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HIERARCHICAL BAYESIAN MODEL FOR SIMULATING THE MECHANICAL BEHAVIOR OF BARE PRINTED CIRCUIT BOARDS WITH FIXING

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Abstract. A method for probabilistic simulation of a bare printed circuit board fixed with bolted joints based on hierarchical Bayesian updating of a numerical model is presented in this paper. The objective is the determination of parameter uncertainties in a set of nominally identical boards and the propagation of these uncertainties to calculate probability distributions for the behavior of the mechanical system. The numerical model of the system is split into models for the circuit board, the bolts and a contact model that are updated separately.

1 INTRODUCTION

Electronic components of electric vehicles are exposed to demanding load conditions including the excitation of mechanical vibrations. These vibrations are a major factor limiting the lifetime of printed circuit boards (PCBs). Typically, numerical simulations are conducted to describe the mechanical behavior of PCBs. Especially the effects of mechanical loads and excitations on PCBs during operation need to be estimated to determine critical loads. For this purpose, it is crucial that the simulation results match the real response of the investigated mechanical system.

Mismatches between simulation and measurements occur due to different sources of uncertainty. Imperfections in the numerical model can arise from uncertainties of the model parameters or from model inadequacies [1], such as a linearization of non-linear behavior. Measurement errors and uncertainties lead to further discrepancy between simulation and experimental results [2]. Model updating is used to calibrate model parameters with the goal of minimizing discrepancies between simulation and measurements [3, 4].

When a set of nominally identical systems is considered, the term parameter uncertainty does not only include deviations between parameter values used for the simulation and true parameter values, but also variations of parameter values within the set of investigated systems. These parameter variations can stem for example from uncertainties in the material composition or uncertainties in the production and assembly process. Hierarchical Bayesian model updating is a technique that considers this expanded definition of parameter uncertainties and provides probability distributions for the parameter values [5]. This approach is used in this work to assess parameter variations in a model of a bare PCB that is fixed to a casing by bolted joints. For this purpose, experimental data of multiple, nominally identical, PCBs is acquisitioned and used to calculate the most probable distributions of the model parameters with the hierarchical Bayesian approach. Potential model inadequacies are not considered in the updating process for the time being. The determined parameter uncertainties are propagated through the numerical model to obtain probabilistic simulations of the PCB with fixing. The objective is to establish an updated Finite Element (FE) model that matches experimental data and is able to predict further measurement data not used in the updating process.

Two sets of measurement data, one for PCBs with free-free boundary conditions and one for the system of PCB and bolted joints, are used for model updating. To account for this, the numerical model of the overall system is split into different models for the PCB, the bolts and the contact between PCB and bolts. The models and their uncertain parameters form a Bayesian network. This modular structure of the system model allows for a flexible use of the updated parameter distributions. For example, the determined parameter distributions of the PCB can later be used to simulate the PCB's behavior with different fixings.

2 BAYESIAN MODEL

Bayesian model updating is a probabilistic method for the inference of model parameters θ based on measured data D of the model output. The relation between the parameters and the measurement data is given by Bayes' theorem

$$p(\theta|D) \propto p(D|\theta)p(\theta) \ .$$

Here, $p(\theta)$ denotes the prior probability distribution of the model parameters that contains initial assumptions on the parameter values before considering the measured data. The posterior distribution $p(\theta|D)$ indicates the probability of the parameter values θ given the measured data D and is proportional to the product of the prior distribution and the likelihood function $p(D|\theta)$ which relates the experimental data to the outputs of a model with input parameters θ. This formulation assumes that the parameters θ have a, a priori unknown, fix value. To account for parameter variations, for example due to uncertainties in the production process or the material composition, a hierarchical Bayesian formulation is advantageous [5]. For this approach, it is assumed that the parameters θ follow a stochastic distribution e.g., a normal distribution

$$\theta \sim N(\mu_\theta, \sigma_\theta)$$

with mean value μ_θ and standard deviation σ_θ. The posterior distribution can then be expressed as

$$p(\boldsymbol{\theta}, \mu_\theta, \sigma_\theta|\boldsymbol{D}) \propto \prod_t p(D_t|\theta_t)p(\theta_t|\mu_\theta, \sigma_\theta)p(\mu_\theta, \sigma_\theta) \ ,$$

where $p(\mu_\theta, \sigma_\theta)$ denotes the hyper-prior distribution for the means and standard deviations [5]. The set of parameters $\boldsymbol{\theta}$ is composed of the parameter values θ_t of every of the nominally identical components that are updated. The same applies to the measurement data $\boldsymbol{D}$ that is composed of the experimental data D_t of every component t.

