Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería

Arctan Power Half-Logistic Model: A Bayesian Approach to Progressively Censored Engineering Data

Amal S. Hassan [1], **Maysaa Elmahi Abd Elwahab** [2], **Mohammed Elgarhy** [3,4,*], **Haifa Alqahtani** [5] and **Rokaya Elmorsy Mohamed** [6]

[1]Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt

[2]Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[3]Faculty of Computers and Information Systems, Egyptian Chinese University, Nasr City, Egypt

[4]Department of Computer Engineering, Biruni University, Istanbul, Turkey

[5]Department of Statistics and Business Analytics, United Arab Emirates University, Al Ain, Abu Dhabi, United Arab Emirates

[6]Department of Mathematics, Statistics and Insurance, Faculty of Management Sciences, Sadat Academy for Management Sciences, Cairo, Egypt

[#]These authors contributed equally to this work

ABSTRACT

The motivation behind the creation of new statistical models is primarily driven by the need to accurately describe complex data and related phenomena. This article introduces the arctan power half logistic distribution, a new and more flexible extension of the power half logistic distribution. The proposed model's hazard rate function is highly versatile, capable of displaying decreasing, J-shaped, or reversed J-shaped patterns. We derive its key statistical properties and investigate its application to progressively Type-II censored data. Parameter estimation is conducted using both maximum likelihood and Bayesian frameworks, the latter incorporating informative and non-informative priors across multiple loss functions. Given the analytical intractability of the posterior distributions, we employ Markov Chain Monte Carlo techniques for numerical approximation. Monte Carlo simulations demonstrate that Bayesian point and interval estimates generally outperform frequentist approaches, maintaining coverage probabilities near 95%. Finally, the model's superiority is validated using a real-world engineering dataset, where it consistently outperforms several established competing distributions.

OPEN ACCESS

Received: 21/02/2026

Accepted: 20/04/2026

DOI
10.23967/j.rimni.2026.10.81008

Keywords:
Arctan generated family
power half logistic
Metropolis-hastings
loss function
stress-strength reliability
entropy measures
MSC: 62F10
62F15

1 Introduction and Motivation

Extending lifetime distributions is a central focus in statistics, with diverse applications in actuarial science, biology, engineering, and medicine. Because classical distributions often lack the flexibility to capture complex real-world phenomena, developing generalized families or generators is essential. By

*Correspondence: Mohammed Elgarhy (m_elgarhy85@sva.edu.eg).

introducing additional scale or shape parameters, these generators significantly enhance the adaptability of pre-existing models. Numerous successful generators have been documented, including the the transformed-transformer-G [1], the Weibull–G [2], the odd reparameterized exponential transformed-G [3], Topp–Leone–Marshall–Olkin–G [4], Type I half-logistic–G [5], generalized transmuted–G [6], Type–I heavy tailed–G [7], the alpha power transformed–G [8], Lomax–G [9], Topp Leone Kumaraswamy–G [10], the exponential alpha power–G [11], the Kavya-Manoharan–G [12], odd Kumaraswamy Fréchet–G [13], and new shifted Lomax–X [14] among others.

Despite their efficacy, traditional generalization techniques often suffer from key limitations, such as reparameterization challenges and complex mathematical derivations when parameter counts increase. Recently, trigonometric transformations have gained attention in probability theory for their versatility and accurate modeling of real-world data. Among these, the Arctan–X (ART–X) generator by Alkhairy et al. [15] is notable, with cumulative distribution function (CDF) and probability density function (PDF) defined as:

$$H(v;\varpi) = \frac{4}{\pi}\arctan\left(G(v;\varpi)\right), \quad v \in \mathbb{R}, \tag{1}$$

$$h(v;\varpi) = \frac{4\,g(v;\varpi)}{\pi\left\{1+[G(v;\varpi)]^2\right\}}, \quad v \in \mathbb{R}, \tag{2}$$

where $G(v;\varpi)$ and $g(v;\varpi)$ are the baseline CDF and PDF of a continuous distribution with parameter vector ϖ.

In comparison to commonly used generators, the arctan transformation has various practical advantages. Firstly, the flexibility of the distribution is improved without increasing the number of parameters, thereby maintaining the simplicity of the model. Secondly, the arctan transformation produces closed-form expressions for the cumulative distribution function as well as the quantile function, which facilitates simulation, estimation, as well as Bayesian computations. Finally, the fact that the arctan transformation is a bounded function guarantees the stability of the likelihood, especially for progressive censoring schemes. The arctan power mechanism, on the other hand, produces a wide range of hazard rate shapes while controlling over-tailedness of the distribution.

A widely used lifetime model in reliability analysis is the half-logistic (HL) distribution, introduced by Balakrishnan [16]. Defined as the absolute value of the logistic distribution, its simplicity and monotonically increasing hazard rate function (HRF) have attracted considerable attention. However, more flexible models are often required to handle diverse data types in lifetime studies. The HL distribution has been extended through various generators and transformations, including the generalized HL model [17], the power HL (PHL) model [18], the McDonald–HL model [19], the exponential HL distribution [20], transmuted–HL model [21], the tan exponentiated odd log-logistic Weibull distribution [22], the generalized half–logistic Poisson distribution (GPHL) [23], sine PHL (SPHL) distribution [24], the Type II HL Ailamujia distribution [25], the sine unit exponentiated HL distribution [26], the power odd Lindley HL distribution [27], inverse PHL distribution [28].

Given the random variable V, the CDF and PDF of the PHL distribution with scale parameter $a > 0$ and shape parameter $b > 0$ are as follows:

$$G(v;\varpi) = 1 - \frac{2}{1+e^{av^b}}, \quad v,a,b > 0, \tag{3}$$

$$g(v;\varpi) = \frac{2ab\,v^{b-1}e^{av^b}}{\left(1+e^{av^b}\right)^2}, \quad a,b,v > 0. \tag{4}$$

The PHL distribution has several limitations: its hazard rate is strictly monotonic, unsuitable for non-monotonic patterns (e.g., bathtub-shaped) common in reliability and survival data; it may fail to capture heavy tails or strong right skewness; and its flexibility is often insufficient for datasets requiring complex tail behavior or multimodality. To overcome these issues, we employ the arctan generator, which introduces trigonometric nonlinearity while preserving closed-form expressions for the cumulative distribution and quantile functions. The resulting arctan–power half logistic (ART-PHL) distribution offers enhanced hazard rate flexibility, improved tail behavior, and tractable analytical properties, making it well suited for censored lifetime data analysis.

In many disciplines, such as engineering, social economics, and pharmacology, censorship is prevalent, especially in the areas of survival and reliability analysis [29–33]. It is challenging to fully observe the sample data in actual production due to time and financial constraints. Despite extensive analysis, traditional censoring systems like Type–I, Type–II, and hybrid schemes are rigid because units cannot be eliminated at will. In light of this, Balakrishnan and Aggarwala [34] suggested the progressive Type II censoring (PT–IIC) scheme as a substitute for the previously mentioned censoring schemes since it allows test units to be removed from lifetime tests whenever something fails. Here is a quick summary of PT–IIC: Suppose that when n identical units are tested, m failures will be observed. A number S_1 of the remaining units $(n-1)$ are chosen at random and eliminated from the experiment at the time of the initial failure $(v_{(1)})$. At the second failure $(v_{(2)})$, a number S_2 of the surviving units $(n-S_1-2)$ are chosen at random and eliminated. The experiment continues until the mth failure $(v_{(m)})$, at which point all remaining $n-m-S_1-S_2-\cdots-S_{m-1}$ units are eliminated. As a result, the PT–IIC contains m observed failure and survival item samples, $S=(S_1,S_2,\ldots,S_m)$ such that $n=m+S_1+\cdots+S_{m-1}$. Notably, it is important to recognize that this form of censorship is restricted to classical Type II censoring (TIIC), wherein $S_1=S_2=\cdots=S_{m-1}=0$ and $S_m=n-m$. It is reduced to a complete sample with no censoring if $n=m$ and $S_i=0$, $i=1,2,\cdots,n$.

The creation of more adaptable and efficient probability distributions remains a top priority in the domains of statistics and probability theory. Since empirical datasets frequently diverge from standard models, expanding classical distributions is essential for achieving accurate data representation. This paper introduces the ART–PHL distribution, a novel lifetime model integrating the PHL distribution with the ART–X generator. The motivations for this proposed model include:

1. The ART–PHL model is easy to formulate, since it doesn't need any extra parameters or special functions. The closed form of its CDF facilitates analysis and modeling.
2. The distribution works well for modeling data that may not fit other popular distributions, such as data that is right–skewed, increasing, J–shaped, decreasing, or reversed J–shaped.
3. Several characteristics of ART–PHL distribution are examined, such as the quantile function (QF), complete and incomplete moments, mean residual life function, mean waiting time, some uncertainty measures, and stress-strength reliability (SSR).
4. Bayesian and classical (maximum likelihood (ML)) methods are used to estimate the unknown parameters, survival function (SF), and HRF of the ART–PHL model using PT–IIC samples.
5. To overcome the difficulties of a theoretical comparison, a simulation study was performed to evaluate the estimators across a wide range of scenarios and sample sizes.
6. For evaluating the capabilities of this novel distribution and showing the practical use of the model, one engineering real-world data set is employed.

Therefore, the ART–PHL distribution is expected to outperform closely related models in the following scenarios: (i) lifetime data with non-monotone or inverted J–shaped hazard rates; (ii) progressively censored experiments with small to moderate sample sizes, where numerical stability is critical; (iii) environmental and reliability data requiring modest tail flexibility without extremely heavy tails; and (iv) empirical applications, such as electronic tubes data. These scenarios demonstrate the practical superiority of ART–PHL over traditional PHL and associated extensions.

The format of this article is as follows: We describe the construction of the ART–PHL distribution in Section 2. Section 3 provides an explanation of the ART–PHL distribution's mathematical and statistical characteristics. Sections 4 and 5 discuss the point and interval estimation of the suggested model. In Section 6, the suggested distribution's adaptability is tested on a real data set. The simulation experiment was designed to evaluate the accuracy of the estimates in Section 7. The article ends with some conclusion in Section 8.

2 The Novel Arctan Model

In this section, a new two–parameter extension of the PHL model, called ART–PHL distribution based on the ART generator, is introduced. The CDF of the ART–PHL distribution is produced by substituting CDF of Eq. (3) in CDF of Eq. (1), as mentioned below:

$$H(v;\varpi) = \frac{4}{\pi}\arctan\left(1-\frac{2}{1+e^{av^{b}}}\right); \quad a,b,v>0, \tag{5}$$

where $\varpi=(a,b)$ is the set of parameters, a and b are the scale and shape parameters, respectively. For the scale $a=1$, the CDF of Eq. (5) provides the ART–half logistic distribution, a new sub-model. The PDF of the ART–PHL distribution is produced by substituting CDF of Eq. (3) and PDF of Eq. (4) in PDF of Eq. (2) as given below:

$$h(v;\varpi) = \frac{8abv^{b-1}e^{av^{b}}}{\pi\left(1+e^{av^{b}}\right)^{2}\left(1+\left[1-\frac{2}{1+e^{av^{b}}}\right]^{2}\right)}; \quad a,b,v>0. \tag{6}$$

The following are the SF, the HRF, and cumulative HRF at time t:

$$\kappa(t;\varpi) = 1-\frac{4}{\pi}\arctan\left(1-\frac{2}{1+e^{at^{b}}}\right),$$

$$\eta(t;\varpi) = \frac{8abt^{b-1}e^{at^{b}}}{\pi\left(1+e^{at^{b}}\right)^{2}\left(1+\left[1-\frac{2}{1+e^{at^{b}}}\right]^{2}\right)\left(1-\frac{4}{\pi}\arctan\left[1-\frac{2}{1+e^{at^{b}}}\right]\right)},$$

and

$$C(t;\varpi) = -\log\left(1-\frac{4}{\pi}\arctan\left[1-\frac{2}{1+e^{at^{b}}}\right]\right).$$

The ART–PHL distribution can accurately depict the positively skewed data, the reverse J-shaped data, and the unimodal data shown in Fig. 1. The shape of the HRF for the ART-PHL distribution is dominated by the shape parameter b, and the scale parameter a affects the shape by controlling the rate. When $b<1$, the HRF is decreasing, it indicates early-life failures. On the other hand, for $b>1$, the shape of the HRF is increasing, indicating either an aging process or wear-out. In the case where b is close to 1, the combined effect of the power function and the bounded arctan function leads to non-monotonic shapes, such as J-shaped and reversed J-shaped. In the non-monotonic case, the scale parameter a has an effect on the sharpness of the turning point. It is not easy to determine the analytical thresholds for transitions among the different shapes, especially in the non-monotonic case, because the HRF expression is complicated. However, based on the numerical and graphical analyses, the transitions between the decreasing and non-monotonic shapes take place when $b = 1$.

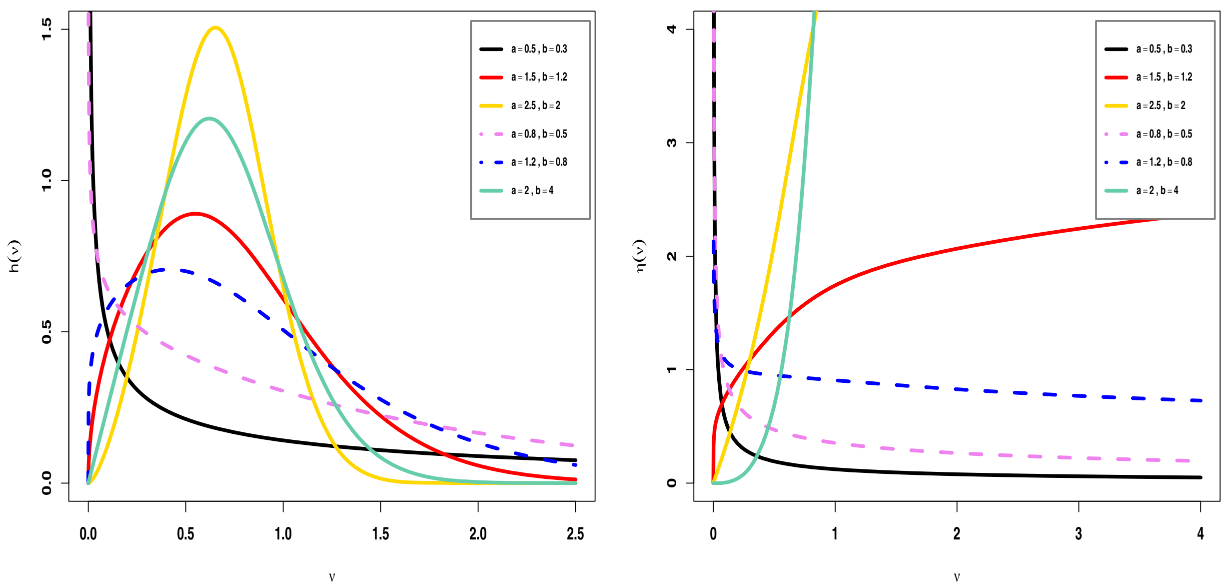

Figure 1: The PDF and HRF of the ART–PHL distribution for different parametric values

3 Fundamental Structural Characteristics

This section determines a number of important mathematical and statistical properties of the suggested ART–PHL model.

