XVI International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS 2021
E. Oñate, D. Peric, M. Chiumenti and Eduardo de Souza Neto (Eds)

TRANSIENT LARGE HEAT ADVECTION IN FRACTURED ROCK: A ZERO-THICKNESS INTERFACE FORMULATION

A. PÉREZ*, I. CAROL* AND P. PRAT*

* Department of Civil and Environmental Engineering
Universitat Politècnica de Catalunya (UPC)
Jordi Girona 1, D2 Building, E-08034 Barcelona, Spain
e-mail: adria.perez@upc.edu, ignacio.carol@upc.edu, pere.prat@upc.edu

Key words: transient large advection, heat transport, zero-thickness interface elements, Péclet number, FEM, Characteristic methods.

Abstract. In a fractured rock mass, the existence of discontinuities may generate preferential paths where the hydraulic flow velocities are frequently much higher than in the porous medium and heat advection tends to dominate over heat diffusion. In these cases, the standard Galerkin FEM method leads to oscillatory results and requires the use of stabilization methods. Thus, the current paper introduces a 3-D formulation to solve the large advection problem for zero-thickness interface elements –which may be used to discretize fractures in a FEM context–, based on a pre-existent Characteristic method. A verification example is presented, showing that the formulation exhibits a good performance to represent the heat transport by the fluid along zero-thickness interface elements.

1 INTRODUCTION

Two of the main mechanisms of heat transport in engineering problems are diffusion and advection. In the FEM context it is well known that if advection dominates over diffusion (Péclet number $Pe > 1$), traditional Galerkin formulations may lead to oscillatory results [1], although in many practical engineering situations, such as the case of geological materials, fluid velocities generally remain small due to the low permeability of the pore system and this problem may be ignored [2].

However, in the presence of preferential paths of fluid circulation this situation may change. The existence of open fractures or cracks may produce fluid velocities much higher in comparison with those found in the surrounding porous medium. This situation can lead to exceeding the limit condition $Pe > 1$. In a FEM context, these preferential paths or cracks may be modelled using double-node zero-thickness interface elements. Thus, the paper discusses these concepts and presents a zero-thickness interface FE formulation for transient large advection, which consists of an implicit extension of Zienkiewicz´s explicit formulation based on characteristics [3], which was originally developed for traditional continuum elements in its explicit form.

2 GOVERNING EQUATIONS

The *transient heat conduction-advection differential equation* may be written as:

$$\rho c\left(\frac{\partial\phi}{\partial t}+\mathbf{v}^{\top}\,\boldsymbol{\nabla}\phi\right)-\boldsymbol{\nabla}^{\top}\,\mathbf{D}^{T}\,\boldsymbol{\nabla}\phi-Q^{T}=0\quad\text{ in }\quad\Omega \tag{1}$$

where ϕ is the temperature (unknown), ρ is the material density, c is the material thermal capacity, $\mathbf{v}=[v_x\quad v_y\quad v_z]^{\top}$ is the velocity vector (usually obtained as the result of the solution of a pore fluid flow problem), $\mathbf{D}^T$ is the thermal conductivity matrix, Q^T is the source term, $\boldsymbol{\nabla}=[\partial/\partial x\quad \partial/\partial y\quad \partial/\partial z]^{\top}$ and $[\ ]^{\top}$denotes the transposed of a vector or a matrix (not to be confused with superscript T which stands for "thermal"). The heat flow vector can be written as:

$$\mathbf{q}^T=-\mathbf{D}^T\boldsymbol{\nabla}\phi\ +\ \rho c\phi\mathbf{v} \tag{2}$$

The boundary and initial conditions are usually defined as:

$$\phi=\bar{\phi}(x,y,x,t)\quad\text{ on }\quad\Gamma_{\phi} \tag{3}$$

$$q_n^T=\bar{q}_n^T(x,y,z,t)\quad\text{ on }\quad\Gamma_{\mathrm{q}} \tag{4}$$

$$\phi(x,y,z,t_0)=\phi_0(x,y,z)\qquad in\qquad \Omega \tag{5}$$

where q_n^T is the flow normal to the boundary and $\bar{q}_n^T$ is the prescribed value of the flow. The flow q_n^T is calculated as the normal projection of the flow vector $\mathbf{q}^T$ on the boundary Γ_{q}. Using Eq. (2) this condition becomes:

$$q_n^T=[\mathbf{n}]^{\top}\mathbf{q}^T=-[\mathbf{n}]^{\top}\mathbf{D}^T\,\boldsymbol{\nabla}\phi\ +\ \rho c\phi v_n \tag{6}$$

where $v_n=[\mathbf{n}]^{\top}\mathbf{v}$ is the normal velocity at the boundary Γ_q. Using Eqs. (4) and (6) the same equation may be finally written as:

$$-\rho c\phi v_n+[\mathbf{n}]^{\top}\mathbf{D}^T\,\boldsymbol{\nabla}\phi+\bar{q}_n^T=0\quad\text{ on }\quad\Gamma_{\mathrm{q}} \tag{7}$$

3 CHARACTERISTIC PROCEDURES TO SOLVE LARGE ADVECTION PROBLEMS

As it is well known [2], the standard Galerkin FEM resolution of Eq. (1) may lead to unstable solutions when the advection dominates the problem ($Pe > 1$). In the last decades, several methodologies have been proposed to solve large advection transient problems, such as the Streamline-Upwind Petrov-Galerkin method –SUPG– [3], the Galerkin Least Squares –GLS– [4], the Characteristic procedures [5-12], the Variational Multiscale Method –VMS– [13], the artificial diffusion method [14], the bubble functions method [15], the Finite Increment Calculus –FIC– [16], or the High Resolution method [17], among others.

