Viscous Damage Model for Timoshenko Beam Structures

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influence of viscosity, thus including damping effects. A structural model which applies these concepts to the analysis of beam structures is developed within the frame of the finite element method. Tangent and secant damping and stiffness matrices of the visco-damage constitutive law are deduced. Numerical examples showing the applicability of the proposed procedure are included.

2 STRUCTURAL MODEL

The structure is modelled using C^0 one dimensional finite elements based on Timoshenko's beam theory, generalized to 3D. The finite elements have three nodes and six degrees of freedom per node. Due to the fact that the constitutive model requires information at any point of the element, a secondary discretization of the cross section of the beam element is necessary. In the plane case, the discretization consists of layers [see Figure 1(a)].

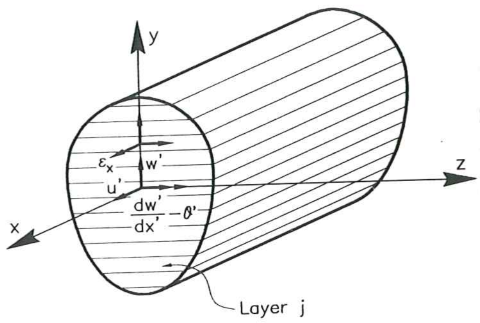


Figure 1(a) Layered 2D Timoshenko beam element.

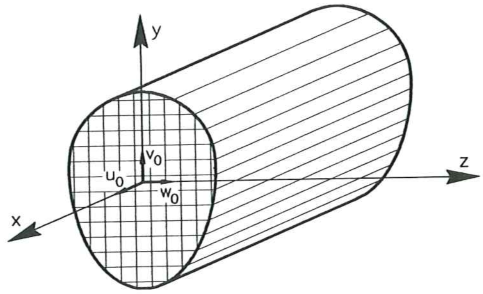


Figure 1(b) 3D Timoshenko beam element discretized with an orthogonal mesh.

In the 3D case, the cross section of the beam is discretized by means of an orthogonal non-homogeneous grid of cells [see Figure 1(b)]. This avoids the formulation of constitutive laws using sectional forces, which is the traditional way to solve the problem, but valid only in certain particular cases and having the additional drawback of lacking precision. The sectional forces are decomposed point by point, layer by layer, in stress tensors which are corrected by using the viscous damage model. The corrected sectional forces are subsequently obtained by integration over the section cells. These forces are then used to compute the residual forces, in order to iterate for equilibrium if necessary.

The relationships between the sectional variables of the problem and the the variables corresponding to a certain point belonging to the mentioned section are described below. A local co-ordinate system is considered for the beam, its longitudinal axis x forming a right triad with the other two axes. The sign convention for translations and rotations is the usual in classical mechanics. The displacement and strain fields are (Oñate 1992)

$$\boldsymbol{u}=\begin{Bmatrix} u(x,y,z)\\ v(x,y,z)\\ w(x,y,z)\end{Bmatrix}=\begin{Bmatrix} u_o+z\theta_y-y\theta_z\\ v_o-z\theta_x\\ w_o+y\theta_x\end{Bmatrix}=\begin{bmatrix}1&0&0&0&z&-y\\0&1&0&-z&0&0\\0&0&1&y&0&0\end{bmatrix}\begin{Bmatrix}u_o(x)\\v_o(x)\\w_o(x)\\\theta_x(x)\\\theta_y(x)\\\theta_z(x)\end{Bmatrix}=\boldsymbol{S}\boldsymbol{u}_o(x) \tag{1}$$

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_x\\ \gamma_{xy}\\ \gamma_{xz}\end{Bmatrix}=\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\end{Bmatrix}=\begin{Bmatrix}\frac{\mathrm{d}u_o}{\mathrm{d}x}+z\frac{\mathrm{d}\theta_y}{\mathrm{d}x}-y\frac{\mathrm{d}\theta_z}{\mathrm{d}x}\\ \frac{\mathrm{d}v_o}{\mathrm{d}x}-\theta_z-z\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\\ \frac{\mathrm{d}w_o}{\mathrm{d}x}+\theta_y+y\frac{\mathrm{d}\theta_x}{\mathrm{d}x}\end{Bmatrix}=\begin{bmatrix}1&0&0&0&z&-y\\0&1&0&-z&0&0\\0&0&1&y&0&0\end{bmatrix} \tag{2}$$
$$\times\left\{\frac{\mathrm{d}u_o}{\mathrm{d}x},\ \frac{\mathrm{d}v_o}{\mathrm{d}x}-\theta_z,\ \frac{\mathrm{d}w_o}{\mathrm{d}x}+\theta_y,\ \frac{\mathrm{d}\theta_x}{\mathrm{d}x},\ \frac{\mathrm{d}\theta_y}{\mathrm{d}x},\ \frac{\mathrm{d}\theta_z}{\mathrm{d}x}\right\}^{\mathrm{T}}=\boldsymbol{S}\,\hat{\boldsymbol{\varepsilon}}$$

where the variables have the following meaning: $\boldsymbol{u}$ – displacement vector of a point belonging to a beam section; $\boldsymbol{\varepsilon}$ – strain vector of a point belonging to a beam section; $\boldsymbol{u}_o$ – displacement vector of the 3D beam finite element corresponding to the central axis of the cross section; $\hat{\boldsymbol{\varepsilon}}$ – generalized strain vector corresponding to the central axis of the beam; $\boldsymbol{S}$ – geometric transformation matrices relating cross section variables with point variables.

The equilibrium equations are written now using the virtual work principle. The internal virtual work L_{int} corresponding to a virtual strain $\delta\boldsymbol{\varepsilon}$ is expressed as

$$\begin{aligned} L_{\text{int}} &= \int_{\mathcal{V}} \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}_{\text{tot}}\,\mathrm{d}V = \int_{\mathcal{V}} \delta\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{\sigma}_{\text{tot}}\,\mathrm{d}V \\ &= \int_o^{\ell} \delta\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\left[\int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\boldsymbol{\sigma}_{\text{tot}}\,\mathrm{d}A\right]\mathrm{d}x = \int_o^{\ell} \delta\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\hat{\boldsymbol{\sigma}}_{\text{tot}}\,\mathrm{d}x \end{aligned} \tag{3}$$

where $\mathcal{V}$ is the volume, $\mathcal{A}$ the surface of the cross section and ℓ the length of the beam element. $\boldsymbol{\sigma}_{\text{tot}}$ are the total stresses at the point level, which are defined in detail later. The total sectional forces $\hat{\boldsymbol{\sigma}}_{\text{tot}}$ have been also introduced in the previous equation as

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_{\text{tot}} &= \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\boldsymbol{\sigma}_{\text{tot}}\,\mathrm{d}A = \int_{\mathcal{A}} \begin{bmatrix}1&0&0&0&z&-y\\0&1&0&-z&0&0\\0&0&1&y&0&0\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_x\\ \tau_{xy}\\ \tau_{xz}\end{Bmatrix}\mathrm{d}A \\ &= \int_{\mathcal{A}} \{\sigma_x \quad \tau_{xy} \quad \tau_{xz} \quad -z\tau_{xy}+y\tau_{xz} \quad z\sigma_x \quad -y\sigma_x\}^{\mathrm{T}}\,\mathrm{d}A \\ &= \{N_x \quad Q_y \quad Q_z \quad T_x \quad M_y \quad M_z\}^{\mathrm{T}} \end{aligned} \tag{4}$$

A sectional density matrix $\hat{\boldsymbol{\rho}}$ can be defined, relating the sectional inertia forces with the acceleration vector $\ddot{\mathbf{u}}_o$ which is calculated by deriving twice equation (1) with respect to time

$$\hat{\boldsymbol{\rho}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\rho\,\boldsymbol{S}\,\mathrm{d}A \tag{5}$$

where ρ is the material density. Equation (5) can be integrated for any distribution of material properties over the beam cross-section.

Following standard finite element procedures, the discrete vector of the internal forces $\boldsymbol{F}_{\rm int}$ is obtained as

$$\boldsymbol{F}_{\rm int} = \int_{\ell} \boldsymbol{B}^{\rm T} \hat{\boldsymbol{\sigma}}_{\rm tot}\, {\rm d}x \tag{6}$$

and the vector of the inertial forces $\boldsymbol{F}_{\rm i}$ is deduced as

$$\boldsymbol{F}_{\rm i} = \int_{\ell} \boldsymbol{N}^{\rm T} \hat{\boldsymbol{\rho}}\, \ddot{\boldsymbol{u}}_{\rm o}\, {\rm d}x \tag{7}$$

where $\boldsymbol{N}$ and $\boldsymbol{B}$ are the shape function and strain matrices which allow to write the following equations:

$$\boldsymbol{u}_{\rm o} = \boldsymbol{N}\boldsymbol{a}\ ; \quad \ddot{\boldsymbol{u}}_{\rm o} = \boldsymbol{N}\ddot{\boldsymbol{a}}\ ; \quad \hat{\boldsymbol{\varepsilon}} = \boldsymbol{B}\boldsymbol{a} \tag{8}$$

in which $\boldsymbol{a}$ is the vector of nodal displacements. The internal forces $\boldsymbol{F}_{\rm i}$ introduced in equation (6) will be analyzed in detail in Section 4, after describing the damage model. Using equations (8), the nodal inertial forces $\boldsymbol{F}_{\rm i}$ become

$$\boldsymbol{F}_{\rm i} = (\int_{\ell} \boldsymbol{N}^{\rm T} \hat{\boldsymbol{\rho}} \boldsymbol{N}\, {\rm d}x)\, \ddot{\boldsymbol{a}} = \boldsymbol{M}\ddot{\boldsymbol{a}} \tag{9}$$

where $\boldsymbol{M}$ is the elemental mass matrix. Using now the expressions of the inertia and internal forces, the equation of motion is formulated as

$$\boldsymbol{M}\ddot{\boldsymbol{a}}(t) + \boldsymbol{F}_{\rm int}(t) = \boldsymbol{F}(t)$$

where $\boldsymbol{F}(t)$ is the vector of the dynamic action.