3.1 Moments Measures

Moments are quantifiable numbers that specify a probability distribution's properties. Moments display a data set's central tendency, variability, skewness (SK), and kurtosis (KU). As shown below, the qth moment of the ART–PHL distribution is obtained using PDF of Eq. (6).

$$\mu_q^{\prime} = \int_0^{\infty} \frac{8abv^{q+b-1}e^{av^b}}{\pi\left(1+e^{av^b}\right)^2\left(1+\left[1-\frac{2}{1+e^{av^b}}\right]^2\right)}\,dv. \tag{7}$$

Employ the following expansion:

$$(1+d)^{-k} = \sum_{j_1=0}^{\infty}(-1)^{j_1}\binom{k+j_1-1}{j_1}d^{j_1},\ |d|<1, \tag{8}$$

in the denominator of Eq. (7):

$$\mu_q' = \sum_{j_1=0}^{\infty} \frac{(-1)^{j_1} 8ab}{\pi} \int_0^{\infty} \left(1 - \frac{2}{1+e^{av^b}}\right)^{2j_1} \frac{v^{q+b-1} e^{av^b}}{\left(1+e^{av^b}\right)^2} dv. \tag{9}$$

Using the binomial expansion in Eq. (9) gives

$$\mu_q' = \sum_{j_1,j_2,j_3=0}^{\infty} \binom{2j_1}{j_2} \binom{1+j_2+j_3}{j_3} \frac{(-1)^{j_1+j_2+j_3} 2^{j_2+3} ab}{\pi} \int_0^{\infty} v^{q+b-1} e^{-(j_2+j_3+1)av^b} dv.$$

Let $z = (j_2 + j_3 + 1)av^b$, then, the qth moment of the ART–PHL distribution is as follows:

$$\mu_q' = \sum_{j_1,j_2,j_3=0}^{\infty} \rho_{j_1,j_2,j_3} \Gamma\left(\frac{q}{b} + 1\right), \tag{10}$$

where $\rho_{j_1,j_2,j_3} = \binom{1+j_2+j_3}{j_3}\binom{2j_1}{j_2} \frac{(-1)^{j_1+j_2+j_3} 2^{j_2+3}}{\pi\left(a[j_2+j_3+1]\right)^{\frac{q}{b}+1}}$, and $\Gamma(\cdot)$ is the gamma function (GaF).

The qth incomplete moment of the ART–PHL distribution is computed as follows:

$$\mu_q(t) = \int_0^t v^q \frac{8abv^{b-1} e^{av^b}}{\pi (1+e^{av^b})^2 \left(1 + \left[1 - \frac{2}{1+e^{av^b}}\right]^2\right)} dv. \tag{11}$$

Using the binomial expansion of Eq. (8) in Eq. (11), then we get

$$\mu_q(t) = \sum_{j_1,j_2,j_3=0}^{\infty} \binom{1+j_2+j_3}{j_3} \binom{2j_1}{j_2} \frac{(-1)^{j_1+j_2+j_3} 2^{j_2+3} ab}{\pi} \int_0^t v^{q+b-1} e^{-(j_2+j_3+1)av^b} dv.$$

Let $z = (j_2 + j_3 + 1)av^b$, then, the qth moment of the ART–PHL model is obtained as follows:

$$\mu_q(t) = \sum_{j_1,j_2,j_3=0}^{\infty} \rho_{j_1,j_2,j_3} \gamma\left(\frac{q}{b} + 1, [j_2 + j_3 + 1]at^b\right),$$

where $\gamma(n,x) = \int_0^x z^{n-1} e^{-z} dz$ is the lower incomplete GaF. The study of measures of inequality (such as Lorenz and Bonferroni curves) benefits from the use of incomplete moments. A system's mean residual life or mean excess function can also be estimated using it. An item's predicted life after reaching a specific age t is determined by the mean residual life function, say $\phi(t)$

$$\phi(t) = \frac{1}{\kappa(t;\boldsymbol{\varpi})} (E[T] - \mu_1[t] - t)$$
$$= \frac{1}{\kappa(t;\boldsymbol{\varpi})} \left\{ \sum_{j_1,j_2,j_3=0}^{\infty} \rho_{j_1,j_2,j_3} \Gamma\left[\frac{1}{b} + 1\right] - \sum_{j_1,j_2,j_3=0}^{\infty} \rho_{j_1,j_2,j_3} \gamma\left[\frac{1}{b} + 1, (j_2 + j_3 + 1)at^b\right] - t \right\}.$$

In order to analyze the actual time of failure of an item that has already failed, the mean waiting time is crucial. Assuming that the failure happened within the interval $[0, t]$, it shows how much time has passed since the object failed. The definition of the mean waiting time, $\bar{\phi}(t)$ is

$$\bar{\phi}(t) = t - \frac{\mu_1(t)}{F(t;\varpi)} = t - \frac{1}{F(t;\varpi)} \sum_{j_1,j_2,j_3=0}^{\infty} \rho_{(j_1,j_2,j_3)} \gamma\left(\frac{1}{b} + 1, [j_2 + j_3 + 1]at^b\right).$$

3.2 Quantile Function

The QF of the ART–PHL distribution is the inverse of the CDF of Eq. (5). The QF helps with statistical analysis and decision-making by being essential in calculating percentiles and confidence intervals. The creation of random numbers that are dispersed from the distribution is one of its many applications. It is also useful for estimating metrics related to shapes (skewness and kurtosis) and central tendency (median). The expression for QF is obtained as follows:

$$p = \frac{4}{\pi}\arctan\left(1 - \frac{2}{1+e^{a(Q(p))^b}}\right); \quad 0 < p < 1 \tag{12}$$

The simplified form of Eq. (12) is

$$e^{a(Q(p))^b} = \frac{2}{1-\tan\left(\frac{p\pi}{4}\right)} = \frac{1+\tan\left(\frac{p\pi}{4}\right)}{1-\tan\left(\frac{p\pi}{4}\right)} \tag{13}$$

Apply the natural logarithm of Eq. (13), the QF of the ART–PHL distribution takes the following expression:

$$Q(p) = \left\{\frac{1}{a}\ln\left[\frac{1+\tan\left(\frac{p\pi}{4}\right)}{1-\tan\left(\frac{p\pi}{4}\right)}\right]\right\}^{1/b}. \tag{14}$$

Setting p to $1/2$, $1/4$, and $3/4$, respectively, in Eq. (14), makes it simple to use the QF to determine the ART-PHL distribution's median $= Q(1/2)$, first quartile $= Q(1/4)$, and third quartile $= Q(3/4)$.

3.3 Stress-Strength Reliability

The SSR, $R = P(Y < V)$, quantifies the probability that a material's strength (V) exceeds the applied stress (Y). It is a critical metric in structural integrity and risk assessment [35–37], representing the balance between resistance and exposure. Reliability and safety decisions are supported by the ART-PHL model's analytical tractability, which allows for effective SSR evaluation in a variety of circumstances.

Assuming that Y and V are independent, let's say that random variable V has the ART–PHL distribution (a_1, b), and Y has the ART–PHL distribution (a_2, b). Then, the SSR is given by:

$$R = P(Y < V) = \frac{2^5}{\pi}\int_0^\infty \left(\frac{a_1 b v^{b-1} e^{a_1 v^b}}{\pi\left(1+e^{a_1 v^b}\right)^2\left(1+\left[1-\frac{2}{1+e^{a_1 v^b}}\right]^2\right)}\right)\arctan\left(1-\frac{2}{1+e^{a_2 v^b}}\right)dv. \tag{15}$$

Using the following expansion $\arctan(z) = \sum_{l=0}^{\infty}(-1)^l \frac{z^{2l+1}}{2l+1}$, $|z| \leqslant 1$ in Eq. (15) provides

$$R = \frac{2^5}{\pi}\int_0^\infty \left(\frac{a_1 b v^{b-1} e^{a_1 v^b}}{\pi\left(1+e^{a_1 v^b}\right)^2\left(1+\left[1-\frac{2}{1+e^{a_1 v^b}}\right]^2\right)}\right)\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(1-\frac{2}{1+e^{a_2 v^b}}\right)^{2l+1}dv. \tag{16}$$

Using the binomial expansion of Eq. (8) into Eq. (16)

$$R = \sum_{l,l_1=0}^{\infty} \frac{(-1)^{l+l_1}}{2l+1} \frac{2^5 a_1 b}{\pi^2} \int_0^{\infty} \left(\frac{v^{b-1} e^{a_1 v^b}}{\left(1+e^{a_1 v^b}\right)^2} \left(1 - \frac{2}{1+e^{a_1 v^b}}\right)^{2l+1} \left(1 - \frac{2}{1+e^{a_2 v^b}}\right)^{2l_1} \right) dv. \quad (17)$$

Using the binomial expansion in Eq. (17)

$$R = \sum_{l,l_1,l_2,l_3=0}^{\infty} \binom{2l_1}{l_3} \binom{2l+1}{l_2} \frac{(-1)^{l+l_1+l_2}}{2l+1} \frac{2^{5+l_2+l_3} a_1 b}{\pi^2} \int_0^{\infty} \left(\frac{v^{b-1} e^{a_1 v^b}}{\left(1+e^{a_1 v^b}\right)^{2+l_2} \left(1+e^{a_2 v^b}\right)^{l_3}} \right) dv.$$

After simplified form the SSR of the ART–PHL distribution is as below:

$$R = \sum_{l_m=0}^{\infty} \Xi_{l_m} \frac{a_1}{a_1(l_2+l_4+1) + a_2(l_3+l_5)},$$

where $\Xi_{l_m} = \binom{2l_1}{l_3}\binom{2l+1}{l_2}\binom{l_3+l_5-1}{l_5}\binom{1+l_2+l_4}{l_4} \frac{(-1)^{l+l_1+l_2}}{2l+1} \frac{2^{5+l_2+l_3}}{\pi^2}, \quad m = 1,2,3,4,5.$

3.4 Uncertainty Measures

Entropy provides information about the quantity of data needed to characterize a random variable by quantifying the uncertainty or unpredictability connected to a probability distribution. In practical applications, entropy metrics are very useful in different fields [38,39]. Lower entropy models may be preferable in regulated engineered systems where predictability is critical, but higher entropy models may be better suited to biological, environmental, or social systems subject to random external effects. Several entropy measures are covered here, including Rényi (RE), Havrda and Charvat (H–C), Tsallis (TS), and Arimoto (AR). The formula of these entropy measures is provided in Table 1.

Table 1: Formulas for different entropy measures of order λ

	Entropy	Expression	Authors
1	RE	$\Im_1 = \frac{1}{1-\lambda} \log\left\{\int_0^{\infty} (h[v;\varpi])^{\lambda}\, dv\right\}; \quad \lambda > 0,\ \lambda \neq 1,$	Rényi [40]
2	H–C	$\Im_2 = \frac{1}{2^{1-\lambda}-1}\left\{\int_0^{\infty} (h[v;\varpi])^{\lambda}\, dv - 1\right\}; \quad \lambda > 0,\ \lambda \neq 1,$	Havrda and Charvát [41]
4	TS	$\Im_3 = \frac{1}{\lambda-1}\left\{1 - \int_0^{\infty} (h[v;\varpi])^{\lambda}\, dv\right\}; \quad \lambda > 0,\ \lambda \neq 1,$	Tsallis [42]
5	AR	$\Im_4 = \frac{\lambda}{1-\lambda}\left\{\left(\int_0^{\infty} [h(v;\varpi)]^{\lambda}\, dv\right)^{1/\lambda} - 1\right\}; \quad \lambda > 0,\ \lambda \neq 1,$	Arimoto [43]

To obtain these measurements of Table 1, the integral $I_N = \int_0^{\infty} (h[v;\varpi])^{\lambda}\, dv$ must be obtained first. Substituting the PDF given in Eq. (6) in the formula ($\Im_1$), then the formula I_N reduces to

$$I_N = \int_0^{\infty} \frac{(8ab)^{\lambda} v^{\lambda(b-1)} e^{\lambda a v^b}}{\pi^{2\lambda} \left(1+e^{av^b}\right)^{2\lambda} \left(1 + \left[1 - \frac{2}{1+e^{av^b}}\right]^2\right)^{\lambda}} dv. \quad (18)$$

Using binomial expansion given in Eq. (8) in integral of Eq. (18) gives:

$$I_N = \sum_{j=0}^{\infty} \binom{\lambda + j - 1}{j} \frac{(8ab)^{\lambda}}{\pi^{2\lambda}} \int_0^{\infty} \frac{v^{\lambda(b-1)} e^{\lambda a v^b}}{\left(1 + e^{a v^b}\right)^{2\lambda}} \left(1 - \frac{2}{1 + e^{a v^b}}\right)^{2j} dv. \tag{19}$$

Using the binomial expansion in Eq. (19) gives

$$I_N = \sum_{j,p=0}^{\infty} \binom{\lambda + j - 1}{j} \binom{2j}{p} \frac{(-1)^p 2^{3\lambda + p} (8)^{\lambda}}{\pi^{2\lambda}} \int_0^{\infty} \frac{(ab)^{\lambda} v^{\lambda(b-1)} e^{-(\lambda + p) a v^b}}{\left(1 + e^{-a v^b}\right)^{2\lambda + p}} dv. \tag{20}$$

Again, by employ expansion of Eq. (8) in the term $(1 + e^{-av^b})^{-(2\lambda+p)}$ of Eq. (20), then we have

$$I_N = \sum_{j,p,q=0}^{\infty} \Omega_{j,p,q} (ab)^{\lambda} \int_0^{\infty} v^{\lambda(b-1)} e^{-(\lambda + p + q) a v^b} dv, \tag{21}$$

where $\Omega_{j,p,q} = \binom{2j}{p} \binom{\lambda + j - 1}{j} \binom{2\lambda + p + q - 1}{q} \frac{(-1)^p 2^{3\lambda + p} (8ab)^{\lambda}}{\pi^{2\lambda}}$. Hence the integral I_N gets the following form

$$I_N = \sum_{j,p,q=0}^{\infty} \frac{\Omega_{j,p,q}\, a^{\lambda} b^{\lambda - 1}}{(a[\lambda + p + q])^{\frac{\lambda(b-1)+1}{b}}} \Gamma\left(\frac{\lambda(b-1)}{b} + \frac{1}{b}\right). \tag{22}$$

Finally, the RE of the ART–PHL distribution is obtained by inserting Eq. (22) in formula ($\Im_1$) given in Table 1 as follows:

$$\Im_1 = \frac{1}{1 - \lambda} \log\left\{ \sum_{j,p,q=0}^{\infty} \frac{\Omega_{j,p,q}\, a^{\lambda} b^{\lambda - 1}}{(a[\lambda + p + q])^{\frac{\lambda(b-1)+1}{b}}} \Gamma\left(\frac{\lambda(b-1)}{b} + \frac{1}{b}\right) \right\}.$$

Eq. (22) is inserted into the formula ($\Im_2$) provided in Table 1 to determine the H–C measure of the ART–PHL distribution:

$$\Im_2 = \frac{1}{2^{1-\lambda} - 1} \left\{ \sum_{j,p,q=0}^{\infty} \frac{\Omega_{j,p,q}\, a^{\lambda} b^{\lambda - 1}}{(a[\lambda + p + q])^{\frac{\lambda(b-1)+1}{b}}} \Gamma\left(\frac{\lambda(b-1)}{b} + \frac{1}{b}\right) - 1 \right\}.$$

To find the TS measure of the ART–PHL model, insert Eq. (22) into the formula ($\Im_3$) presented in Table 1:

$$\Im_3 = \frac{1}{\lambda - 1} \left\{ 1 - \sum_{j,p,q=0}^{\infty} \frac{\Omega_{j,p,q}\, a^{\lambda} b^{\lambda - 1}}{(a[\lambda + p + q])^{\frac{\lambda(b-1)+1}{b}}} \Gamma\left(\frac{\lambda(b-1)}{b} + \frac{1}{b}\right) \right\}.$$

Similarly, as illustrated below, the AR entropy is produced using Eq. (22) and formula ($\Im_4$) in Table 1:

$$\Im_4 = \frac{\lambda}{1 - \lambda} \left\{ \left(\sum_{j,p,q=0}^{\infty} \frac{\Omega_{j,p,q}\, a^{\lambda} b^{\lambda - 1}}{[a(\lambda + p + q)]^{\frac{\lambda(b-1)+1}{b}}} \Gamma\left(\frac{\lambda(b-1)}{b} + \frac{1}{b}\right) \right)^{1/\lambda} - 1 \right\}.$$

4 Maximum Likelihood Method

In this section, the ML estimation procedure is used to address the ART–PHL distribution parameter estimation problem, together with its SF and HRF based on the PT–IIC scheme. This section gives both point estimators and approximate confidence intervals (ACIs) for parameters, SF, and HRF.

