | 0.15273[3] | 0.21755[15] |
| 20 | $D_{abs}$ | 0.05433[2] | 0.05559[3] | 0.05818[9] | 0.05781[6] | 0.05863[10] | 0.05805[8] | 0.05698[4] | 0.05798[7] | 0.06372[11] | 0.06474[13] | 0.06414[12] | 0.05431[1] | 0.07142[14] | 0.05767[5] | 0.07298[15] |
| 20 | $D_{max}$ | 0.08814[2] | 0.0886[3] | 0.09482[10] | 0.08975[4] | 0.09236[7] | 0.09458[9] | 0.0899[5] | 0.09315[8] | 0.1005[11] | 0.10136[12] | 0.10375[13] | 0.08369[1] | 0.11258[14] | 0.09074[6] | 0.11618[15] |
| 20 | ASAE | 0.0441[3] | 0.05219[7] | 0.05844[10] | 0.04604[4] | 0.06406[11] | 0.07766[13] | 0.05374[8] | 0.04231[2] | 0.05693[9] | 0.05003[5] | 0.03504[1] | 0.10907[15] | 0.07685[12] | 0.05109[6] | 0.07963[14] |
| 20 | $\sum$ Ranks | 63[6] | 50[4] | 102[12] | 33[2] | 72[9] | 111[14] | 52[5] | 83[10] | 69[8] | 66[7] | 92[11] | 24[1] | 108[13] | 39[3] | 116[15] |
| 55 | BIAS($\hat{\eta}$) | 0.44009[6] | 0.4439[7] | 0.54508[14] | 0.41097[3] | 0.50638[13] | 0.59692[15] | 0.47585[11] | 0.42945[5] | 0.35547[2] | 0.45007[9] | 0.44484[8] | 0.08223[1] | 0.49834[12] | 0.41959[4] | 0.4603[10] |
| 55 | BIAS($\hat{\gamma}$) | 0.01785[2] | 0.01884[5] | 0.02184[10] | 0.018[3] | 0.02074[9] | 0.02235[12] | 0.02001[7] | 0.02015[8] | 0.02228[11] | 0.01894[6] | 0.02506[13] | 0.01781[1] | 0.02607[15] | 0.01843[4] | 0.02507[14] |
| 55 | MSE($\hat{\eta}$) | 0.34527[8] | 0.35082[9] | 0.57023[14] | 0.27509[3] | 0.46923[13] | 0.69876[15] | 0.42055[12] | 0.32442[6] | 0.26477[2] | 0.33113[7] | 0.35684[11] | 0.04327[1] | 0.35661[10] | 0.31475[4] | 0.3176[5] |
| 55 | MSE($\hat{\gamma}$) | 0.00054[4] | 0.00057[6] | 0.00079[11] | 0.00049[1] | 7e-04[9] | 0.00084[12] | 0.00064[7] | 0.00066[8] | 0.00076[10] | 0.00055[5] | 0.00102[15] | 0.00051[2] | 0.00098[14] | 0.00053[3] | 9e-04[13] |
| 55 | MRE($\hat{\eta}$) | 0.12574[6] | 0.12683[7] | 0.15574[14] | 0.11742[3] | 0.14468[13] | 0.17055[15] | 0.13596[11] | 0.1227[5] | 0.10156[2] | 0.12859[9] | 0.1271[8] | 0.02349[1] | 0.14238[12] | 0.11988[4] | 0.13151[10] |
| 55 | MRE($\hat{\gamma}$) | 0.08925[2] | 0.09419[5] | 0.1092[10] | 0.09001[3] | 0.10369[9] | 0.11174[12] | 0.10006[7] | 0.10073[8] | 0.1114[11] | 0.0947[6] | 0.12528[13] | 0.08905[1] | 0.13036[15] | 0.09214[4] | 0.12534[14] |
| 55 | $D_{abs}$ | 0.03561[5] | 0.03484[3] | 0.0362[7] | 0.03433[2] | 0.03801[10] | 0.03666[9] | 0.03607[6] | 0.03551[4] | 0.03924[12] | 0.03872[11] | 0.04086[13] | 0.03297[1] | 0.04534[15] | 0.03637[8] | 0.04312[14] |
| 55 | $D_{max}$ | 0.05702[4] | 0.05587[3] | 0.05978[8] | 0.05458[2] | 0.06127[10] | 0.06097[9] | 0.05855[7] | 0.05757[5] | 0.06296[12] | 0.06142[11] | 0.06631[13] | 0.05147[1] | 0.07299[15] | 0.05777[6] | 0.06967[14] |
| 55 | ASAE | 0.01892[4] | 0.02014[6] | 0.02557[10] | 0.01769[2] | 0.02564[11] | 0.02932[15] | 0.02036[7] | 0.01803[3] | 0.02203[8] | 0.02289[9] | 0.01601[1] | 0.0273[12] | 0.02833[14] | 0.02005[5] | 0.02779[13] |
| 55 | $\sum$ Ranks | 41[3] | 51[5] | 97[11] | 22[2] | 102[12] | 113[14] | 75[9] | 52[6] | 69[7] | 73[8] | 93[10] | 21[1] | 120[15] | 42[4] | 109[13] |
| 110 | BIAS($\hat{\eta}$) | 0.29163[5] | 0.28724[4] | 0.36332[14] | 0.27753[3] | 0.32579[12] | 0.39853[15] | 0.29817[7] | 0.29711[6] | 0.26954[2] | 0.31214[10] | 0.30078[9] | 0.06094[1] | 0.3271[13] | 0.31476[11] | 0.29874[8] |
| 110 | BIAS($\hat{\gamma}$) | 0.01197[3] | 0.01247[4] | 0.0148[10] | 0.01185[1] | 0.01441[9] | 0.01556[12] | 0.01334[5] | 0.01402[8] | 0.01501[11] | 0.01378[7] | 0.01788[14] | 0.01195[2] | 0.01719[13] | 0.01348[6] | 0.01866[15] |