2.1 Division of structure into components

Sankararaman et al. proposed a framework for uncertainty quantification in engineering systems for multi-level models [6]. These multi-level models can consist for instance of multiple models with each model describing different parts or physics of the overall system. The dependencies between the different models and their uncertain model inputs, parameters and outputs can be described by a Bayesian network [6]. In this work, we use this framework to quantify uncertainty in a mechanical system that consists of multiple components. The mechanical system is defined by the FE models of the components and the contacts between the components. Each of the models depends on uncertain parameters: mass and stiffness parameters for the components and additional contact parameters. A hierarchical approach is used for the definition of the uncertain parameters to account for parameter variations in a set of multiple nominally identical systems. The parameters θ are supposed to follow a normal distribution with mean values μ_θ and standard deviations σ_θ. Figure 1 shows the resulting Bayesian network for a system consisting of two components. Hierarchical Bayesian model updating is used to determine probability density functions for the model parameters θ given measurement data of the two components and the overall system.

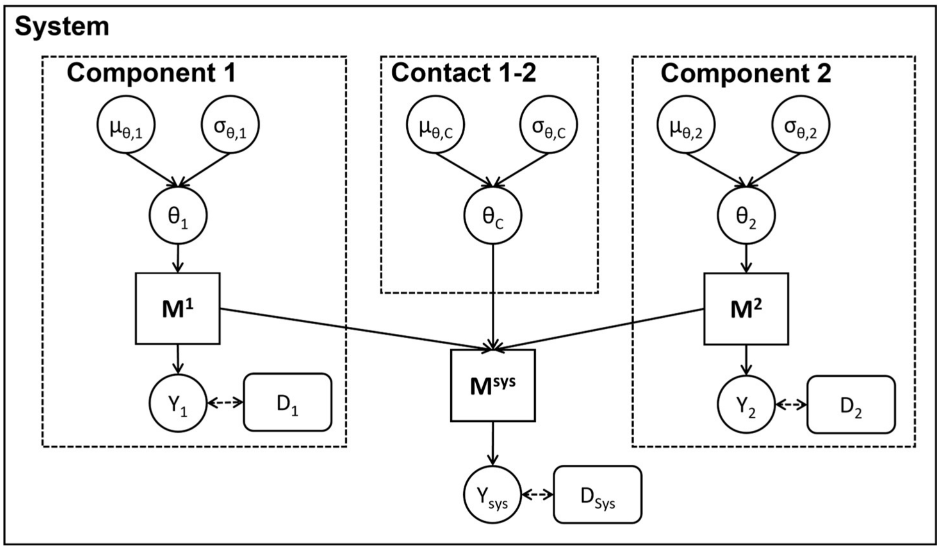


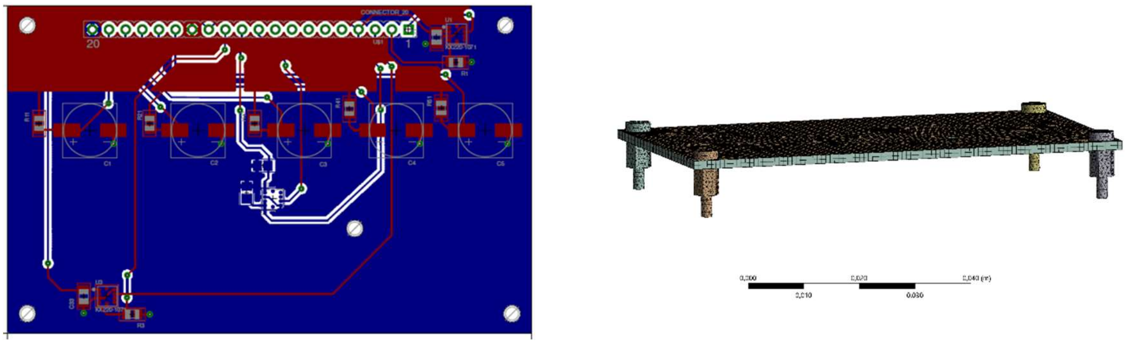
Figure 1: Bayesian network of a 2-component system with a set of uncertain parameters $\boldsymbol{\theta}$, model outputs $\boldsymbol{Y}$, measurement data $\boldsymbol{D}$ for every model $\boldsymbol{M}$

The division of the overall system into components allows to incorporate different types of measurement data to the inference of model parameters. Assuming that the model error of the overall system is bigger than the errors of the less complex models of the individual components, the use of component measurement data might lead to more precise results for the model parameter distributions. Moreover, the modular structure makes it possible to change one of the components while keeping the measurement data and inference results of the components not affected by these changes.

