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SAFETY ASSESSMENT OF HISTORIC MASONRY STRUCTURES BY LIMIT ANALYSIS AND DETERMINISTIC PARTIAL SAFETY FACTORS

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Abstract. *Safety assessment of historic masonry structures is a complex problem mainly due to the mechanical characteristics of their material. In the 50's it was shown that Standard Limit Analysis is suitable for that type of structures and has proven effectiveness for simplified assessment as long as sliding collapse does not occur. This can be formulated as an optimization problem with the intention of calculating the bounds of the load factor, the maximum for static formulations and the minimum for kinematics. In the static case, it is generally assumed that a load factor lower than the referred of the onset of collapse is a safe load factor, but this assumption is false. The collapse due to the lack of stability may occur by increase or decrease of the load factor.*

This work presents an alternative to load factor determination to evaluate the safety of masonry structures. The possibility to incorporate one or more safety coefficients is presented applying a deterministic partial safety factor method. An important difficulty for this purpose is that usually these partial coefficients are applied to variables that are referred to the origin of coordinates. This would be appropriate for materials with similar mechanical behaviour under tension and compression stresses, but it is not the case for the typical materials employed in masonry structures like stones, bricks or similar. Materials with non-symmetric tension-compression behaviour have the origin of coordinates over the yield surface or very close to it. For this reason, the origin can hardly be considered as a safe reference point.

The method proposed in this work consists of the calculation of the interior point further of the yield surface and considers it as the safest point. Considering that point as the origin of coordinates, the deterministic partial safety factors can be calculated.

1 INTRODUCTION

In 1952 Kooharian [1] proved that Limit Analysis theorems were suitable for masonry structures, although in 1953 Drucker [2] proved that these theorems were not appropriated for cases of sliding collapse. The Standard Limit Analysis is an excellent method to evaluate the resistance and stability of the cases in which sliding do not occur [3], it is especially useful for study in a simplified way the short-time behaviour. In fact, this is probably one of the best methods to evaluate the limit resistance of one structure obtaining the failure strength and the global stability, both of them from basic data as the geometry of the structure and loads. On the contrary, this method is not appropriate to evaluate the mechanical behaviour considering long-time effects, for this issue there are sophisticated tools, although it is important to note that it requires much more specific data.

Heyman has developed “manual” analysis based on these theorems, and he successfully did it because that analysis method perfectly connects with the graphic and non-graphic traditional analysis based on pressure lines.

The Limit Analysis and the linear and non-linear Mathematical Programming have been always closely related. Charnes et al. [4] matched the equivalence between two dual Linear Programming problems with the static and kinematic principles of plastic collapse in trusses. Dorn [5] and Charnes et al. [6] extended the equivalence to structural frames. From the 70’s, Limit Analysis using Linear Programming experienced a great development mainly due to the increase of the use of computers. Some examples of this phenomenon can be highlighted, as the application by Anderheggen [7] of a medium continuous with finite elements method of a rigid plastic material, or the application to masonry structures modelled by rigid blocks with unilateral contact by Livesley [8] using a static formulation, or the application of a kinematic formulation by Gilbert et al [9].

The application of lower bound theorem or static method, is used to predict if structures are stables and resistant under specific load case. This is based on the “manual” analysis mentioned above, that consists of obtaining one solution of the equilibrium between whatever internal forces with the load case. It is important to note that this simulated structural response does not have to be the “real” one, but these internal forces cannot violate the constraints of the material at any point. In other words, if a solution that allows to achieve the equilibrium and does not violate the constraints is possible, the structure “finds” it.

It is a different topic how to obtain the variation of the security if the loads or the proprieties of the structure vary. The application of these theorems is not easy if the materials are heterogenous, anisotropic, and with tensile strength values near to zero or even zero. The topic is complicated if the theorems are applied in combination with Mathematic Programming. Their names, lower bound theorem and the upper bound theorem, can be misled.

It is a known fact in these structures that it is more dangerous to reduce the loads than to increase them, nevertheless this phenomenon is sometimes ignored with serious consequences (fig. 1).