| 110 | MSE($\hat{\eta}$) | 0.14548[4] | 0.13856[3] | 0.21942[14] | 0.11886[2] | 0.18334[13] | 0.28982[15] | 0.14945[7] | 0.15146[8] | 0.14808[5] | 0.15762[9] | 0.14915[6] | 0.01654[1] | 0.16431[12] | 0.15827[10] | 0.15954[11] |
| 110 | MSE($\hat{\gamma}$) | 0.00023[2.5] | 0.00025[4] | 0.00035[10.5] | 0.00022[1] | 0.00032[8] | 4e-04[12] | 0.00029[5] | 0.00032[8] | 0.00035[10.5] | 3e-04[6] | 0.00051[14] | 0.00023[2.5] | 0.00044[13] | 0.00032[8] | 0.00053[15] |
| 110 | MRE($\hat{\eta}$) | 0.08332[5] | 0.08207[4] | 0.10381[14] | 0.07929[3] | 0.09308[12] | 0.11387[15] | 0.08519[7] | 0.08489[6] | 0.07701[2] | 0.08918[10] | 0.08594[9] | 0.01741[1] | 0.09346[13] | 0.08993[11] | 0.08536[8] |
| 110 | MRE($\hat{\gamma}$) | 0.05986[3] | 0.06233[4] | 0.07402[10] | 0.05926[1] | 0.07205[9] | 0.07779[12] | 0.06669[5] | 0.07011[8] | 0.07504[11] | 0.06891[7] | 0.08939[14] | 0.05974[2] | 0.08595[13] | 0.0674[6] | 0.0933[15] |
| 110 | $D_{abs}$ | 0.02366[2] | 0.02452[4] | 0.02563[8] | 0.0244[3] | 0.02496[6] | 0.02624[9] | 0.02456[5] | 0.02514[7] | 0.0285[13] | 0.02742[11] | 0.02802[12] | 0.0229[1] | 0.03015[14] | 0.02715[10] | 0.03397[15] |
| 110 | $D_{max}$ | 0.03812[2] | 0.03925[4] | 0.04231[8] | 0.03859[3] | 0.04066[6] | 0.04329[9] | 0.03994[5] | 0.0408[7] | 0.0455[12] | 0.04381[11] | 0.0459[13] | 0.03557[1] | 0.04885[14] | 0.04335[10] | 0.05436[15] |
| 110 | ASAE | 0.01138[4] | 0.01309[6] | 0.01546[11] | 0.01115[3] | 0.01497[10] | 0.01803[15] | 0.01256[5] | 0.01093[2] | 0.0145[9] | 0.01446[8] | 0.01088[1] | 0.01638[12] | 0.01691[13] | 0.01319[7] | 0.01767[14] |
| 110 | $\sum$ Ranks | 30.5[3] | 37[4] | 98.5[12] | 20[1] | 84[9] | 117[15] | 51[5] | 59[6] | 74.5[7] | 87[10] | 90[11] | 23.5[2] | 116[14] | 78[8] | 114[13] |
| 170 | BIAS($\hat{\eta}$) | 0.2202[3] | 0.25674[11] | 0.27467[13] | 0.22273[4] | 0.28992[14] | 0.32004[15] | 0.25221[10] | 0.24787[8] | 0.23126[5] | 0.26473[12] | 0.2454[7] | 0.05553[1] | 0.25018[9] | 0.24004[6] | 0.18307[2] |
| 170 | BIAS($\hat{\gamma}$) | 0.00993[3] | 0.01022[4] | 0.01211[11] | 0.00914[1] | 0.01203[10] | 0.01222[12] | 0.0106[5] | 0.01198[9] | 0.0119[8] | 0.0108[6] | 0.01555[15] | 0.00984[2] | 0.01436[14] | 0.01125[7] | 0.01382[13] |
| 170 | MSE($\hat{\eta}$) | 0.08309[4] | 0.10518[10] | 0.12254[13] | 0.07672[3] | 0.14159[14] | 0.16663[15] | 0.10489[9] | 0.09896[7] | 0.10277[8] | 0.11181[12] | 0.09374[6] | 0.01154[1] | 0.1097[11] | 0.08948[5] | 0.06752[2] |
| 170 | MSE($\hat{\gamma}$) | 0.00016[2.5] | 0.00017[4] | 0.00023[10] | 0.00013[1] | 0.00023[10] | 0.00024[12] | 0.00018[5.5] | 0.00023[10] | 0.00022[7.5] | 0.00018[5.5] | 0.00038[15] | 0.00016[2.5] | 0.00032[14] | 0.00022[7.5] | 3e-04[13] |
| 170 | MRE($\hat{\eta}$) | 0.06291[3] | 0.07335[11] | 0.07848[13] | 0.06364[4] | 0.08283[14] | 0.09144[15] | 0.07206[10] | 0.07082[8] | 0.06607[5] | 0.07564[12] | 0.07011[7] | 0.01587[1] | 0.07148[9] | 0.06858[6] | 0.05231[2] |
| 170 | MRE($\hat{\gamma}$) | 0.04967[3] | 0.05111[4] | 0.06057[11] | 0.04572[1] | 0.06014[10] | 0.0611[12] | 0.05298[5] | 0.05992[9] | 0.05948[8] | 0.054[6] | 0.07774[15] | 0.04922[2] | 0.07178[14] | 0.05623[7] | 0.0691[13] |
| 170 | $D_{abs}$ | 0.01861[2] | 0.01981[5] | 0.02027[6] | 0.01807[1] | 0.02118[9] | 0.02054[8] | 0.01946[4] | 0.0205[7] | 0.02321[12] | 0.02227[11] | 0.02443[13] | 0.01876[3] | 0.02678[15] | 0.02201[10] | 0.02603[14] |