3 APPLICATION ON PRINTED CIRCUIT BOARDS

The presented approach for a hierarchical Bayesian model for mechanical systems with multiple components is applied to a bare printed circuit board that is fixed with bolted joints. The goal is to quantify parameter uncertainties due to variations in the material properties and to use the determined parameter distributions to conduct probabilistic simulations of the PCB. The parameter uncertainties are determined by hierarchical Bayesian model updating of the PCB's numerical model using experimental data of several nominally equal PCB.

The examined PCB is pictured in Figure 2. It is an 80mm x 50mm rectangular plate with a thickness of approximately 1.5mm, weighing approximately 13 grams. The PCB mainly consists of FR-4 material, which is a glass-reinforced epoxy laminate with orthotropic mechanical properties. Thin layers of copper are laminated to the bottom and the top side of the PCB. The FR-4 material of the PCB is modeled using finite element theory in ANSYS. The copper structures are integrated into the numerical model by importing the ECAD file of the PCB. The material properties of each element are then calculated with a homogenization method [7].



(a) (b)

Figure 2: (a) Plan of copper layers of the PCB, (b) FE model of the PCB with bolted joints

The PCB is fixed with four bolted joints of size M2.5 at the four corners. The bolts are made of steel and modeled as solid objects. To enable a linear solution, bonded contacts between the bolts and the PCB are assumed. The contact stiffness is defined by a constant value.

In the following, we will first present the measurement data used for model updating, then select the model parameters that we consider to be uncertain and define the Bayesian network with the uncertain parameters of the PCB.

3.1 Experimental data

Experimental modal data is used for the updating of the PCB model. Measurements are performed on a total of seven different, but nominally equal, PCB. First, modal data of the PCBs with free-free boundary conditions is collected. This is done by placing the PCB on foam material and exciting it with an automatic modal hammer. The acquisitioned data includes the eigenfrequencies in the range up to 4500 Hz (first eight eigenfrequencies) and the corresponding mode shapes measured with a scanning laser vibrometer. Five measurements per PCB are conducted to decrease the impact of aleatory measurement uncertainty on the model updating. Some of the eigenmodes are not recognized in every measurement. To ensure that similar data is used for all seven PCB, only modes that are measured in every measurement are considered for model updating. This condition applies to four eigenmodes: the first torsional mode, the first bending mode in transverse direction and the first two bending modes in longitudinal direction. The remaining data is used for validation of the updating results.

Further measurements are performed on PCBs fixed with bolted joints. For this purpose, the PCB is mounted on a shaker using the bolts and is excited up to a frequency of 4000 Hz. The first seven eigenmodes lie within this frequency range and are measured with a scanning laser vibrometer. A total of eighteen measurements are performed on two of the seven PCBs with different bolts and are used for model updating. Additionally, twelve measurements are performed on four other PCB for validation purpose.

Since modal data is not sufficient to determine mass and stiffness properties at the same time, the weights of the seven PCBs are determined using an electric scale. These measurements are also included into model updating.

3.2 Choice of parameters for model updating

The numerical model of the PCB depends on many different parameters. However, not all these parameters can be used for model updating since every open parameter increases the complexity and thus the computation time. Furthermore, it may not be possible to determine some combinations of parameters with the present data if several sets of parameter values yield identical modal properties. Therefore, the number of updating parameters must be reduced to a small number.

The parameters that determine the PCB behavior include geometric quantities (length, width, and height of the PCB) and material properties of both FR-4 and copper. Measurements at the seven PCBs indicate that there is no significant variation in length, width, and height of the plates and that these three quantities are represented accurately in the initial model. Therefore, the geometric quantities are not considered for model updating. Variations in the geometry of the copper structures may also cause uncertainty in the PCB behavior. However, an analysis of uncertainty in these structures would be very complex and only be possible by the introduction of many parameters. Therefore, uncertainty in the copper structure is also neglected.