Let $v_{1:m:n}, v_{2:m:n}, \ldots, v_{m:m:n}$, $1 \leq m \leq n$ represent the PT–IIC sample of size m extracted from a sample of n with scheme $S_1, S_2, \ldots, S_m$, $m < n$ derived from the ART–PHL distribution with CDF and PDF of Eqs. (5) and (6). The likelihood function under the PT–IIC sample is given by:

$$L(\boldsymbol{\varpi}; \underline{v}) = C \prod_{i=1}^{m} h(v_{i:m:n}; \boldsymbol{\varpi})(1 - H[v_{i:m:n}; \boldsymbol{\varpi}])^{S_i}, \quad m < n, \tag{23}$$

where $C = n(n-1-S_1)(n-2-S_1-S_2)\ldots(n-\sum_{i=1}^{m-1}(S_i+1))$. The likelihood function is obtained by inserting the expressions for the CDF and PDF from Eqs. (5) and (6) into Eq. (23) as follows:

$$L(\boldsymbol{\varpi}) \propto (ab)^m \prod_{i=1}^{m} \frac{v_i^{b-1} e^{av_i^b}}{(1+e^{av_i^b})^2\{1+(M[a,b])^2\}} \left\{1 - \frac{4}{\pi}\arctan(M[a,b])\right\}^{S_i}, \tag{24}$$

where $M(a,b) = \left(1 - \dfrac{2}{1+e^{av_i^b}}\right)$, and v_i is written for the simplified form of $v_{i:m:n}$. The log-likelihood of Eq. (24) is as below:

$$l(\boldsymbol{\varpi}) \propto m\ln(ab) + (b-1)\sum_{i=1}^{m}\ln v_i + a\sum_{i=1}^{m} v_i^b - 2\sum_{i=1}^{n}\ln(1+e^{av_i^b}) - \sum_{i=1}^{m}\ln\{1+(M[a,b])^2\}$$

$$+ \sum_{i=1}^{m} S_i \ln\left\{1 - \frac{4}{\pi}\arctan(M[a,b])\right\}. \tag{25}$$

Based on log-likelihood Eq. (25), the first partial derivatives of the parameters a and b are as follows:

$$\begin{aligned}\frac{\partial l(\varpi)}{\partial a} = \frac{m}{a} + \sum_{i=1}^{m} v_i^b - 2\sum_{i=1}^{n}\frac{v_i^b}{1+e^{-av_i^b}} - \sum_{i=1}^{m}\frac{2M(a,b)\,M'_a(a,b)}{1+\left(M(a,b)\right)^2} \\ - \sum_{i=1}^{m}\frac{S_i}{1-\frac{4}{\pi}\arctan(M(a,b))} \times \frac{4}{\pi}\frac{M'_a(a,b)}{1+\left(M(a,b)\right)^2}\end{aligned} \tag{26}$$

and,

$$\begin{aligned}\frac{\partial l(\boldsymbol{\varpi})}{\partial b} = \frac{m}{b} + \sum_{i=1}^{m}\ln v_i + a\sum_{i=1}^{m} v_i^b \ln v_i - 2\sum_{i=1}^{n}\frac{av_i^b\ln v_i}{1+e^{-av_i^b}} - \sum_{i=1}^{m}\frac{2M(a,b)M'_b(a,b)}{1+(M[a,b])^2} \\ - \sum_{i=1}^{m}\frac{4S_i}{\pi\left\{1-\frac{4}{\pi}\arctan(M[a,b])\right\}}\frac{M'_b(a,b)}{1+(M[a,b])^2},\end{aligned} \tag{27}$$

where $M'_a(a,b) = \dfrac{\partial M(a,b)}{\partial a} = \dfrac{2e^{av_i^b}v_i^b}{(1+e^{av_i^b})^2}$ and $M'_b(a,b) = \dfrac{\partial M(a,b)}{\partial b} = \dfrac{2ae^{av_i^b}v_i^b\ln v_i}{(1+e^{av_i^b})^2}$.

The estimators $\hat{a}$ of a and $\hat{b}$ of b will be computed numerically after setting the Eqs. (26) and (27) to zero using nonlinear optimization software due to the lack of a closed-form solution for these equations. Furthermore, the SF estimator $\hat{\kappa}(t; \boldsymbol{\varpi})$ of $\kappa(t; \boldsymbol{\varpi})$ and the HRF estimator $\hat{\eta}(t; \boldsymbol{\varpi})$ of $\eta(t; \boldsymbol{\varpi})$ are evaluated using the invariance property of the ML estimators:

$$\hat{\kappa}(t;\boldsymbol{\varpi}) = 1 - \frac{4}{\pi}\arctan\left(1 - \frac{2}{1+e^{\hat{a}t^{\hat{b}}}}\right),$$

and,

$$\hat{\eta}(t;\boldsymbol{\varpi}) = \frac{8\hat{a}\hat{b}t^{\hat{b}-1}e^{\hat{a}t^{\hat{b}}}}{\pi\left(1+e^{\hat{a}t^{\hat{b}}}\right)^2\left(1+\left[1-\frac{2}{1+e^{\hat{a}t^{\hat{b}}}}\right]^2\right)\left(1-\frac{4}{\pi}\arctan\left[1-\frac{2}{1+e^{\hat{a}t^{\hat{b}}}}\right]\right)}.$$

It is noteworthy to mention here that despite the incorporation of trigonometric generators in the ART-PHL distribution, the ML estimators of the parameters a and b are identifiable and unique. The identifiability of the parameters is ensured under standard regularity conditions of the log-likelihood function provided by Eq. (25) for the ART-PHL distribution. In fact, the ART-PHL distribution is a two-parameter family of densities, and the log-likelihood function is not invariant under distinct permutations of the parameters. The existence of the ML estimates (MLEs) of the parameters is visually confirmed by the global maxima of the log-likelihood profiles (see Fig. 2) and the well-separated unimodal log-likelihood surface plots over the parameter space (see Fig. 3) of the ART-PHL distribution. In fact, the log-likelihood surface plots over the parameter space (Fig. 3) of the ART-PHL distribution clearly indicate the presence of only one dominant peak.

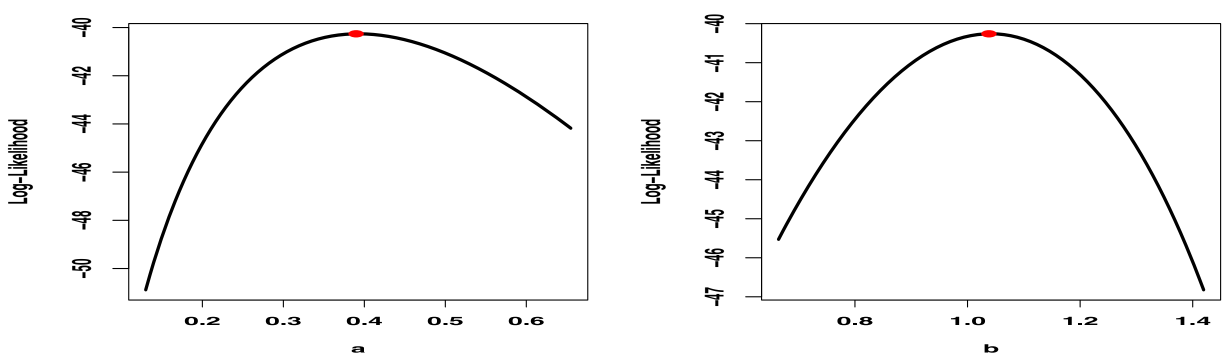

Figure 2: Global maxima of log-likelihood values by parameters in complete sample for the electronic tubes dataset

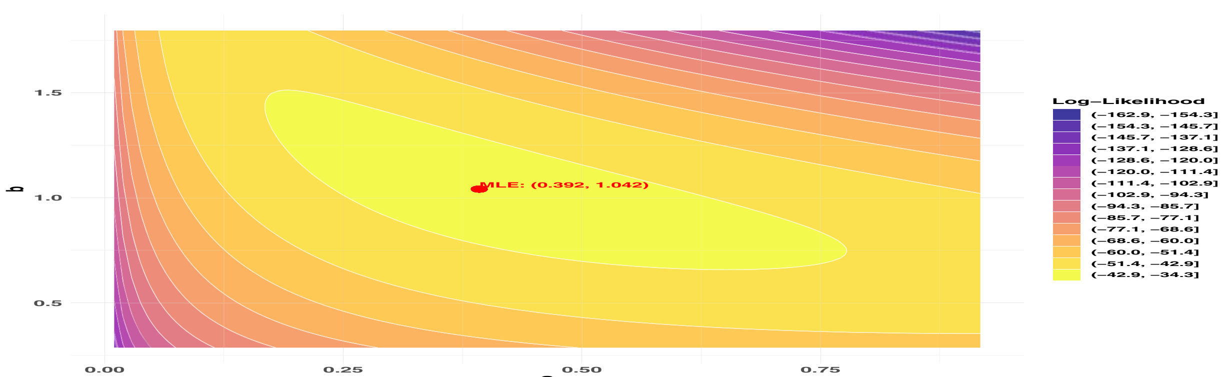

Figure 3: Contour plot for parameters of ART-PHL distribution for the electronic tubes dataset

The asymptotic normal distribution of the ML estimators is the most widely used technique for establishing confidence intervals for the parameters, as stated by Vander Wiel and Meeker [44]. The entries of the inverse of the Fisher information matrix (FIM) $I_{ij} = E\left((-\partial^2 l(\boldsymbol{\varpi}))/(\partial\varpi_i\partial\varpi_j)\right)$, $i,j = 1,2$ and $\boldsymbol{\varpi} = (\varpi_1,\varpi_2) = (a,b)$ provide the asymptotic variances and covariances of the ML estimators. However, it is challenging to find the exact closed forms for the aforementioned expectations. Consequently, the observed FIM $\hat{I}_{ij} = \left((-\partial^2 l(\boldsymbol{\varpi}))/(\partial\varpi_i\partial\varpi_j)\right)\Big|_{\boldsymbol{\varpi}=\hat{\boldsymbol{\varpi}}}$, which is obtained by dropping the expectation operator (see [45]). The second partial derivatives of the log-likelihood function are the entries in the observed FIM. Consequently, the observed FIM $\hat{I}(\hat{a},\hat{b})$ is inverted to yield the asymptotic variance–covariance matrix for the ML estimators.

$$\hat{I}^{-1}(\hat{a},\hat{b}) = \left.\begin{pmatrix} \dfrac{-\partial^2 l[\boldsymbol{\varpi}]}{\partial a^2} & \dfrac{-\partial^2 l[\boldsymbol{\varpi}]}{\partial a\partial b} \\ \dfrac{-\partial^2 l[\boldsymbol{\varpi}]}{\partial b\partial a} & \dfrac{-\partial^2 l[\boldsymbol{\varpi}]}{\partial b^2} \end{pmatrix}^{-1}\right|_{\hat{\boldsymbol{\varpi}}=(\hat{a},\hat{b})} = \begin{pmatrix} \mathrm{var}[\hat{a}] & \mathrm{cov}[\hat{a},\hat{b}] \\ \mathrm{cov}[\hat{b},\hat{a}] & \mathrm{var}[\hat{b}] \end{pmatrix}_{a=\hat{a},b=\hat{b}}.$$

As indicated by Lawless [46], under certain regularity conditions, $(\hat{a},\hat{b})$ is roughly distributed as a multivariate normal with mean and covariance matrix $\hat{I}^{-1}(\hat{a},\hat{b})$. It is now possible to get the $100(1-\delta)\%$ ACIs for parameters a and b as shown below: $\hat{a} \pm z_{\delta/2}\sqrt{\mathrm{var}(\hat{a})}$, and $\hat{b} \pm z_{\delta/2}\sqrt{\mathrm{var}(\hat{b})}$, where $z_{\delta/2}$ is the upper $(\delta/2)$th percentile of the standard normal distribution.