| 170 | $D_{max}$ | 0.02995[3] | 0.03204[5] | 0.03323[6] | 0.02901[1] | 0.03473[9] | 0.03415[8] | 0.03156[4] | 0.03351[7] | 0.037[12] | 0.03551[11] | 0.03974[13] | 0.02919[2] | 0.04297[15] | 0.03506[10] | 0.04127[14] |
| 170 | ASAE | 0.00904[4] | 0.01003[5] | 0.012[11] | 0.00879[3] | 0.01176[9] | 0.01441[15] | 0.01011[6] | 0.00876[2] | 0.0118[10] | 0.01115[8] | 0.00874[1] | 0.01232[12] | 0.01344[13] | 0.01046[7] | 0.01369[14] |
| 170 | $\sum$ Ranks | 27.5[3] | 59[5] | 94[12] | 19[1] | 99[13] | 112[14] | 58.5[4] | 67[7] | 75.5[8] | 83.5[9] | 91[11] | 26.5[2] | 113[15] | 65.5[6] | 89[10] |
| 240 | BIAS($\hat{\eta}$) | 0.19412[5] | 0.20476[8] | 0.22794[13] | 0.18557[3] | 0.23422[14] | 0.2795[15] | 0.21032[11] | 0.19533[7] | 0.19028[4] | 0.20899[9] | 0.20941[10] | 0.04955[1] | 0.19524[6] | 0.2109[12] | 0.14647[2] |
| 240 | BIAS($\hat{\gamma}$) | 0.00817[1.5] | 0.00844[4] | 0.00967[9] | 0.00817[1.5] | 0.0097[10.5] | 0.01059[13] | 0.00914[7] | 0.00908[6] | 0.0097[10.5] | 0.00895[5] | 0.01246[14] | 0.0083[3] | 0.01253[15] | 0.00919[8] | 0.01037[12] |
| 240 | MSE($\hat{\eta}$) | 0.06253[5] | 0.06757[6] | 0.08465[13] | 0.05275[3] | 0.09248[14] | 0.13138[15] | 0.07203[10] | 0.06233[4] | 0.07011[9] | 0.06802[7] | 0.06863[8] | 0.00997[1] | 0.07266[11] | 0.07683[12] | 0.04027[2] |
| 240 | MSE($\hat{\gamma}$) | 0.00011[2.5] | 0.00012[4.5] | 0.00015[9] | 1e-04[1] | 0.00015[9] | 0.00018[13] | 0.00013[6.5] | 0.00013[6.5] | 0.00015[9] | 0.00012[4.5] | 0.00025[15] | 0.00011[2.5] | 0.00023[14] | 0.00016[11] | 0.00017[12] |
| 240 | MRE($\hat{\eta}$) | 0.05546[5] | 0.0585[8] | 0.06512[13] | 0.05302[3] | 0.06692[14] | 0.07986[15] | 0.06009[11] | 0.05581[7] | 0.05436[4] | 0.05971[9] | 0.05983[10] | 0.01416[1] | 0.05578[6] | 0.06026[12] | 0.04185[2] |
| 240 | MRE($\hat{\gamma}$) | 0.04086[2] | 0.0422[4] | 0.04833[9] | 0.04083[1] | 0.04849[10.5] | 0.05297[13] | 0.04572[7] | 0.04542[6] | 0.04849 [10.5] | 0.04476[5] | 0.06229[14] | 0.04148[3] | 0.06263[15] | 0.04594[8] | 0.05183[12] |
| 240 | $D_{abs}$ | 0.01555[1] | 0.01618[3] | 0.01697[7] | 0.01645[4] | 0.01736[8] | 0.01777[9] | 0.01674[6] | 0.01673[5] | 0.01887[12] | 0.01874[11] | 0.01974[14] | 0.01575[2] | 0.02168[15] | 0.01791[10] | 0.01918[13] |
| 240 | $D_{max}$ | 0.02518[2] | 0.02615[3] | 0.02787[7] | 0.02617[4] | 0.02844[8] | 0.02969[11] | 0.02707[5.5] | 0.02707[5.5] | 0.02996[12] | 0.02963[10] | 0.03218[14] | 0.02458[1] | 0.03501[15] | 0.02873[9] | 0.03047[13] |
| 240 | ASAE | 0.0075[2] | 0.00836[6] | 0.00999[10] | 0.00752[3] | 0.01003[11] | 0.01256[15] | 0.00825[5] | 0.0073[1] | 0.00995[9] | 0.00956[8] | 0.00757[4] | 0.01032[12] | 0.01065[14] | 0.009[7] | 0.01058[13] |
| 240 | $\sum$ Ranks | 25[1] | 45.5[4] | 89[11] | 37.5[3] | 98[12] | 118[15] | 68[7] | 47[5] | 79[8] | 67.5[6] | 102[13] | 25.5[2] | 110[14] | 88[10] | 80[9] |
| 320 | BIAS($\hat{\eta}$) | 0.16012[4] | 0.18273[10] | 0.196[13] | 0.1632[5] | 0.19744[14] | 0.22881[15] | 0.17967[8] | 0.17962[7] | 0.17112[6] | 0.18265[9] | 0.18373[11] | 0.04902[1] | 0.11981[2] | 0.18815[12] | 0.1391[3] |
| 320 | BIAS($\hat{\gamma}$) | 0.0069[1] | 0.00768[6] | 0.00855[9] | 0.00743[4] | 0.0086[11] | 0.00903[12] | 0.00736[3] | 0.00816[7] | 0.0082[8] | 0.00764[5] | 0.01135[15] | 0.00712[2] | 0.00973[14] | 0.00857[10] | 0.00919[13] |
| 320 | MSE($\hat{\eta}$) | 0.04183[4] | 0.05346[10] | 0.06292[13] | 0.04213[5] | 0.06647[14] | 0.08551[15] | 0.05145[7] | 0.04965[6] | 0.05625[11] | 0.05292[9] | 0.05258[8] | 0.00822[1] | 0.03264[3] | 0.05783[12] | 0.0324[2] |