Characteristic methods lead to stabilization parameters similar to those obtained with the SUPG and the GLS, but are obtained with a methodology conceptually based on the wave nature of the equations, which "is much more direct, intuitive and fully justifies the numerical procedures" [1].

3.1 Introduction to the Characteristic Methods

From Eq. (1) it is possible to write the differential equation for the conduction-advection transient problem in indicial notation as follows

$$\rho c\left(\frac{\partial \phi}{\partial t}+v_i \frac{\partial \phi}{\partial x_i}\right)-\frac{\partial}{\partial x_i}\left(k_{ij}^T \frac{\partial \phi}{\partial x_j}\right)-Q^T(x_i)=0 \tag{8}$$

In order to simplify the numerical solution and determine the main behaviour patterns of this equation, a change of the independent variable x to x' is proposed. Considering a moving coordinate system x' defined by

$$dx_i{}'=dx_i-v_i dt \tag{9}$$

and the material derivative of the temporal derivative of Eq. (8) can be written as

$$\left.\frac{\partial \phi}{\partial t}\right|_{x\equiv const}=-v_i\cdot\frac{\partial \phi}{\partial x_i{}'}+\left.\frac{\partial \phi}{\partial t}\right|_{x'\equiv const} \tag{10}$$

Performing a change of the independent variable x_i to $x_i{}'$ on Eq. (8), the differential equation becomes simply:

$$\rho c\frac{\partial \phi}{\partial t}-\frac{\partial}{\partial x_i{}'}\left(k_{ij}^T \frac{\partial \phi}{\partial x_i{}'}\right)-Q^T(x_i{}')=0 \tag{11}$$

Equation (9) has the same form as the traditional diffusion equation but formulated on the moving coordinate system x', where the advection term has disappeared. This type of equations can be discretized *along the characteristics* with the standard Galerkin spatial approximation, as usual for pure diffusion problems but along a moving coordinate system x'.

The coordinate system of Eq. (9) describes the so-called *characteristic directions* of the problem, and it is a moving coordinate system that depends on the velocity of the fluid particles in a time-increment Δt, which determines the distance travelled by the particle in that time-increment. This concept leads to the so-called *Characteristic-based* methods [1].

From this concept, different methods have been developed, such as the Mesh Updating Method [5], the Galerkin-Characteristics Method [6] or the Taylor-Characteristics Method [7]. However, these methods are "computationally complex and time-consuming" [1]. Thus, a simple alternative method was proposed by [18]: the simple explicit Characteristic-Galerkin method, that avoids the difficulties of the previous methods but is conditionally stable.

Thus, and for simplicity, this paper starts from the simple Characteristic-Galerkin method proposed by [18], but with a time-integration scheme unconditionally stable.

3.2 An implicit Characteristic-Galerkin Method (ICG Method)

This section develops the extension of the simple explicit Characteristic method proposed by [18] to an implicit scheme. The first step of this procedure consists of the time discretization of Eq. (11) along the characteristic, using the Finite Difference Method. Developing this part for a one-dimensional problem:

$$\rho c\frac{1}{\Delta t}\left(\phi^{n+1}|_x-\phi^n|_{(x-\delta)}\right)\approx\theta\left.\left[\frac{\partial}{\partial x}\left(k^T\frac{\partial \phi}{\partial x}\right)-Q^T\right]^{n+1}\right|_x+(1-\theta)\left.\left[\frac{\partial}{\partial x}\left(k^T\frac{\partial \phi}{\partial x}\right)-Q^T\right]^{n}\right|_{(x-\delta)} \tag{12}$$

where $0\le\theta\le 1$; n indicates the n-th time-increment, and δ is the distance travelled by the particle in the x direction (Figure 1) and can be expressed as

$$\delta=\bar{v}\Delta t \tag{13}$$

where $\bar{v}$ is the average value of v along the characteristic, which can be approximated as described in [19] by the expression

$$\bar{v} = \frac{v^{n+1} + v^n|_{(x-\delta)}}{2} \tag{14}$$

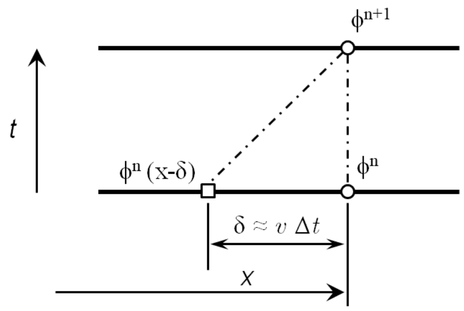

Figure 1: Basis of the characteristic-Galerkin method, that shows the distance travelled by the particle (δ) from time n to time n+1 [1].