From this point, this work is limited to methods with static formulation. In this way, this problem has been approached with “manual” analysis for linear structures with the sole objective of checking the stability using the geometry safety coefficient.

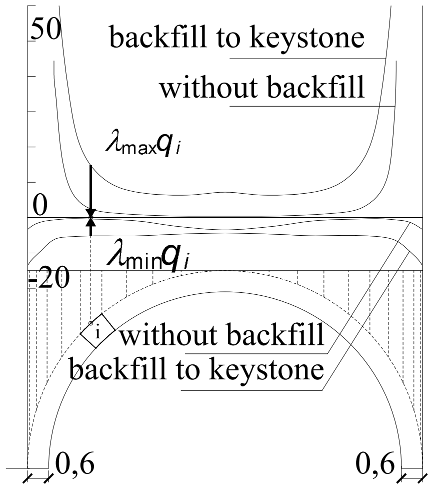

Figure 1: Influence lines for a point load with backfill to keystone and without backfill.

2 STRUCTURE MODELLING

This work proposes a model of a masonry structure with a unilateral material, considering both blocks and the interfaces, with little or no tensile strength, high compressive strength and non-existence or impeded sliding. In the simplest model the tensile strength is voided and in the model of Heyman [10] the compressive strength is not limited. The assumption of considering zero tensile strength has been traditionally applied and it works in favor of the safety. The use of limited compressive resistance considering the possibility of crushing has not been traditional applied, but it appears in researches as Delbecq [11] or Livesley [8]. Respect to the sliding, the Standard Limit Analysis method requires that it be voided. In this way, will be supposed the friction enough to impede the sliding, or the necessary characteristics to avoid it. The sliding collapse has been studied by several authors [12].

Therefore, the mechanical behaviour of the material is modelized as a plastic-rigid, regardless elastic strains. In this way, it is a realistic simplification because the plastic strains that occurs at the point of yield constrains are much superior than elastic ones.

Figure 2 shows an example of axial force (*N*) and moment (*M*) applied on a generic section and a stress distribution over this surface. This distribution can be considered for a section, real or virtual, with a unilateral and rigid-plastic material as described before.

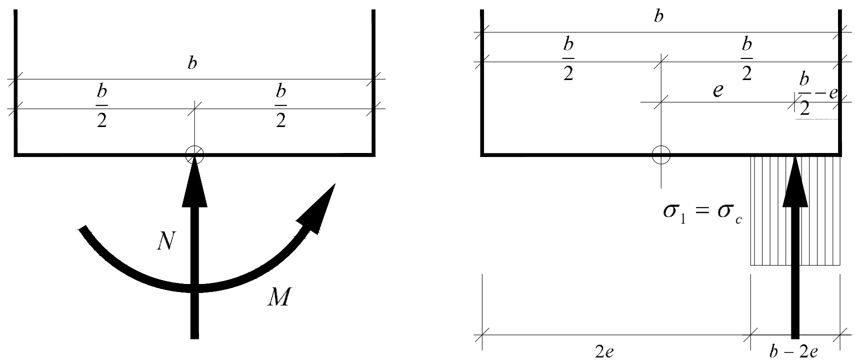

Figure 2: Generalized stresses and stresses in a contact surface of a masonry-like structure**.**

It is possible to define a limit surface or yield surface using the variables of axial force *N* and moment *M*, as can be observed in figure 3. The stresses represented into the limit surface of figure 3 are safe points. In the states represented in the strict limit, the collapse is about to begin. Every stress state outside the yield surface has not physical meaning or, in other words, they are not possible. It is important to note that this limit surface must to be convex so that Limit Analysis can be applied. Further, the elemental deformations produced when the variables reach one point of the limit surface or "plastic multipliers" must be perpendicular to the limit surface in that point.