(Continued)

**Table 6 (continued)**

| n | Est. | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 320 | MSE($\hat{\gamma}$) | 8e-05$^{\{1.5\}}$ | 1e-04$^{\{6\}}$ | 0.00011$^{\{8\}}$ | 9e-05$^{\{4\}}$ | 0.00012$^{\{10\}}$ | 0.00013$^{\{11.5\}}$ | 9e-05$^{\{4\}}$ | 0.00011$^{\{8\}}$ | 0.00011$^{\{8\}}$ | 9e-05$^{\{4\}}$ | 2e-04$^{\{15\}}$ | 8e-05$^{\{1.5\}}$ | 0.00015$^{\{14\}}$ | 0.00014$^{\{13\}}$ | 0.00013$^{\{11.5\}}$ |
| 320 | MRE($\hat{\eta}$) | 0.04575$^{\{4\}}$ | 0.05221$^{\{10\}}$ | 0.056$^{\{13\}}$ | 0.04663$^{\{5\}}$ | 0.05641$^{\{14\}}$ | 0.06537$^{\{15\}}$ | 0.05133$^{\{8\}}$ | 0.05132$^{\{7\}}$ | 0.04889$^{\{6\}}$ | 0.05219$^{\{9\}}$ | 0.05249$^{\{11\}}$ | 0.014$^{\{1\}}$ | 0.03423$^{\{2\}}$ | 0.05376$^{\{12\}}$ | 0.03974$^{\{3\}}$ |
| 320 | MRE($\hat{\gamma}$) | 0.0345$^{\{1\}}$ | 0.03842$^{\{6\}}$ | 0.04274$^{\{9\}}$ | 0.03713$^{\{4\}}$ | 0.04299$^{\{11\}}$ | 0.04513$^{\{12\}}$ | 0.03678$^{\{3\}}$ | 0.04082$^{\{7\}}$ | 0.041$^{\{8\}}$ | 0.03821$^{\{5\}}$ | 0.05675$^{\{15\}}$ | 0.03559$^{\{2\}}$ | 0.04867$^{\{14\}}$ | 0.04287$^{\{10\}}$ | 0.04597$^{\{13\}}$ |
| 320 | $D_{abs}$ | 0.01352$^{\{1\}}$ | 0.01458$^{\{5\}}$ | 0.01481$^{\{6\}}$ | 0.01417$^{\{3\}}$ | 0.01507$^{\{8\}}$ | 0.01546$^{\{9\}}$ | 0.0143$^{\{4\}}$ | 0.01495$^{\{7\}}$ | 0.01653$^{\{11\}}$ | 0.01599$^{\{10\}}$ | 0.01814$^{\{14\}}$ | 0.01364$^{\{2\}}$ | 0.01926$^{\{15\}}$ | 0.01674$^{\{12\}}$ | 0.01738$^{\{13\}}$ |
| 320 | $D_{max}$ | 0.02175$^{\{2\}}$ | 0.02366$^{\{5\}}$ | 0.02427$^{\{7\}}$ | 0.02281$^{\{3\}}$ | 0.02459$^{\{8\}}$ | 0.02555$^{\{10\}}$ | 0.02318$^{\{4\}}$ | 0.02426$^{\{6\}}$ | 0.02638$^{\{11\}}$ | 0.02546$^{\{9\}}$ | 0.02951$^{\{14\}}$ | 0.02123$^{\{1\}}$ | 0.03028$^{\{15\}}$ | 0.02675$^{\{12\}}$ | 0.02763$^{\{13\}}$ |
| 320 | ASAE | 0.00645$^{\{4\}}$ | 0.0073$^{\{6\}}$ | 0.00881$^{\{11\}}$ | 0.00644$^{\{3\}}$ | 0.00863$^{\{10\}}$ | 0.01053$^{\{15\}}$ | 0.00713$^{\{5\}}$ | 0.00626$^{\{1\}}$ | 0.0084$^{\{9\}}$ | 0.00805$^{\{8\}}$ | 0.00643$^{\{2\}}$ | 0.00896$^{\{12\}}$ | 0.00945$^{\{14\}}$ | 0.00768$^{\{7\}}$ | 0.00918$^{\{13\}}$ |
| 320 | $\sum Ranks$ | 32.5$^{\{1\}}$ | 67$^{\{6\}}$ | 83$^{\{10\}}$ | 46$^{\{3\}}$ | 94$^{\{12.5\}}$ | 108.5$^{\{15\}}$ | 56$^{\{5\}}$ | 50$^{\{4\}}$ | 72$^{\{7\}}$ | 78$^{\{8\}}$ | 100$^{\{14\}}$ | 33.5$^{\{2\}}$ | 87$^{\{11\}}$ | 94$^{\{12.5\}}$ | 78.5$^{\{9\}}$ |