As stated before, the cross section of the beam is discretized using an orthogonal grid. Each rectangle of the grid may have different size and different material. The rectangles are defined by their corners and it is assumed that all the stresses have a linear variation over each cell of the grid. This implies solving a system of four equations with three unknowns, defining the equation of the plane which approximates by minimum squares the variation of each component of the stress tensor. The same grid can be used to calculate all the other characteristics of the cross section.

3 VISCOUS DAMAGE CONSTITUTIVE MODEL

3.1 General concepts

The solution of beam structures subjected to seismic actions beyond the linear behaviour has been usually treated using: (a) Theories based on plastic hinge formation (Massonet and Save 1966). This approach has the drawback of admiting that the damage of a structure point is dominated by bending criteria, which is true only for some very particular structures. (b) Simulation of beam structures based on the concept of plastification bending moment. This procedure is based on formulating simplified curvature–bending moment constitutive laws (Clough *et al.* 1965, Aoyama and Sugano 1968).

The last formulations started from representing the behaviour of materials in an approximate form based mainly on experimental studies. Today, it is required that these formulations be thermodynamically sustainable. Between those which meet this latter requirement, the so-called continuous damage theory is generally accepted as an alternative in the most complex constitutive formulations (DiPasquale and Cakmak 1989, Oliver *et al.* 1990). An application of this model to the dynamic case can be seen in Mazars (1991) where a column discretized in plane finite elements, subjected to seismic action, is calculated. The damage models have a rigourous but relatively simple formulation strictly based on thermodynamics. They deal with the non-linear behaviour by means of one or more internal variables called damage variables which indicate the loss of secant stiffness of the material and are normalized to a unit value which corresponds to maximum damage. Figure 2 shows a unidimensional representation of the behaviour of a point within a damaged material (Oliver *et al.* 1990).

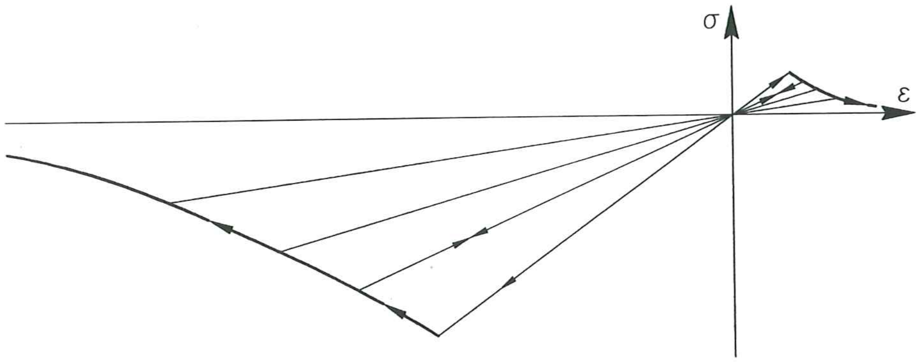


Figure 2 Local damage behaviour.

The model presented herein is a 3D damage constitutive model based on solid mechanics and it has a single internal variable. Therefore it is a local isotropic damage model and it is based on Kachanov's theory (1958). Many ideas innherent to the model have been taken from the work of Simó and Ju (1987), Lubliner *et al.* (1989) and Oliver *et al.* (1990). This formulation is a compromise between the complexity implied in modeling concrete behaviour and the versatility needed when dealing with dynamic problems. This insures accurate results and low cost solutions for the non-linear problems to be solved.

The numerical treatment of viscoelastic phenomena in materials can be followed in detail in Lubliner (1990) and Simó and Hughes (1995). The damping effect of the beam structure was simulatated in this paper by using a model consisting of a damper placed in parallel with the structure.

3.2 Characteristics of the damage model

Free energy and constitutive law

The model is formulated in the material configuration, for thermodynamically stable problems, with no temperature time variation. For this specific case the following mathematical form for the free energy is assumed, where the non-damaged elastic part is expressed as a scalar quadratic function of tensorial arguments

$$\Psi(\boldsymbol{\varepsilon};d) = (1-d)\Psi_{\mathrm{o}}(\boldsymbol{\varepsilon}) = (1-d)\left(\frac{1}{2\,m_{\mathrm{o}}}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}^{\mathrm{o}}\right) = (1-d)\left(\frac{1}{2\,m_{\mathrm{o}}}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{o}}\boldsymbol{\varepsilon}\right) \quad (10)$$

In (10) the strain tensor $\boldsymbol{\varepsilon}$ is the free variable of the problem, d $(0 \le d \le 1)$ is the internal damage variable, m_{o} is the density in the material configuration and $\boldsymbol{C}^{\mathrm{o}}$ is the stiffness tensor of the material in the initial undamaged state.

For stable thermical state problems the Clasius Planck dissipation inequality is valid, whose local lagrangian form is

$$\dot{\Xi}_{\mathrm{m}} = \frac{1}{m_{\mathrm{o}}}\boldsymbol{\sigma}^{\mathrm{T}}\dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \ge 0 \quad (11)$$

$$\dot{\Xi}_{\mathrm{m}} = \left(\frac{1}{m_{\mathrm{o}}}\boldsymbol{\sigma}^{\mathrm{T}} - \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\right)\dot{\boldsymbol{\varepsilon}} - \frac{\partial\Psi}{\partial d}\dot{d} \ge 0 \quad (12)$$

This expression for the dissipation rate $\dot{\Xi}_{\mathrm{m}}$ allows the following two considerations:

a) In order to guarantee the unconditional fulfilment of the Clasius Planck inequation, the multiplier of $\dot{\boldsymbol{\varepsilon}}$ which represents an arbitrary temporal variation of the free variable, must be null. This condition provides the constitutive law of the damage problem:

$$\frac{1}{m_{\mathrm{o}}}\boldsymbol{\sigma}^{\mathrm{T}} - \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = 0 \Rightarrow \boldsymbol{\sigma} = m_{\mathrm{o}}\left\{\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}}\right\}^{\mathrm{T}} = (1-d)\boldsymbol{C}^{\mathrm{o}}\boldsymbol{\varepsilon} = \boldsymbol{C}^{\mathrm{s}}\boldsymbol{\varepsilon} \quad (13)$$

where $\boldsymbol{C}^{\mathrm{s}}$ is the secant stiffness tensor.

b) Inserting the last equation into (12), the dissipation is now given by

$$\dot{\Xi}_{\mathrm{m}} = -\frac{\partial\Psi}{\partial d}\dot{d} = \Psi_{\mathrm{o}}\dot{d} \ge 0 \quad (14)$$

As Ψ_{o} is always positive, equation (14) states that the damage rate $\dot{d}$ cannot be negative, i.e. the damage level can only stay constant or increase and never decrease.

Damage yield criterion

The damage yield criterion is defined as a function of the free energy of the undamaged material, expressed in terms of the undamaged principal stresses σ_i^{o}, as

$$F = K(\boldsymbol{\sigma}^{\mathrm{o}})\sqrt{2m_{\mathrm{o}}\Psi_{\mathrm{o}}} - 1 = \frac{K(\boldsymbol{\sigma}^{\mathrm{o}})}{\sqrt{E^{\mathrm{o}}}}\sqrt{\sum_{i=1}^{3}(\sigma_i^{\mathrm{o}})^2} - 1 \leq 0 \tag{15}$$

where the terms of the above equation have the following meaning:

$$K(\boldsymbol{\sigma}^{\mathrm{o}}) = \frac{r}{\sqrt{2m_{\mathrm{o}}\left(\Psi_{\mathrm{t}}^{\mathrm{o}}\right)_{\mathrm{L}}}} + \frac{1-r}{\sqrt{2m_{\mathrm{o}}\left(\Psi_{\mathrm{c}}^{\mathrm{o}}\right)_{\mathrm{L}}}}\;; \qquad r = \frac{\sum_{i=1}^{3}\langle\sigma_i^{\mathrm{o}}\rangle}{\sum_{i=1}^{3}|\sigma_i^{\mathrm{o}}|}$$

$$2m_{\mathrm{o}}\left(\Psi_{\mathrm{t,c}}^{\mathrm{o}}\right)_{\mathrm{L}} = \sum_{i=1}^{3}\langle\pm\sigma_i^{\mathrm{o}}\rangle\,\varepsilon_i\;; \qquad \left(\Psi_{\mathrm{o}}\right)_{\mathrm{L}} = \left(\Psi_{\mathrm{t}}^{\mathrm{o}}\right)_{\mathrm{L}} + \left(\Psi_{\mathrm{c}}^{\mathrm{o}}\right)_{\mathrm{L}}$$

