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DEVELOPMENT OF LOW-COST SOLVER FOR INCOMPRESSIBLE VISCOUS FLUID FLOW BASED ON THE FUNDAMENTAL AND PARTICULAR SOLUTIONS OF DIFFERENTIAL OPERATOR

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Abstract. *In industrial numerical simulation of the complex and/or large-scale fluid flow, the computation cost must be reduced. In order to develop numerical low-cost solver for incompressible fluid flow problems based on BEM an effective scheme of DRM is proposed.*

1 INTRODUCTION

The contemporary problem of the large-scale computational fluid dynamics along with the proportion of complexity and scale up of the object size will be the cost for computation as time and energy. This comes from the finite resources for the computation, so we need to manage accuracy and cost of CFD. The ideas to solve this problem are for example, the increase of the power of CPU, acceleration of calculation by the software and algorithm, and the decrease of the computational points or grids by use of something like the Boundary Element Method (BEM) as the low-cost solver.

In this work, to manage between accuracy and cost of computational incompressible viscous fluid flow, we selected BEM which decreases computational points. For this aim, so far, there was BEM for incompressible viscous fluid flow using Dual Reciprocity Method (DRM) [1] which convert domain integration to boundary integration by use of fundamental and particular solutions. To solve continuity and momentum equations, Florez and Power [2] used mass conservative interpolation. That method got a good agreement with the results of Ghia [3], but did not solve continuity and momentum equations simultaneously. To avoid complexity we implemented the DRM into the continuity and momentum simultaneous solver [4] directly. And we found the differential order consistency in DRM makes accuracy and pressure convergence better.

2 FORMULATION

2.1 Governing equations

The governing equations are given by Navier-Stokes (NS) equation and continuity condition or incompressible condition in the vector form:

$$\mu\Delta\mathbf{u} - \nabla p = \mathbf{f} = \rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) \tag{1}$$

$$div\,\mathbf{u} = \nabla\cdot\mathbf{u} = 0 \tag{2}$$

where **u** is velocity, p is pressure, μ is viscosity, ρ is density and **f** is the inhomogeneous function by the nonlinear convective term. With use of the constitutive equation (3), equation (1) becomes equation (4). In case of steady, we get equations (5) and its matrix form (6).

$$\tau_{ij} = -\frac{1}{\mu}p\delta_{ij} + u_{i,j} + u_{j,i} \tag{3}$$

$$\mu\tau_{ij,j} = \rho(\partial_t u_i + u_j u_{i.j}) \tag{4}$$

$$u_{i,jj} + u_{j,ij} - \frac{1}{\mu}p_j\delta_{ij} = \frac{1}{\nu}u_j u_{i.j} \tag{5}$$

$$\begin{pmatrix} \Delta + D_1^2 & D_1D_2 & -\frac{1}{\mu}D_1 \\ D_2D_1 & \Delta + D_2^2 & -\frac{1}{\mu}D_2 \\ D_1 & D_2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} \frac{1}{\nu}(uD_1u + vD_2u) \\ \frac{1}{\nu}(uD_1v + vD_2v) \\ 0 \end{pmatrix} \tag{6}$$

2.2 Inverse formulation

Equation (6) is expressed in the tensor form as equation (7) by using tensor form of differential operator L_{ij}.

$$L_{ij}u_j = b_i, i = 1,2,3\,,\; u_3 = p \tag{7}$$

Integral form of equation (7) is equation (8).

$$\int_\Omega L_{ij}u_j \cdot v_i^* d\Omega = \int_\Omega b_i \cdot v_i^* d\Omega \tag{8}$$

By the integration by parts, we get equation (9).

$$\int_\Gamma (\tau_1 v_1^* + \tau_2 v_2^* - \sigma_1 u - \sigma_2 v) d\Gamma + \int_\Omega u_j \mathcal{L}_{ij} v_i^* d\Omega = \int_\Omega b_i \cdot v_i^* d\Omega \tag{9}$$

where $\mathcal{L}_{ij}$ is the adjoint differential operator of L_{ij}, τ_α is traction given by equation (10) and σ_α is pseud-traction given by equation (11).