In addition, to construct the ACIs of SF ($\kappa(t;\boldsymbol{\varpi})$) and HRF ($\eta(t;\boldsymbol{\varpi})$), we must obtain their variances. As a result, we employ the delta method described by Greene [47] to obtain approximate estimates of the variance of $\hat{\kappa}(t;\boldsymbol{\varpi})$ and $\hat{\eta}(t;\boldsymbol{\varpi})$. Using this method, the variance of $\hat{\kappa}(t;\boldsymbol{\varpi})$ and $\hat{\eta}(t;\boldsymbol{\varpi})$ can be roughly calculated by $\mathrm{var}(\hat{\kappa}[t;\boldsymbol{\varpi}]) = (D)^T(\hat{I}^{-1}[\hat{a},\hat{b}])(D)$, and $\mathrm{var}(\hat{\eta}[t;\boldsymbol{\varpi}]) = (D^*)^T(\hat{I}^{-1}[\hat{a},\hat{b}])(D^*)$,

where $D = \left(\dfrac{\partial\kappa[t;\boldsymbol{\varpi}]}{\partial a}, \dfrac{\partial\kappa[t;\boldsymbol{\varpi}]}{\partial b}\right)$, $D^* = \left(\dfrac{\partial\eta[t;\boldsymbol{\varpi}]}{\partial a}, \dfrac{\partial\eta[t;\boldsymbol{\varpi}]}{\partial b}\right)$, and $\hat{I}^{-1}(\hat{a},\hat{b})$ is the approximation of covariance matrix. The two-sided $100(1-\delta)\%$ ACI of $\kappa(t;\boldsymbol{\varpi})$ and $\eta(t;\boldsymbol{\varpi})$ are determined as follows: $\hat{\kappa}(t;\boldsymbol{\varpi}) \pm z_{\delta/2}\sqrt{\mathrm{var}(\hat{\kappa}[t;\boldsymbol{\varpi}])}$ and $\hat{\eta}(t;\boldsymbol{\varpi}) \pm z_{\delta/2}\sqrt{\mathrm{var}(\hat{\kappa}[t;\boldsymbol{\varpi}])}$.

5 Bayesian Estimation

This section includes Bayesian estimators along with corresponding credible intervals of the unknown parameters (a,b), SF, and HRF based on the PT–IIC scheme. The Bayesian estimators of parameters, SF, and HRF are discussed in two scenarios, gamma (informative, IP) and uniform (non-IP, NIP), under different loss functions, including the squared error loss function (SELF), linear exponential loss function (LLX), and minimum expected loss function (MLF). Assuming that a and b are independent parameters that follow gamma prior distributions. The joint prior distribution of parameters is given by

$$g(a,b) \propto a^{\vartheta_1-1}b^{\vartheta_2-1}e^{-(\beta_1 a+\beta_2 b)}; \quad \vartheta_i,\beta_i > 0,\ i = 1,2, \tag{28}$$

where ϑ_i and β_i, $i = 1,2$ are the hyper-parameters. The Bayesian estimators are obtained in the case of NIP when the chosen hyperparameters approach zero. Combining the likelihood function (24) with the prior (28) via Bayes' theorem yields the posterior distribution of the parameters a and b, as given below:

$$\Pi(a, b \mid \underset{\sim}{v}) \propto a^{\vartheta_1+m-1} b^{\vartheta_2+m-1} \exp\left(-b\left[\beta_2 - \sum_{i=1}^{m} \ln v_i\right] - a\left[\beta_1 - \sum_{i=1}^{m} v_i^b\right]\right)$$
$$\times \prod_{i=1}^{m} \left\{1 + (M[a,b])^2\right\}^{-1} \left(1 + e^{a v_i^b}\right)^{-2} \left\{1 - \frac{4}{\pi} \arctan(M[a,b])\right\}^{S_i}.$$

The conditional posterior densities of a and b can be written as follows:

$$\Pi_1(a \mid b, \underset{\sim}{v}) \propto a^{\vartheta_1+m-1} e^{-a\left(\beta_1 - \sum_{i=1}^{m} v_i^b\right)} \prod_{i=1}^{m} \left\{1 + (M[a,b])^2\right\} \left(1 + e^{a v_i^b}\right)^{-2} \left\{1 - \frac{4}{\pi} \arctan(M[a,b])\right\}^{S_i}, \tag{29}$$

and,

$$\Pi_2(b \mid a, \underset{\sim}{v}) \propto b^{\vartheta_2+m-1} e^{-b\left(\beta_2 - \sum_{i=1}^{m} \ln v_i\right)} \prod_{i=1}^{m} \left\{1 + (M[a,b])^2\right\} \left(1 + e^{a v_i^b}\right)^{-2} \left\{1 - \frac{4}{\pi} \arctan(M[a,b])\right\}^{S_i}. \tag{30}$$

The Metropolis-Hastings (M–H) algorithm must be used to generate a and b from the conditional distributions in Eqs. (29) and (30), since they cannot be reduced to well-known distributions using conventional methods. For more information, see [48] and [49]. Here, the goal is to reduce the rejection rate as much as feasible. The following algorithm relies on the M–H algorithm, which is based on selecting the normal distribution as a proposal distribution. It is used to construct the credible intervals for a and b as well as to find the Bayesian estimators. The following algorithm is required in order to apply the M–H technique:

1. Set initial value of $\boldsymbol{\varpi}$ as $\boldsymbol{\varpi} = (a^{(0)}, b^{(0)}) = (\hat{a}, \hat{b})$.
2. Set the counter i to 1.
3. Generate a^* and b^* from normal distribution $N(\hat{a}, \hat{\sigma}_{\hat{a}}^2)$ and $N(\hat{b}, \hat{\sigma}_{\hat{b}}^2)$, respectively.
4. Compute $\psi_1 = \dfrac{\pi(a^* \mid b^{(i-1)}, \nu)}{\pi(a^{(i-1)} \mid b^{(i-1)}, \nu)}$ and $\psi_2 = \dfrac{\pi(b^* \mid a^{(i-1)}, \nu)}{\pi(b^{(i-1)} \mid a^{(i-1)}, \nu)}$.
5. Generate random samples u_1 and u_2 from a uniform distribution $U(0, 1)$.
6. If both $u_1 < \min(1, \psi_1)$ and $u_2 < \min(1, \psi_2)$ set $a^{(i)} = a^*$ and $b^{(i)} = b^*$; otherwise, set $a^{(i)} = a^{(i-1)}$ and $b^{(i)} = b^{(i-1)}$.
7. Obtain $\kappa^i(t; \boldsymbol{\varpi})$ and $\eta^i(t; \boldsymbol{\varpi})$.
8. Repeat steps 2–7 H times to obtain $\boldsymbol{\varpi}^{(i)} \equiv (a^{(i)}, b^{(i)}, \kappa^{(i)}[t; \boldsymbol{\varpi}], \eta^{(i)}[t; \boldsymbol{\varpi}])$ for $i = 1, 2, \ldots, H$.
9. Find the Bayes estimate (BE) of $\boldsymbol{\varpi} \equiv (a, b, \kappa[t; \boldsymbol{\varpi}], \eta[t; \boldsymbol{\varpi}])$ under SELF, LLX, and MLF as

$$\tilde{\boldsymbol{\varpi}}_{SELF} = \frac{1}{H - H_0} \sum_{i=H_0+1}^{H} \tilde{\boldsymbol{\varpi}}^{(i)}, \quad \tilde{\boldsymbol{\varpi}}_{LLF} = -\frac{1}{\tau} \ln\left(\frac{1}{H - H_0} \sum_{i=H_0+1}^{H} e^{-\tau \tilde{\boldsymbol{\varpi}}^{(i)}}\right),$$

$$\tilde{\boldsymbol{\varpi}}_{MLF} = \frac{\left(\frac{1}{H-H_0} \sum_{i=H_0+1}^{H} \tilde{\boldsymbol{\varpi}}^{(i)}\right)^{-1}}{\left(\frac{1}{H-H_0} \sum_{i=H_0+1}^{H} \tilde{\boldsymbol{\varpi}}^{(i)}\right)^{-2}},$$

where H is the total number of MCMC iterations and H_0 is the burn-in period.

10. Obtain the Bayesian credible interval (BCI) via loss functions based on $\boldsymbol{\varpi} \equiv (a, b, \kappa[t; \boldsymbol{\varpi}], \eta[t; \boldsymbol{\varpi}])$ as
    (a) Arrange $\boldsymbol{\varpi}^{(i)}$ as $\boldsymbol{\varpi}^{[H_0+1]}, \boldsymbol{\varpi}^{[H_0+2]}, \ldots, \boldsymbol{\varpi}^{[H]}$.
    (b) Obtain the $100(1-\delta)\%$ BCI as $\left\{\boldsymbol{\varpi}^{([H-H_0]\delta/2)}, \boldsymbol{\varpi}^{([H-H_0][1-\delta/2])}\right\}$.

6 Real Data Analysis

In order to provide illustrative examples and evaluate the statistical efficiency of ML and Bayesian estimation methods for the ART–PHL distribution under different PT–IIC schemes, an actual dataset is examined. This dataset includes twenty lifetime values (in hundreds of hours) of electronic tubes from the engineering field. The objective of an electron tube is to use electrons in a sealed container constructed of glass or metal to conduct electricity in a vacuum or gas. Ref. [50] initially published this data set, as follows: 0.1415, 0.5937, 2.3467, 3.1356, 3.5681, 0.3484, 1.1045, 2.4651, 3.2259, 3.7287, 0.3994, 1.7323, 2.6155, 3.4177, 9.2817, 0.4174, 1.8348, 2.7425, 3.5551, 9.3208.

A comprehensive data inspection is provided through the descriptive statistics in Table 2.

Table 2: Descriptive analysis in the dataset

Dataset	Sample size	Mean	Q (1/2)	Variance	SK	KU	Range	Minimum	Maximum
	20	2.799	2.540	6.417	1.572	5.063	9.179	0.142	9.321

It is crucial to first determine whether the ART–PHL distribution is appropriate for the given data set. The MLEs and their standard errors (SEs) for the parameters (a, b) are calculated. The Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS) test statistic, Anderson-Darling (A^*) test statistic, and Cramer-von Mises (W^*) test statistic, and their corresponding p-values are among the goodness-of-fit criteria that are assessed. By using these criteria, the ART–PHL distribution's suitability is evaluated in relation to other distributions, including the exponentiated HL distribution (EHL), transmuted–HL (THL) distribution,SPHL distribution, Poisson generalized HL (PGHL) distribution (PGHL) [51], generalized half–logistic Poisson distribution (GHLP) [52], and PHL distribution. It shows that the ART–PHL distribution has the greatest p-value and the lowest values for other statistics; thus, the ART–PHL model fits the electronic tubes data set quite satisfactorily and is the best choice among others. The goodness-of-fit statistics and estimated parameters are shown in Tables 3 and 4.

Table 3: MLEs with their SEs of the electronic tubes data

Distribution	$\hat{a}$	SE($\hat{a}$)	$\hat{b}$	SE($\hat{b}$)	$\hat{c}$	SE($\hat{c}$)
ART-PHL	0.3924	0.0221	1.042	0.0888	–	–
EHL	0.4980	0.0411	1.0037	0.1102	–	–
THL	2.0119	0.0030	1.6651	0.1383	–	–
SPHL	0.3299	0.0221	0.9422	0.0887	–	–
PGHL	37.8827	0.1121	0.0287	0.2175	1.1466	0.0446
GHLP	0.4341	0.4270	0.9127	0.8591	1.1406	0.5157
PHL	0.4979	0.0411	1.0037	0.1102	–	–

Table 4: Model adequacy and goodness of fit test statistics for the electronic tubes data

Distribution	AIC	BIC	CAIC	HQIC	A^*	p-value	W^*	p-value	KS	p-value
ART–PHL	84.5189	86.5104	90.5104	84.9077	0.5756	0.6693	0.0902	0.6402	0.1673	0.5733
EHL	84.6194	86.6109	90.6109	85.0082	0.5818	0.6632	0.0972	0.6362	0.1710	0.5458
THL	84.6196	86.6111	90.6111	85.0084	0.5824	0.6645	0.0973	0.6378	0.1711	0.5432
SPHL	84.7829	86.7744	90.7744	85.1717	1.2970	0.2331	0.2511	0.1872	0.2942	0.0499
PGHL	86.6107	89.5979	95.5979	87.1938	0.5913	0.6541	0.0961	0.6093	0.1682	0.5663
GHLP	86.5429	89.5301	95.5301	87.1260	0.5958	0.6594	0.0948	0.6159	0.1678	0.5694
PHL	84.6194	86.6109	90.6109	85.0082	0.5818	0.6632	0.0972	0.6362	0.1710	0.5458

Fig. 2 shows the log-likelihood function of the parameters for the ART-PHL distribution as an example, which shows the existence and uniqueness of MLEs. Fig. 3 provide the contour plots of the log-likelihood function of a and b based on the complete dataset to solve this problem. It indicates that the most suitable starting values of a and b are close to $\hat{a}$ and $\hat{b}$, respectively. In addition, it indicates that the MLEs $\hat{a}$ and $\hat{b}$ exist and are unique. Fig. 4 displays the fitted density and CDF of the given data set. It is noticeable that the computed curves suit the empirical objects quite well.

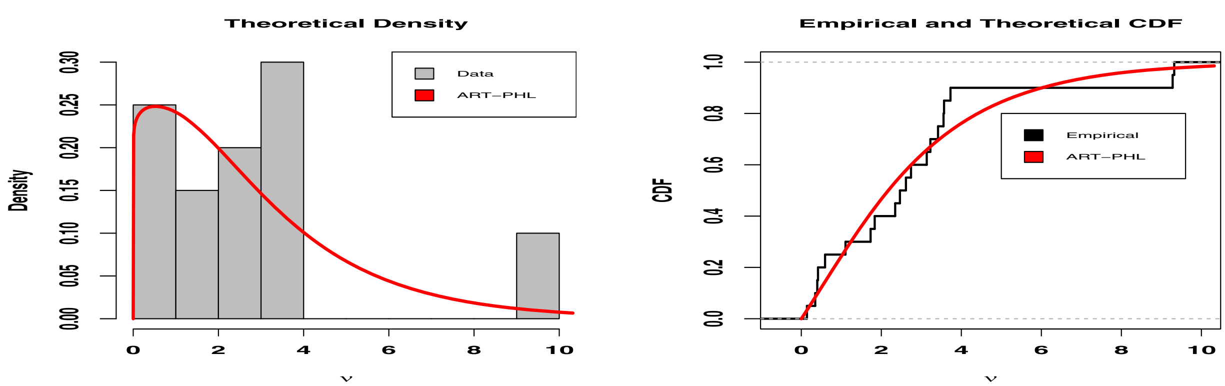

Figure 4: Fitted density and CDF of the ART–PHL distribution for the electronic tubes dataset

Using the original dataset, we generate some PT–IIC samples with $m = 10$, using three different sampling schemes. The generated data and the corresponding censored schemes are reported as follows:

- **Scheme I:** 0.1415 2.7425 3.1356 3.2259 3.4177 3.5551 3.5681 3.7287 9.2817 9.3208, and **S**: 10, 0, 0, 0, 0, 0, 0, 0, 0.
- **Scheme II:** 0.1415 0.3484 0.3994 0.4174 0.5937 1.1045 1.7323 1.8348 2.3467 2.4651, and **S**: 0, 0, 0, 0, 0, 0, 0, 0, 0, 10.
- **Scheme III:** 0.1415 1.7323 3.1356 3.2259 3.4177 3.5551 3.5681 3.7287 9.2817 9.3208, and **S**: 5, 5, 0, 0, 0, 0, 0, 0, 0, 0.