**Table 7:** Partial and overall ranks for all estimation methods of our proposed model

| Parameter | $n$ | MLE | ADE | CVME | MPSE | OLSE | RTADE | WLSE | LTADE | MSADE | MSALDE | ADSOE | KE | MSSDE | MSSLDE | MSLNDE |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\eta = 0.8, \gamma = 0.5$ | 20 | 2.0 | 1.0 | 8.0 | 4.0 | 6.0 | 7.0 | 3.0 | 9.5 | 11.0 | 12.0 | 14.0 | 9.5 | 13.0 | 5.0 | 15.0 |
| | 55 | 1.0 | 2.0 | 7.0 | 4.0 | 6.0 | 5.0 | 3.0 | 11.0 | 12.0 | 9.0 | 13.5 | 8.0 | 13.5 | 10.0 | 15.0 |
| | 110 | 3.0 | 2.0 | 7.0 | 1.0 | 5.0 | 6.0 | 4.0 | 8.0 | 12.0 | 11.0 | 13.0 | 10.0 | 15.0 | 9.0 | 14.0 |
| | 170 | 2.0 | 4.0 | 5.0 | 1.0 | 9.0 | 8.0 | 3.0 | 7.0 | 12.0 | 11.0 | 15.0 | 10.0 | 14.0 | 6.0 | 13.0 |
| | 240 | 2.0 | 3.0 | 9.0 | 1.0 | 4.0 | 8.0 | 5.0 | 7.0 | 12.0 | 11.0 | 15.0 | 6.0 | 14.0 | 10.0 | 13.0 |
| | 320 | 1.0 | 3.0 | 7.0 | 4.0 | 5.0 | 8.0 | 2.0 | 6.0 | 12.0 | 11.0 | 15.0 | 10.0 | 13.0 | 9.0 | 14.0 |
| $\eta = 1.5, \gamma = 0.85$ | 20 | 6.0 | 2.0 | 12.0 | 1.0 | 4.0 | 13.0 | 5.0 | 7.0 | 8.0 | 9.0 | 14.0 | 11.0 | 10.0 | 3.0 | 15.0 |
| | 55 | 4.0 | 1.0 | 10.0 | 2.0 | 7.0 | 12.0 | 3.0 | 5.0 | 13.0 | 11.0 | 9.0 | 8.0 | 14.0 | 6.0 | 15.0 |
| | 110 | 3.0 | 2.0 | 8.0 | 1.0 | 4.0 | 9.5 | 6.0 | 5.0 | 13.0 | 12.0 | 11.0 | 9.5 | 15.0 | 7.0 | 14.0 |
| | 170 | 1.0 | 4.0 | 7.0 | 3.0 | 6.0 | 9.5 | 2.0 | 5.0 | 12.0 | 11.0 | 13.0 | 8.0 | 14.0 | 9.5 | 15.0 |
| | 240 | 2.5 | 2.5 | 8.5 | 1.0 | 7.0 | 11.0 | 4.0 | 5.0 | 13.0 | 10.0 | 12.0 | 8.5 | 15.0 | 6.0 | 14.0 |
| | 320 | 1.0 | 2.0 | 6.0 | 3.0 | 7.0 | 10.5 | 4.0 | 5.0 | 12.0 | 8.5 | 13.0 | 10.5 | 14.0 | 8.5 | 15.0 |
| $\eta = 2.5, \gamma = 1.5$ | 20 | 5.0 | 2.0 | 12.0 | 1.0 | 8.5 | 13.0 | 8.5 | 7.0 | 6.0 | 10.0 | 11.0 | 3.0 | 14.0 | 4.0 | 15.0 |
| | 55 | 1.5 | 5.0 | 10.0 | 1.5 | 9.0 | 13.0 | 4.0 | 6.0 | 12.0 | 8.0 | 11.0 | 7.0 | 15.0 | 3.0 | 14.0 |
| | 110 | 3.0 | 2.0 | 8.0 | 1.0 | 10.0 | 12.0 | 5.0 | 4.0 | 13.0 | 9.0 | 11.0 | 6.0 | 15.0 | 7.0 | 14.0 |
| | 170 | 1.0 | 4.0 | 10.0 | 2.0 | 7.0 | 12.0 | 3.0 | 5.0 | 13.0 | 9.0 | 11.0 | 8.0 | 15.0 | 6.0 | 14.0 |
| | 240 | 1.0 | 2.0 | 9.0 | 4.0 | 8.0 | 12.0 | 3.0 | 5.0 | 13.0 | 10.0 | 11.0 | 6.0 | 14.0 | 7.0 | 15.0 |
| | 320 | 1.0 | 3.0 | 8.0 | 2.0 | 10.0 | 13.0 | 4.5 | 4.5 | 12.0 | 9.0 | 11.0 | 6.0 | 14.0 | 7.0 | 15.0 |
| $\eta = 0.4, \gamma = 3.0$ | 20 | 2.0 | 3.0 | 12.0 | 6.5 | 8.0 | 9.0 | 4.0 | 11.0 | 6.5 | 10.0 | 15.0 | 1.0 | 13.0 | 5.0 | 14.0 |
| | 55 | 2.0 | 3.5 | 11.0 | 3.5 | 8.0 | 6.0 | 5.0 | 9.0 | 10.0 | 12.0 | 15.0 | 1.0 | 13.0 | 7.0 | 14.0 |
| | 110 | 2.0 | 3.0 | 9.0 | 4.0 | 11.0 | 6.0 | 5.0 | 7.0 | 12.0 | 10.0 | 15.0 | 1.0 | 13.0 | 8.0 | 14.0 |
| | 170 | 2.0 | 3.0 | 11.0 | 5.0 | 9.0 | 8.0 | 4.0 | 6.0 | 12.0 | 10.0 | 15.0 | 1.0 | 13.0 | 7.0 | 14.0 |
| | 240 | 2.0 | 3.0 | 9.0 | 4.0 | 10.0 | 7.0 | 5.0 | 6.0 | 12.0 | 11.0 | 15.0 | 1.0 | 13.0 | 8.0 | 14.0 |
| | 320 | 3.0 | 4.0 | 7.0 | 2.0 | 8.0 | 6.0 | 5.0 | 10.0 | 11.0 | 12.0 | 15.0 | 1.0 | 14.0 | 9.0 | 13.0 |
| $\eta = 3.5, \gamma = 0.2$ | 20 | 6.0 | 4.0 | 12.0 | 2.0 | 9.0 | 14.0 | 5.0 | 10.0 | 8.0 | 7.0 | 11.0 | 1.0 | 13.0 | 3.0 | 15.0 |
| | 55 | 3.0 | 5.0 | 11.0 | 2.0 | 12.0 | 14.0 | 9.0 | 6.0 | 7.0 | 8.0 | 10.0 | 1.0 | 15.0 | 4.0 | 13.0 |
| | 110 | 3.0 | 4.0 | 12.0 | 1.0 | 9.0 | 15.0 | 5.0 | 6.0 | 7.0 | 10.0 | 11.0 | 2.0 | 14.0 | 8.0 | 13.0 |
| | 170 | 3.0 | 5.0 | 12.0 | 1.0 | 13.0 | 14.0 | 4.0 | 7.0 | 8.0 | 9.0 | 11.0 | 2.0 | 15.0 | 6.0 | 10.0 |
| | 240 | 1.0 | 4.0 | 11.0 | 3.0 | 12.0 | 15.0 | 7.0 | 5.0 | 8.0 | 6.0 | 13.0 | 2.0 | 14.0 | 10.0 | 9.0 |
| | 320 | 1.0 | 6.0 | 10.0 | 3.0 | 12.5 | 15.0 | 5.0 | 4.0 | 7.0 | 8.0 | 14.0 | 2.0 | 11.0 | 12.5 | 9.0 |
| $\sum$ Ranks | | 71.0 | 94.0 | 278.5 | 74.5 | 244.0 | 311.5 | 135.0 | 199.0 | 319.5 | 294.5 | 383.5 | 161.0 | 412.5 | 210.5 | 411.0 |
| Overall Rank | | 1 | 3 | 9 | 2 | 8 | 11 | 4 | 6 | 12 | 10 | 13 | 5 | 15 | 7 | 14 |