$$\tau_\alpha = \tau_{\alpha\beta} n_\beta = (D_\alpha u_\beta + D_\beta u_\alpha - \frac{1}{\mu} \delta_{\alpha\beta} u_3) n_\beta \tag{10}$$

$$\sigma_\alpha = (v_{\alpha,\beta}^* + v_{\beta,\alpha}^* - \delta_{\alpha\beta} v_3^*) n_\beta \tag{11}$$

By use of pseud-traction tensor V^*_{ki} given by equation (12) and pseud-traction tensor Σ^*_{ki} given by equation (13),

$$v_i^* = V_{ki}^* e_k \tag{12}$$

$$\sigma_i^* = \Sigma_{ki}^* e_k \tag{13}$$

where pseud-traction tensor is given by equation (14),

$$\Sigma_{\alpha\beta}^* = (V_{\alpha\beta,\gamma}^* + V_{\alpha\gamma,\beta}^* - \delta_{\beta\gamma} V_{\alpha 3}^*) n_\gamma \tag{14}$$

we get inverse formulation (15,16) or matrix formation (17).

$$u_\lambda = \int_\Gamma (\tau_\alpha V_{\lambda\alpha}^* - \Sigma_{\lambda\alpha}^* u_\alpha) d\Gamma - \int_\Omega b_i \cdot V_{\lambda i}^* d\Omega, \quad \lambda = 1, 2, i = 1, 2, \alpha = 1, 2 \tag{15}$$

$$p_3 = \int_\Gamma (\tau_\alpha V_{3\alpha}^* - \Sigma_{3\alpha}^* u_\alpha) d\Gamma - \int_\Omega b_i \cdot V_{3i}^* d\Omega, \quad i = 1, 2, \alpha = 1, 2 \tag{16}$$

$$\mathbf{Kv} - \mathbf{Gt} = -\mathbf{f} \tag{17}$$

where **v** is velocity vector including pressure given by equation (18), **t** is traction vector given by equation (19) , **K** is matrix given by equation (20), **G** is matrix given by equation (21) and **f** is matrix given by equation (22).

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ p \end{pmatrix} \tag{18}$$

$$\mathbf{t} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ 0 \end{pmatrix} \tag{19}$$

$$\mathbf{K} = \Sigma^*_{k\alpha}\Delta\Gamma_\alpha \tag{20}$$

$$\mathbf{G} = V^*_{k\alpha}\Delta\Gamma_\alpha \tag{21}$$

$$\mathbf{f} = \begin{pmatrix} \frac{1}{\nu}(uD_1u + vD_2u) \\ \frac{1}{\nu}(uD_1v + vD_2v) \\ 0 \end{pmatrix} \tag{22}$$

To avoid singularity in calculating diagonal terms, we use regularized form (23), which can be got from the inverse formulation (15,16).

$$\int_\Gamma \tau_\alpha V^*_{\lambda\alpha} d\Gamma - \int_\Gamma \Sigma^*_{\lambda\alpha}(u_\alpha(x) - u_\alpha(y)) d\Gamma = \int_\Omega b_i \cdot V^*_{\lambda i} d\Omega \tag{23}$$

2.3 Effective scheme for DRM

In DRM, inhomogeneous term expressed by equation (24), with use of the radial function **f** (25) as series of distance r between field point **x** and source point **y**. The corresponding particular solutions are given by equation (26).

$$\mathbf{f} = \mathbf{F}_1\alpha \tag{24}$$

$$f(r) = c_0 1 + c_1 r + c_2 r^2 + c_3 r^3, r = |\mathbf{x} - \mathbf{y}| \tag{25}$$

$$\psi = \mu\left(c_0\frac{r^4}{8^2} + c_1\frac{r^5}{15^2} + c_2\frac{r^6}{24^2} + c_3\frac{r^7}{35^2}\right) \tag{26}$$