For each PT–IIC data, the point and interval estimators of a, b, $\kappa(t;\varpi)$ and $\eta(t;\varpi)$ at $t = 2$ are obtained; see Tables 5 and 6, respectively. Since we have no prior information about ART–PHL distribution, the Bayesian results are approximated using the M–H algorithm based on NIP, i.e., $\vartheta_i =$

$\beta_i = 0.0001$, $i = 1, 2$. To run the MCMC sampler, the initial guess of a and b is taken as MLE, and then the first 2000 (of 10,000) iterations are taken as burn-in. Tables 5 and 6 indicate that the computed estimates of parameters, SF, and HRF based on the MCMC approaches performed more satisfactorily than those obtained based on the classical approach with respect to minimum SE and average interval length (AIL) values. Plots of parameter (a) and SF diagnostics are shown in Figs. 5 and 6. Following the burn-in period, the trace plot shows good mixing. It appears that the posterior distribution for parameter (a) is right-skewed and unimodal and for SF is fairly symmetrical and unimodal; also, the cumulative mean stabilizes, indicating convergence. The decreasing autocorrelation, which implies that the samples become more independent with time, adds even more credibility to the analysis. Overall, the MCMC chain appears to be reliable for estimation and posterior inference.

Table 5: MLE and BEs with different schemes for the electronic tubes dataset

Shemes	m		ML		Bayesian							
					SELF		LLX ($\tau = -2$)		LLX ($\tau = 2$)		MLF	
		Parameter	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
I	10	a	0.154525	0.095666	0.636830	0.004856	0.638358	0.004828	0.635282	0.004887	0.631754	0.005013
		b	1.351447	0.321236	0.872928	0.005884	0.874009	0.005861	0.871847	0.005906	0.870395	0.005959
		$\kappa(2)$	0.755244	0.100728	0.386421	0.003437	0.387248	0.003465	0.385604	0.003411	0.382369	0.003348
		$\eta(2)$	0.208193	0.074345	0.480699	0.006424	0.482774	0.006401	0.478622	0.006451	0.471649	0.006665
	20	a	0.392483	0.131207	0.649322	0.004720	0.650623	0.004703	0.648006	0.004740	0.645116	0.004830
		b	1.041792	0.188735	0.882629	0.005662	0.883679	0.005649	0.881580	0.005675	0.880216	0.005714
		$\kappa(2)$	0.533868	0.092656	0.375222	0.003164	0.375873	0.003183	0.374580	0.003146	0.371925	0.003105
		$\eta(2)$	0.373276	0.087839	0.500007	0.006076	0.501835	0.006065	0.498178	0.006091	0.492410	0.006239
II	10	a	0.374902	0.135415	0.384164	0.007212	0.397541	0.007826	0.371633	0.006685	0.319152	0.006099
		b	0.932219	0.282921	0.917340	0.018856	0.956288	0.019962	0.878260	0.017706	0.825619	0.018263
		$\kappa(2)$	0.579081	0.100753	0.587621	0.003415	0.595795	0.003281	0.579375	0.003570	0.556970	0.004365
		$\eta(2)$	0.289333	0.121789	0.282381	0.005687	0.295032	0.006261	0.271143	0.005255	0.208836	0.005040
	20	a	0.392483	0.131207	0.414981	0.007010	0.428687	0.007602	0.401955	0.006453	0.353170	0.006015
		b	1.041792	0.188735	1.019256	0.010071	1.041892	0.010194	0.996967	0.010074	0.973644	0.010646
		$\kappa(2)$	0.533868	0.092656	0.533478	0.003735	0.540060	0.003652	0.526939	0.003819	0.507635	0.004268
		$\eta(2)$	0.373276	0.087839	0.364066	0.002202	0.371430	0.002322	0.357089	0.002114	0.325863	0.002293
III	10	a	0.129273	0.077458	0.205425	0.006211	0.212976	0.006575	0.198401	0.005871	0.145879	0.005194
		b	1.414659	0.318470	1.188508	0.017984	1.232865	0.018778	1.143292	0.017369	1.107516	0.018115
		$\kappa(2)$	0.784815	0.086567	0.731779	0.004452	0.737831	0.004323	0.725503	0.004589	0.713080	0.005017
		$\eta(2)$	0.186553	0.065449	0.197995	0.001812	0.201988	0.001914	0.194194	0.001725	0.160418	0.001780
	20	a	0.392483	0.094762	0.414981	0.007010	0.428687	0.007602	0.401955	0.006453	0.353170	0.006015
		b	1.041792	0.217313	1.019256	0.010071	1.041892	0.010194	0.996967	0.010074	0.973644	0.010646
		$\kappa(2)$	0.533868	0.085605	0.533478	0.003735	0.540060	0.003652	0.526939	0.003819	0.507635	0.004268
		$\eta(2)$	0.373276	0.089219	0.364066	0.002202	0.371430	0.002322	0.357089	0.002114	0.325863	0.002293

Table 6: The interval and AIL of a, b, $\kappa(\nu;\boldsymbol{\varpi})$, and $\eta(\nu;\boldsymbol{\varpi})$ at different schemes for the electronic tubes dataset

Shemes	m	ML			Bayesian		
		Lower	Upper	AIL	Lower	Upper	AIL
I	10	0.043146	0.248029	0.335264	0.130105	0.172276	0.042171
		0.770002	2.258999	1.488997	1.381813	1.602755	0.220942
		0.586401	0.970020	0.383619	0.744352	0.784879	0.040527
		0.058684	0.355482	0.296798	0.196891	0.217799	0.020907
	20	0.073065	0.464601	0.391537	0.270234	0.295426	0.025192
		0.883336	1.758237	0.874901	1.278644	1.342944	0.064300
		0.427714	0.775094	0.347381	0.593386	0.610773	0.017388
		0.202104	0.558708	0.356604	0.369323	0.378786	0.009463
II	10	0.081514	3.131460	3.049946	1.540398	1.950895	0.410497
		0.684872	2.862126	2.177253	1.691139	1.891289	0.200150
		−0.041467	0.051955	0.093422	0.041580	0.065668	0.024088
		−5.886795	15.627257	21.514052	5.777322	5.987535	0.210213
	20	0.488084	1.114851	0.626767	0.787147	0.807074	0.019927
		0.698857	1.675407	0.976550	1.167318	1.209955	0.042637
		0.031463	0.375597	0.344134	0.213790	0.225249	0.011459
		0.202102	1.927622	1.725519	1.065027	1.139847	0.074820
III	10	−0.021735	0.251497	0.273231	0.119231	0.150822	0.031590
		0.864483	2.205252	1.340769	1.416182	1.600595	0.184413
		0.629680	0.954100	0.324420	0.766965	0.797047	0.030082
		0.062275	0.326720	0.264446	0.186887	0.201517	0.014630
	20	0.077633	0.449092	0.371458	0.263952	0.286895	0.022942
		0.898234	1.750087	0.851853	1.283786	1.344184	0.060398
		0.439899	0.775464	0.335565	0.600433	0.616525	0.016092
		0.198382	0.548115	0.349732	0.362285	0.371356	0.009071

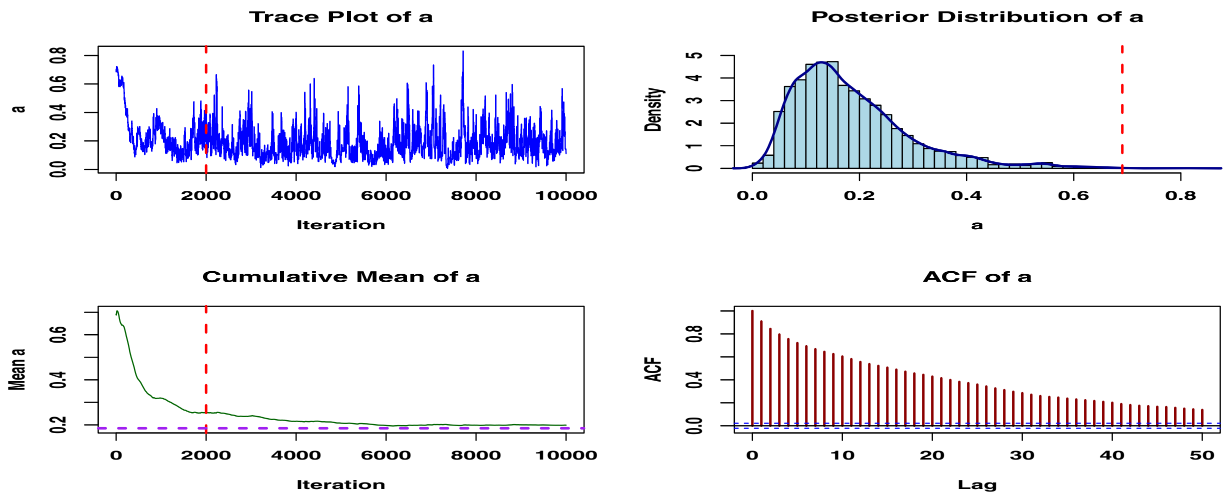

Figure 5: The MCMC diagnostic plots of a estimate for the electronic tubes dataset

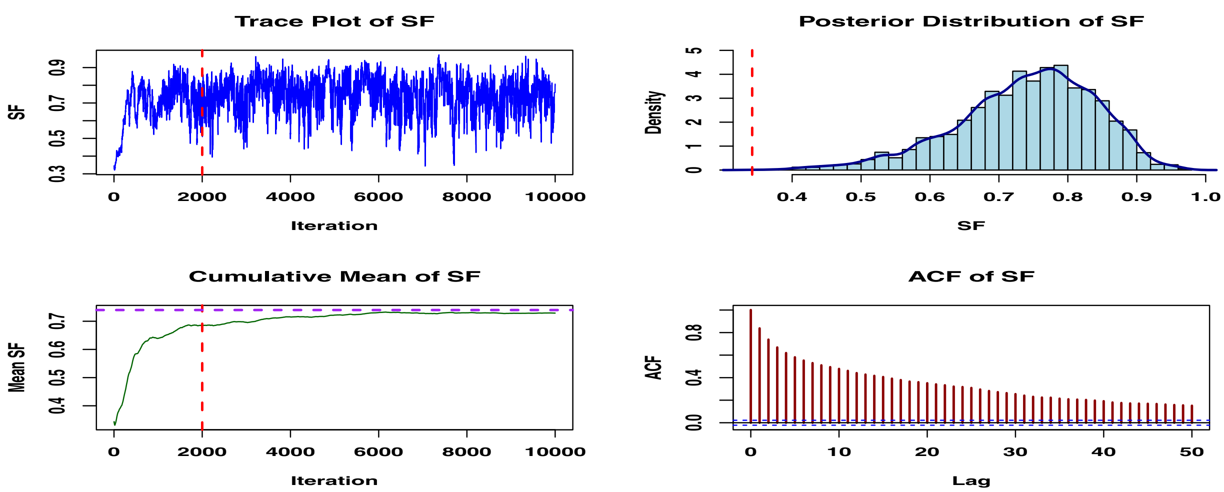

Figure 6: The MCMC diagnostic plots of SF estimate for the electronic tubes dataset

7 Evaluating Outcomes through Simulation

In this section, the effectiveness of the proposed estimating techniques is demonstrated. According to the algorithm proposed by [53] for various levels of n (total test items), m (number of failures), and **S** (progressive design), large 1000 PT–IIC samples are simulated. For the ART–PHL distribution, two distinct parameter scenarios are considered as:

Set1 ($\equiv a = 0.5, b = 1.5$) and Set2 ($\equiv a = 2.5, b = 2.0$). The selected parameter values are designed to identify decreasing, increasing, and non-monotone-shaped hazard rate functions that are widely encountered in reliability and survival data. In particular, small values of the shape parameter are used to model early life failure behavior, while larger values correspond to aging and wear-out mechanisms. By tuning the scale parameter, we can investigate the impact of time scaling on estimator performance in short- and long-time systems. These options avoid an artificial setting and take more realistic modeling into account.

For each set of parameters and using different $n = 50, 100, 150$, different m and $\mathbf{S}$, the MLEs and BEs using MCMC under SELF, LLX at $\tau = (-2, 2)$ and MLF of a, b, $\kappa(t;\varpi)$ and $\eta(t;\varpi)$ are calculated. We use mission time $t = 0.8$ for the estimates of the SF and HRF.

Three progressive modes are also taken into consideration in order to evaluate the consequences of the removals S_i, $i = 1, 2, \ldots, m$, for given (n, m), specifically:

Scheme I: Early removal $S_1 = n - m$, $S_2 = S_3 = \ldots = S_m = 0$, simulates studies where units are removed early (e.g., pilot testing, early withdrawal).

Scheme II: Delay in removal $S_1 = S_2 = \ldots = S_{m-1} = 0$, $S_m = n - m$ (TIIC), it corresponds to the classical life-testing wherein all units left over are removed at the mth failure.

Scheme III: Equally spaced removal $S_1 = S_2 = (n - m)/2$, $S_3 = S_4 = \ldots = S_m = 0$, this scheme corresponds to scenarios with non-overlapping removals (e.g., periodic inspection or multi-stage experiment).

Once the simulated PT–IIC samples are obtained, the classical (point and 95% interval) estimates are provided. In BEs, after 10,000 MCMC sample iterations, the first 2000 observations are removed to ignore the effect of the selection of initial guesses. The acceptance rates of the M–H algorithm were tracked in all simulations. An acceptance rate of 20% to 50% is typically considered optimal for successful sampling. In our analysis, the average acceptance rate was around 30%, which falls within the recommended range and ensures proper chain mixing.

To calculate the appropriate hyper-parameters for the independent joint prior, two scenarios for prior distributions are considered. In the first scenario, an IP is employed where the estimates and the variance-covariance matrix from the ML technique are obtained. Ref. [54] referred to this technique as the elicitation of hyperparameters. Equating the gamma priors' mean and variance yields the following expression for the estimated hyper-parameters:

$$\frac{\vartheta_i}{\beta_i} = H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i, \quad \frac{\vartheta_i}{\beta_i^2} = (H-1)^{-1}\sum_{i=1}^{H}\left(\hat{\varpi}^i - H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i\right)^2.$$

Now, on solving the above two equations, the estimated hyperparameters can be written as:

$$\vartheta_i = \frac{\left(H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i\right)^2}{(H-1)^{-1}\sum_{i=1}^{H}\left(\hat{\varpi}^i - H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i\right)^2}, \quad \text{and} \quad \beta_i = \frac{H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i}{(H-1)^{-1}\sum_{i=1}^{H}\left(\hat{\varpi}^i - H^{-1}\sum_{i=1}^{H}\hat{\varpi}^i\right)^2}.$$

In the second scenario, an NIP is considered; where hyper-parameter values are set to $\vartheta_1 = \vartheta_2 = \beta_1 = \beta_2 = 0.0001$. To analyze the performance of the theoretical results of a, b, $\kappa(t;\varpi)$ and $\eta(t;\varpi)$, in terms of their values, the absolute biases (ABs) and the mean squared errors (MSEs) are computed. Furthermore, AIL and coverage probability (CP) in % are used to compare intervals when $\delta = 5\%$. All computations were performed using the R 4.4.2 program. The outcomes of the simulation are displayed in Tables 7–15. Taking into account the simulation findings that were achieved, the following conclusions can be drawn:

- The precision of all obtained estimates increases as n (or m) rises or $\sum_{i=1}^{m} S_i$ decreases (see Tables 7–15).
- Comparing the proposed point approaches, it is evident that:
    1. The Bayesian approach in both IP and NIP outperforms the ML approach in terms of ABs, MSEs, and AIL for estimating a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$ (see Tables 7–13).