The material properties of the PCB model consist of mass and stiffness variables. The mass matrix depends on the specific masses of FR-4 material ρ_{FR4} and copper ρ_{Cu}. FR-4 is modeled as an orthotropic material. Hence, nine independent parameters determine the material stiffness. This number is reduced to four due to the plate characteristics of the PCB. These four parameters are the Young's moduli E_x and E_y, the shear modulus G_{xy} and the Poisson coefficient ν_{xy}. Copper is modeled as an isotropic material, resulting in two additional stiffness parameters E_{Cu} and ν_{Cu}. A sensitivity analysis based on an analytic calculation of gradients [8] is conducted to determine the influence of these parameters on the four eigenfrequencies used for updating of the PCB in free-free boundary conditions (see Table 1).

The sensitivity analysis shows that the influence of the Poisson coefficient ν_{xy} on the eigenfrequencies is limited. Thus, this parameter will be neglected for model updating. The same applies to ν_{Cu}. A change of the copper stiffness E_{Cu} has a nearly equal effect on all frequencies, as well as a change of the specific masses ρ_{FR} and ρ_{Cu}. Therefore, an inference of these three parameters will be difficult to obtain with the available modal data. However, with the additional mass data, the inference of one mass parameter becomes possible. We choose to update the total PCB mass m instead of one of the specific masses for simplicity reasons, assuming that the proportion between the mass of FR-4 and the copper mass is constant over all seven examined PCBs.

Table 1: Gradients of eigenfrequencies for PCB with free-free boundary conditions (values $\frac{\Delta EF}{\Delta\theta}$ in $\frac{\%}{\%}$)

	Eigenfrequency 1 (Torsion)	Eigenfrequency 2 (Bending X)	Eigenfrequency 4 (Bending Y)	Eigenfrequency 5 (Bending X)
E_x	0.03	0.77	0.00	0.64
E_y	0.01	0.01	0.77	0.08
G_{xy}	0.67	0.00	0.00	0.04
ν_{xy}	0.002	-0.005	0.007	-0.022
E_{Cu}	0.27	0.21	0.23	0.22
ρ_{FR4}	-0.88	-0.89	-0.88	-0.88
ρ_{Cu}	-0.11	-0.10	-0.10	-0.10

Similarly, a gradient calculation for the eigenfrequencies of the PCB fixed with bolts is performed to determine the updating parameters of the bolts and the contact model. Since the bolts are much stiffer than the PCB, the variation of mass and stiffness parameters of the bolts only has a minor impact on the measured modal data. Therefore, the bolts are modeled with an invariable numerical model. However, the contact stiffnesses between the PCB and the bolts are considered for model updating. We choose to have one contact stiffness parameter K_{Con}. This stiffness is applied to all four bolted joints for the contact between the PCB and the bolt head and for the contact between the PCB and the nut.

3.3 Network of uncertain parameters

All parameters that were chosen to be modeled as uncertain parameters are represented in a Bayesian network (Figure 3). The model of the PCB $\boldsymbol{M}^{PCB}$ is governed by the three stiffness parameters E_x, E_y and G_{xy} of FR-4 material, as well as the PCB mass m. The PCB model calculates the modal properties Y_{PCB} of the PCB with free-free boundary conditions which correspond to the measured eigenmodes D_{PCB}. No uncertain parameter for the numerical model of the four bolts $\boldsymbol{M}^{B}$ was selected. The model of the overall system $\boldsymbol{M}^{sys}$ is then composed by the individual models of the PCB and the bolts. The contact between the PCB and the bolts is defined by the contact stiffness K_{Con}.

The model parameters are all supposed to follow normal distributions. Since measurement data of seven different PCBs are used, seven PCB parameter sets are determined during model updating. The measurement data used for updating of the overall system D_{sys} consists of 18 different measurements on two PCBs. It is assumed that the contact stiffness K_{Con} varies from measurement to measurement e.g., due to possible variations of the tightening torques.

Model updating is done in two steps. First, the parameters of the PCBs are determined with the experimental data in free-free boundary conditions D_{PCB} and the measured masses D_m. In a second step, the probability distributions of the contact stiffness are inferred from the modal data of the PCBs fixed with bolted joints D_{sys} using the previously updated parameter values of the PCBs. The resulting normal distributions of all model parameters and their covariances are used for a probabilistic simulation of the PCB with bolted joints. For this purpose, the model parameter distributions are first determined by sampling from the individual distributions of parameter means and standard deviations. It is supposed that the contact stiffness K_{Con} is independent from the mass and stiffness parameters of the PCB while covariances between the PCB parameters are considered. Based on these parameter distributions, Monte Carlo distributions of the system model $\boldsymbol{M}^{sys}$ are then conducted to obtain probability distributions of the model outputs, for example the system's eigenfrequencies.