Velocities are also expressed with using radial functions as (27). In conventional DRM, matrix F1 was adopted in determining the unknowns. On the other hand we use matrix F1 for the unknowns α but matrix F2 for the derivative of velocity as shown in equation (28). This effective scheme leads to consistency of order of radial function. Then we select the coefficients in equations (25,26) as (c0,c1,c2,c3) = (1,1,0,0) for F1 and (c01,c11,c21,c31) = (1,1,1,0) for F2.

$$\mathbf{v} = \mathbf{F}_2 \beta \tag{27}$$

$$\frac{\partial \mathbf{v}}{\partial x} = \frac{\partial \mathbf{F}_2}{\partial x} \mathbf{F}_2^{-1} \mathbf{v} \tag{28}$$

RMD formation becomes equation (29), where unknowns α is given by equation (30), the matrix **S** for the inhomogeneous term in DRM is given by equation (31). By this way, we get solvable equation (32).

$$\mathbf{Kv} - \mathbf{Gt} = -\left(\mathbf{K}\hat{\mathbf{v}} - \mathbf{G}\hat{\mathbf{t}}\right) \alpha \tag{29}$$

$$\alpha = \frac{1}{\nu} \mathbf{F}_1^{-1} \left(\mathbf{V}_x \frac{\partial \mathbf{F}_2}{\partial x} + \mathbf{V}_y \frac{\partial \mathbf{F}_2}{\partial y} \right) \mathbf{F}_2^{-1} \mathbf{v}^n \tag{30}$$

$$\mathbf{S} = \frac{\rho}{\mu} \left(\mathbf{K}\hat{\mathbf{v}} - \mathbf{G}\hat{\mathbf{t}}\right) \mathbf{F}_1^{-1} \left(\mathbf{V}_x \frac{\partial \mathbf{F}_2}{\partial x} + \mathbf{V}_y \frac{\partial \mathbf{F}_2}{\partial y} \right) \mathbf{F}_2^{-1} \tag{31}$$

$$(\mathbf{K} + \mathbf{S})\mathbf{v} = \mathbf{Gt} \tag{32}$$

To avoid complexity, we calculate velocity and pressure simultaneously, so velocity vector **v** includes pressure but to avoid the singularity in integral expression of pressure, we calculate pressure only for on inner point p_I as shown in equation (33).

$$\mathbf{v} = \begin{pmatrix} u_B \\ u_I \\ v_B \\ v_I \\ p_I \end{pmatrix}, \mathbf{t} = \begin{pmatrix} \tau_1 \\ 0 \\ \tau_2 \\ 0 \\ 0 \end{pmatrix} \tag{33}$$

3 NUMERICAL TESTS

The numerical tests are demonstrated for the steady cavity flow with the length of 1m and driven wall speed 1m/s. The viscosity, density of the fluid and numbers of boundary Γ and inner Ω points are shown in Table 1. The results are shown in Figure 1 to 4 along with Reynolds numbers, with comparing the case of F1=F2 and present case. Figure 5 is the distribution of velocity comparing with the results of Ghia [3] in Re=10^3 and Kakuda [5] in Re=0 and above 2. Figure 6 is the minimum pressure in iteration as the index of convergence. These show that present DRM scheme with the consistency of order of radial function in differentiating velocity in DRM bring good accuracy and better convergence of pressure.

Table 1: Computational condition for the numerical tests

case	Re	μ PaS	ρ kgm^{-3}	Γ	Ω	DRM
A	10^0	1	1	20x4	19x19	$F_1 = F_2$
B	10^1	0.1	1	20x4	19x19	$F_1 = F_2$
C	10^2	0.01	1	20x4	19x19	$F_1 = F_2$
D	10^3	0.001	1	20x4	19x19	$F_1 = F_2$
A1	10^0	1	1	20x4	19x19	present
B1	10^1	0.1	1	20x4	19x19	present
C1	10^2	0.01	1	20x4	19x19	present
D1	10^3	0.001	1	20x4	19x19	present