In these equations $\left(\Psi_{\mathrm{t,c}}^{\mathrm{o}}\right)_{\mathrm{L}}$ represent the part of the free energy developed when the traction/compression limit is reached and $\langle\pm x\rangle = \frac{1}{2}(|x| \pm x)$ is the Mc-Auley's function. Taking into account that the traction/compression strengthes are $f_{\mathrm{t}} = \left(\Psi_{\mathrm{t}}^{\mathrm{o}}E^{\mathrm{o}}\right)_{\mathrm{L}}^{1/2}$ and $f_{\mathrm{c}} = \left(\Psi_{\mathrm{c}}^{0}E^{0}\right)_{\mathrm{L}}^{1/2}$, respectively, the damage yield function can be written, according to Figure 3, as

$$F = \bar{\sigma} - f_{\mathrm{c}} = [1 + r(n-1)]\sqrt{\sum_{i=1}^{3}(\sigma_i^{\mathrm{o}})^2} - f_{\mathrm{c}} \leq 0 \tag{16}$$

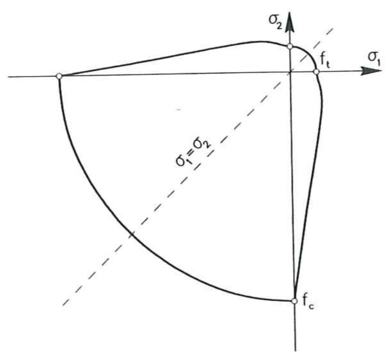


Figure 3 Damage yield function in the principal plane $\sigma_1 - \sigma_2$.

with $n = f_{\mathrm{c}}/f_{\mathrm{t}}$. This damage yield function, expressed in the non-damaged principal stresses space, allows a great number of choices. The advantage of the yield criterium (16) is that any yield function F can be used always as long as it is homogenous and of first order in stresses (i.e. Mohr-Coulomb, Drucker-Prager, Lubliner *et al.* (1989), etc). The form given by equation (16) fulfils the above requirements; besides, it is simple and yields satisfactory results within the range of assumptions made for this model and therefore will be used henceforward as the scalar expression defining $\bar{\sigma}$. An expression entirely equivalent to (16) proposed by Simó (1987) with the aim of simplifying the mathematical deduction of the damage variable of the model is the following:

$$\bar{F} = G(\bar{\sigma}) - G(f_c) \leq 0 \tag{17}$$

where $G(\chi)$ is a scalar monotonic function to be determined. Its shape will be chosen conveniently for the subsequent development of the damage model.

Evolution of the damage variable

The following evolution law is used to deduce the damage internal variable evolution rule:

$$\dot{d} = \dot{\eta}\frac{\partial \bar{F}}{\partial \bar{\sigma}} = \dot{\eta}\frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}} \tag{18}$$

where $\dot{\eta}$ is a non-negative scalar denominated damage consistency parameter, analogous to the plastic consistency parameter $\dot{\lambda}$ in standard plasticity theory.

Similarly to plasticity, a yielding rule $\bar{F} = 0$ and a consistency rule $\dot{\bar{F}} = 0$ for a point subjected to a damaging process are defined. The yielding rule and the properties of $G(\chi)$ allow to write $G(\bar{\sigma}) - G(f_c) = 0$, what implies $\bar{\sigma} = f_c$ and consequently

$$\frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}} = \frac{\mathrm{d}G(f_c)}{\mathrm{d}f_c} \tag{19}$$

From the condition of consistency —that means percistency on the damage yield surface— and from the properties of function $G(\chi)$, the following equation is deduced:

$$\frac{\partial \bar{F}}{\partial \bar{\sigma}}\dot{\bar{\sigma}} + \frac{\partial \bar{F}}{\partial f_c}\dot{f}_c = \frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\dot{\bar{\sigma}} - \frac{\mathrm{d}G(f_c)}{\mathrm{d}f_c}\dot{f}_c = 0 \tag{20}$$

and the use of (19) allows to write $\dot{\bar{\sigma}} = \dot{f}_c$. Equation (20) can be now rewritten and leads to

$$\frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\dot{\bar{\sigma}} = \frac{\mathrm{d}G(f_c)}{\mathrm{d}f_c}\dot{f}_c = \frac{\mathrm{d}G(f_c)}{\mathrm{d}f_c}\frac{df_c}{\mathrm{d}(d)}\dot{d} = \frac{\mathrm{d}G(f_c)}{\mathrm{d}(d)}\dot{\eta}\frac{d\bar{G}(\bar{\sigma})}{\mathrm{d}\bar{\sigma}} \tag{21}$$

$$\dot{\bar{\sigma}} = \frac{\mathrm{d}G(f_c)}{\mathrm{d}(d)}\dot{\eta} \tag{22}$$

Conveniently choosing $G(f_c)$ as the function which describes the evolution of the damage $[d = G(f_c)]$, the damage consistency parameter $\dot{\eta}$ can be expressed as

$$\dot{\eta} = \dot{\bar{\sigma}} = \dot{f}_c = \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}^{\mathrm{o}}}\dot{\boldsymbol{\sigma}}^{\mathrm{o}} = \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}^{\mathrm{o}}}\boldsymbol{C}^{\mathrm{o}}\,\dot{\boldsymbol{\varepsilon}} \tag{23}$$

Substituting this equation into (18) and (14), the following expressions which formulate the temporal evolution of the damage and dissipation variables are obtained:

$$\dot{d} = \frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\,\dot{\bar{\sigma}} \tag{24}$$

$$\dot{\Xi}_{\mathrm{m}} = \Psi_{\mathrm{o}}\dot{G}(\bar{\sigma}) = \Psi_{\mathrm{o}}\frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\dot{\bar{\sigma}} = \Psi_{\mathrm{o}}\frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\frac{\partial\bar{\sigma}}{\partial\boldsymbol{\sigma}^{\mathrm{o}}}\boldsymbol{C}^{\mathrm{o}}\,\dot{\boldsymbol{\varepsilon}} \tag{25}$$

The loading/unloading condition is derivated from the relations of Kuhn-Tucker formulated for problems with unilateral restrictions: (a) $\dot{\eta} \geq 0$; (b) $\bar{F} \leq 0$ and (c) $\dot{\eta}\bar{F} = 0$. From these, if $\bar{F} < 0$, then the third condition imposes $\dot{\eta} = 0$ and, if $\dot{\eta} > 0$, then the same condition requires that $\bar{F} = 0$.

Definition of function G

From the different alternatives for defining function $G(\chi)$, the following was chosen

$$G(\chi) = 1 - \frac{\bar{G}(\chi)}{\chi} \tag{26}$$

where $\bar{G}(\chi)$ describes a function so that it gives for $\chi = \chi^*$ the compression initial yield tension $\bar{G}^*$ and for $\chi \to \infty$ the final strength $\bar{G} \to 0$. Thus, by running all the deformation path, the point will have dissipated an energy equivalent to the specific fracture energy. In our work, the exponential function proposed by Oliver *et al.* (1990), which is shown in Figure 4, was used

$$\bar{G}(\chi) = \chi^*\,\mathrm{e}^{A\left(1-\frac{\chi}{\chi^*}\right)}\ ; \quad G(\chi) = 1 - \frac{\chi^*}{\chi}\,\mathrm{e}^{A\left(1-\frac{\chi}{\chi^*}\right)} \tag{27}$$

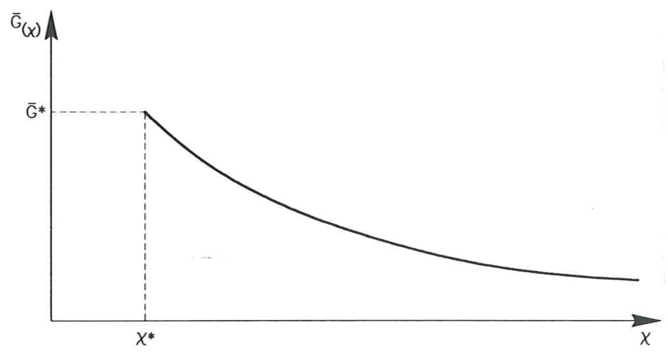


Figure 4 Representation of the chosen $G(\chi)$.

For a uniaxial traction process under monotonically increasing load, the temporal dissipation change is given by (14), with $\bar{\sigma} = n\sigma_{\mathrm{t}}$ and $\bar{\Psi}_{\mathrm{o}} = \frac{1}{2}\varepsilon_{\mathrm{t}}E^{\mathrm{o}}\varepsilon_{\mathrm{t}} = \frac{(\sigma_{\mathrm{t}})^2}{2E^{\mathrm{o}}} =$

$\frac{\bar{\sigma}^2}{2n^2 E^{\circ}}$. Integrating (14) in time we can calculate the total dissipated energy at the end of the uniaxial traction process as

$$\Xi_{\mathrm{t}}^{\max} = \int_{\bar{\sigma}^*}^{\infty} \frac{\bar{\sigma}^2}{2m_{\mathrm{o}} n^2 E^{\circ}} \frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}} \mathrm{d}\bar{\sigma} = \int_{\bar{\sigma}^*}^{\infty} \frac{\bar{\sigma}^2}{2m_{\mathrm{o}} n^2 E^{\circ}} \mathrm{d}G(\bar{\sigma}) \tag{28}$$

and after operating we get

$$\Xi_{\mathrm{t}}^{\max} = \frac{(\bar{\sigma}^*)^2}{m_{\mathrm{o}} n^2 E^{\circ}} \left[\frac{1}{2} + \frac{1}{A}\right] \tag{29}$$

giving

$$A = \frac{1}{\frac{\Xi_{\mathrm{t}}^{\max} m_{\mathrm{o}} n^2 E^{\circ}}{(\bar{\sigma}^*)^2} - \frac{1}{2}} \tag{30}$$

where $\bar{\sigma}^*$ is the initial damage stress. Parameter A is never negative, as the material must dissipate at least the energy accumulated when reaching the initial damage stress $\bar{\sigma}^*$. Making the same hypotheses for a uniaxial compression process and postulating that parameter A must be the same in both cases, it is deduced that

$$A = \frac{1}{\frac{\Xi_{\mathrm{c}}^{\max} E^{\circ}}{(\bar{\sigma}^*)^2} - \frac{1}{2}} \tag{31}$$

and, as paramerer A is the same as in (30)

$$\Xi_{\mathrm{c}}^{\max} = n^2 \Xi_{\mathrm{t}}^{\max} \tag{32}$$

The value of traction maximum dissipation $\Xi_{\mathrm{t}}^{\max}$ is an input of the problem and is equal to the fracture energy density g_{f}, parameter derived from fracture mechanics as $g_{\mathrm{f}} = G_{\mathrm{f}}/l_{\mathrm{c}}$, where G_{f} is the fracture energy and l_{c} is the characteristic length of the fractured domain (Lubliner *et al.* 1989).