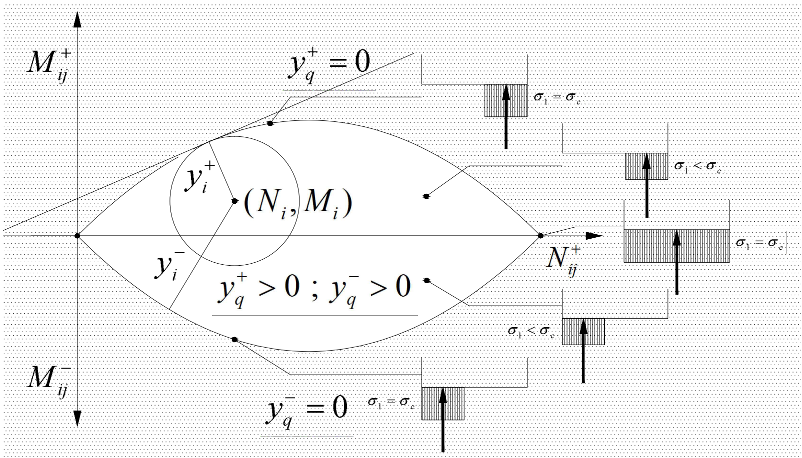

Figure 3: Yield surface M-N of a masonry-like structure.

The effect of the tangential stress *V* can be incorporated by means of the corresponding yield strength surface *N*. If this effect produces sliding the Standard Limit Analysis could not be applied. For this reason, as it has been explained before, it is supposed enough friction to impede that the sliding occurs.

The terminology used in this work has been defined below. B^t corresponds to the equilibrium matrix, *s* is the vector of internal force, *g* is the vector of permanent loads, λ correspond to the load factor, *q* is the vector of variable loads, *y* is the vector of "slack referred to limit stress" and L^t is yield matrix that consist of a matrix of suitable coefficients that relates internal forces to those mentioned slacks. The vector *y* must to be positive, and from a geometric point of view, corresponds to the distance from one stress point to the limit surface.

The method proposed in this article is based on a static formulation in which every valid solution must be according to Eq. (1).

$$\text{s.a.} \quad \left.\begin{array}{rcl} \mathbf{B^t s} & = & \mathbf{g}+\lambda\mathbf{q} \\ \mathbf{-L^t s} & = & \mathbf{y} \\ \mathbf{y} & \geq & \mathbf{0} \end{array}\right\} \equiv \mathbf{y} \in \left\{\mathrm{E}_y\right\} \tag{1}$$

The Standard Limit Analysis can be applied in the cases in which the material is valid according to the previous conditions and the sliding do not conditionate the collapse mode. In these cases, the typical approach to obtain a solution of the problem that shows the collapse mode consists of the calculation of the maximum value of the load factor applicable to live loads, for which the collapse begins Eq.(2) or Eq. (3), and considering that any value of load factor lower than this value is a safe solution. The lower the load factor, the greater the safety. In this way, the maximum load factor does the work of global safety factor over the variable loads. The cases with λ_{max}=0 show the structure is about to collapse with only permanent loads. The cases with λ_{max}=1 show that the structure is about to collapse with concomitant permanent and variable loads without safety coefficients. On the other hand, the cases with λ_{max}>1 show that the structure is able to resist with an increase of variable load before collapse occurs.

$$\max \lambda \quad \text{s.a.} \left\{ \mathbf{B^t s} = \mathbf{f} = \mathbf{g} + \boldsymbol{\lambda}\mathbf{q} \ ; \ -\mathbf{L^t s} = \mathbf{y} \ ; \ \mathbf{y} \geq \mathbf{0} \right\} \tag{2}$$

$$\max \lambda \text{ s.a. } \mathbf{y} \in \left\{ \mathrm{E}_y \right\} \tag{3}$$

The second and third constraints of Eq. (2) define a convex polyhedron, and in most closed cases, which are called the yield surface. It is obvious that in problems that include stability there is both an upper and a lower bound for λ, and therefore the methods specifically limited in the obtainment of a λ_{max} are not suitable.