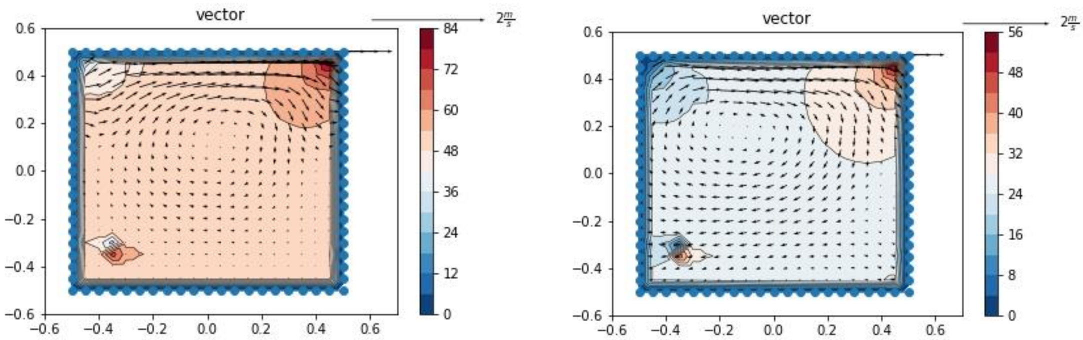

Figure 1: Velocity and pressure field for steady cavity flow (Re=1), left: F1=f2, right: present

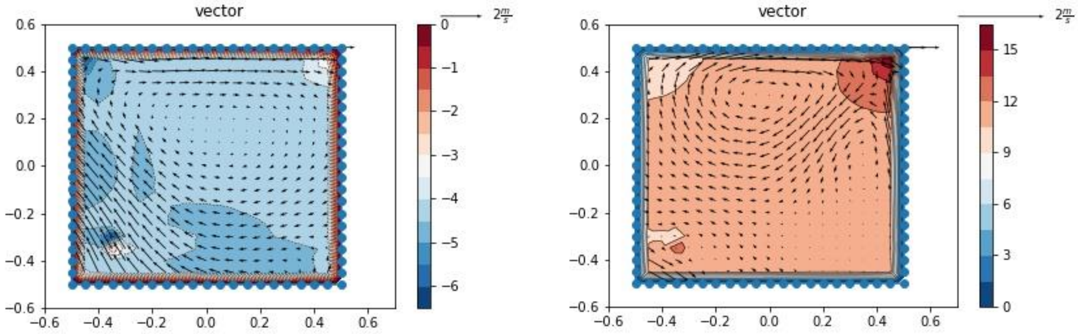

Figure 2: Velocity and pressure field for steady cavity flow (Re=10), left: F1=f2, right: present

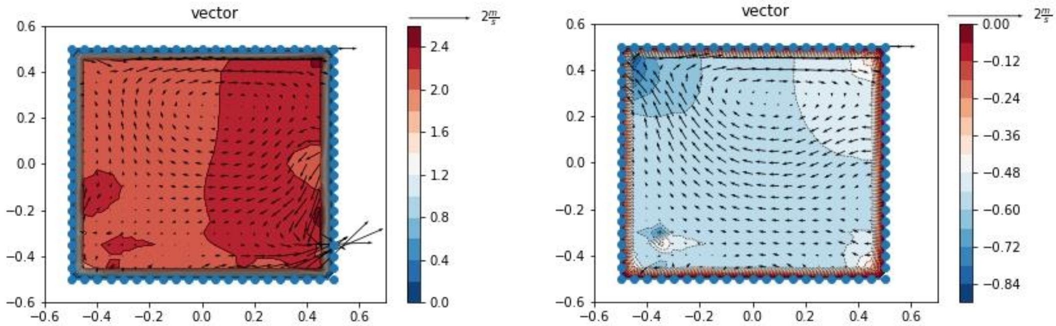

Figure 3: Velocity and pressure field for steady cavity flow (Re=100), left: F1=f2, right: present