## 8 Illustration with Practical Data

In this section, we analyze two practical data sets to evaluate and compare the performance of the proposed models in explaining real-world scenarios as well as the flexibility of the proposed model. For comparison, the following probability distributions were taken into account: UPSD, beta distribution (BD), Topp Leone distribution (TLD) by [28], exponentiated Topp-Leone distribution (ETLD) by [29], Kumaraswamy distribution (KD) by [30] and unit Burr-XII distribution (UBD) by [31]. To assess the usefulness of the new model, we employ several model selection and goodness-of-fit criteria, including the negative twice log-likelihood value ($-2\log l$), Akaike's Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike's Information Criterion (CAIC), and Hannan–Quinn Information Criterion (HQIC). In addition, we use the Kolmogorov–Smirnov (KS), Anderson–Darling (A), and Cramér–von Mises (CvM) test statistics along with their corresponding $p$-values to evaluate the overall fit of the distributions.

### *8.1 Data Set I*

This dataset represents the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania, over 20 four-year periods from 1890 to 1969. The dataset was previously analyzed by [32]. The flood level data are 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

Table 8 presents the MLEs along with their corresponding standard errors (SEs) for the parameters of the fitted models. In Table 8, $P = (P_1, P_2)$ denotes the MLEs of the model parameters. On the other hand, Table 9 summarizes the model selection criteria for the failure time dataset. Fig. 3 displays the fitted PDFs and CDFs, respectively.

**Table 8:** The MLEs and SEs of the parameters for the data set I

| Distribution | P1 | P2 | SE (P1) | SE (P2) |
|---|---|---|---|---|
| UPSD | 1.3957 | 3.7116 | 0.2235 | 0.6604 |
| TLD | 2.2449 | — | 0.5019 | — |
| ETLD | 4.6079 | 4.0458 | 0.9493 | 1.4746 |
| KD | 3.3638 | 11.7932 | 0.6016 | 5.3453 |
| BD | 6.7594 | 9.1140 | 2.3542 | 3.1886 |
| UBD | 1.6505 | 4.8044 | 0.3713 | 0.9094 |

**Table 9:** The comparison statistics for the data set I

| Distribution | $-2\log\ell$ | AIC | BIC | CAIC | HQIC | KS | A | CvM | $p$-Value (KS) | $p$-Value (A) | $p$-Value (CvM) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| UPSD | −32.4001 | −28.4015 | −26.4100 | −27.6956 | −28.0127 | 0.1282 | 0.3237 | 0.0469 | 0.8974 | 0.9184 | 0.8994 |
| TLD | −14.7364 | −12.7364 | −11.7407 | −12.5142 | −12.5420 | 0.3351 | 2.8771 | 0.5770 | 0.0223 | 0.0321 | 0.0244 |
| ETLD | −27.1904 | −23.1904 | −21.1989 | −22.4845 | −22.8016 | 0.2063 | 0.8146 | 0.1433 | 0.3618 | 0.4689 | 0.4132 |
| KD | −25.7371 | −21.7371 | −19.7456 | −21.0312 | −21.3483 | 0.2109 | 0.9325 | 0.1636 | 0.3355 | 0.3934 | 0.3525 |
| BD | −28.1310 | −24.1302 | −22.1387 | −23.4243 | −23.7415 | 0.1988 | 0.7329 | 0.1236 | 0.4076 | 0.5301 | 0.4844 |
| UBD | −29.1977 | −25.1977 | −23.2062 | −24.4918 | −24.8089 | 0.1796 | 0.5998 | 0.0893 | 0.5385 | 0.6460 | 0.6446 |

Based on the results reported in Table 9, the UPSD distribution is identified as the best fitting model for the first dataset in terms of all comparison criteria. Hence, the UPSD distribution can be regarded as a more suitable and effective model compared to its competing distributions, offering a strong alternative for modeling this dataset.