3 THE PROPOSED METHOD

The most advanced method to evaluate the safety of a masonry structure consist of the development of a simulation, for example using Monte Carlo type, with aleatory values of loads, the data of the initial geometry considering the possibility of its variation, and mechanical characteristics. This method is an alternative to use the partial safety factors and corresponds to the probabilistic methods of level III. This possibility is very attractive from a theoretical point of view. However, its real application is difficult due to the lack of data. This is one of the reason why the Limit Analysis is used. In already published works [13] is exposed that would be needed to obtain the distribution of the frequencies of the involved parameters, as loads, resistances, etc. Even in the case to obtain them, it would be necessary to determine the number of replicates per assay to obtain the necessary confidence level.

This article is focused on the evaluation methods with level I of safety. They are deterministic methods in which the results are modified by a safety margin, and sometimes they are developed by means of partial safety coefficients.

The concepts are described below in an example in order to clarify the exposition. The own limitations of the example are obvious, but the clarification and the graphical representation are important advantage. On the contrary, the accuracy is lower.

3.1 Case-Study

This case-study consists of the example of a simple beam of a rigid-plastic material laid on a rigid block, as is show in figure 4. There are one permanent force F_1 and one variable force F_2 applied in both extremes. In this case, the beam is able to resist an ultimate bending moment M_u, that depends on the material. The permanent force is similar to dead loads g and the variable one is similar to live loads q, conditioned by a load factor λ. The geometric dimensions and the location of the forces as can be observed in figure 4.

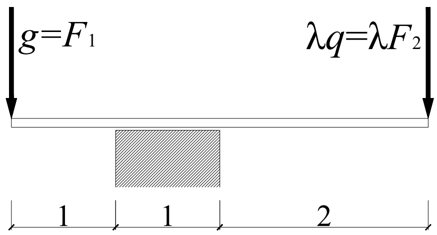

Figure 4: The example of the case-study.

The static variables of this example are de loads g, λq, and the constraints are defined according to the stability limit cases (Eq. 4, 5) and the resistance restrictions (Eq. 6, 7).

These constraints can be graphically represented by the lines that represent the static variables g, λq, as figures 5 shows, limiting two half-spaces: one in white defines the area where constraints are valid, and other one coloured in grey that defines the area where the constraints are unacceptable.

The intersection of all these half-spaces defines a n-dimensional convex polyhedron, in the example a polygon.

$$F_1 \geq \lambda F_2 \tag{4}$$
$$F_1 \leq 3\lambda F_2 \tag{5}$$
$$M_u \geq F_1 \tag{6}$$
$$\frac{M_u}{2} \geq \lambda F_2 \tag{7}$$

Inside this polyhedron which, if it is closed is called a polytope, are all solutions that meet all restrictions, in the example it is the white polygon. In the envelope of the polyhedron are the solutions for which some limit constraints have been reached and, in the case of the example, the collapse is about to begin. Outside, in the gray zone, there are no possible solutions from the physical point of view.

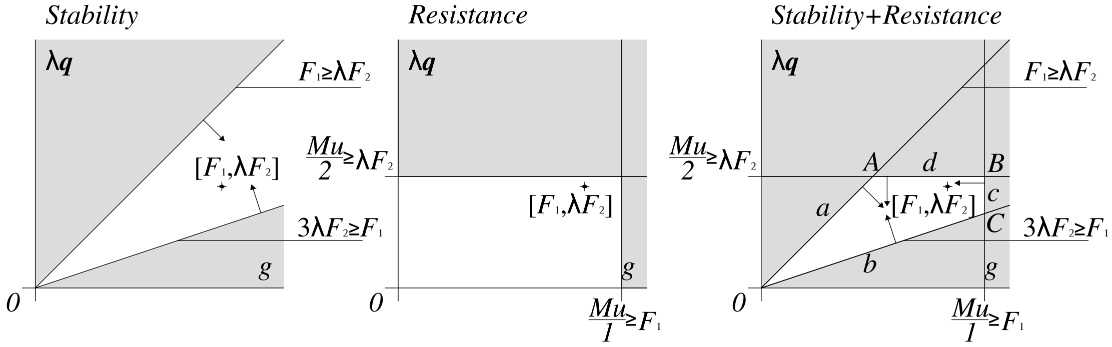

Figure 5: Yield surfaces λq-g of case-study.