In order to obtain the solution value at time $n+1$ it is necessary to evaluate the equations at time n but at the position $x-\delta$ (position at where the particle that is currently at x for time $n+1$, was at time n). The functions at $x-\delta$ can be approximated by using Taylor expansions as follows:

$$\phi^n|_{(x-\delta)} \approx \phi^n - \delta\frac{\partial \phi^n}{\partial x} + \frac{\delta^2}{2}\frac{\partial^2\phi^n}{\partial x^2} + O(\Delta t^3) \tag{15}$$

$$\frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right)^n\Bigg|_{(x-\delta)} \approx \frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right)^n - \delta\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right)^n\right] + O(\Delta t^2) \tag{16}$$

$$Q^{T^n}\Big|_{(x-\delta)} \approx Q^{T^n} - \delta\frac{\partial Q^{T^n}}{\partial x} + O(\Delta t^2) \tag{17}$$

$$v^n|_{(x-\delta)} \approx v^n - \Delta t\, v^n\frac{\partial V^n}{\partial x} + O(\Delta t^2) \tag{18}$$

The average value of v along the characteristics (Eq. 14) can be approximated by leaving out the second term of Eq. (18) as

$$\bar{v} = \frac{v^{n+1} + v^n|_{(x-\delta)}}{2} \approx \frac{v^{n+1}+v^n}{2} = v^{n+1/2} \tag{19}$$

And substituting Eqs. (15)-(19) in Eq. (12) the following is obtained:

$$\begin{aligned}\rho c\frac{1}{\Delta t}(\phi^{n+1}-\phi^n) &= \\ &= -\rho c\, v^{n+1/2}\frac{\partial\phi^n}{\partial x} + \frac{\rho c}{2}\Delta t\, v^{n+1/2}\, v^{n+1/2}\frac{\partial^2\phi^n}{\partial x^2} + \theta\left[\frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right) - Q^T\right]^{n+1} \\ &+ (1-\theta)\left[\frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right) - Q^T\right]^n \\ &- (1-\theta)\left[\Delta t\, v^{n+1/2}\left(\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(k^T\frac{\partial\phi}{\partial x}\right)\right\} - \frac{\partial Q^{T^n}}{\partial x}\right)\right]\end{aligned} \tag{20}$$

From Eq. (20), the final FE equations of the ICG method are obtained by leaving out the third order terms and performing the spatial discretization using the standard FEM Galerkin weighting. Boundary terms are the same as defined in Eq. (7).

The resulting time-integration scheme is always stable when $0.5 \leq \theta \leq 1$. Additionally, the Courant condition must be satisfied for each element of the FEM mesh [1], which fixes a maximum time-increment to be applied due to the advection effect:

$$\mathrm{C} = \frac{|v|\Delta t}{h_{car}} \leq 1 \qquad \Delta t_{c,max} = \frac{h_{car}}{v} \tag{21}$$

where h_{car} is the characteristic length of the element. Finally, the time-increment condition due to the diffusion problem [1] must also be satisfied:

$$\Delta t_{d,max} = \rho c \frac{h_{car}^2}{2k^T} \tag{22}$$

4 ZERO-THICKNESS FORMULATION: LARGE ADVECTION PROBLEM USING A SIMPLE IMPLICIT CHARACTERISTIC METHOD

This section develops a formulation of the thermal problem with large advection for zero-thickness interface elements by using the ICG method presented in Section 3.

It is assumed that the discontinuity is surrounded by a continuum medium and, as in the hydraulic flow described by [20], the thermal flow can leak from the discontinuity to the continuum medium and vice versa.

4.1 Double-nodded zero-thickness interface elements

This work follows the definition of zero-thickness interface element originally proposed for purely mechanical problems in [21]. In this type of elements one of the dimensions is collapsed and the integration is reduced one order; lines for 2D and surfaces for 3D problems (Figure 2).

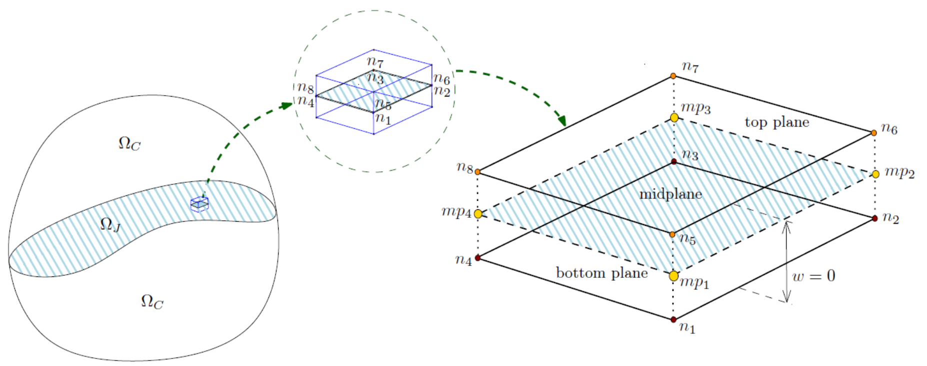

Figure 2: Double-nodded zero-thickness interface element. Example of a quadrilateral interface element, where n_i are the element nodes and mp_i the mid-plan nodes [24]

The formulation of this kind of elements for flow or diffusion problems is composed of two terms: the longitudinal flow along the interface and the transversal flow across the same, as described by [20, 22-24].

The longitudinal (or in-plane) problem is formulated along the mid-plane surface in terms of the mid-points (Figure 2). Once the results are obtained at the mid-points, the temperatures at the nodes of the element are obtained assuming that the temperature at each mid-point is the average of the temperature of the two original nodes of the interface. Additional to the longitudinal flow, the existence of a discontinuity may also represent a resistance to the temperature flow in the transversal direction, which would result in a localized temperature drop across the interface. It is assumed that the potential thermal drop at the mid-plane nodes is defined as the difference of potential between the two nodes of the interface.

It is also assumed that the large advection affects only the longitudinal flow of the interfaces and not the transversal one.