### *8.2 Data Set II*

The second data set used for the practical illustration concerns with the comparison of the SC 16 and P3 unit capacity factor estimation techniques and was previously reported by [33] and [34]. The dataset is obtained as 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006.

Table 10 presents the MLEs along with their corresponding SEs for the parameters of the fitted models. In Table 10, $P = (P_1, P_2)$ denotes the MLEs of the model parameters. On the other hand, Table 11 summarizes the model selection criteria for the failure time dataset. Fig. 4 displays the fitted PDFs and CDFs, respectively.

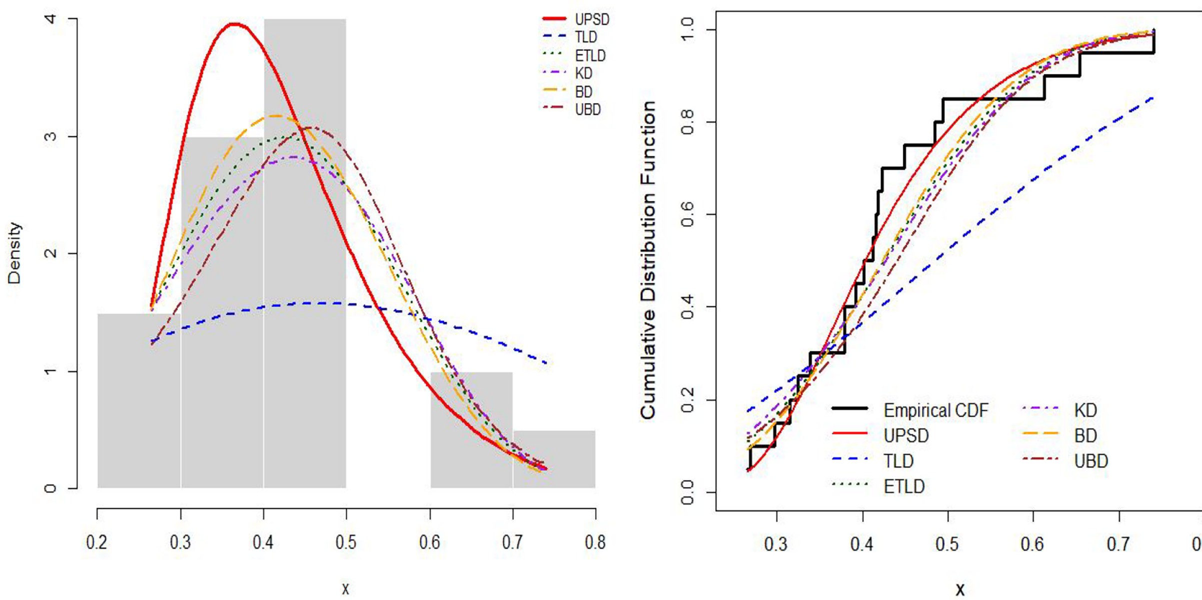


**Figure 3:** The fitted PDFs and CDFs for the data set I

**Table 10:** The MLEs and SEs of the parameters for the data set II

| Distribution | P1 | P2 | SE (P1) | SE (P2) |
|---|---|---|---|---|
| UPSD | 0.7540 | 0.9941 | 0.1331 | 0.1489 |
| TLD | 0.5942 | — | 0.1239 | — |
| ETLD | 0.4623 | 0.6806 | 0.1389 | 0.1717 |
| KD | 0.5043 | 1.1861 | 0.1287 | 0.3264 |
| BD | 0.4869 | 1.1679 | 0.1208 | 0.3576 |
| UBD | 1.5049 | 0.7848 | 0.2951 | 0.1919 |

**Table 11:** The comparison statistics for the data set II

| Distribution | $-2\log\ell$ | AIC | BIC | CAIC | HQIC | KS | A | CvM | $p$-Value (KS) | $p$-Value (A) | $p$-Value (CvM) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| UPSD | −19.7981 | −15.7984 | −13.5274 | −15.1984 | −15.2273 | 0.1621 | 0.6977 | 0.1108 | 0.5811 | 0.5600 | 0.5378 |
| TLD | −16.2302 | −14.2302 | −13.0947 | −14.0397 | −13.9446 | 0.1689 | 1.5271 | 0.1734 | 0.3270 | 0.1703 | 0.5272 |
| ETLD | −18.7827 | −14.7826 | −12.5116 | −14.1826 | −14.2115 | 0.1933 | 0.8022 | 0.1369 | 0.4346 | 0.4780 | 0.3563 |
| KD | −19.3416 | −15.3415 | −13.0705 | −14.7415 | −14.7704 | 0.1789 | 0.6984 | 0.1159 | 0.4529 | 0.5592 | 0.5157 |
| BD | −19.2149 | −15.2149 | −12.9439 | −14.6149 | −14.6437 | 0.1836 | 0.6998 | 0.1189 | 0.4202 | 0.5573 | 0.5031 |
| UBD | −11.7456 | −7.7455 | −5.4746 | −7.1455 | −7.1744 | 0.2393 | 1.2136 | 0.2221 | 0.1433 | 0.2620 | 0.2290 |