2. The performance of the $\kappa(t;\varpi)$ estimate behaved well in both Bayesian and ML approaches across all censoring schemes compared to other estimates.

- Comparing the interval approaches, it is evident that: The BCI approach in both IP and NIP of a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$, has overall lower AILs and higher CPs compared to those obtained using the ACI approach in most cases.
- It is noteworthy that most of the CPs were high, ranging around 95%.
- In most cases, the BEs of a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$ under LLX for $\tau = 2$ perform better than those under SELF, LLX for $\tau = -2$ and MLF. Whereas the BEs of a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$ under MLF are the worst among all methods (see Tables 8–13).
- In the majority of cases, the CPs of the ACI estimates are closer to the nominal level, while the CPs of the BCI estimates are slightly above the nominal level.
- Comparing the proposed schemes, it is clear that the point estimates of a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$ for both the ML and the Bayesian approaches in Scheme III behaved better than others in most circumscriptions.
- In the majority of cases, regarding the interval estimates of a, b, $\kappa(t;\varpi)$, and $\eta(t;\varpi)$, the associated asymptotic/credible interval estimates, they become well under Scheme III compared to others.
- The efficiency of the BEs of the ART–PHL distribution parameters is higher than that of the MLEs.
- Larger sample sizes may result in more accurate estimates, but they may also increase computing needs. Some estimate methods, especially those that use MCMC, may take longer and use more computer power when dealing with bigger samples.
- The BEs based on LLX with positive weights yields superior BEs compared to those utilizing negative weights.

Table 7: ABs, MSEs, AIL, and CP for ART–PHL distribution for MLEs at different estimates, SF, and HRF

			Scheme I							
n	m	Estimate	$set1 = (a = 0.5, b = 1.5)$				$set2 = (a = 2.5, b = 2)$			
			AB	MSE	AIL	CP	AB	MSE	AIL	CP
50	20	a	0.0108	0.0216	0.5751	95.9	0.2378	0.6739	3.0816	96.3
		b	0.0722	0.0785	1.0616	96.3	0.0963	0.1395	1.4155	96.3
		$\kappa(t;\varpi)$	0.0022	0.0048	0.2709	97.5	0.0083	0.0069	0.3329	97.5
		$\eta(t;\varpi)$	0.0179	0.0234	0.6421	97.5	0.4936	0.8133	5.7847	97.5
	30	a	0.0035	0.0147	0.4749	97	0.1524	0.3364	2.1948	94.9
		b	0.0716	0.0621	0.9361	96.2	0.0955	0.1104	1.248	96.2
		$\kappa(t;\varpi)$	0.0038	0.0034	0.2239	97.5	0.0036	0.0047	0.2712	97.5
		$\eta(t;\varpi)$	0.0023	0.0147	0.4651	97.5	0.3433	0.6411	4.6025	97.5

(Continued)

Table 7 (continued)

100	20	a	0.0164	0.0203	0.5548	96	0.2237	0.6284	2.9824	95.8
		b	0.0544	0.0635	0.9652	96.5	0.0726	0.113	1.287	96.5
		$\kappa(t;\varpi)$	0.0051	0.0044	0.2634	97.5	0.0089	0.0068	0.3303	97.5
		$\eta(t;\varpi)$	0.0233	0.023	0.6411	97.5	0.4478	0.5276	5.5696	97.5
	50	a	0.0006	0.0089	0.3701	97	0.0752	0.1577	1.5292	95.5
		b	0.0355	0.0295	0.6586	96.2	0.0473	0.0524	0.8782	96.2
		$\kappa(t;\varpi)$	0.0014	0.0021	0.1751	97.5	0.0014	0.0029	0.2042	97.5
		$\eta(t;\varpi)$	0.0023	0.0088	0.3735	97.5	0.1661	0.487	3.1491	97.5
	70	a	0.0002	0.0059	0.3023	97	0.0577	0.103	1.2382	95.3
		b	0.026	0.023	0.5856	96.4	0.0347	0.0408	0.7808	96.4
		$\kappa(t;\varpi)$	0.0009	0.0014	0.1458	97.5	0.0019	0.0018	0.1682	97.5
		$\eta(t;\varpi)$	0.0016	0.0055	0.2994	97.5	0.1276	0.4403	2.5616	97.5
150	50	a	0.0007	0.0086	0.3639	97	0.0736	0.1563	1.5233	95.5
		b	0.032	0.0274	0.6367	96.3	0.0426	0.0487	0.849	96.3
		$\kappa(t;\varpi)$	0.0007	0.002	0.1726	97.5	0.0016	0.0028	0.2036	97.5
		$\eta(t;\varpi)$	0.0034	0.0086	0.367	97.5	0.1595	0.615	3.1099	97.5
	70	a	0.0006	0.0057	0.2966	97	0.0569	0.1026	1.2361	95.5
		b	0.0237	0.0216	0.5684	96.4	0.0317	0.0384	0.7579	96.4
		$\kappa(t;\varpi)$	0.0005	0.0014	0.1421	97.5	0.002	0.0018	0.1687	97.5
		$\eta(t;\varpi)$	0.0024	0.0054	0.2954	97.5	0.1237	0.4332	2.5642	97.5
	100	a	0.0006	0.0046	0.2655	97.3	0.0402	0.0649	0.987	95.8
		b	0.0203	0.0161	0.4916	96.6	0.027	0.0287	0.6555	96.6
		$\kappa(t;\varpi)$	0.0009	0.0011	0.1282	97.5	0.0012	0.0013	0.1426	97.5
		$\eta(t;\varpi)$	0.0008	0.0042	0.259	97.5	0.0899	0.2682	2.0237	97.5
	130	a	0.0019	0.0034	0.2286	97.5	0.0309	0.048	0.8506	96.3
		b	0.011	0.0121	0.4292	97.1	0.0147	0.0215	0.5723	97.1
		$\kappa(t;\varpi)$	0.0005	0.0008	0.1152	97.5	0.0015	0.001	0.1207	97.5
		$\eta(t;\varpi)$	0.0024	0.003	0.208	97.5	0.0633	0.201	1.7388	97.5
						Scheme II				
n	m	Estimate	$set1=(a=0.5,b=1.5)$				$set1=(a=2.5,b=2)$			
			AB	MSE	AIL	CP	AB	MSE	AIL	CP
50	20	a	0.0711	0.0377	0.5144	95.7	0.2604	0.9814	3.7656	95.9
		b	0.1487	0.1705	1.4739	95.9	0.1349	0.1822	1.5953	95.9
		$\kappa(t;\varpi)$	0.023	0.0044	0.2304	97.5	0.0043	0.0071	0.3164	97.5
		$\eta(t;\varpi)$	0.4079	0.2384	0.7317	97.5	0.5811	0.9919	5.3423	97.5
	30	a	0.0628	0.0308	0.4072	97	0.1162	0.3126	2.1544	95.4
		b	0.1349	0.1795	1.1669	95.8	0.0791	0.106	1.2441	95.8
		$\kappa(t;\varpi)$	0.0215	0.2236	0.1992	97.5	0.0004	0.0048	0.293	97.5
		$\eta(t;\varpi)$	0.8218	0.0129	0.4678	97.5	0.2585	0.4279	4.1823	97.5

(Continued)

Table 7 (continued)

100	20	a	0.0664	0.044	0.7807	95.7	0.246	0.8845	3.576	96.4
		b	0.1456	0.1695	1.5104	95.9	0.1037	0.135	1.3886	95.9
		$\kappa(t;\varpi)$	0.0165	0.0043	0.2569	97.5	0.0057	0.0071	0.3187	97.5
		$\eta(t;\varpi)$	0.1262	0.1384	1.2086	97.5	0.5301	0.9294	5.1375	97.5
	50	a	0.0013	0.0058	0.298	97.1	0.0697	0.1774	1.6363	94.7
		b	0.0553	0.0478	0.8299	96.2	0.0038	0.0379	0.7668	96.2
		$\kappa(t;\varpi)$	0.0016	0.0014	0.1468	97.5	0.0037	0.0028	0.208	97.5
		$\eta(t;\varpi)$	0.0087	0.0075	0.3316	97.5	0.1156	0.648	3.2641	97.5
	70	a	0.0007	0.0045	0.2631	97	0.0653	0.1264	1.3768	95.7
		b	0.0339	0.0313	0.6809	96.2	0.0236	0.0419	0.8011	96.2
		$\kappa(t;\varpi)$	0.0014	0.0011	0.1309	97.5	0.0035	0.0019	0.1644	97.5
		$\eta(t;\varpi)$	0.0024	0.0044	0.2678	97.5	0.132	0.5614	2.5166	97.5
150	50	a	0.0083	0.0054	0.2875	96.6	0.0679	0.176	1.6311	94.4
		b	0.0582	0.0505	0.8511	96.4	0.0004	0.0345	0.7316	96.4
		$\kappa(t;\varpi)$	0.0007	0.0011	0.1239	97.5	0.0038	0.0028	0.2081	97.5
		$\eta(t;\varpi)$	0.0201	0.0106	0.4094	97.5	0.1098	0.6336	3.2228	97.5
	70	a	0.0025	0.0033	0.2257	96	0.0642	0.1255	1.3728	96
		b	0.0374	0.0346	0.7144	96.1	0.0198	0.0384	0.7677	96.1
		$\kappa(t;\varpi)$	0.0005	0.0008	0.1129	97.5	0.0037	0.0019	0.1648	97.5
		$\eta(t;\varpi)$	0.008	0.0048	0.2663	97.5	0.127	0.548	2.4828	97.5
	100	a	0.0009	0.0033	0.2264	96.9	0.0456	0.0741	1.0575	95.8
		b	0.0272	0.0225	0.5786	96.4	0.0304	0.0309	0.6822	96.4
		$\kappa(t;\varpi)$	0.0013	0.0009	0.1175	97.5	0.0014	0.0014	0.1348	97.5
		$\eta(t;\varpi)$	0.0016	0.0032	0.2271	97.5	0.1023	0.305	2.2037	97.5
	130	a	0.0014	0.0031	0.2169	97.7	0.0328	0.0584	0.9432	96
		b	0.0126	0.0142	0.4655	96.8	0.0262	0.0243	0.6059	96.8
		$\kappa(t;\varpi)$	0.0003	0.0008	0.1087	97.5	0.0004	0.0012	0.1313	97.5
		$\eta(t;\varpi)$	0.0021	0.0027	0.1967	97.5	0.0775	0.2421	1.7827	97.5

Scheme III

n	m	Estimate	$set1=(a=0.5,b=1.5)$				$set2=(a=2.5,b=2)$			
			AB	MSE	AIL	CP	AB	MSE	AIL	CP
50	20	a	0.0129	0.0206	0.5603	96.1	0.2452	0.6942	3.1231	96.1
		b	0.0677	0.0727	1.0234	96.3	0.0903	0.1292	1.3646	96.3
		$\kappa(t;\varpi)$	0.0031	0.0045	0.2619	97.5	0.0095	0.0069	0.3326	97.5
		$\eta(t;\varpi)$	0.0207	0.0231	0.6286	97.5	0.5044	0.8761	5.923	97.5
	30	a	0.0027	0.0143	0.4683	97	0.1539	0.339	2.2021	94.9
		b	0.0696	0.0598	0.9193	96.3	0.0927	0.1063	1.2258	96.3
		$\kappa(t;\varpi)$	0.0034	0.0033	0.2192	97.5	0.004	0.0047	0.2718	97.5
		$\eta(t;\varpi)$	0.0033	0.0145	0.4687	97.5	0.3447	0.4457	4.6215	97.5

(Continued)

Table 7 (continued)

100	20	a	0.0195	0.0188	0.5329	95.5	0.2333	0.6551	3.0396	95.9
		b	0.0478	0.0552	0.9022	96.5	0.0637	0.0981	1.2029	96.5
		$\kappa(t;\varpi)$	0.0064	0.004	0.2527	97.5	0.0105	0.0068	0.3299	97.5
		$\eta(t;\varpi)$	0.0274	0.0228	0.6186	97.5	0.4608	0.5958	5.5597	97.5
	50	a	0	0.0086	0.364	96.9	0.0757	0.1587	1.534	95.4
		b	0.0339	0.0281	0.6436	96	0.0452	0.0499	0.8582	96.1
		$\kappa(t;\varpi)$	0.0011	0.002	0.1741	97.5	0.0016	0.0029	0.2039	97.5
		$\eta(t;\varpi)$	0.003	0.0086	0.3702	97.5	0.1656	0.4291	3.151	97.5
	70	a	0	0.0059	0.3002	97	0.0577	0.1033	1.2398	95.4
		b	0.0254	0.0225	0.5803	96.4	0.0339	0.0401	0.7737	96.4
		$\kappa(t;\varpi)$	0.0008	0.0014	0.1453	97.5	0.002	0.0018	0.1682	97.5
		$\eta(t;\varpi)$	0.0018	0.0055	0.2987	97.5	0.1271	0.4399	2.569	97.5
150	50	a	0.0015	0.0082	0.3545	96.8	0.0742	0.1578	1.5303	95.5
		b	0.0296	0.0253	0.6126	96.3	0.0395	0.0449	0.8168	96.3
		$\kappa(t;\varpi)$	0.0003	0.0019	0.1678	97.5	0.0019	0.0028	0.2027	97.5
		$\eta(t;\varpi)$	0.0044	0.0084	0.3571	97.5	0.1585	0.4145	3.1078	97.5
	70	a	0.0011	0.0056	0.2922	97	0.0569	0.1031	1.2391	95.3
		b	0.0224	0.0206	0.5566	96.3	0.0299	0.0367	0.7422	96.3
		$\kappa(t;\varpi)$	0.0002	0.0013	0.1404	97.5	0.0022	0.0018	0.1682	97.5
		$\eta(t;\varpi)$	0.0029	0.0053	0.2941	97.5	0.1225	0.4119	2.5784	97.5
	100	a	0.0005	0.0045	0.2637	97.3	0.0402	0.065	0.9877	95.7
		b	0.0199	0.0158	0.4871	96.6	0.0265	0.0281	0.6495	96.6
		$\kappa(t;\varpi)$	0.0008	0.0011	0.1268	97.5	0.0013	0.0013	0.1423	97.5
		$\eta(t;\varpi)$	0.0009	0.0041	0.2539	97.5	0.0896	0.2678	2.0222	97.5
	130	a	0.0019	0.0034	0.2282	97.4	0.0308	0.048	0.8508	96.3
		b	0.0109	0.012	0.4281	97.1	0.0145	0.0214	0.5708	97.1
		$\kappa(t;\varpi)$	0.0006	0.0008	0.1143	97.5	0.0015	0.001	0.1207	97.5
		$\eta(t;\varpi)$	0.0025	0.003	0.2082	97.5	0.0632	0.2009	1.7389	97.5