3.2 Range of stability

Figure 5 shows that the non-stable solutions coloured in grey can occur both increase and decrease of load factors. The simplest way to obtain both higher and lower limits consists of achieving them according to Eq. 8, 9 and define margin between them (λ_{min}, λ_{max}), thus obtaining the stability range of the system loads.

$$\lambda_{\max} = \max \lambda \quad \text{s.a.} \ \left\{ \mathbf{B^t s = f = g + \lambda q} \ ; \ \mathbf{-L^t s = y} \ ; \ \mathbf{y \geq 0} \right\} \tag{8}$$

$$\lambda_{\min} = \min \lambda \quad \text{s.a.} \ \left\{ \mathbf{B^t s = f = g + \lambda q} \ ; \ \mathbf{-L^t s = y} \ ; \ \mathbf{y \geq 0} \right\} \tag{9}$$

This process has been fully developed [12, 14], and it can be graphically represented, as shows figure 6. The straight segment (F_1, $\lambda_{min}F_2$), (F_1, $\lambda_{max}F_2$) builds a polytope with lesser dimensions in this case 1 and inside it, there are all admissible static solutions, where F_1 does not vary.

Any value of load factor into the stability range is a safe solution. The higher distance from the point to the both limits, the greater the safety. In this way it is possible to define a safety coefficient γ for a load factor according to the distance from the point to distances to the average value of the extreme values (Eq. 10).

$$\gamma = \arg\max \left\{ \frac{\lambda_{\max} - \lambda_{\text{med}}}{\lambda - \lambda_{\text{med}}}, \frac{\lambda_{\text{med}} - \lambda_{\min}}{\lambda_{\text{med}} - \lambda} \right\} \tag{10}$$

This method supposes an advance of current knowledge. Although some of the possibilities has been proven before [14], it turns unsafe in cases where the point is near to the yield surface in the direction of another variable than λ.

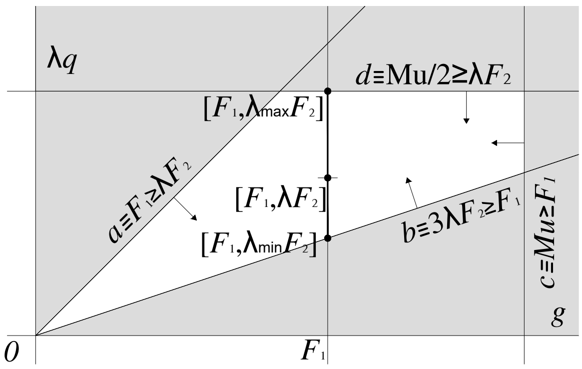

Figure 6: Stability interval.

3.3 Contraction in a fixed value of the yield surface

It is important to have standardized analysis methods. In this way, the safety range of linear structures has been traditionally evaluated using the Geometric Safety Coefficient (GSC) where the evaluation is limited to the stability. This GSC can be interpreted as the possible verification to draw a pressure line between the lines of the own geometry of the contracted structure (figure 7).

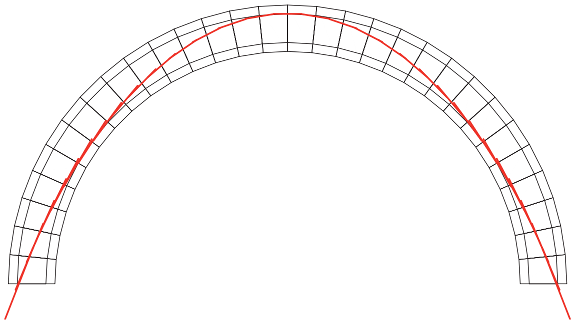

Figure 7: Geometric representation of the Geometric Safety Coefficient.

It is not possible to solve that problem directly with a Linear Program, but it is also possible to solve it iteratively with this tool. It can be achieved increasing and/or reducing the GSC in a process that finishes when the structure remain stable at maximum GSC. In those cases, the resistance is usually checked in the last step.