4.2 Longitudinal flow

The longitudinal flow vector $\mathbf{q}_{\mathrm{L}}^{T}$ defined by the conduction-advection equation in the transient problem can be written from Eq. (2) as:

$$\mathbf{q}_{\mathrm{L}}^{T} = -k_{L}^{T}\, \boldsymbol{\nabla}_{J}\, \phi \, + \, \rho c \phi \mathbf{v}_{\mathrm{L}} \tag{23}$$

where k_{L}^{T} is the longitudinal thermal conductivity; $\boldsymbol{\nabla}_{J} = \left[\frac{\partial}{\partial l_1} \;\; \frac{\partial}{\partial l_2}\right]^{\mathsf{T}}$ is the partial differential operator for the local in-plane axis; $\mathbf{v}_{\mathrm{L}}$ is the longitudinal velocity (known) field at the mid-plane of the interface element; ϕ is the temperature field; ρ and c are the density and the thermal capacity of the material filling the interface, respectively; and $\{n, l_1, l_2\}$ is the local orthogonal coordinate system, where l_1 and l_2 are the local directions of the interface along the mid-plane.

By imposing heat conservation in the longitudinal direction of a differential interface element (Figure 3) and assuming Eq. (23) the governing differential equation is obtained:

$$r_N \rho c \left(\frac{\partial \phi}{\partial t} + [\mathbf{v}_{\mathrm{L}}]^{\mathsf{T}} \boldsymbol{\nabla}_J \phi\right) - r_N \left[\boldsymbol{\nabla}_J\right]^{\mathsf{T}} k_L^T\, \boldsymbol{\nabla}_J \phi = 0 \qquad \text{in} \quad \Omega_J \tag{24}$$

where r_N is the interface aperture and Ω_J is the local mid-plane longitudinal coordinate defined by the system $\{n, l_1, l_2\}$. Source terms are left out for simplicity.

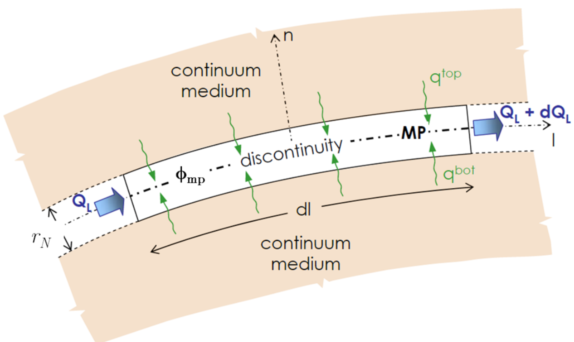

Figure 3: Thermal flow through a differential interface element [20]

The complete Neumann boundary condition may be written from [25] as:

$$-r_N \rho c \phi v_{L,N} + [\mathbf{n}]^{\mathsf{T}} r_N\, k_L^T\, \boldsymbol{\nabla}_J \phi \, + \bar{q}_N^T \qquad \text{on} \quad \Gamma_{\mathrm{q}} \tag{25}$$

where $v_{L,N} = [\mathbf{n}]^{\top}\mathbf{v}_{\mathrm{L}}$ is the normal longitudinal component of the velocity at the boundary Γ_{q}, and $\bar{q}_N^T$ is the prescribed value of the heat flow.

In order to solve the problem using the simple implicit Characteristic-Galerkin method (Section 3), the differential equation for the one-dimensional transient problem *along the characteristics* (the advective term of the equation disappears) can be written from Eq. (24) and Eq. (12) as:

$$r_N \rho c \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial l'}\left(r_N\, k_L^T\, \frac{\partial \phi}{\partial l'}\right) \tag{26}$$

where l is the local coordinate system for the one-dimensional problem $\{l, n\}$ and $dl' = dl - v_L\, dt$.

The first step consists on the time discretization of Eq. (26) using the FDM and the concepts developed in Section 3 (ICG method):

$$r_N \rho c \frac{1}{\Delta t}\left(\phi^{n+1}|_x - \phi^n|_{(x-\delta)}\right) \approx\approx \theta \left[\frac{\partial}{\partial l}\left(r_N\, k_L^T\, \frac{\partial \phi}{\partial l}\right)\right]^{n+1}\Bigg|_x + (1-\theta)\left[\frac{\partial}{\partial l}\left(r_N\, k_L^T\, \frac{\partial \phi}{\partial l}\right)\right]^{n}\Bigg|_{(x-\delta)} \tag{27}$$

Substituting Eqs. (15)-(20) into Eq. (27) and leaving out third-order terms results in:

$$\begin{aligned} r_N \rho c \frac{1}{\Delta t}(\phi^{n+1} - \phi^n) &= \\ &= -r_N \rho c\, v_L^{n+1/2} \frac{\partial \phi^n}{\partial l} + \frac{1}{2} r_N \rho c \Delta t\, v_L^{n+1/2}\, v_L^{n+1/2} \frac{\partial^2 \phi^n}{\partial l^2} + \theta \left[\frac{\partial}{\partial l}\left(r_N\, k_L^T \frac{\partial \phi}{\partial l}\right)\right]^{n+1} \\ &+ (1-\theta)\left[\frac{\partial}{\partial l}\left(r_N\, k_L^T \frac{\partial \phi}{\partial l}\right)\right]^{n} \end{aligned} \tag{28}$$

The extension of the above formulation to 3D is automatic by considering a local coordinate system such as $\boldsymbol{l}_i = (l_1, l_2)$ and the corresponding nodes and mid-points.