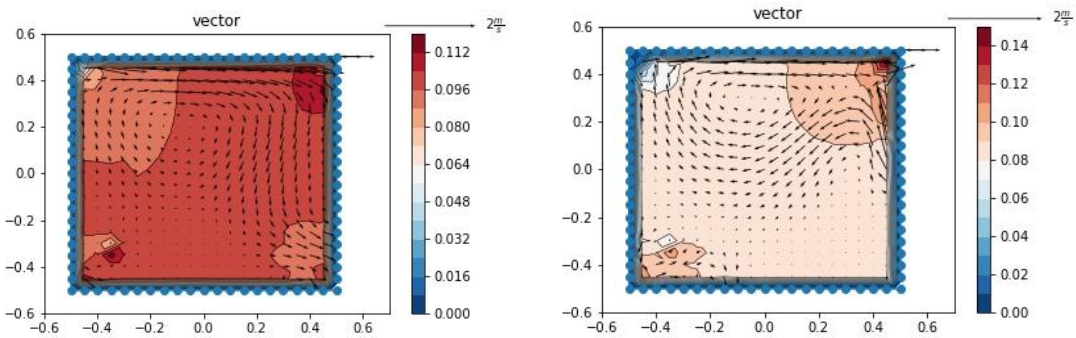

Figure 4: Velocity and pressure field for steady cavity flow (Re=1000), left: F1=f2, right: present

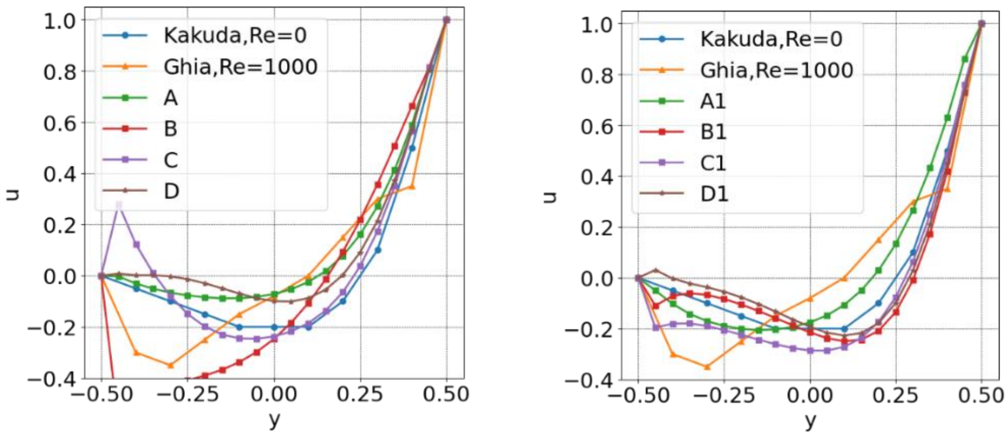

Figure 5: Horizontal velocities at the vertical cross section, left: F1=f2, right: present

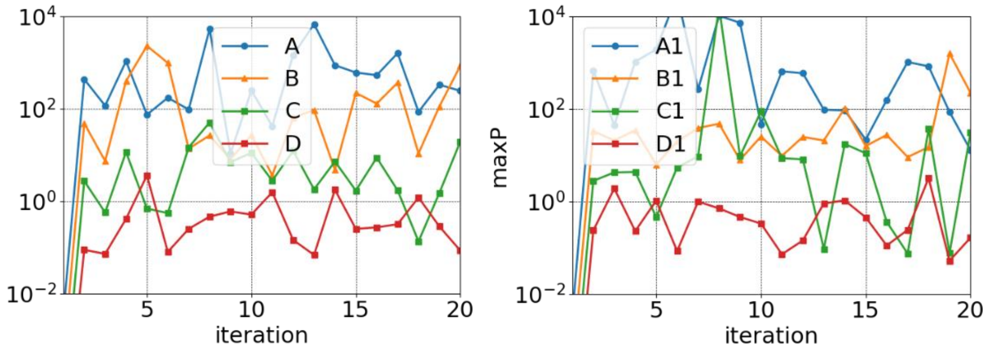

Figure 6: max Pressure along iteration, left: F1=f2, right: present

4 CONCLUSIONS

- To manage between accuracy and cost of computational incompressible viscous fluid flow, we selected BEM which decreases computational points.
- An effective scheme in DRM to treat nonlinear convective term was proposed and brought about accuracy and convergence of numerical solutions.

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