This article is focused on alternative methods, more general and applicable to any type of structure. Then, from now on this work is oriented to methods about the contraction of the yield surface. In other words, methods based on the distances from a safe stress state point to the yield surface. These types of problems can be solved in the cases of convex envelopes as Convex Optimization problems, and the simplest cases as the case-study with polyhedral envelop can be solved with Linear Programming.

General methods to solve these type of problems have been already described, for example by Boyd et al. [15]. In that work the vector y as the components of the geometric distances from the point of the reference stress to the yield limits is defined, as can be observed in the left of figure 8. This work takes advantage of this vector.

For all these reasons, one alternative easily implementable consists of the calculation of the minimum of the distances that forms a stress point to the nearly point of the yield surface. This calculation can be achieved with a new condition $h \leq y_i$ and solving *max h* and *min* y_i= max h. Eq.(11).

$$\min y_i = \max h \quad \text{s.a.} \ \left\{\lambda = \lambda_1 ; 0 \leq h \leq y_i ; \mathbf{B}^t\mathbf{s} = \mathbf{g} + \lambda\mathbf{q} \ ; \ -\mathbf{L}^t\mathbf{s} = \mathbf{y} \ ; \ \mathbf{y} \geq \mathbf{0}\right\} \tag{11}$$

The set of points which $y_i \geq h$ defines a contracted yield surface, as can be observed in the right of figure 8. This envelope is defined by the restriction $y_i = h$.

Every point inside this envelope represents a stress state that is separated at least at a distance h from the nearly limit.

In this way, the method proposed in this work allows with only one Linear Programming problem to guarantee that almost one stress point that corresponds to the load factor $\lambda = \lambda_1$ is separated one distance with respect to the nearly yield surface. Also, this method allows to know which limit constraint is the nearest. The inconvenient of this method is that it does not specify the distance. Thereby, it is not shown if the distance is small or large with respect to the size of the yield surface.

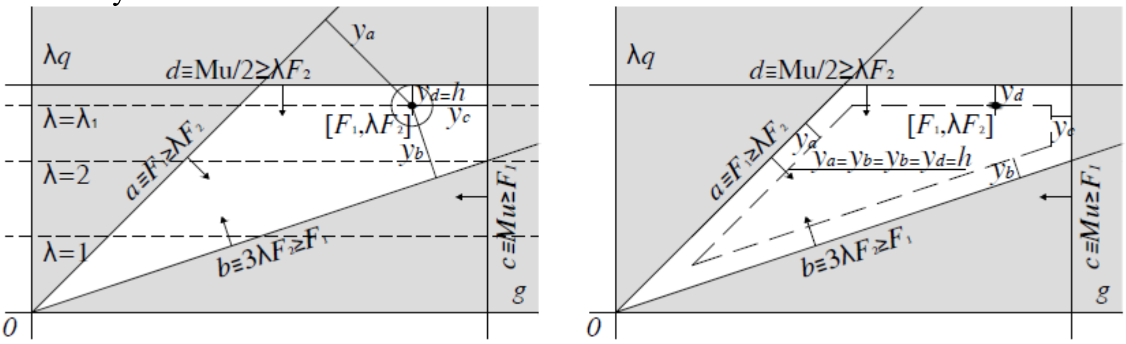

Figure 8: Parallel contraction of the yield surface.

3.4 Homothetic contraction of the yield surface

A most ambitious proposal is based on the implementation of the work of Cervera [16] and it consists of contracting homothetically the yield surface, with the homothetic centre in an interior point that is hypothetically the safest. This proposal is not yet implemented for masonry structures but the idea is very similar with the partial safety factors method, as is described by Ditlevsen [17].

In the left image of figure 9 shows a contracted limit of the yield surface as has already shown[17]. This yield surface, or limit state surface according to Ditlevsen, delimits all the inside points "enough safe" according to the partial safety factors previously established. The main difficulty to apply this methods in masonry structures consists of the ubication of the origin to which to refer the partial safety factors.

In the left image of figure 9 only the resistance restrictions are represented. In this case, the safest static solution is the near to the origin (0,0). Nevertheless, in the right image of figure 9 shows that the safest point (the further of the yield surface) is far from the origin. Generally this occurs including resistance and stability restrictions.