Leaving out the third order terms and performing the spatial discretization, applying the Divergence Theorem to the diffusion terms and using the FEM with the standard Galerkin weighting ($w = N_J^{(1)}, N_J^{(2)}, \ldots$) results in

$$\mathbf{Q}_{\mathrm{Lmp}} = \left(\frac{1}{\Delta t}\mathbf{C}_{\mathrm{Lmp}} + \theta \mathbf{K}_{\mathrm{Lmp}}\right)\Delta \boldsymbol{\phi}_{\mathrm{mp}}^n + \left(\mathbf{K}_{\mathrm{Lmp}} + \mathbf{K}_{\mathbf{v},\mathrm{Lmp}} + \frac{1}{2}\mathbf{K}_{\mathbf{s},\mathrm{Lmp}}\right)\boldsymbol{\phi}_{\mathrm{mp}}^n \tag{29}$$

where

$$\mathbf{C}_{\mathrm{Lmp}} = r_N \int_{\Omega_J} \rho c\, [\mathbf{N}_J]^{\top}\, \mathbf{N}_J\, \mathrm{d}\Omega_J \tag{30}$$

$$\mathbf{K}_{\mathrm{Lmp}} = r_N \int_{\Omega_J} \left([\mathbf{B}_J]^{\top}\, k_L^T\, \mathbf{B}_J\right) \mathrm{d}\Omega_J \tag{31}$$

$$\mathbf{K}_{\mathbf{v},\mathrm{Lmp}} = r_N \int_{\Omega_J} \rho c\, \left([\mathbf{N}_J]^{\top}\, \mathbf{v}_{\mathrm{L}}\, \mathbf{B}_J\right) \mathrm{d}\Omega_J \tag{32}$$

$$\mathbf{K}_{\mathbf{s},\mathrm{Lmp}} = r_N\, \Delta t \int_{\Omega_J} \rho c\, \left([\mathbf{B}_J]^{\top}\, [\mathbf{v}_{\mathrm{L}}]^{\top}\mathbf{v}_{\mathrm{L}}\, \mathbf{B}_J\right) \mathrm{d}\Omega_J \tag{33}$$

and

$$\mathbf{B}_J = \boldsymbol{\nabla}_J \mathbf{N}_J \tag{34}$$

$$\boldsymbol{\phi}_{\mathrm{mp}} = \left[\phi_{mp}^{(1)}\ \phi_{mp}^{(2)}\ \dots\ \phi_{mp}^{(m)}\right]^{\mathsf{T}} \qquad \mathbf{v}_{\mathrm{L}} = \left[v_{L}^{(1)}\ v_{L}^{(2)}\ \dots\ v_{L}^{(m)}\right]^{\mathsf{T}} \tag{35}$$

In order to obtain the final FEM formulation related to the nodes of the mesh (and not to the mid-points), it is assumed that the temperature at the mid-plane is the average of the nodal temperatures. As described accurately by [25], the temperature at the mid-plane of the interface can be written in terms of the element's nodal temperatures as

$$\boldsymbol{\phi}_{\mathrm{mp}} = \frac{1}{2}\,[\mathbf{I}_m\ \ \mathbf{I}_m]\,\boldsymbol{\phi}_{\mathrm{e}} = \widetilde{\mathbb{T}}_{\mathrm{L}}\,\boldsymbol{\phi}_{\mathrm{e}} \tag{36}$$

where $\widetilde{\mathbb{T}}_{\mathrm{L}} = \frac{1}{2}\,[\mathbf{I}_m\ \ \mathbf{I}_m]$ is the longitudinal transference matrix, $\mathbf{I}_m$ is the $m \times m$ identity matrix and $\boldsymbol{\phi}_{\mathrm{e}}$ is the element's temperature vector. Finally, Eq. (29) has to be written referred to the element nodes [25]:

$$\mathbf{Q}_{\mathrm{L_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{L}}\right]^{\mathsf{T}} \mathbf{Q}_{\mathrm{L_{mp}}} \tag{37}$$

All the matrices and vectors defined at the mid-plane of the interface have to be transformed using the transference matrix to obtain element matrices and vectors by substituting Eqs. (36) and (37) into Eq. (29):

$$\mathbf{Q}_{\mathrm{L_e}} = \left(\frac{1}{\Delta t}\mathbf{C}_{\mathrm{L_e}} + \theta\mathbf{K}_{\mathrm{L_e}}\right)\Delta\boldsymbol{\phi}_{\mathrm{e}}^{n} + \left(\mathbf{K}_{\mathrm{L_e}} + \mathbf{K}_{\mathbf{v},\mathrm{L_e}} + \frac{1}{2}\mathbf{K}_{\mathbf{s},\mathrm{L_e}}\right)\boldsymbol{\phi}_{\mathrm{e}}^{n} \tag{38}$$

with

$$\mathbf{C}_{\mathrm{L_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{L}}\right]^{\mathsf{T}} \mathbf{C}_{\mathrm{L_{mp}}}\,\widetilde{\mathbb{T}}_{\mathrm{L}} \quad ; \quad \mathbf{K}_{\mathrm{L_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{L}}\right]^{\mathsf{T}} \mathbf{K}_{\mathrm{L_{mp}}}\,\widetilde{\mathbb{T}}_{\mathrm{L}} \tag{39}$$

$$\mathbf{K}_{\mathbf{v},\mathrm{L_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{L}}\right]^{\mathsf{T}} \mathbf{K}_{\mathbf{v},\mathrm{L_{mp}}}\,\widetilde{\mathbb{T}}_{\mathrm{L}} \quad ; \quad \mathbf{K}_{\mathbf{s},\mathrm{L_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{L}}\right]^{\mathsf{T}} \mathbf{K}_{\mathbf{s},\mathrm{L_{mp}}}\,\widetilde{\mathbb{T}}_{\mathrm{L}} \tag{40}$$