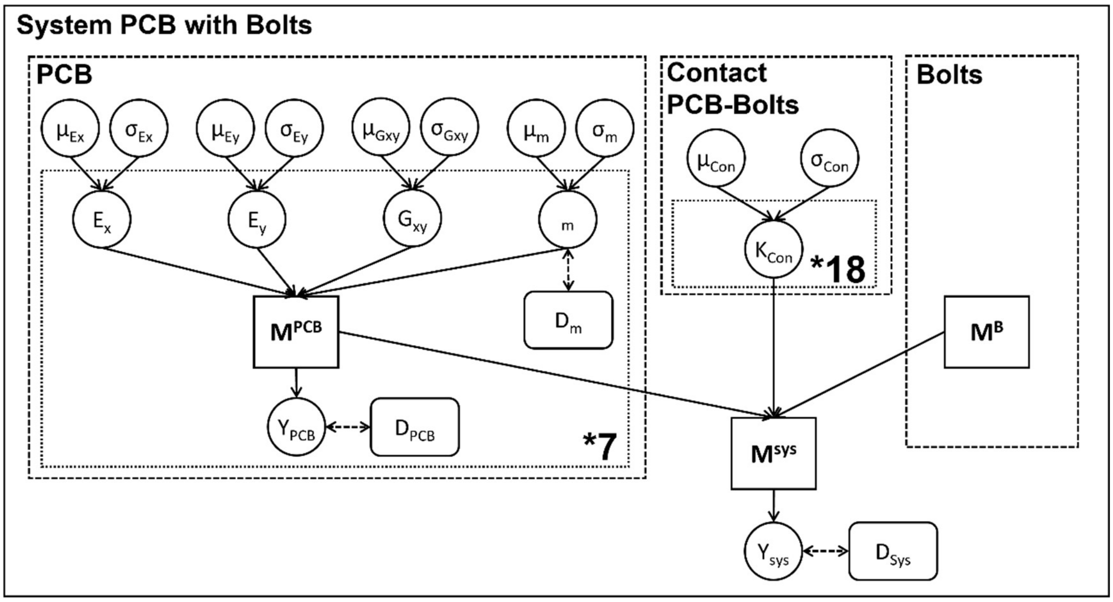


Figure 3: Bayesian network regrouping all uncertain parameters of the system of PCB and bolts

3.4 Model updating

The Bayesian inference of the model parameters is performed using the sequential Monte Carlo algorithm of PyMC3 [9]. This algorithm is based on the Transitional Markov Chain Monte Carlo [10] and the CATMIP algorithm [11]. Since these methods require a multitude of model evaluations, the computation time may be important when using complex numerical models. Therefore, a parametrically reduced model of the PCB is generated with a modal reduction technique [12]. The reduced models provide precise results for parameter changes of up to 20 % around the initial parameter values. Since the rough location of the parameter values is initially known and the main goal of the model updating is the determination of parameter uncertainties, the reduced models are sufficient for this application.

Three different likelihood functions are used for the updating of the PCB without bolts (eigenfrequencies, eigenvectors and masses) and two for the system of PCB and bolts (eigenfrequencies and eigenvectors). The likelihood functions for the modal parameters are determined as proposed by Vanik et al. [13]. The formulation of the likelihood function for the measured PCB masses is based on a Gaussian error model with zero mean. As described in the previous section, all parameters are supposed to follow normal distributions. Correlations between the mass and stiffness parameters of the PCB model are assumed. Therefore, the distributions of these parameters are modeled with a multivariate normal distribution. The prior distributions for the mean values are chosen to be normal distributions with the initial parameter values as prior means, while half-normal distributions are chosen as prior distributions for the parameter standard deviations. The correlation matrix of the multivariate normal distribution is modeled with a Lewandowski-Kurowicka-Joe (LKJ) prior distribution [14].