The mode of how to obtain the reference point and how to evaluate the safety of a structure is discussed below.

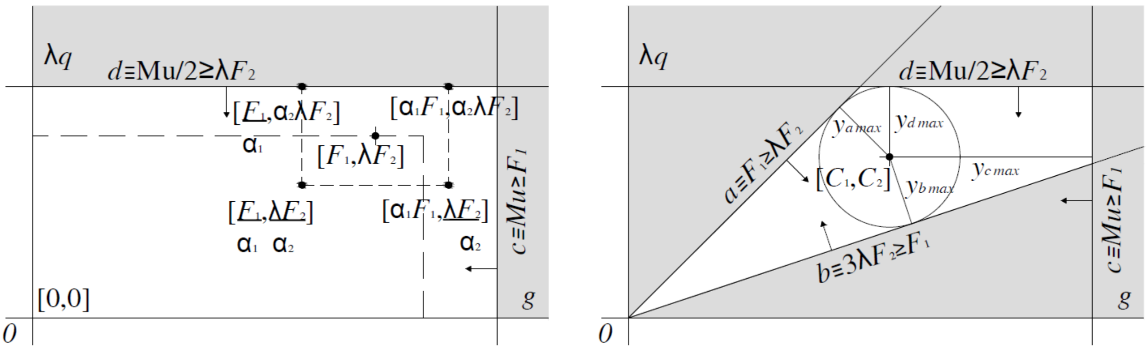

Figure 9: Partial safety factor and point farthest from the yield surface.

3.5 Deterministic partial safety factors

The Chebyshev centre is the inner point that is as far away as possible from a convex envelope, Boyd [15]. It is also the centre of the largest hypersphere, or n-dimensional sphere, contained within this envelope (right image of figure 9) and for that reason it also receives the name of “ball centre”. When this envelope is a polyhedron, as in this case, or it can become one by linearization, the problem can be solved by Linear Programming, Eq.(12), very similar to that of Eq.(11). This process consists of the obtainment of the maximum value of h. In this case this value matches with the radio of the maximum hypersphere that can be included into the polyhedron.

$$\max h \;\; \text{s.a.} \; \left\{ 0 \le h \le y_i ; \; \mathbf{-L^t s = y} \; ; \; \mathbf{y \ge 0} \right\} \tag{12}$$

The solution of the Linear Program will gived both the value of h and the different y_{imax}, these y_{imax} are the maximum distances from the centre that has been used as the origin to the corresponding restrictions. Being this the furthest point from the yield surface and, therefore, the safest.

If the centre of the safe area is known, the maximum distances can be directly obtained replacing the values of the vector s corresponding to the centre of the equation of the vector y. This occurs for example when the yield surface is symmetric with respect to two perpendicular axis. In the same way, it could be obtained for other point referring it as “the most safe point”.

According to the left image of figure 9, it can be observed that the safety margins are proportional to some coefficients and to the distance to the origin and, in this case, to the safest point.

This problem can be solved by Linear Program. If a deterministic partial safety factor γ_i is used, that corresponds to a constraint i as the relation between the distance from the centre to the i constraint and the different between this relation with the distance from the selected point to the yield surface. It can be observed in the left of figure 10 and it is represented in equation 13, adding a new variable α_i, according to equation 14. The equation 15 defines the resulting Linear Program.

$$\gamma_i = \frac{y_{i\max}}{y_{i\max} - y_i} \tag{13}$$

$$\alpha_i = \frac{y_i}{y_{i\max}} \; ; \; 0 \le \alpha_i \le 1 \tag{14}$$

$$\max h \;\; \text{s.a.} \; \left\{ \mathbf{B^t s = g + \lambda q} \; ; \; \mathbf{-L^t s = y} \; ; \; \mathbf{y \ge 0} \; ; 0 \le h \le \alpha_i \; ; \; 0 \le \alpha_i \le 1 \right\} \tag{15}$$