4.3 Transversal flow

The transverse heat flow q_N^T may be written by a simple discrete version of Fick's law as:

$$q_N^T = \check{k}_N^T\ \check{\phi}_N \tag{41}$$

where $\check{k}_N^T$ is the transversal thermal conductivity of the interface and $\check{\phi}_N$ is the localized thermal drop across the interface, which is defined by the temperature difference between the two surfaces of the interface as follows:

$$\check{\phi}_N = \phi_{bot} - \phi_{top} \tag{42}$$

It is important to note that the large advection affects only the longitudinal flow and not the transversal one. Thus, as described accurately by [24] the final FEM formulation for the transversal flow may be written as:

$$\mathbf{Q}_{\mathrm{N_e}} = \mathbf{K}_{\mathrm{N_e}}\,\boldsymbol{\phi}_{\mathrm{e}} \tag{43}$$

$$\mathbf{K}_{\mathrm{N_e}} = \left[\widetilde{\mathbb{T}}_{\mathrm{N}}\right]^{\mathsf{T}} \left(\int_{\Omega_J} \left[\mathbf{N}_J\right]^{\mathsf{T}} \check{k}_N^T\,\mathbf{N}_J\,\mathrm{d}\Omega_J\right)\widetilde{\mathbb{T}}_{\mathrm{N}} \tag{44}$$

where $\widetilde{\mathbb{T}}_{\mathrm{N}} = [-\mathbf{I}_m\ \ \mathbf{I}_m]$ is the transference transversal matrix and $\mathbf{K}_{\mathrm{N_e}}$ is the transversal thermal conductivity matrix of the interface element.

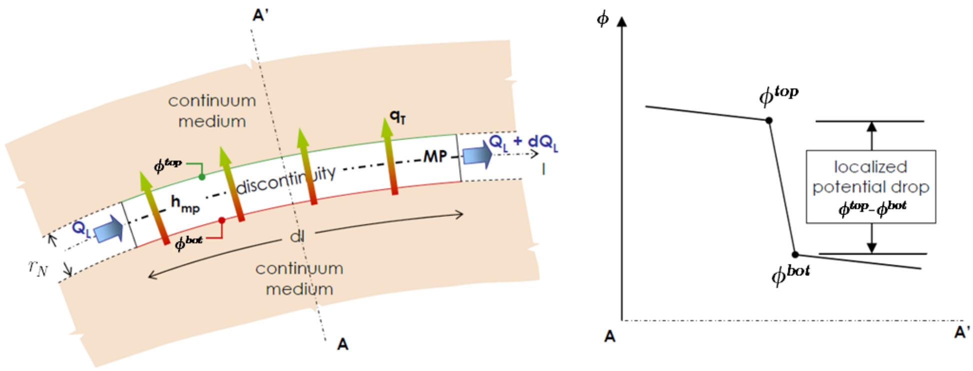

Figure 4: Scheme of the potential thermal drop across a differential interface element [20]

4.4 Integrated formulation: longitudinal and transversal flow

To obtain an integrated conductivity matrix it is necessary to combine the longitudinal and the transversal flow as a sum of both, performing first the time discretization of the transversal flow by means of the FDM. The final element equation may be written in the following form:

$$\left(\frac{1}{\Delta t}\mathbf{C}_{\mathrm{L_e}} + \theta\left(\mathbf{K}_{\mathrm{L_e}} + \mathbf{K}_{\mathrm{N_e}}\right)\right)\Delta\boldsymbol{\phi}_{\mathrm{e}}^{n} = -\left(\mathbf{K}_{\mathrm{L_e}} + \mathbf{K}_{\mathrm{N_e}} + \mathbf{K}_{\mathrm{v,L_e}} + \frac{1}{2}\mathbf{K}_{\mathrm{s,L_e}}\right)\boldsymbol{\phi}_{\mathrm{e}}^{n} \tag{45}$$

where the element matrices are defined by Eqs. (30-33) and Eqs. (39-40).

Finally, the global matrices and vectors of the system of equations are obtained by the assembly of the contribution of each element of the mesh. Boundary conditions are left out for simplicity. Conditions defined by Eqs. (21) and (22) must be satisfied.

5 VERIFICATION EXAMPLE

The objective of this example is to compare the solutions between the standard Galerkin FEM method and the ICG Method (Section 3) and verify that the latter provides stable solutions when $Pe > 1$ in a 2-D simple transient example with zero-thickness interfaces.

The geometry consists of a domain of 400×250 m with an interface at the symmetry axis, composed by 600 continuum elements and 20 zero-thickness interface elements, with an aperture of $r_N = 0.01$ m. The material thermal parameters for the analysis are: $k^T = 7.0 \cdot 10^{-6}$ J/(m °C s), $k_L^T = 7.0 \cdot 10^{-8}$ J/(m °C s), $k_T^T = 1.00$ J/(m^2 °C s), $\rho c_{interfaces} = 0.001$ J/(m^3 °C) and $\rho c_{continuum} = 0.10$ J/(m^3 °C). The boundary conditions are: 100°C at the left side boundary and 0°C at the right one (both Dirichlet).

In order to reproduce the heat transport, a known and constant velocity field of $v_L = 0.19$ m/s from left to right is imposed along the discontinuity, as a preferential path throughout the continuous medium.