Tangent constitutive law

From (13), the variation of the stress tensor and finally the unsymmetric tangent constitutive tensor $\boldsymbol{C}^{\mathrm{D}}$ of the damage model can be deduced as

$$\delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{S}}\,\delta\boldsymbol{\varepsilon} + \delta\boldsymbol{C}^{\mathrm{S}}\,\boldsymbol{\varepsilon}\ ; \qquad \delta\boldsymbol{C}^{\mathrm{S}} = \frac{\partial \boldsymbol{C}^{\mathrm{S}}}{\partial d}\delta d = -\boldsymbol{C}^{\mathrm{o}}\,\delta d \tag{33}$$

$$\delta\boldsymbol{\sigma} = \boldsymbol{C}^{\mathrm{D}}\,\delta\boldsymbol{\varepsilon} = \left[(1-d)\,\boldsymbol{I} - \frac{dG(\bar{\sigma})}{d\bar{\sigma}}\boldsymbol{\sigma}^{\mathrm{o}} \frac{\partial\bar{\sigma}}{\partial\boldsymbol{\sigma}^{\mathrm{o}}}\right] \boldsymbol{C}^{\mathrm{o}}\,\delta\boldsymbol{\varepsilon} = (\boldsymbol{I} - \boldsymbol{D})\,\boldsymbol{C}^{\mathrm{o}}\,\delta\boldsymbol{\varepsilon} \tag{34}$$

where

$$\boldsymbol{C}^{\mathrm{D}} = (\boldsymbol{I} - \boldsymbol{D})\,\boldsymbol{C}^{\mathrm{o}}$$

In these equations, $\boldsymbol{I}$ is the identity matrix of the same order as $\boldsymbol{C}^{\mathrm{o}}$ and $\boldsymbol{D}$ is a non-symetric matrix, depending only on the stress vector $\boldsymbol{\sigma}^{\mathrm{o}}$ in the undamaged material, as the damage variable is also implicitely related with the mentioned stress vector through the equivalent stress $\bar{\sigma}$.

3.3 Visco-elastic effects

The effect of damping on the beam structure is now considered by placing a damper in parallel with the structure. Both the structure and the damper undergo the same deformation $\boldsymbol{\varepsilon}$, so that the total stress $\boldsymbol{\sigma}_{\mathrm{tot}}$ of the system will be the sum of a structural stress $\boldsymbol{\sigma}$ and a viscous stress $\boldsymbol{\sigma}_{\mathrm{vis}}$, i.e.

$$\boldsymbol{\sigma}_{\mathrm{tot}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}_{\mathrm{vis}} = \boldsymbol{C}^{\mathrm{s}}\boldsymbol{\varepsilon} + \boldsymbol{\eta}^{\mathrm{s}}\dot{\boldsymbol{\varepsilon}} \tag{35}$$

where the secant viscous constitutive matrix $\boldsymbol{\eta}^{\mathrm{s}}$ is defined by

$$\boldsymbol{\eta}^{\mathrm{s}} = \frac{\eta}{E^{\mathrm{o}}}\,\boldsymbol{C}^{\mathrm{s}} = \alpha\,\boldsymbol{C}^{\mathrm{s}} \tag{36}$$

In this equation η is the onedimensional viscous parameter and α is the relaxation time, defined as the time needed by the structure-damper system to reach a stable configuration in the undamaged state.

With this assumptions, the behaviour of the system under virtual variations in strains and strain velocities can be obtained as

$$\delta\boldsymbol{\sigma}_{\mathrm{tot}} = \delta\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}_{\mathrm{vis}} = \boldsymbol{C}^{D}\delta\boldsymbol{\varepsilon} + \alpha\left(\boldsymbol{C}^{\mathrm{s}}\delta\dot{\boldsymbol{\varepsilon}} + \delta\boldsymbol{C}^{\mathrm{s}}\dot{\boldsymbol{\varepsilon}}\right) = \boldsymbol{C}^{D}\delta\boldsymbol{\varepsilon} + \alpha\left(\boldsymbol{C}^{\mathrm{s}}\delta\dot{\boldsymbol{\varepsilon}} - \boldsymbol{C}^{\mathrm{o}}\dot{\boldsymbol{\varepsilon}}\,\delta d\right) \tag{37}$$

Introducing $\boldsymbol{\sigma}^{\mathrm{o}}_{\mathrm{vis}} = \alpha\boldsymbol{C}^{\mathrm{o}}\dot{\boldsymbol{\varepsilon}}$ and using relation (34), the visco-elastic incremental strain-stress relation is obtained as

$$\delta\boldsymbol{\sigma}_{\mathrm{tot}} = \boldsymbol{C}^{\mathrm{D}}_{\mathrm{vis}}\delta\boldsymbol{\varepsilon} + \alpha\boldsymbol{C}^{\mathrm{s}}\delta\dot{\boldsymbol{\varepsilon}} = (\boldsymbol{I} - \boldsymbol{D}_{\mathrm{vis}})\boldsymbol{C}^{\mathrm{o}}\delta\boldsymbol{\varepsilon} + \alpha\boldsymbol{C}^{\mathrm{s}}\delta\dot{\boldsymbol{\varepsilon}} \tag{38}$$

where $\boldsymbol{D}_{\mathrm{vis}}$ takes the following value:

$$\boldsymbol{D}_{\mathrm{vis}} = d\,\boldsymbol{I} + \frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}\left(\boldsymbol{\sigma}^{\mathrm{o}} + \boldsymbol{\sigma}^{\mathrm{o}}_{\mathrm{vis}}\right) \otimes \frac{\partial\bar{\sigma}}{\partial\boldsymbol{\sigma}^{\mathrm{o}}} \tag{39}$$

4 APPLICATION OF THE VISCO-DAMAGE CONSTITUTIVE LAW TO THE STRUCTURAL MODEL

The secant and tangent form of the visco-damage constitutive model are deduced in this section. The first one is required because it is used in the integration of the mentioned constitutive model; the second one allows to deduce the constitutive tensor needed in establishing the tangent stiffness and damping matrices.

4.1 Secant equilibrium law

Considering equation (35), the sectional forces —equation (4)— can be expressed as

$$\hat{\boldsymbol{\sigma}}_{\text{tot}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{\sigma}_{\text{tot}}\, \mathrm{d}A = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C}^{\mathrm{s}} \boldsymbol{\varepsilon} \mathrm{d}A + \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{\eta}^{\mathrm{s}} \dot{\boldsymbol{\varepsilon}} \mathrm{d}A \tag{40}$$

where

$$\boldsymbol{\sigma} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C}^{\mathrm{s}} \boldsymbol{\varepsilon} \mathrm{d}A, \qquad \boldsymbol{\sigma}_{\text{vis}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{\eta}^{\mathrm{s}} \dot{\boldsymbol{\varepsilon}} \mathrm{d}A \tag{41}$$

Substituting now $\boldsymbol{\varepsilon}$ from equation (2) and its derivative in (40), the sectional forces become

$$\hat{\boldsymbol{\sigma}}_{\text{tot}} = \Big(\int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C}^{\mathrm{s}} \boldsymbol{S}\, \mathrm{d}A\Big) \hat{\boldsymbol{\varepsilon}} + \Big(\int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{\eta}^{\mathrm{s}} \boldsymbol{S}\, \mathrm{d}A\Big) \dot{\hat{\boldsymbol{\varepsilon}}} \tag{42}$$

This equation can be written in the following compact form

$$\hat{\boldsymbol{\sigma}}_{\text{tot}} = \hat{\boldsymbol{C}}^{\mathrm{s}} \hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\eta}}^{\mathrm{s}} \dot{\hat{\boldsymbol{\varepsilon}}} \tag{43}$$

where the definitions

$$\hat{\boldsymbol{C}}^{\mathrm{s}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{C}^{\mathrm{s}} \boldsymbol{S} \mathrm{d}A, \qquad \hat{\boldsymbol{\eta}}^{\mathrm{s}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \boldsymbol{\eta}^{\mathrm{s}} \boldsymbol{S} \mathrm{d}A$$

have been used. Recalling now equation (8), the derivative of the generalized strain vector is $\dot{\hat{\boldsymbol{\varepsilon}}} = \boldsymbol{B}\dot{\boldsymbol{a}}$ and the sectional forces can be written in the following form:

$$\hat{\boldsymbol{\sigma}}_{\text{tot}} = \hat{\boldsymbol{C}}^{\mathrm{s}} \boldsymbol{B} \boldsymbol{a} + \hat{\boldsymbol{\eta}}^{\mathrm{s}} \boldsymbol{B} \dot{\boldsymbol{a}} \tag{44}$$