4 RESULTS OF MODEL UPDATING AND SIMULATION

In this section, the updating results and the following probabilistic simulation of the PCB will be presented. First, the resulting parameter distributions of the PCB in free-free boundary conditions are shown and then used for the probabilistic simulation. The simulation results are compared to further experimental data for validation purpose. The updating and simulation results of the PCB with bolted joints are discussed in the same way.

4.1 Updating of PCB with free-free boundary conditions

The results of the Bayesian inference for the PCB with free-free boundary conditions are shown in Figure 4. For the resulting probability distributions of the four model parameters, standard deviations between 1.5 % and 2 % are observed. It has to be noted that these standard deviations are higher than the actual inherent parameter variability since estimation uncertainties of parameter means and standard deviations also contribute [15]. However, the mean values of the posterior distributions of the variances of the parameters amount to more than 90% of the variance of the sampled parameter distributions for all four parameters. Thus, the biggest part of the variance of the sampled parameter distributions stems from true parameter variability.

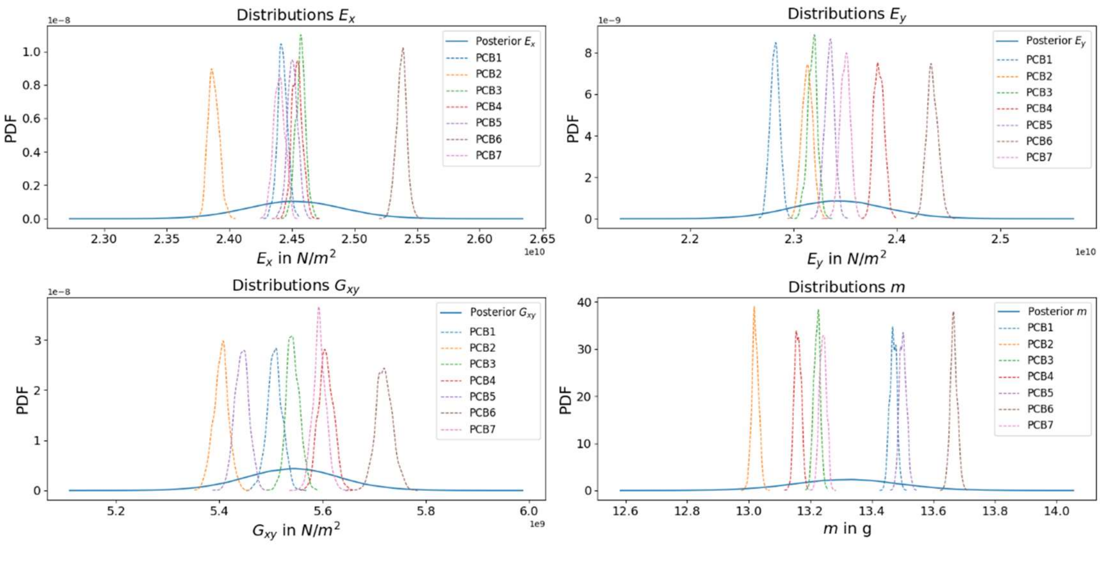


Figure 4: Posterior distributions of the PCB parameters (for all seven PCB and their common normal distribution)

By Monte Carlo simulation of the PCB's numerical model with the determined parameter distributions, probability distributions of the eigenfrequencies are obtained. These are compared to the experimental data in Table 2. The simulated distributions match the measured eigenfrequencies well in the case of the four eigenmodes used for model updating. Especially the mean values of the measured eigenfrequencies are predicted with high precision. The simulations of the eigenmodes 1 (torsion) and 4 (y-bending) are closer to the measurements than those of the eigenmodes 2 and 5 (x-bending). The latter two eigenmodes both mainly depend on E_x (see Table 1). However, the measured distributions for the two eigenfrequencies cannot be perfectly matched with the same distribution for E_x. While the mean values of the eigenfrequency distributions are close to the measured means, the estimated standard deviations differ from the experimental results especially for the second eigenmode. For validation purpose, the simulated distributions of the eigenfrequencies are also compared to the measurements of the remaining eigenmodes. Here too, a good agreement between the experimental data and the simulation is reached. The deviation of the simulated means und measured means of the validation data is similar to those of the updating data for three of the four validation eigenfrequencies. Only for eigenfrequency 3, a higher deviation of the mean value of about 0.5 % is observed. The standard deviations are estimated with a lower precision than the means. One possible reason for this is the measurement uncertainty that is different for every eigen mode. This variation of the measurement uncertainty is not taken into account during model updating.