The time-increments applied during the analysis are $\Delta t = 70$ s, and consequently the Courant number is smaller than 1 ($C = 0.67$), a necessary condition to obtain a stable solution. As the element characteristic length is equal to 20 meters, Péclet number $Pe = 273$ (large advection).

Figure 4 shows the results of the transient analysis for different time-steps, first using the standard Galerkin weighting, which leads to oscillatory results, and then using the ICG method, which leads to the stable correct solution.

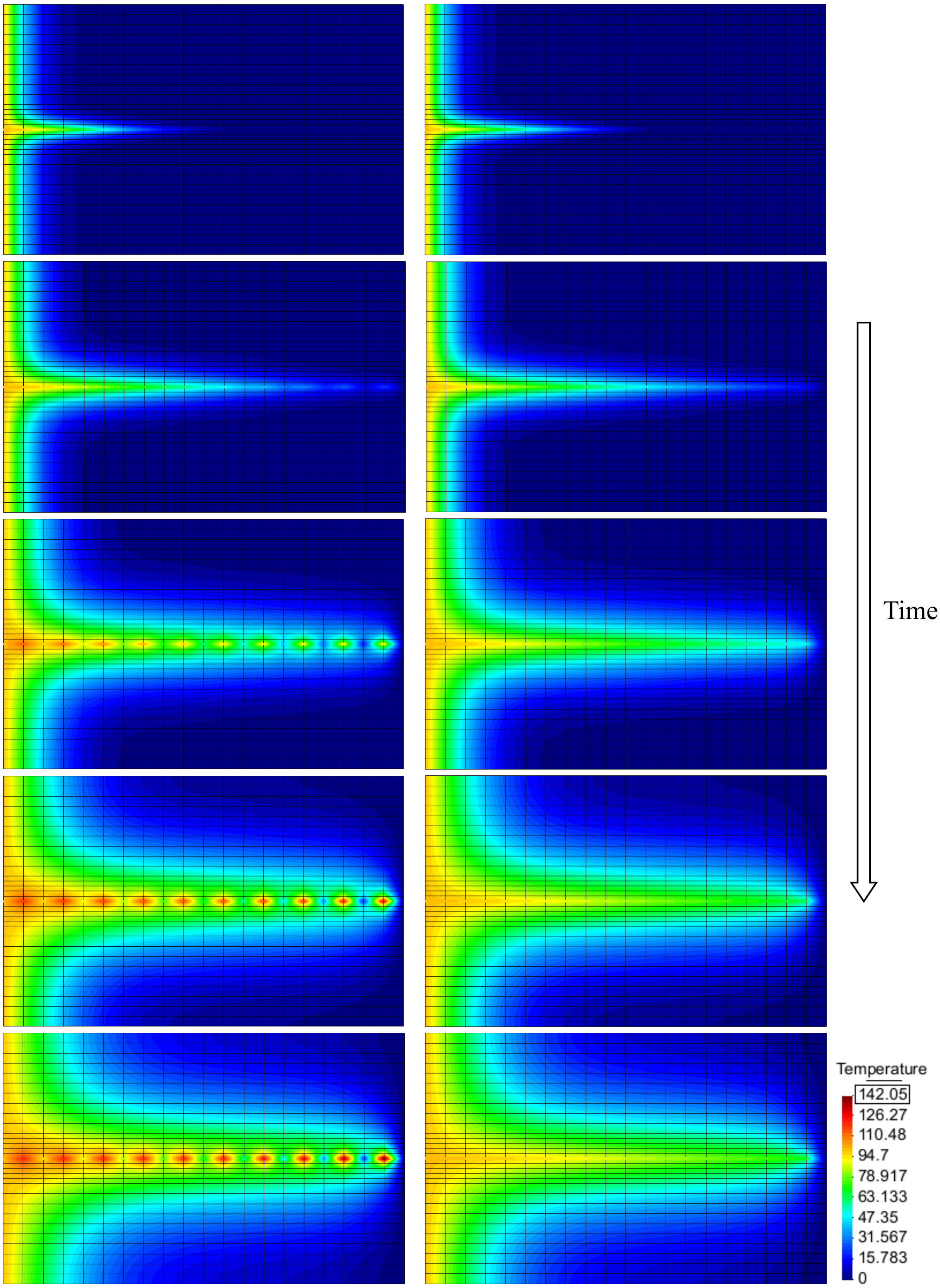

Figure 5: Transient thermal solution for Pe = 273: (left) Oscillatory results using the standard Galerkin weighting, and (right) correct solution using the ICG method (section 3).

6 CONCLUDING REMARKS

This paper describes a numerical FEM solution of the heat flow transient problem for zero-thickness interface elements, where a known velocity field produces thermal transport with large advection. First, a simple Characteristic methodology to solve large advection problems has been reviewed [18]. Then, a formulation of the large advection heat problem for zero-thickness interface elements has been developed, using for this purpose the implicit Characteristic methodology reviewed in the previous section. Finally, a verification example has been presented to show that this method is suitable for this type of elements when the advection is dominant for transient problems. Current work aims at the extension of the formulation presented to different stabilization methodologies to solve large advection problems, such as the GLS [4], the FIC [16] or the High Resolution method [17], among others.

ACKNOWLEDGEMENTS

This work was partially supported by research grant BIA2016-76543-R from MEC (Madrid), which includes European FEDER funds, and 2017SGR-1153 from Generalitat de Catalunya (Barcelona). The first author also acknowledges his FPU scholarship from MEC (Madrid).