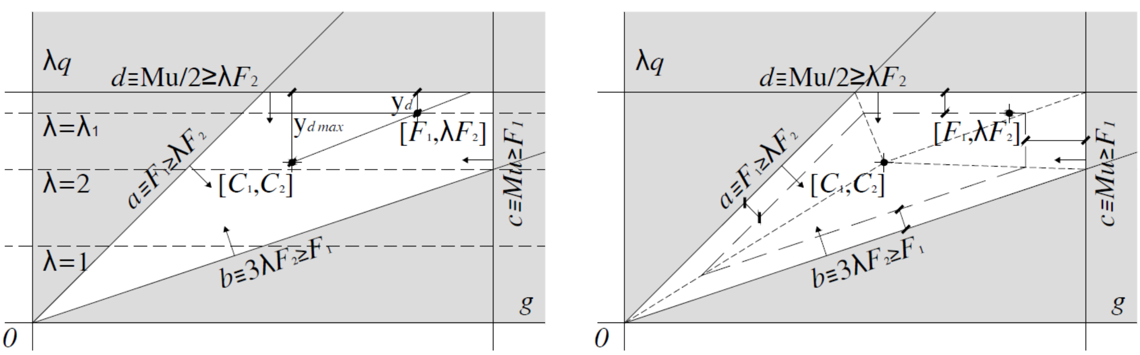

Figure 10: Homothetic contraction of the yield surface.

When the Linear Program is solved the value of h is obtained, along with α_i and together the rest of involved variables. Therefore, the values of γ_i that correspond to each constraint i, as the global minimum of the deterministic partial safety factor $\gamma_{i\min}$ can be deduced through the equation 16.

$$\gamma_i = \frac{1}{1-\alpha_i} \; ; \gamma_{i\min} = \frac{1}{1-h} \tag{16}$$

It is possible to apply a different focus of the same problem. The geometric place of the points with the same γ_{min} coefficient is a homothetic surface with the yield surface, and with the centre of the homothecy, according to Chebyshew, that is represented on the right image of figure 10. It can be applied too in other point considering it as the safest point. It can be established a specific value to the γ_{min} coefficient, in this way a value of h is obtained and hence the separation between the yield surface with the constrained of yield surface. With this "enough safety" surface similar operations could be performed as if it were the original one.

4 DISCUSSION

In order to guarantee that the structure "can find" the solution with higher safety coefficient it is an indispensable requirement that the principles of Standard Limit Analysis are applicable. Thus, if the beginning of the collapse is influenced by sliding, this principles are not applicable.

The implementation of the methods proposed in this work for a yield surface as presented in figure 3 has been carried out. However, its exposition exceeds the objectives of this article.

The work presented in this paper has been focused in the comparison between the maximum safety that can be obtained for a structure for each restriction with the safety referred to in the mentioned case-study. However, this study paradoxically shows that the partial safety factor referred to the rocking constraint decreases when the compressive strength of the material increases. This occurs with the masonry structures typical yield surface.

This work also shows that it is possible to select other points as the centre of the homothecy, or even consider other different coefficients if the case requires it. Both processes have been developed in this article, although in this last case the "enough safety surface" is not a homothetic of the yield surface.

5 CONCLUSIONS

The Limit Analysis by means of Linear Programming consists of the calculation of the maximum of the load factor that increases the live loads to produce the collapse of a structure. In some cases, it is unsafe to evaluate the safety of the masonry structures applying this analysis. When the external variables loads, or any of them, produces a stabilizing phenomenon, a decrease of these loads can produce the collapse of the structure. This is the main conclusion that can be highlighted from this work, because it also affects to other methods that approach the estimation of the safety of masonry structures.

The partial safety coefficients, and specially the lower value of them obtained by the descripted method, guarantee that for a given system of loads (both fixed and variables) there is a solution corresponding.

The method presented in this article underestimates the safety factor in the cases of masonry structures with high compressive strength and in cases in which the collapse mode is the rocking. In spite of the method remains safe, it would be necessary to continue the research for these cases and other similar ones. It could be interesting to expand research about the points that should be considered as the "safest points" and how to estimate the values of the safe coefficients referred to the different collapse modes that can occur in masonry structures.

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