Table 8: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 1 $\equiv$ $(a = 0.5, b = 1.5)$ for Scheme I

			IP								NIP							
n	m	Estimate	SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF		SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0071	0.0072	0.0119	0.0074	0.0023	0.007	0.012	0.0072	0.0098	0.0133	0.0164	0.0138	0.0033	0.0128	0.0163	0.0134
		b	0.0231	0.0216	0.0316	0.0224	0.0147	0.021	0.0119	0.0212	0.0105	0.0327	0.0213	0.0336	0.0001	0.0322	0.0038	0.0329
		$\kappa(t;\varpi)$	0.0018	0.0014	0.0008	0.0014	0.0027	0.0015	0.0042	0.0015	0.0036	0.0027	0.0023	0.0027	0.0048	0.0027	0.007	0.0028
		$\eta(t;\varpi)$	0.0123	0.01	0.0191	0.0106	0.0056	0.0094	0.013	0.0095	0.0115	0.0173	0.0205	0.0185	0.0027	0.0164	0.0219	0.017
	30	a	0.0023	0.005	0.0015	0.0051	0.006	0.0049	0.0174	0.0053	0.0061	0.0106	0.0116	0.0109	0.0007	0.0103	0.0155	0.0106
		b	0.0318	0.0178	0.0396	0.0186	0.024	0.0171	0.0215	0.0172	0.0172	0.0298	0.027	0.0307	0.0073	0.0292	0.004	0.0297
		$\kappa(t;\varpi)$	0.0025	0.0011	0.0032	0.0011	0.0017	0.0011	0.0005	0.0011	0.002	0.0023	0.0009	0.0022	0.0031	0.0023	0.0049	0.0024
		$\eta(t;\varpi)$	0.0025	0.0066	0.0075	0.0069	0.0025	0.0064	0.0169	0.0065	0.0072	0.0124	0.0143	0.0131	0.0003	0.0119	0.0192	0.0122
100	20	a	0.0098	0.0068	0.0145	0.0071	0.0051	0.0066	0.0088	0.0067	0.0096	0.0131	0.0161	0.0137	0.0031	0.0127	0.0164	0.0133
		b	0.0188	0.0194	0.027	0.02	0.0106	0.0189	0.0079	0.0191	0.0101	0.0314	0.0206	0.0322	0.0003	0.0309	0.0039	0.0315
		$\kappa(t;\varpi)$	0.0031	0.0013	0.0022	0.0013	0.004	0.0014	0.0055	0.0014	0.0034	0.0026	0.0022	0.0026	0.0047	0.0027	0.0068	0.0028
		$\eta(t;\varpi)$	0.015	0.0097	0.0217	0.0103	0.0084	0.0091	0.0098	0.009	0.0117	0.0174	0.0206	0.0186	0.0028	0.0165	0.0218	0.0171
	50	a	0.0013	0.003	0.0014	0.003	0.0039	0.003	0.0118	0.0031	0.0047	0.0076	0.0088	0.0078	0.0007	0.0075	0.0112	0.0077
		b	0.0207	0.0098	0.0267	0.0103	0.0147	0.0095	0.0128	0.0095	0.0144	0.0226	0.0228	0.0232	0.006	0.0222	0.0032	0.0225
		$\kappa(t;\varpi)$	0.0015	0.0006	0.002	0.0006	0.0009	0.0006	7.66E-05	0.0006	0.0017	0.0017	0.0008	0.0017	0.0025	0.0017	0.0039	0.0018
		$\eta(t;\varpi)$	0.0018	0.0038	0.0052	0.004	0.0016	0.0037	0.0112	0.0038	0.0049	0.0082	0.0097	0.0085	5.63E-05	0.008	0.0135	0.0082
	70	a	0.0006	0.0017	0.0015	0.0018	0.0026	0.0017	0.0086	0.0018	0.0046	0.0058	0.008	0.006	0.0013	0.0057	0.0086	0.0058
		b	0.0159	0.0074	0.0211	0.0077	0.0108	0.0072	0.0091	0.0072	0.0126	0.0202	0.0203	0.0207	0.0049	0.0198	0.0024	0.02
		$\kappa(t;\varpi)$	0.0009	0.0004	0.0014	0.0004	0.0005	0.0004	0.0002	0.0004	0.0018	0.0014	0.0011	0.0014	0.0025	0.0014	0.0037	0.0014
		$\eta(t;\varpi)$	0.0017	0.0022	0.0042	0.0023	0.0008	0.0022	0.0079	0.0022	0.0041	0.0056	0.008	0.0058	0.0003	0.0055	0.0104	0.0056
150	50	a	0.0005	0.0029	0.0021	0.0029	0.0031	0.0029	0.0109	0.003	0.0047	0.0075	0.0087	0.0077	0.0007	0.0073	0.0111	0.0075
		b	0.0191	0.0093	0.025	0.0096	0.0133	0.009	0.0114	0.009	0.0142	0.0218	0.0225	0.0224	0.006	0.0214	0.0032	0.0217
		$\kappa(t;\varpi)$	0.0011	0.0006	0.0016	0.0006	0.0006	0.0006	0.0003	0.0006	0.0016	0.0017	0.0008	0.0017	0.0025	0.0017	0.0038	0.0017
		$\eta(t;\varpi)$	0.0024	0.0038	0.0058	0.0039	0.0009	0.0036	0.0104	0.0037	0.005	0.0081	0.0099	0.0084	0.0002	0.0079	0.0134	0.0081
	70	a	0.0001	0.0017	0.0019	0.0017	0.002	0.0016	0.008	0.0017	0.0045	0.0057	0.0078	0.0058	0.0011	0.0055	0.0087	0.0056
		b	0.0147	0.0069	0.0197	0.0072	0.0096	0.0068	0.008	0.0068	0.0127	0.0195	0.0203	0.0201	0.0051	0.0191	0.0026	0.0193
		$\kappa(t;\varpi)$	0.0007	0.0004	0.0011	0.0004	0.0003	0.0004	0.0004	0.0004	0.0017	0.0013	0.001	0.0013	0.0024	0.0013	0.0036	0.0014
		$\eta(t;\varpi)$	0.0021	0.0021	0.0045	0.0022	0.0004	0.0021	0.0074	0.0021	0.0042	0.0056	0.008	0.0057	0.0004	0.0054	0.0103	0.0055
	100	a	0.0012	0.0012	0.0003	0.0012	0.0028	0.0012	0.0074	0.0013	0.0025	0.0045	0.0053	0.0046	0.0002	0.0045	0.0082	0.0046
		b	0.0138	0.0049	0.0179	0.005	0.0096	0.0047	0.0083	0.0047	0.013	0.0164	0.0198	0.0169	0.0061	0.0159	0.0039	0.0161
		$\kappa(t;\varpi)$	0.0011	0.0003	0.0015	0.0003	0.0008	0.0003	0.0003	0.0003	0.0009	0.0011	0.0003	0.0011	0.0015	0.0011	0.0025	0.0011
		$\eta(t;\varpi)$	0.0007	0.0015	0.0026	0.0016	0.0012	0.0015	0.0065	0.0016	0.0021	0.0041	0.005	0.0042	0.0008	0.004	0.0091	0.0041
	130	a	0.0012	0.0008	0.0024	0.0009	1.87E-05	0.0008	0.0037	0.0009	0.0039	0.0037	0.0063	0.0038	0.0016	0.0037	0.0052	0.0037
		b	0.0069	0.0036	0.0104	0.0037	0.0034	0.0035	0.0023	0.0036	0.0076	0.0138	0.0138	0.0142	0.0015	0.0135	0.0005	0.0136
		$\kappa(t;\varpi)$	0.0002	0.0002	0	0.0002	0.0005	0.0002	0.0009	0.0002	0.0017	0.0009	0.0012	0.0009	0.0022	0.0009	0.0031	0.0009
		$\eta(t;\varpi)$	0.0023	0.0011	0.0037	0.0011	0.0008	0.0011	0.0033	0.0011	0.0029	0.0031	0.0053	0.0032	0.0005	0.0031	0.0062	0.0031

Table 9: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 2 $\equiv$ $(a = 2.5, b = 2.0)$ for Scheme I

			IP								NIP							
n	m	Estimate	SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF		SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0057	0.0386	0.0068	0.0388	0.0181	0.0385	0.0158	0.0391	0.0042	0.0426	0.0163	0.043	0.0079	0.0426	0.0056	0.0431
		b	0.0124	0.03	0.0226	0.0306	0.0023	0.0295	0.0022	0.0299	0.0061	0.0386	0.0061	0.0389	0.0182	0.0387	0.0185	0.0393
		$\kappa(t;\varpi)$	0.0031	0.0013	0.0035	0.0013	0.0027	0.0013	0.0001	0.0012	0.0029	0.0013	0.0033	0.0013	0.0025	0.0013	0.0002	0.0013
		$\eta(t;\varpi)$	0.0039	0.1556	0.0481	0.1612	0.0399	0.1548	0.0193	0.1571	0.0345	0.1574	0.0143	0.1598	0.0828	0.1608	0.0605	0.1614
	30	a	0.007	0.0367	0.0053	0.0368	0.0192	0.0366	0.0169	0.0372	0.0071	0.0387	0.0186	0.0392	0.0045	0.0386	0.0023	0.039
		b	0.0224	0.0268	0.0321	0.0277	0.0126	0.0261	0.0126	0.0264	0.0013	0.0372	0.0131	0.0378	0.0104	0.0371	0.0106	0.0377
		$\kappa(t;\varpi)$	0.0033	0.0012	0.0037	0.0012	0.0029	0.0011	0.0005	0.0011	0.0037	0.0012	0.0041	0.0012	0.0033	0.0012	0.0007	0.0011
		$\eta(t;\varpi)$	0.0202	0.1407	0.0627	0.1481	0.0219	0.1377	0.0019	0.1406	0.0283	0.1509	0.0197	0.1534	0.076	0.1537	0.0539	0.1543
100	20	a	0.0058	0.0383	0.0067	0.0389	0.0183	0.0382	0.0159	0.0392	0.0041	0.0423	0.0162	0.0427	0.0079	0.0423	0.0057	0.0428
		b	0.0105	0.0276	0.0203	0.0282	0.0007	0.0272	0.0007	0.0275	0.0058	0.0372	0.0061	0.0375	0.0177	0.0372	0.018	0.0378
		$\kappa(t;\varpi)$	0.0029	0.0013	0.0033	0.0013	0.0026	0.0012	4.33E-05	0.0012	0.003	0.0012	0.0034	0.0012	0.0026	0.0012	0.0001	0.0012
		$\eta(t;\varpi)$	0.003	0.1527	0.0468	0.158	0.0405	0.152	0.0201	0.1542	0.0338	0.1565	0.0148	0.1591	0.0819	0.1597	0.0596	0.1603
	50	a	0.0071	0.0319	0.0175	0.0324	0.0035	0.0317	0.0014	0.032	0.0074	0.0349	0.0044	0.0349	0.0193	0.0352	0.0171	0.0355
		b	0.0175	0.0184	0.0256	0.019	0.0094	0.0179	0.0094	0.0181	0.0009	0.0299	0.0114	0.0304	0.0096	0.0297	0.0097	0.0301
		$\kappa(t;\varpi)$	0.0025	0.0009	0.0028	0.0009	0.0022	0.0009	2.50E-05	0.0009	0.0036	0.0011	0.0039	0.0011	0.0032	0.0011	0.0007	0.001
		$\eta(t;\varpi)$	0.0191	0.1119	0.0575	0.1189	0.0193	0.1083	0.0008	0.1109	0.0261	0.1378	0.0198	0.1403	0.0717	0.1401	0.0504	0.1406
	70	a	0.0115	0.0272	0.0213	0.0278	0.0018	0.0269	0.0037	0.0271	0.006	0.032	0.0054	0.0322	0.0175	0.0322	0.0153	0.0324
		b	0.0153	0.0148	0.0227	0.0153	0.0079	0.0145	0.008	0.0146	0.0002	0.0273	0.0101	0.0277	0.0097	0.0271	0.0098	0.0275
		$\kappa(t;\varpi)$	0.0013	0.0008	0.0016	0.0008	0.001	0.0008	0.001	0.0007	0.0031	0.0009	0.0034	0.0009	0.0027	0.0009	0.0004	0.0009
		$\eta(t;\varpi)$	0.0264	0.096	0.0624	0.1027	0.0097	0.0923	0.0077	0.0947	0.0225	0.1317	0.0223	0.1345	0.0668	0.1335	0.0461	0.1341
150	50	a	0.0068	0.0319	0.0173	0.0324	0.0037	0.0316	0.0017	0.0319	0.0073	0.0349	0.0046	0.0349	0.0191	0.0352	0.0169	0.0354
		b	0.0168	0.0174	0.0247	0.018	0.0089	0.017	0.009	0.0171	0.0013	0.029	0.0116	0.0295	0.0091	0.0288	0.0092	0.0292
		$\kappa(t;\varpi)$	0.0025	0.0009	0.0028	0.0009	0.0021	0.0009	4.03E-05	0.0009	0.0036	0.0011	0.0039	0.0011	0.0032	0.001	0.0007	0.001
		$\eta(t;\varpi)$	0.0185	0.1106	0.0567	0.1174	0.0198	0.1072	0.0014	0.1097	0.0251	0.137	0.0206	0.1396	0.0705	0.1392	0.0493	0.1397
	70	a	0.0113	0.0272	0.021	0.0278	0.0016	0.0269	0.0035	0.0272	0.0059	0.0321	0.0055	0.0323	0.0173	0.0322	0.0152	0.0325
		b	0.0146	0.014	0.0218	0.0145	0.0074	0.0137	0.0074	0.0139	0.0004	0.0265	0.0102	0.0269	0.0093	0.0263	0.0094	0.0266
		$\kappa(t;\varpi)$	0.0013	0.0008	0.0016	0.0008	0.001	0.0007	0.001	0.0007	0.0031	0.0009	0.0034	0.0009	0.0027	0.0009	0.0004	0.0009
		$\eta(t;\varpi)$	0.0256	0.0952	0.0614	0.1018	0.0102	0.0916	0.0071	0.094	0.0216	0.131	0.0229	0.1338	0.0658	0.1328	0.0451	0.1334
	100	a	0.0123	0.0218	0.021	0.0222	0.0035	0.0215	0.0052	0.0217	0.0037	0.0295	0.0071	0.0298	0.0146	0.0295	0.0125	0.0298
		b	0.0147	0.0104	0.0212	0.0107	0.0083	0.0101	0.0083	0.0102	0.0035	0.0233	0.0128	0.0239	0.0057	0.0229	0.0057	0.0232
		$\kappa(t;\varpi)$	0.0008	0.0006	0.0011	0.0006	0.0006	0.0006	0.0012	0.0006	0.0028	0.0008	0.0032	0.0008	0.0025	0.0008	0.0003	0.0008
		$\eta(t;\varpi)$	0.0293	0.077	0.0621	0.0827	0.0035	0.0739	0.0124	0.0758	0.0132	0.1226	0.0291	0.1266	0.0554	0.1228	0.0355	0.1239
	130	a	0.012	0.0176	0.02	0.018	0.004	0.0174	0.0056	0.0176	0.002	0.0274	0.0083	0.0277	0.0124	0.0274	0.0104	0.0276
		b	0.0076	0.008	0.0133	0.0082	0.0019	0.0079	0.0019	0.008	0.001	0.0203	0.0077	0.0207	0.0096	0.0201	0.0096	0.0203
		$\kappa(t;\varpi)$	5.95E-06	0.0005	0.0002	0.0005	0.0002	0.0005	0.0018	0.0005	0.002	0.0007	0.0023	0.0007	0.0017	0.0007	0.0003	0.0007
		$\eta(t;\varpi)$	0.0229	0.0622	0.053	0.0664	0.0072	0.0602	0.0073	0.0614	0.0134	0.1143	0.0272	0.1182	0.0539	0.1142	0.0348	0.1153