REFERENCES

[1] Zienkiewicz, O. & Taylor, R., *The Finite Element Method: Vol 3. Fluid Dynamics*. Fifth ed. Oxford: Butterworth-Heinemann (2000).

[2] Huysmans, M. & Dassargues, A. Review of the use of Péclet numbers to determine the relative importance of advection and diffusion in low permeability environments. *Hydrogeology Journal*. **13**:895-904, (2004).

[3] Hughes, T.J.R, & Brooks, A.N. *Finite Elements in Fluids, Vol. IV*: A theoretical framework for Petrov-Galerkin, methods with discontinuous weighting functions: application to the streamline upwind procedure. Eds: R.H. Gallagher, D.M. Norrie, J.T. Oden, O.C. Zienkiewicz. Wiley, Chichester (1982).

[4] Nguyen, N. & Reynen, J. A space-time least square finite element scheme for advection-diffusion equations. *Comp. Meth. in Appl. Mech. and Eng.* **42**:331-42 (1984).

[5] Adey, R. & Brebbia, C. Finite element solution of effluent dispersion. *Numerical Methods in Fluid Mechanics*. Southampton (UK): Pentech Press, pp. 325-54 (1974).

[6] Morton, K. Generalized Galerkin methods for hyperbolic problems. *Comp. Meth. in Appl. Mech. and Eng.* **52**:857-71 (1985).

[7] Donea, J. A Taylor-Galerkin method for convective transport problems. *Int. J. Numer. Methods Engrg.* **20**:101-119 (1984).

[8] Ewing, R. & Russell, T. *Advances in Computation Methods for PDE's:* Multistep Galerkin methods along characteristics for convection-diffusion problems. Rutgers University, Brunswick. Eds. R. Vichnevetsky and R.S. Stepleman, pp. 28-36. Vol. IV, IMACS (1981).

[9] Douglas, J. & Russell, T.F. Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with the finite element or finite difference procedures. *SIAM J. Numerical Analysis.* **19**:871–885 (1982).

[10] Pironneau, O. On the transport diffusion algorithm and its application to the Navier-Stokes equation. *Numerical Math.* **38**:309-32 (1982).

[11] Bercovier, M., Pironneau, O., Harbani, Y. & Leveine, E. *The Mathematics of Finite*

Elements and Applications: Characteristics and finite element methods applied to equation of fluids. Academic Press, London: ed. J.R. Whiteman, **5**:471-8 (1982).
[12] Bercovier, M., Pironneau, O. & Sastri, V. Finite elements and characteristics for some parabolic-hyperbolic problems. *Applied Math. Modelling*. **7**:89-96 (1983).
[13] Hughes, T.J.R. Multiscale phenomena: Green's function, the Dirichlet-to- Neumann formulation, subgrid scale models, bubbles and the origins of stabilized formulations, *Comput. Methods Appl. Mech. Engrg*. **127**:387-401 (1995).
[14] Löhner, R., Morgan, K. & Zienkiewicz, O. An adaptive finite element procedure for compressible high speed flows. *Comp. Meth. in Appl. Mech. and Eng.* **51**:441-65 (1985)
[15] Brezzi, F., Bristeau, M.O., Franca L.P., Mallet, M. & Rogé, G. A relationship between stabilized finite element methods and the Galerkin method with bubble functions. *Comput. Methods Appl. Mech. Engrg.* **96**:117-129 (1992).
[16] Oñate, E. & Idelsohn, S. A mesh free finite point method for advective-diffusive transport problems. *Computational Mechanics*. **21**:283-292 (1998).
[17] Nadukandi, P., Oñate, E. & Garcia, J. A high-resolution P-G method for the 1D convection-diffusion-reaction Problem. *Comp. Meth. Appl. Mech. Eng*. **199**:525-546 (2010)
[18] Zienkiewicz, O., Löhner, R., Morgan, K. & Nakazawa, S. Finite element in fluid mechanics - a decade of progress. *Finite Elements in Fluids*. Wiley, Chichester (UK): ed R.H. Gallagher et al. Volume 5, chapter 1, pp 1-26. (1984)
[19] Zienkiewicz, O. & Codina, R. A general algorithm for compressible and incompressible flows. Part I: the split, characteristic based scheme, *Int. J. Numer. Methods Fluids.* **20**:869-885 (1995).
[20] Segura, J. *Coupled HM analysis using zero-thickness interface elements with double nodes.* PhD Thesis. Universitat Politècnica de Catalunya, Barcelona (2007).
[21] Goodman, R., Taylor, R. & Brekke, T.,. A model for the mechanics of jointed rock. *ASCE Journal of the Soil Mechanics and Foundations Division,* **94**:637–659, (1968).
[22] Segura, J., Carol, I. On zero-thickness interface elements for diffusion problems. *Int. J. Num. Anal.Meth.Geomech.,* **28**:947-962 (2004).
[23] Segura, J., Carol, I. Coupled H-M analysis using zero-thickness interface elements with double nodes. Part I: Theoretical model, and Part II: Verification and application. *Int. J. Num. Anal.Meth.Geomech.,* **32**:2083-2123 (2008).
[24] Garolera, D. *Zero-thickness interface elements in petroleum geomechanics: sand production and hydraulic fracture.* PhD Thesis. Universitat Politècnica de Catalunya, Barcelona (2017).
[25] Pérez, A. *THM coupling with large advection in fractured rock masses using zero-thickness interface elements.* PhD. Thesis. Universitat Politècnica de Catalunya, Barcelona (2018).