Finally, the vector of the internal forces —equation (7)— can be rewritten as

$$\boldsymbol{F}_{\text{int}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}} \hat{\boldsymbol{\sigma}}_{\text{tot}}\, \mathrm{d}x = \Big(\int_{\ell} \boldsymbol{B}^{\mathrm{T}} \hat{\boldsymbol{C}}^{\mathrm{s}} \boldsymbol{B}\, \mathrm{d}x\Big) \boldsymbol{a} + \Big(\int_{\ell} \boldsymbol{B}^{\mathrm{T}} \hat{\boldsymbol{\eta}}^{\mathrm{s}} \boldsymbol{B}\, \mathrm{d}x\Big) \dot{\boldsymbol{a}} \tag{45}$$

Introducing the notations

$$\boldsymbol{\mathcal{K}}_{\text{SEC}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}} \hat{\boldsymbol{C}}^{\mathrm{s}} \boldsymbol{B}\, \mathrm{d}x, \qquad \boldsymbol{\mathcal{D}}_{\text{SEC}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}} \hat{\boldsymbol{\eta}}^{\mathrm{s}} \boldsymbol{B}\, \mathrm{d}x$$

equation (45) is rewritten as

$$\boldsymbol{F}_{\text{int}} = \boldsymbol{\mathcal{K}}_{\text{SEC}}\, \boldsymbol{a} + \boldsymbol{\mathcal{D}}_{\text{SEC}}\, \dot{\boldsymbol{a}} \tag{46}$$

4.2 Tangent equilibrium law

The variation of the sectional forces can be expressed starting from equation (4) in the following form:

$$\delta\hat{\boldsymbol{\sigma}}_{\text{tot}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\, \delta\boldsymbol{\sigma}_{\text{tot}}\, \mathrm{d}A \tag{47}$$

Writing now the variation of the total stresses $\boldsymbol{\sigma}$ of the system using the equations (37) and (38)

$$\delta\boldsymbol{\sigma}_{\text{tot}} = \delta\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}_{\text{vis}} = (\boldsymbol{I} - \boldsymbol{D}_{\text{vis}})\boldsymbol{C}^{\text{o}}\, \delta\boldsymbol{\varepsilon} + \alpha\boldsymbol{C}^{\text{s}}\, \delta\dot{\boldsymbol{\varepsilon}} \tag{48}$$

equation (47) becomes

$$\delta\hat{\boldsymbol{\sigma}}_{\text{tot}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} \Big[(\boldsymbol{I} - \boldsymbol{D}_{\text{vis}})\boldsymbol{C}^{\text{o}}\Big] \delta\boldsymbol{\varepsilon}\, \mathrm{d}A + \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\, \alpha\boldsymbol{C}^{\text{s}}\, \delta\dot{\boldsymbol{\varepsilon}}\, \mathrm{d}A \tag{49}$$

Substituting $\boldsymbol{\varepsilon}$ given by (2) and its derivative in (49), the variation of the sectional forces takes the form

$$\delta\hat{\boldsymbol{\sigma}}_{\text{tot}} = \Big[\int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} (\boldsymbol{I} - \boldsymbol{D}_{\text{vis}})\boldsymbol{C}^{\text{o}}\, \boldsymbol{S}\, \mathrm{d}A\Big]\, \delta\hat{\boldsymbol{\varepsilon}} + \Big[\int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\, \alpha\boldsymbol{C}^{\text{s}}\, \boldsymbol{S}\, \mathrm{d}A\Big] \delta\dot{\hat{\boldsymbol{\varepsilon}}} \tag{50}$$

which, using similar developments as those used in Section 4.1, can be written as

$$\delta\hat{\boldsymbol{\sigma}}_{\text{tot}} = \hat{\boldsymbol{C}}^{\text{D}}\, \boldsymbol{B}\delta\boldsymbol{a} + \hat{\boldsymbol{\eta}}^{\text{s}} \boldsymbol{B}\delta\dot{\boldsymbol{a}} \tag{51}$$

where

$$\hat{\boldsymbol{C}}^{\text{D}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}} (\boldsymbol{I} - \boldsymbol{D}_{\text{vis}})\boldsymbol{C}^{\text{o}}\, \boldsymbol{S}\, \mathrm{d}A, \qquad \hat{\boldsymbol{\eta}}^{\text{s}} = \int_{\mathcal{A}} \boldsymbol{S}^{\mathrm{T}}\, \alpha\boldsymbol{C}^{\text{s}}\, \boldsymbol{S}\, \mathrm{d}A \tag{52}$$

Using these equations, the variation of the internal forces vector is expressed in the following form

$$\delta\boldsymbol{F}_{\text{int}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}}\, \delta\hat{\boldsymbol{\sigma}}_{\text{tot}}\, \mathrm{d}x = \Big(\int_{\ell} \boldsymbol{B}^{\mathrm{T}}\, \hat{\boldsymbol{C}}^{\text{D}}\, \boldsymbol{B}\, \mathrm{d}x\Big)\, \delta\boldsymbol{a} + \Big(\int_{\ell} \boldsymbol{B}^{\mathrm{T}}\, \hat{\boldsymbol{\eta}}^{\text{s}} \boldsymbol{B}\, \mathrm{d}x\Big)\, \delta\dot{\boldsymbol{a}} \tag{53}$$

Introducing the notations

$$\boldsymbol{\mathcal{K}}_{\text{TAN}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}}\, \hat{\boldsymbol{C}}^{\text{D}}\, \boldsymbol{B}\, \mathrm{d}x, \qquad \boldsymbol{\mathcal{D}}_{\text{TAN}} = \int_{\ell} \boldsymbol{B}^{\mathrm{T}}\, \hat{\boldsymbol{\eta}}^{\text{s}} \boldsymbol{B}\, \mathrm{d}x$$

equation (53) is finally written as

$$\delta\boldsymbol{F}_{\text{int}} = \boldsymbol{\mathcal{K}}_{\text{TAN}}\, \delta\boldsymbol{a} + \boldsymbol{\mathcal{D}}_{\text{TAN}}\, \delta\dot{\boldsymbol{a}} \tag{54}$$

5 GLOBAL DAMAGE INDICES

The starting point for deducing a global structural damage index is equation (10), which relates the damaged part of the free energy Ψ with the non-damaged elastic free energy Ψ_{o}. In order to find a global index, a similar expression is deduced by integrating (10) over the entire volume of the structure as follows:

$$\Psi = (1-d)\Psi_{o} \Rightarrow W_{P} = \int_{V} \Psi \mathrm{d}V = \int_{V} (1-d)\Psi_{o} \mathrm{d}V = (1-D) W_{P}^{o} \qquad (55)$$

where D is the global damage index, $W_{P}^{o} = \int_{V} \Psi_{o} \mathrm{d}V$ is the total potential energy of the structure considered as undamaged and W_{P} is the total potential energy corresponding to the actual damaged state. Solving equation (55) for D, the following final relation is obtained:

$$D = 1 - \frac{W_{P}}{W_{P}^{o}} = \frac{\int_{V} \Psi_{o} \mathrm{d}V - \int_{V} (1-d)\Psi_{o} \mathrm{d}V}{\int_{V} \Psi_{o} \mathrm{d}V} = \frac{\int_{V} d\,\Psi_{o} \mathrm{d}V}{\int_{V} \Psi_{o} \mathrm{d}V} \qquad (56)$$

If a damage index for a part or member of the structure is needed (such as floors, columns, etc) the integration will be performed only over that specific part.

In a finite element scheme, the damage index D_{P} of a beam cross-section is given by a similar expression obtained by integrating (10) over the cross-section of the beam, with $\Psi_{o} = \frac{1}{2}\boldsymbol{\varepsilon}^{T}\boldsymbol{\sigma}_{tot}^{o}$ and $\boldsymbol{\varepsilon} = \boldsymbol{S}\hat{\boldsymbol{\varepsilon}}$, i.e.

$$D_{P} = 1 - \frac{\hat{\boldsymbol{\varepsilon}}^{T}\hat{\boldsymbol{\sigma}}_{tot}}{\hat{\boldsymbol{\varepsilon}}^{T}\hat{\boldsymbol{\sigma}}_{tot}^{o}} \; ; \qquad \hat{\boldsymbol{\sigma}}_{tot} = \int_{A} \boldsymbol{S}^{T} \boldsymbol{\sigma}_{tot} \, \mathrm{d}A = \int_{A} (1-d) \boldsymbol{S}^{T} \boldsymbol{\sigma}_{tot}^{o} \, \mathrm{d}A \qquad (57)$$

where $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\sigma}}_{tot}$ are the generalized strains and stresses in that beam cross-section, respectively.

In general, the global damage index will take the following form:

$$D = 1 - \frac{\sum_{(e)} \boldsymbol{a}^{(e)\,T} \int_{\ell^{(e)}} \boldsymbol{B}^{(e)\,T} \hat{\boldsymbol{\sigma}}_{tot}^{(e)} \mathrm{d}s}{\sum_{(e)} \boldsymbol{a}^{(e)\,T} \int_{\ell^{(e)}} \boldsymbol{B}^{(e)\,T} \hat{\boldsymbol{\sigma}}_{tot}^{o\,(e)} \mathrm{d}s} \qquad (58)$$

where the sum is performed over the beam elements for which the global damage index is calculated. This damage index is similar to that proposed by DiPasquale and Cakmak (1989).