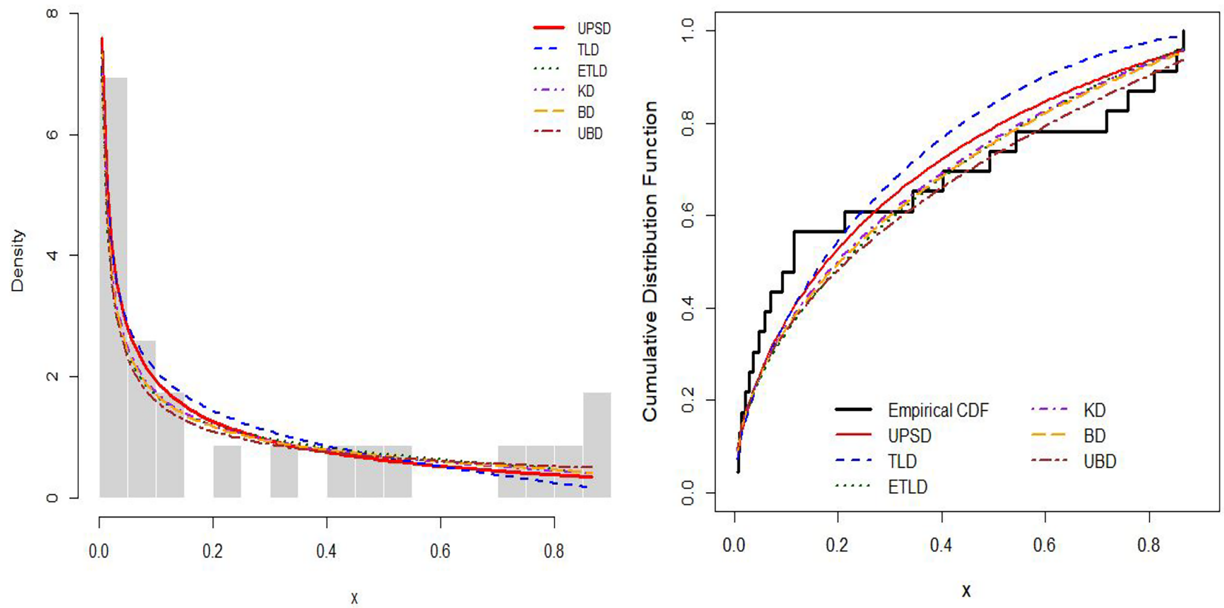


**Figure 4:** The fitted PDFs and CDFs for the data set II

Based on the results reported in Table 11, the UPSD distribution is found to be the best fitting model for the first dataset in terms of all comparison criteria. In this regard, UPSD distribution is considered more flexible and superior to its competitors and a perfect alternative to its analyzed competitors in this dataset.

## 9 Conclusion and Future Work

This work proposes and examines a novel limited probability distribution, the UPSD. The model was created by applying a logarithmic transformation to the PSD, thereby broadening its domain to the unit interval (0, 1). The resultant two-parameter family offers a highly adaptable framework for modeling bounded data that display skewness or atypical tail behavior. Analytical derivations were conducted for the CDF, PDF, moments, quantile function, and reliability measures. The analytical tractability of the UPSD renders it a significant enhancement to the category of bounded distributions. The paper also examined Tsallis entropy and its connections to other established entropy measures, including Shannon, Rényi, Arimoto, and extropy. These formulations provide a better understanding of the proposed distribution's information-theoretic properties and uncertainty characteristics. We developed and examined various classical and modern methods for parameter estimation, including MLE, AD, CvM, MPS, and LS, along with their variants. A comprehensive Monte Carlo simulation analysis was conducted to evaluate the finite-sample performance of these estimators in terms of bias, MSE, and goodness-of-fit metrics. The findings demonstrated that the MLE method consistently produced the most efficient and stable estimates, followed by the MPS method, thereby validating their suitability for practical applications. The proposed UPSD model demonstrates considerable applicability across various scientific and technical fields when the variable of interest is intrinsically limited to the unit interval. Some examples are modeling proportions in biology, survival probability in reliability studies, success rates in economics, and normalized indices in the social and environmental sciences. The UPSD is a viable alternative to well-known distributions, such as the Beta, Kumaraswamy, and Topp-Leone distributions, because it is flexible and easy to interpret.

**Future Work:** There are many ways to build on this research. A promising direction is the development of multivariate extensions of the UPSD, using suitable copula frameworks to model the interdependence among bounded random variables. Another possible direction is to extend the model to regression contexts, allowing parameters to depend on covariates to improve data fitting. You can explore Bayesian estimation methods to combine existing information and estimate the uncertainty in the results. Additionally, practical applications of the UPSD across diverse datasets in hydrology, finance, reliability, and biostatistics will further validate its versatility and utility. Future research may examine goodness-of-fit metrics, model selection criteria, and generalized entropy-based metrics tailored to the UPSD framework. The UPSD is a stable, easy-to-analyze, and flexible tool for modeling bounded data. This opens up many possibilities for both theoretical and practical research.

**Acknowledgments:** Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R734), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

**Funding Statement:** This research was funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R734), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

**Author Contributions:** Ohud A. Alqasem: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. Jondeep Das: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. I. A. Husseiny: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. Lamis M. Alamoudi: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. Maryam Ibrahim Habadi: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. Ahmed M. Gemeay: Writing—original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing—review, Investigation. All authors reviewed and approved the final version of the manuscript.

**Availability of Data and Materials:** The data that support the findings of this study are available within the article.

**Ethics Approval:** Not applicable. No human participants were involved in this study.

**Conflicts of Interest:** The authors declare no conflict of interest.

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