Table 2: Comparison between experimentally measured eigenfrequencies (EF) and simulated eigenfrequency distributions with updated parameter distributions (MU = Model Updating)

	Eigenmodes used for MU				Eigenmodes not used for MU			
	EF1	EF2	EF4	EF5	EF3	EF6	EF7	EF8
Measured mean [Hz]	778.5	954.4	2402	2562	1872	2966	3537	4116
Simulated mean [Hz]	778.4	953.8	2402	2565	1862	2968	3535	4115
Error	-0.01%	-0.07%	-0.01%	+0.14%	-0.56%	+0.06%	-0.06%	-0.02%
Measured standard deviation [Hz]	5.8	4.5	21.1	14.5	14.3	25.2	21.3	26.5
Simulated standard deviation [Hz]	5.9	5.9	22.4	15.8	12.6	22.4	22.1	29.9
Error	+1.9%	+31.0%	+6.3%	+8.7%	-11.4%	-11.1%	+3.6%	+12.9%

4.2 Updating of the overall system with PCB and bolted joints

The posterior distributions of the contact stiffness are obtained by model updating using the experimental data of the PCB fixed with bolted joints and shown in Figure 5. Compared to the results of the PCB parameters, more parameter uncertainty is observed. The standard deviation of the resulting probability distribution is 8.5 % of the contact stiffness mean value. Reasons for this relatively high parameter uncertainty could be variations in the positioning of the bolts or variations of the torque that is applied to the bolts.

The model updating of the contact model was done using nine measurements of each of the first two PCBs. Additionally, three measurements were performed for each of the PCB three to six. This experimental data is used for validation of the updating results. Figure 6 shows a comparison of the measurements with the simulation results for each PCB, using the previously determined stiffness and mass distributions of each PCB and the contact stiffness uncertainty. Since the parameter values of a specific PCB were determined with a relatively low estimation uncertainty, the variance in these simulation results mainly stems from uncertainties in the contact stiffness.

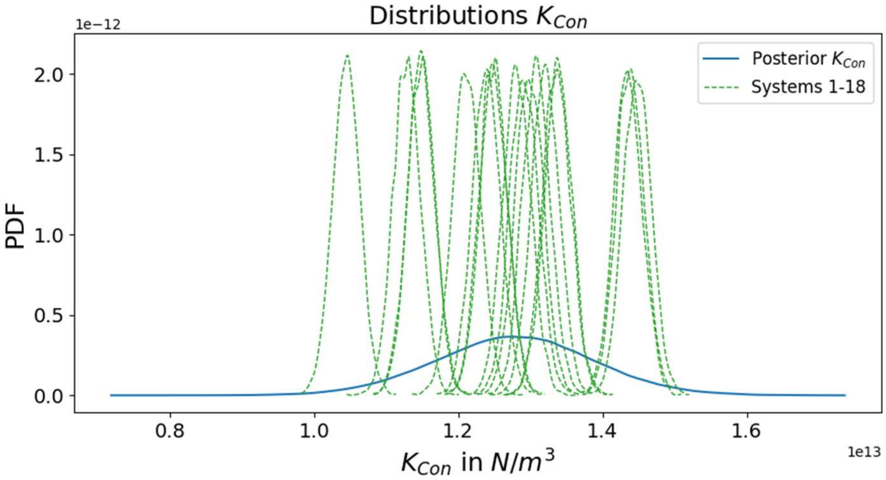


Figure 5: Posterior distributions of the contact stiffness (for all 18 measurements of PCB with bolted joints and the common contact stiffness distribution)

Even for the first two PCB that were used for the updating of the contact stiffness distribution, deviations between the simulated eigenfrequency distributions and the measured eigenfrequencies are observed. Especially for the eigenfrequencies 2 and 3, discrepancies between simulations and measurements of up to 30 Hz (about 1.5 %) are reached. This implies that the numerical model of the PCB with bolted joints is not able to explain all aspects of the experimental data due to model inadequacies. Since the validation of the simulation results for the PCB in free-free boundary conditions provided a good accordance with experimental data, potential model inadequacies most likely stem from the modeling of the bolted joints.

However, it is observed that the deviations between the simulation results and the measurements are not significantly higher for the PCBs used for validation (numbers 3-6) than for the PCBs used for model updating. Moreover, in most cases the measured data deviates in the same direction from the simulation results for a specific eigenfrequency. These observations indicate that the determined contact stiffness uncertainties can be applied to further PCBs.