6 NUMERICAL IMPLEMENTATION

The implementation process of the visco-damage model in a finite element computer program is explained in Tables 1, 2 and 3. The block scheme of Table 3 is called within Table 2 for evaluating the constitutive characteristics of the model. Table 2 is called in Table 1, within the point **B.III**, to compute the sectional forces and the tangent and secant constitutive tensors.

Table 1 Nonlinear time integration scheme (Newmark).

▶ **A. First iteration (passage from time instant i to time instant $i+1$)**

▷ **Update relevant matrices**

$$\mathcal{K}_{\text{SEC}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{C}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{K}_{\text{TAN}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{C}}^{\text{D}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{D}_{\text{SEC}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{\eta}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{D}_{\text{TAN}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{\eta}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x$$

▷ **Compute**

$$\hat{\boldsymbol{K}} = \frac{1}{\beta\,\Delta t^2}\,\boldsymbol{M} + \frac{\gamma}{\beta\,\Delta t}\,\mathcal{D}_{\text{TAN}} + \mathcal{K}_{\text{TAN}}$$

$$\hat{\boldsymbol{F}}_{i+1}^{(1)} = \boldsymbol{F}(t_{i+1}) + \boldsymbol{M}\left[\frac{1}{\beta\,\Delta t}\,\dot{\boldsymbol{a}}_i + \left(\frac{1}{2\,\beta} - 1\right)\ddot{\boldsymbol{a}}_i\right] - (\mathcal{D}_{\text{TAN}} - \mathcal{D}_{\text{SEC}})\,\dot{\boldsymbol{a}}_i - \mathcal{D}_{\text{TAN}}\left[\left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{a}}_i + \left(1 - \frac{\gamma}{2\,\beta}\right)\Delta t\,\ddot{\boldsymbol{a}}_i\right] - \mathcal{K}_{\text{SEC}}\,\boldsymbol{a}_i$$

▷ **Calculate the first aproximations for the time instant $i+1$:**

$$\Delta\boldsymbol{a}_{i+1}^{(1)} = \hat{\boldsymbol{K}}^{-1}\,\hat{\boldsymbol{F}}_{i+1}^{(1)}$$

$$\begin{aligned}
\ddot{\boldsymbol{a}}_{i+1}^{(1)} &= \frac{1}{\beta\,\Delta t^2}\,\Delta\boldsymbol{a}_{i+1}^{(1)} - \frac{1}{\beta\,\Delta t}\,\dot{\boldsymbol{a}}_i - \left(\frac{1}{2\,\beta} - 1\right)\ddot{\boldsymbol{a}}_i \\
\dot{\boldsymbol{a}}_{i+1}^{(1)} &= \frac{\gamma}{\beta\,\Delta t}\,\Delta\boldsymbol{a}_{i+1}^{(1)} + \left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{a}}_i + \left(1 - \frac{\gamma}{2\,\beta}\right)\Delta t\,\ddot{\boldsymbol{a}}_i \\
\boldsymbol{a}_{i+1}^{(1)} &= \Delta\boldsymbol{a}_{i+1}^{(1)} + \boldsymbol{a}_i
\end{aligned}$$

▶ **B. Second and subsequent iterations (seeking the equilibrium for the time instant $i+1$)**

Loop over global convergence interations: j^{th} iteration

▷ **I. Update relevant matrices**

$$\mathcal{K}_{\text{SEC}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{C}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{K}_{\text{TAN}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{C}}^{\text{D}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{D}_{\text{SEC}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{\eta}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x;\ \mathcal{D}_{\text{TAN}} = \int_\ell \boldsymbol{B}^{\text{T}} \hat{\boldsymbol{\eta}}^{\text{S}} \boldsymbol{B}\,\mathrm{d}x$$

$$\hat{\boldsymbol{K}} = \frac{1}{\beta\,\Delta t^2}\,\boldsymbol{M} + \frac{\gamma}{\beta\,\Delta t}\,\mathcal{D}_{\text{TAN}} + \mathcal{K}_{\text{TAN}}$$

$$\hat{\boldsymbol{F}}_{i+1}^{(j+1)} = \boldsymbol{F}_{\text{int}}(t_{i+1}) - \boldsymbol{M}\,\ddot{\boldsymbol{a}}_{i+1}^{(j)} - \mathcal{D}_{\text{SEC}}\,\dot{\boldsymbol{a}}_{i+1}^{(j)} - \mathcal{K}_{\text{SEC}}\,\boldsymbol{a}_{i+1}^{(j)}$$

▷ **II. If the residual forces norm $||\hat{\boldsymbol{F}}_{i+1}^{(j+1)}|| \leq \varepsilon$, end of iteretions and beginning of the computations in the next time step. If not, proceed calculating:**

$$\delta\boldsymbol{a}_{i+1}^{(j+1)} = \hat{\boldsymbol{K}}^{-1}\,\hat{\boldsymbol{F}}_{i+1}^{(j+1)}$$

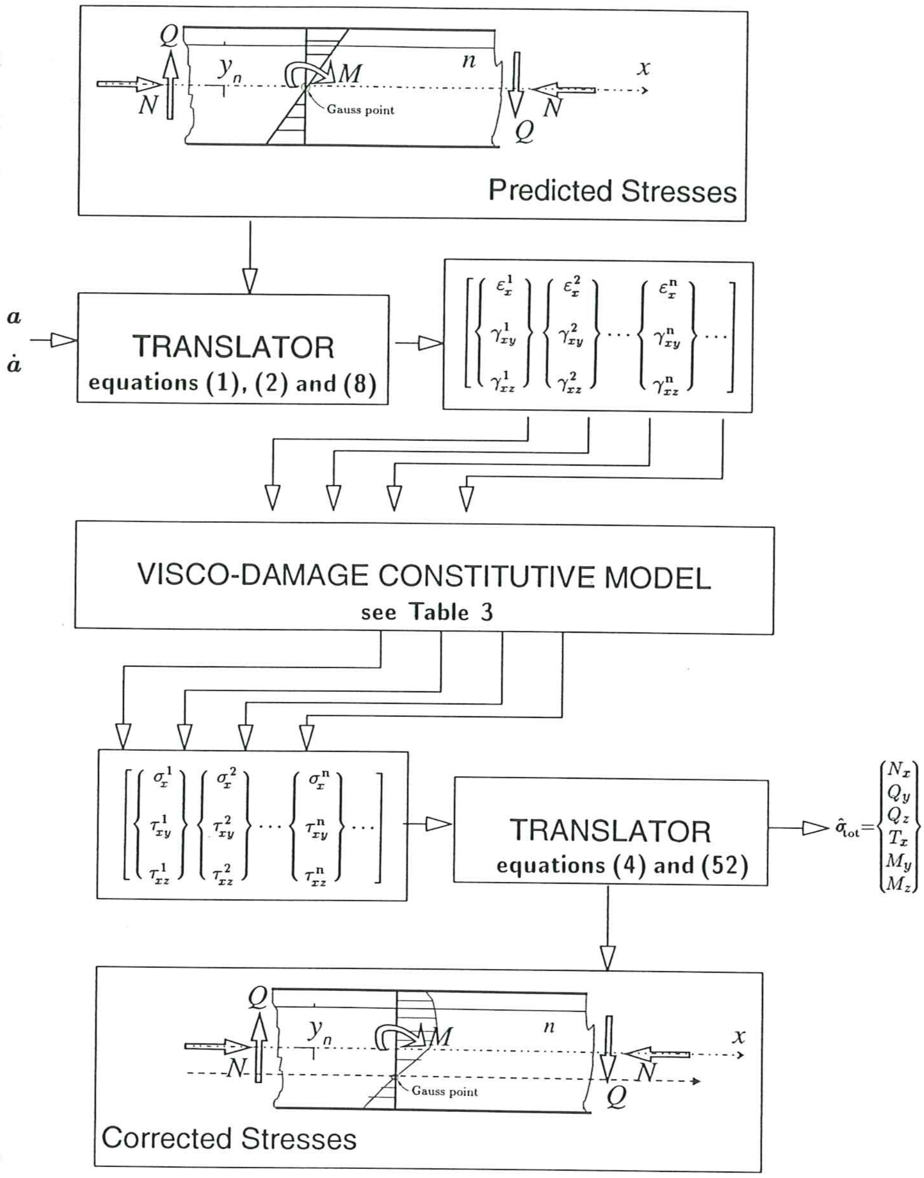
$$\begin{aligned}
\ddot{\boldsymbol{a}}_{i+1}^{(j+1)} &= \frac{1}{\beta\,\Delta t^2}\,\delta\boldsymbol{a}_{i+1}^{(j+1)} + \ddot{\boldsymbol{a}}_{i+1}^{(j)} \\
\dot{\boldsymbol{a}}_{i+1}^{(j+1)} &= \frac{\gamma}{\beta\,\Delta t}\,\delta\boldsymbol{a}_{i+1}^{(j+1)} + \dot{\boldsymbol{a}}_{i+1}^{(j)} \\
\boldsymbol{a}_{i+1}^{(j+1)} &= \delta\boldsymbol{a}_{i+1}^{(j+1)} + \boldsymbol{a}_{i+1}^{(j)}
\end{aligned}$$

▷ **III. Compute the sectional forces and the tangent and secant constitutive tensors**

Cross sectional forces decomposition at each Gauss point (see Table 2)

▷ **IV. Back to step I**

Table 2 Cross-sectional forces decomposition for each Gauss point.



7 NUMERICAL EXAMPLES

Example 1

The simulation of the evolution of the damage process in a reinforced concrete plane frame (Figure 5) subjected to dynamic loading is first analyzed.

The frame is 9 meters high and 6 meters wide and has three levels. The columns have a 30 cm×30 cm cross-section of reinforced concrete with a 4.35% steel ratio. The horizontal beams are 40 cm thick and 30 cm wide, with a steel ratio of 5.3%. The structure was discretized in 45 quadratic three-noded beam finite elements having two Gauss points each. Thus, the resulting dynamic model has 87 nodes with three degrees of freedom per node. Each element is one meter long and has the cross-section divided in 20 layers of equal thickness.

Table 3 Visco-damage constitutive model at layer level n.

1. **Input: strains and strain velocities**

$$\Longrightarrow \boxed{\varepsilon_i^{(j)}} \quad \boxed{\dot{\varepsilon}_i^{(j)}}$$

2. **Compute the predicted non-damaged stresses for the load step i and the global convergence iteration j**

$$\boxed{(\boldsymbol{\sigma}^{\mathrm{o}})_i^{(j)} = (\boldsymbol{C}^{\mathrm{o}})_1^{(0)} \boldsymbol{\varepsilon}_i^{(j)}}$$

3. **General form to integrate the damage constitutive equation (Euler Backward Scheme)**
Loop over inner convergence iterations: k^{th} iteration

$$\text{for: } k = 1 \Longrightarrow \boldsymbol{\sigma}_i^{(j,0)} = (\boldsymbol{\sigma}^{\mathrm{o}})_i^{(j)}$$

$$\circledast \qquad \boldsymbol{\sigma}_i^{(j,k)} = (1 - d_i^{(j,k)})\boldsymbol{\sigma}_i^{(j,k-1)}$$

$$\bar{\sigma}_i^{(j,k)} = \bar{\sigma}\left(\boldsymbol{\sigma}_i^{(j,k)}\right)$$

If $F(\bar{\sigma}, d) \leq 0$ [equation (17)] $\Longrightarrow$ no damage $\Longrightarrow$ GOTO 4

⇓
else
||
damage
⇓

$$(\Delta d)_i^{(j,k)} = (\Delta \eta)_i^{(j,k)} \cdot \left(\frac{\partial G}{\partial \bar{\sigma}}\right)_i^{(j,k)}$$

$$(d)_i^{(j,k)} = (d)_i^{(j,k-1)} + (\Delta d)_i^{(j,k)}$$

$$(\boldsymbol{D}_{\mathrm{vis}})_i^{(j,k)} = \left[d\,\boldsymbol{I} + \frac{\mathrm{d}G(\bar{\sigma})}{\mathrm{d}\bar{\sigma}}(\boldsymbol{\sigma}^{\mathrm{o}} + \boldsymbol{\sigma}_{\mathrm{vis}}^{\mathrm{o}}) \otimes \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}^{\mathrm{o}}}\right]_i^{(j,k)}$$

$$\left(\boldsymbol{C}_{\mathrm{vis}}^{\mathrm{D}}\right)_i^{(j,k)} = \left[(\boldsymbol{I} - \boldsymbol{D}_{\mathrm{vis}})\boldsymbol{C}^{\mathrm{o}}\right]_i^{(j,k)}$$

$k = k + 1$ *Go back to* $\circledast$

4. **Compute the visco-elastic part of the stresses and the total stresses**

$$\boxed{(\boldsymbol{\eta}^{\mathrm{S}})_i^{(j,k)} = \alpha\left(\boldsymbol{C}^{\mathrm{S}}\right)_i^{(j,k)}}$$

$$\boxed{(\boldsymbol{\sigma}_{\mathrm{tot}})_i^{(j)} = (\boldsymbol{\sigma})_i^{(j)} + (\boldsymbol{\eta}^{\mathrm{S}}\dot{\boldsymbol{\varepsilon}})_i^{(j,k)}}$$

STOP

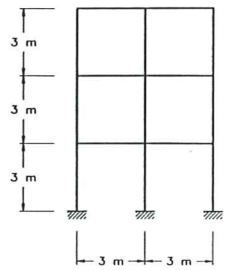


Figure 5 Geometry of the studied frame.

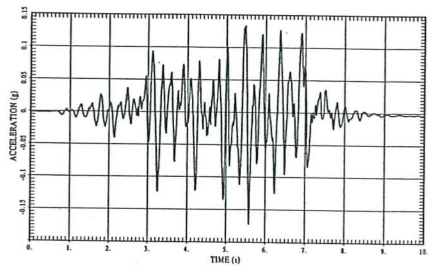


Figure 6 Synthetic seismic accelerogram corresponding to case (a).

The 2nd and 19th layer are made of steel and the rest of concrete. The steel ratio was controlled by modifing the width of the steel layers. The state of the material is checked at the interface between layers and afterwards interpolated linearly across each layer. This gives 40 check points per cross-section in each Gauss point. The materials have the following properties: *(a) steel* $E = 2.1 \times 10^6\,\mathrm{daN/cm^2}$, $\sigma^o = 4,200\,\mathrm{daN/cm^2}$, $\nu = 0.25$, $\rho = 8\,\mathrm{g/cm^3}$; *(b) concrete* $E = 2.0 \times 10^5\,\mathrm{daN/cm^2}$, $\sigma^o = 300\,\mathrm{daN/cm^2}$, $\nu = 0.17$, $\rho = 2.5\,\mathrm{g/cm^3}$.

The equations of motion governing the dynamic behaviour of the structure have been solved using the Newmark algorithm with $\beta = 0.25$ and $\gamma = 0.5$. The initial stiffness method was chosen as nonlinear solution scheme due to the negative definition of the tangent stiffness matrix when softening effects occur. The time step used was a thirtieth of the fundamental period of the structure. As the integration of the constitutive law can be done analytically, an explicit formula [equation (20)] was used for the local damage index thus reducing remarkably the solution cost.

The structure was calculated in two load cases: (a) subjected to a synthetic earthquake accelerogram (Figure 6) having a predominant frequency of 4 Hz and a maximum amplitude of 0.175 g and (b) subjected to the same accelerogram with doubled amplitudes. This allows the simulation of the structural behaviour firstly in a less damaged state [Figure 7(a)] and finally in a generally collapsed state [Figures 7(b) and 8].

Figures 7(a) and 7(b) show the distribution of the sectional damage as given by equation (53). The damage is located at the joints of the columns with the floors, this being precisely the expected damage localization for this type of structure and load. As the frame is to fail mainly by damage of the columns at their joint with the base floor, the damage plots confirm this prognosed behaviour too. The results of Figure 9(a) correspond to the undamped case, while in those of Figure 9(b) the damping effects are included through a value for the relaxation time $\alpha = 0.001\,\mathrm{s}$.

The results of Figure 9 show that the maximum sectional damage D_p at the base of the columns is practically equal to the global damage of the entire structure. This fact ratifies the choice of the global damage index as the ratio between the potential energy which the structure cannot undertake in the damaged state and the potential energy that the structure should undertake if it were undamaged. The first floor damage is slightly higher than the global damage of the structure as this floor is the most affected, while the second and third

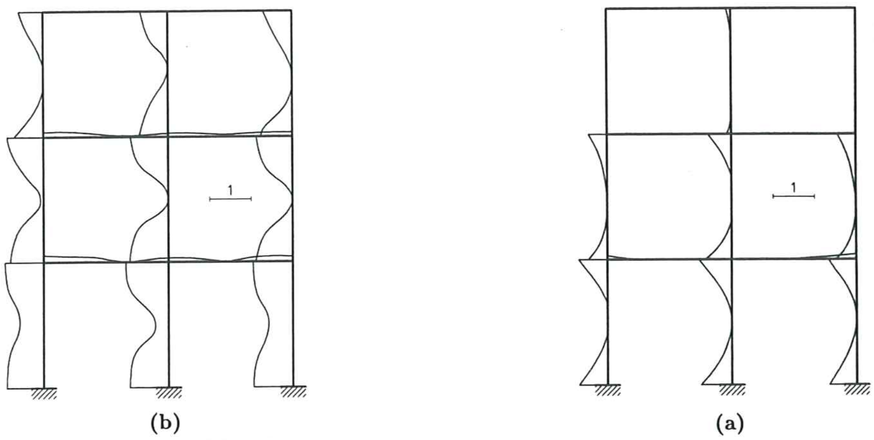


Figure 7 Distribution of sectional damage D_p all over the structure. Case (a), accelerogram with an amplitude of 0.175 g. Case (b), accelerogram with an amplitude of 0.35 g.

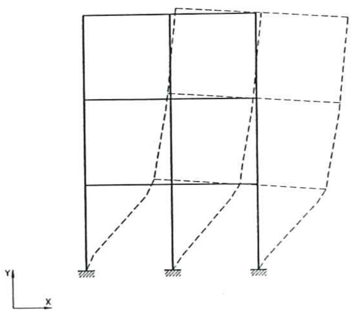


Figure 8 Deformed configuration at collapse.