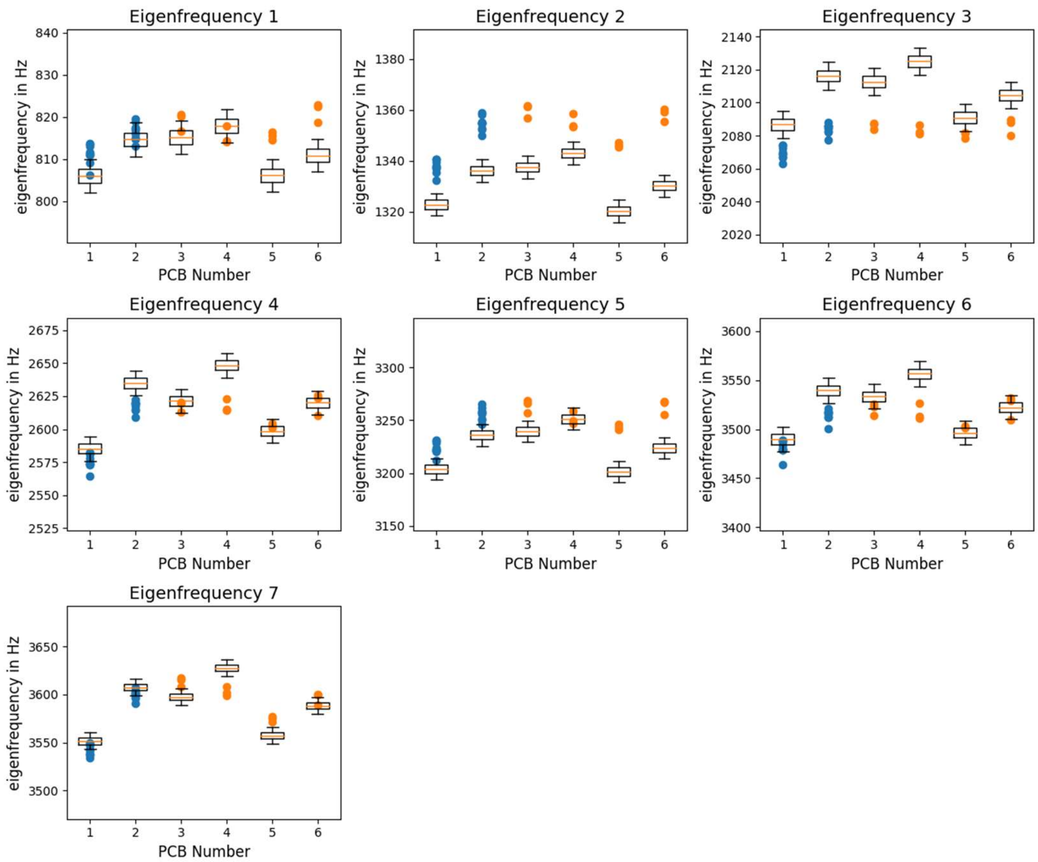


Figure 6: Comparison of measurements used for updating (blue points) and validation measurements (orange points) to simulated eigenfrequency distributions (box from lower to upper quartile, whiskers at 5 and 95% percentiles, median in orange) for individual PCB

5 CONCLUSIONS

Model updating of a mechanical system consisting of a printed circuit board fixed with bolted joints was conducted using a hierarchical Bayesian model. The updating was done in two steps, first for the model parameters of the PCB and then for the contact parameters. This method allowed to identify probability distributions of the parameters and to determine parameter variations in a set of multiple PCBs caused by production or assembly uncertainties. The determined parameter distributions were then used for probabilistic simulations of the mechanical system to assess the system behavior. Validation with experimental data of a PCB in free-free boundary conditions shows that the determined probability distributions of the PCB parameters explain the variations in the mechanical behavior observed within the measurements. However, if the PCB is fixed with bolted joints, the simulation results deviate from experimental data. These discrepancies are attributed to model inadequacies of the contact model between the PCB and the bolts that were not taken into consideration in this work. To achieve a better match with measurement data of the PCB with bolted joints, a more realistic model for the bolts and the contacts needs to be found. Alternatively, a quantification of model inadequacies could be performed if improvements in the contact model are not sufficient to achieve adequate accuracy of the numerical model.

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