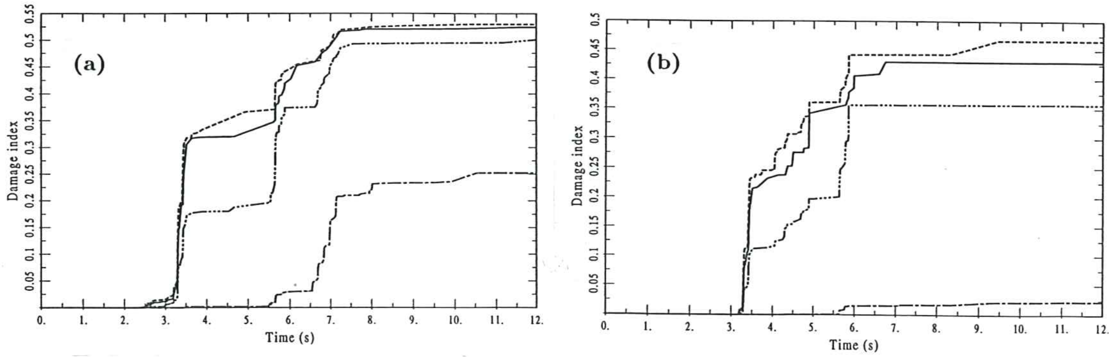


Figure 9 Global and floor damage indices. —— global; - - - - - first floor; —···· second floor; —-— third floor. Case (a), damping not considered. Case (b), damping considered, relaxation time $\alpha = 0.001\,\mathrm{s}$.

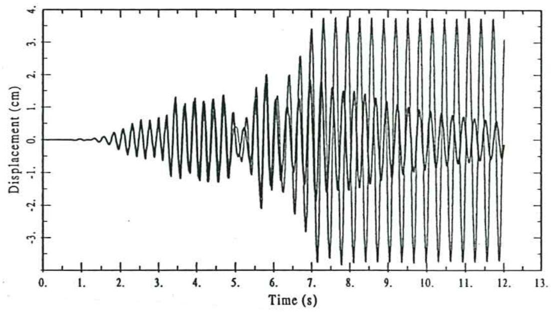


Figure 10 3rd floor displacements, with and without damping.

floors follow in decreasing order as the damage reduces with height. The effect of viscous damping is reducing amplitudes and damage levels (Figures 9(b) and 10). This is in agreement with the real behaviour of structures in a dynamic environment, where the materials display increased strengths and nonconservative energy dissipation.

Example 2

The described methodology has been also used to simulate the behaviour of a 3D frame subjected to the same synthetic accelerogram of Figure 6, acting in the x direction.

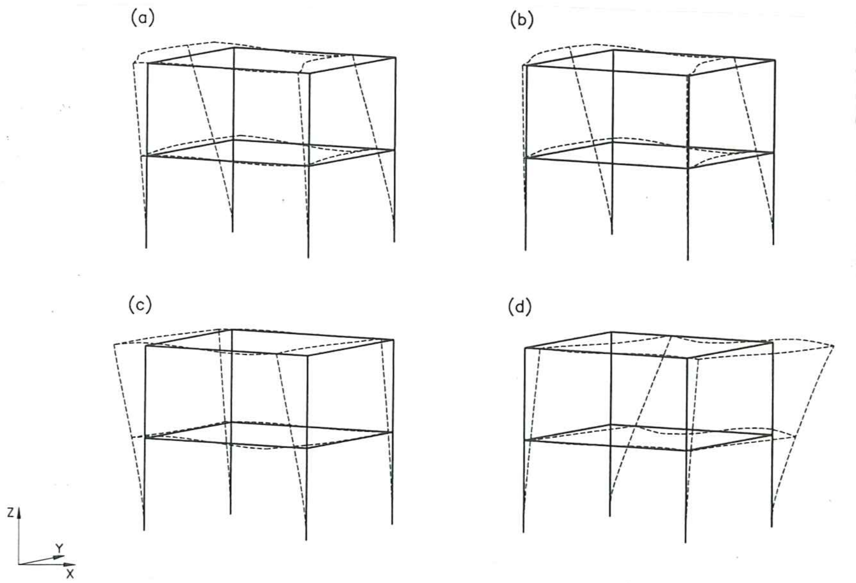


Figure 11 Deformed shapes of the frame at different time instants during the earthquake.

The frame has two floors, is 6 m high and has a squared base of 6 m. The columns have a 30 cm×30 cm squared reinforced concrete cross-section. The

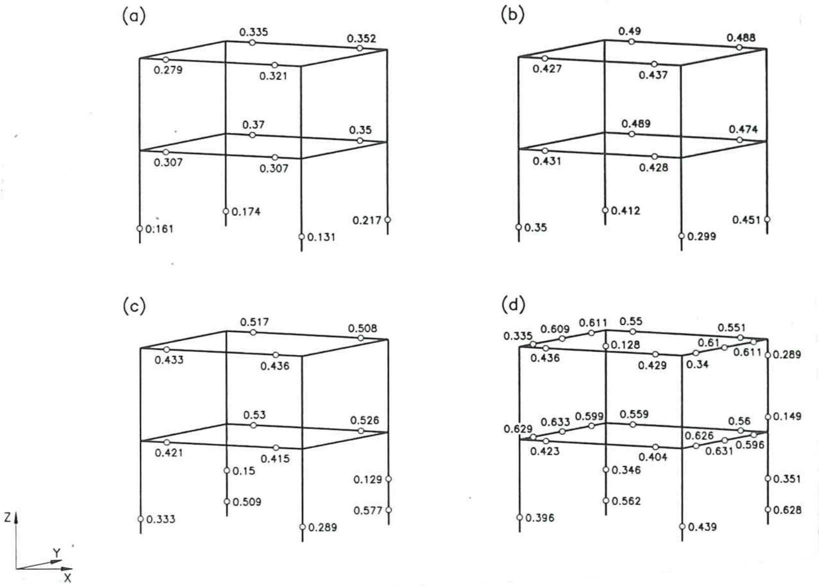


Figure 12 Evolution of the structural damage level.

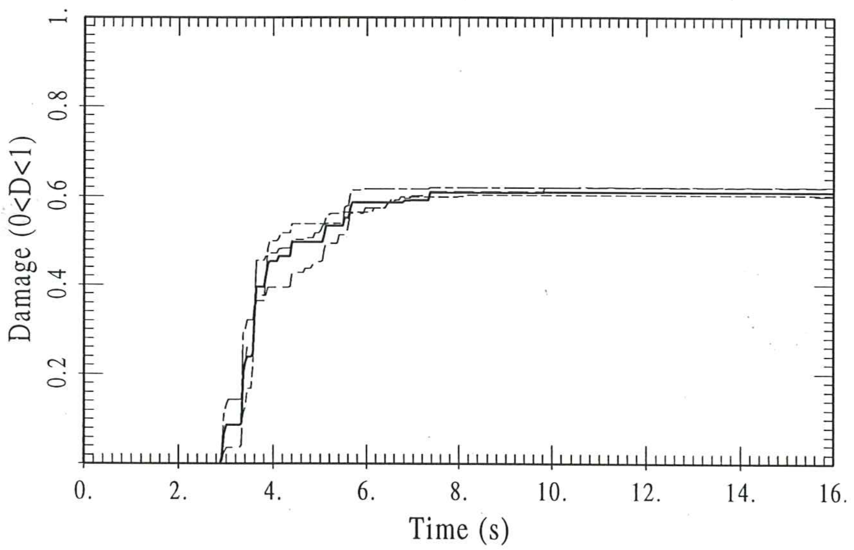


Figure 13 Evolution of the global and floor damage indices.——global;—-—columns of the first floor;-—-—-beams of the first floor;-----second floor.

horizontal beams are 30 cm thick and 15 cm wide. All the bars have an 8% of steel, located symetrically at the corners, with a concrete cover of 3 cm. The density of the concrete has been increased, to take into account the effect of the inertia of the entire floor. The beams placed at one of the sides of the structure (vertical plane $y = 6$ m) have double density, thus the effect of the global structural torsion being simulated.

The structure was discretized in 48 quadratic three-noded beam finite elements with two Gauss points each and the resulted dynamic model has 92 nodes with six degrees of freedom per node. The elements corresponding to the columns are one meter long and those corresponding to the beams are two meters long. All the cross-sections are discretized by means of a 10×10 grid, appearing thus 400 grid corners per Gauss point at which the state of the material is checked. The materials have the same properties as in Example 1.

The equations of motion have been solved using Newmark's step by step algorithm for $\beta = 0.25$ and $\gamma = 0.5$. Four deformed shapes of the structure during the earthquake can be seen in Figure 11. Figure 12 shows the distribution of the sectional damage in the structural elements. The beams are damaged first, due to their higher inertia and smaller stiffness. The columns are highly damaged at their lower part, as expected. It can be observed in Figure 13 that both floors are damaged almost simultaneously and that the three compared damage indices have a similar time evolution.

8 CONCLUSIONS

The visco-damage constitutive model developed has proved to have good performance in describing the nonlinear behaviour of reinforced concrete building structures under dynamic load. The model has been incorporated in a finite element scheme using 2D and 3D Timoshenko beam elements discretized in a grid of rectangles of concrete and steel in order to approximate the nonlinear behaviour of reinforced concrete beams. A global damage index was deduced from the local damage index supplied by the constitutive model.

A reinforced concrete building structure, under both non viscous and viscous regimes, subjected to seismic actions, has been solved and satisfactory results were obtained. It is shown that the effect of considering the viscosity is of great importance. An interesting property of the global damage index is that of allowing the decision of the state of the structure in what regards its failure mechanisms. The model permits the identification of the mechanism of collapse by observing the local damage indices and continuous comparison with the global one. When, during a damaging process, the global index gets close to the maximum local damage and the rest of the points of the structure stop degradating, the critical points of the structure has been indentified. The failure of these points leads to the formation of a failure mechanism, i.e. collapse of the structure. This is important from an engineering structural retrofitting point of view.

The model, in its present form, has two major drawbacks: first, it does not provide information about remanent deformation, which is a well-known feature of non-linear materials and second, it does not discriminate between traction and compression damage, thus being unable the simulate "crack closure". These two problems are currently under study and solutions are already in sight.

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