Table 10: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 1 $\equiv$ $(a = 0.5, b = 1.5)$ for Scheme II

			IP								NIP							
n	m	Estimate	SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF		SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0131	0.006	0.0175	0.0062	0.0088	0.0057	0.0041	0.0058	0.0107	0.0118	0.0169	0.0123	0.0045	0.0114	0.014	0.0118
		b	0.023	0.0303	0.033	0.0313	0.0129	0.0295	0.0096	0.0299	0.0026	0.0392	0.0147	0.0398	0.0094	0.0389	0.0138	0.0401
		$\kappa(t;\varpi)$	0.0042	0.0011	0.0034	0.0011	0.005	0.0011	0.0063	0.0012	0.0038	0.0022	0.0026	0.0022	0.005	0.0022	0.0069	0.0023
		$\eta(t;\varpi)$	0.0201	0.0102	0.027	0.0111	0.0134	0.0096	0.0048	0.0092	0.0136	0.0193	0.023	0.0207	0.0043	0.0181	0.0212	0.0186
	30	a	0.0002	0.0038	0.0031	0.0039	0.0034	0.0037	0.0132	0.0039	0.0074	0.0087	0.0123	0.009	0.0025	0.0084	0.0121	0.0086
		b	0.0333	0.024	0.0424	0.0251	0.0242	0.0231	0.0214	0.0232	0.0132	0.0354	0.0243	0.0363	0.0021	0.0348	0.0017	0.0356
		$\kappa(t;\varpi)$	0.0016	0.0008	0.0023	0.0008	0.001	0.0008	1.78E-06	0.0008	0.0025	0.0018	0.0016	0.0018	0.0035	0.0018	0.0051	0.0019
		$\eta(t;\varpi)$	0.0052	0.0057	0.01	0.006	0.0005	0.0055	0.0127	0.0055	0.0088	0.0119	0.0157	0.0126	0.002	0.0113	0.0169	0.0115
100	20	a	0.0236	0.01	0.029	0.0105	0.0183	0.0095	0.0029	0.0092	0.0094	0.0131	0.0159	0.0136	0.003	0.0126	0.0166	0.0133
		b	0.0241	0.0303	0.0342	0.0314	0.014	0.0295	0.0107	0.0299	0.0029	0.0382	0.0091	0.0387	0.0149	0.0381	0.0193	0.0393
		$\kappa(t;\varpi)$	0.008	0.0017	0.007	0.0016	0.0089	0.0017	0.0104	0.0017	0.0028	0.0022	0.0017	0.0021	0.004	0.0022	0.0059	0.0022
		$\eta(t;\varpi)$	0.0361	0.0186	0.045	0.0204	0.0273	0.0171	0.0046	0.0162	0.0148	0.0244	0.0255	0.0264	0.0043	0.0228	0.0242	0.0234
	50	a	0.001	0.002	0.003	0.002	0.0011	0.002	0.0072	0.002	0.0055	0.0054	0.0089	0.0055	0.0021	0.0053	0.0078	0.0053
		b	0.025	0.0161	0.0326	0.0168	0.0174	0.0156	0.015	0.0156	0.0124	0.0287	0.0224	0.0294	0.0024	0.0283	0.001	0.0288
		$\kappa(t;\varpi)$	0.0008	0.0004	0.0012	0.0004	0.0003	0.0004	0.0003	0.0004	0.0018	0.0012	0.0012	0.0011	0.0025	0.0012	0.0036	0.0012
		$\eta(t;\varpi)$	0.0051	0.0031	0.0082	0.0032	0.0021	0.003	0.0063	0.003	0.0067	0.0076	0.0114	0.0079	0.002	0.0073	0.0111	0.0073
	70	a	0.0006	0.0013	0.0011	0.0014	0.0023	0.0013	0.0073	0.0014	0.0043	0.0044	0.0071	0.0045	0.0014	0.0043	0.007	0.0044
		b	0.019	0.0104	0.0252	0.0108	0.0128	0.0101	0.0108	0.0102	0.0128	0.0244	0.0217	0.025	0.0039	0.0239	0.001	0.0242
		$\kappa(t;\varpi)$	0.0011	0.0003	0.0014	0.0003	0.0007	0.0003	0.0002	0.0003	0.0016	0.0011	0.001	0.0011	0.0022	0.0011	0.0032	0.0011
		$\eta(t;\varpi)$	0.0021	0.0019	0.0043	0.0019	0.0001	0.0019	0.0064	0.0019	0.0039	0.0047	0.0074	0.0049	0.0005	0.0046	0.0092	0.0046
150	50	a	0.0054	0.0019	0.0074	0.0019	0.0034	0.0018	0.0026	0.0018	0.0066	0.0053	0.0099	0.0055	0.0032	0.0052	0.0066	0.0052
		b	0.025	0.0174	0.0328	0.0181	0.0172	0.0168	0.0147	0.0169	0.0111	0.0294	0.0214	0.0301	0.0009	0.029	0.0027	0.0296
		$\kappa(t;\varpi)$	0.001	0.0004	0.0006	0.0004	0.0014	0.0004	0.002	0.0004	0.002	0.001	0.0014	0.001	0.0026	0.001	0.0036	0.001
		$\eta(t;\varpi)$	0.0108	0.0034	0.0141	0.0036	0.0076	0.0033	0.0014	0.0031	0.0096	0.0093	0.015	0.0099	0.0044	0.0089	0.0101	0.0088
	70	a	0.0018	0.001	0.0033	0.001	0.0004	0.001	0.0038	0.001	0.0047	0.0035	0.0073	0.0036	0.0022	0.0034	0.0054	0.0035
		b	0.019	0.0124	0.0257	0.0129	0.0123	0.0121	0.0101	0.0121	0.0127	0.026	0.0222	0.0268	0.0032	0.0255	0.0001	0.0259
		$\kappa(t;\varpi)$	0.0001	0.0002	0.0004	0.0002	0.0002	0.0002	0.0007	0.0002	0.0015	0.0008	0.001	0.0008	0.0021	0.0008	0.0029	0.0008
		$\eta(t;\varpi)$	0.0052	0.0018	0.0073	0.0019	0.003	0.0018	0.0029	0.0017	0.0059	0.0051	0.0095	0.0053	0.0023	0.005	0.0077	0.0049
	100	a	0.0011	0.0009	0.0001	0.0009	0.0024	0.0009	0.0061	0.001	0.0024	0.0032	0.0046	0.0032	0.0002	0.0031	0.0062	0.0032
		b	0.0182	0.0074	0.0234	0.0077	0.0129	0.0072	0.0111	0.0072	0.014	0.0205	0.0222	0.0213	0.0058	0.02	0.0031	0.0202
		$\kappa(t;\varpi)$	0.0013	0.0002	0.0016	0.0002	0.001	0.0002	0.0006	0.0002	0.0007	0.0008	0.0003	0.0008	0.0012	0.0008	0.002	0.0008
		$\eta(t;\varpi)$	0.0015	0.0013	0.0031	0.0013	0.0001	0.0013	0.0047	0.0013	0.0022	0.0032	0.0048	0.0033	0.0003	0.0031	0.0074	0.0032
	130	a	0.0009	0.0008	0.002	0.0008	0.0002	0.0008	0.0036	0.0008	0.0037	0.0033	0.0058	0.0033	0.0016	0.0032	0.0046	0.0033
		b	0.0082	0.0044	0.0121	0.0046	0.0042	0.0044	0.0029	0.0044	0.0079	0.0158	0.0147	0.0163	0.0011	0.0155	0.0011	0.0156
		$\kappa(t;\varpi)$	0.0001	0.0002	0.0002	0.0002	0.0003	0.0002	0.0007	0.0002	0.0016	0.0008	0.0011	0.0008	0.0021	0.0008	0.0029	0.0009
		$\eta(t;\varpi)$	0.002	0.001	0.0034	0.001	0.0007	0.001	0.0032	0.001	0.0025	0.0027	0.0047	0.0028	0.0003	0.0027	0.0059	0.0027

Table 11: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 2 $\equiv$ (a = 2.5, b = 2.0) for Scheme II

n	m	Estimate	IP								NIP							
			SELF		LLX($\tau = -2$)		LLX($\tau = 2$)		MLF		SELF		LLX($\tau = -2$)		LLX($\tau = 2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0067	0.0395	0.0059	0.0397	0.0193	0.0394	0.0169	0.0402	0.0031	0.0465	0.0095	0.0465	0.0157	0.0464	0.0134	0.0474
		b	0.0016	0.0349	0.013	0.0353	0.0097	0.0347	0.0098	0.0352	0.0215	0.0283	0.0315	0.0293	0.0115	0.0276	0.0116	0.0279
		$\kappa(t;\varpi)$	0.0033	0.0011	0.0036	0.0011	0.0029	0.0011	0.0003	0.0011	0.0041	0.0013	0.0045	0.0013	0.0037	0.0013	0.0011	0.0012
		$\eta(t;\varpi)$	0.0304	0.1636	0.0194	0.1666	0.0796	0.1669	0.057	0.1678	0.0031	0.1736	0.0437	0.1785	0.0495	0.1742	0.0279	0.1763
	30	a	0.0079	0.0373	0.0044	0.0374	0.0201	0.0376	0.0179	0.0379	0.0064	0.0416	0.0183	0.042	0.0055	0.0414	0.0032	0.0419
		b	0.0038	0.034	0.0076	0.0344	0.0153	0.034	0.0154	0.0345	0.0123	0.0296	0.0224	0.0303	0.0021	0.0292	0.0021	0.0295
		$\kappa(t;\varpi)$	0.0038	0.0011	0.0042	0.0011	0.0034	0.0011	0.0009	0.0011	0.0035	0.0012	0.0038	0.0012	0.0031	0.0012	0.0006	0.0011
		$\eta(t;\varpi)$	0.0245	0.1605	0.0247	0.1639	0.0731	0.1634	0.0507	0.1643	0.0035	0.1744	0.0435	0.1791	0.0502	0.1754	0.0285	0.1774
100	20	a	0.0069	0.0371	0.0057	0.0393	0.0194	0.0393	0.017	0.0397	0.001	0.0457	0.0115	0.0457	0.0134	0.046	0.0112	0.0464
		b	0.0015	0.0278	0.0088	0.0282	0.0117	0.0277	0.0118	0.028	0.0127	0.0246	0.0219	0.0252	0.0034	0.0242	0.0034	0.0244
		$\kappa(t;\varpi)$	0.0034	0.0011	0.0038	0.0011	0.003	0.0011	0.0005	0.001	0.0044	0.0013	0.0048	0.0013	0.004	0.0012	0.0015	0.0012
		$\eta(t;\varpi)$	0.0268	0.1598	0.0225	0.1627	0.0756	0.1628	0.053	0.1635	0.0191	0.1586	0.0642	0.1663	0.0256	0.1559	0.0045	0.1591
	50	a	0.0081	0.0365	0.0039	0.0365	0.0201	0.0368	0.0178	0.0371	0.0083	0.0391	0.0196	0.0397	0.003	0.0388	0.0008	0.0392
		b	0.002	0.0264	0.0118	0.0269	0.0077	0.0263	0.0078	0.0265	0.0192	0.0204	0.0272	0.021	0.0111	0.02	0.0111	0.0201
		$\kappa(t;\varpi)$	0.0037	0.0009	0.004	0.0009	0.0034	0.0009	0.0011	0.0009	0.0026	0.0009	0.0029	0.0009	0.0022	0.0009	0.0001	0.0009
		$\eta(t;\varpi)$	0.0188	0.1517	0.029	0.1554	0.0662	0.1535	0.0442	0.1546	0.025	0.1502	0.0685	0.1588	0.0183	0.1464	0.0023	0.1498
	70	a	0.0076	0.0346	0.004	0.0347	0.0192	0.0348	0.017	0.0351	0.0106	0.0347	0.0212	0.0354	0.0001	0.0343	0.0021	0.0346
		b	0.0009	0.0258	0.0087	0.0261	0.0106	0.0257	0.0107	0.026	0.0156	0.0164	0.0232	0.0169	0.008	0.0161	0.0081	0.0162
		$\kappa(t;\varpi)$	0.0038	0.0009	0.0041	0.0009	0.0034	0.0009	0.0012	0.0009	0.0026	0.0009	0.0029	0.001	0.0023	0.0009	0.0002	0.0009
		$\eta(t;\varpi)$	0.022	0.1503	0.0258	0.1538	0.0693	0.1523	0.0473	0.1533	0.0252	0.1421	0.0677	0.1508	0.0171	0.138	0.0031	0.1414
150	50	a	0.0081	0.0355	0.0039	0.0356	0.02	0.0359	0.0178	0.0361	0.0087	0.0369	0.0197	0.0375	0.0024	0.0365	0.0002	0.0369
		b	0.0026	0.022	0.0114	0.0223	0.0061	0.0218	0.0062	0.022	0.0177	0.018	0.0251	0.0185	0.0103	0.0176	0.0103	0.0177
		$\kappa(t;\varpi)$	0.0032	0.0008	0.0035	0.0008	0.0028	0.0008	0.0006	0.0008	0.0013	0.0008	0.0016	0.0008	0.001	0.0008	0.0009	0.0008
		$\eta(t;\varpi)$	0.0194	0.1496	0.0277	0.1531	0.066	0.1516	0.0444	0.1526	0.0265	0.1333	0.0677	0.1418	0.0146	0.129	0.0051	0.1323
	70	a	0.0071	0.0334	0.0044	0.0335	0.0186	0.0336	0.0165	0.0338	0.0125	0.031	0.0226	0.0316	0.0023	0.0306	0.0043	0.0309
		b	0.0001	0.021	0.0087	0.0212	0.0086	0.0209	0.0087	0.0211	0.0141	0.0146	0.0211	0.015	0.007	0.0143	0.0071	0.0144
		$\kappa(t;\varpi)$	0.0033	0.0008	0.0036	0.0008	0.003	0.0008	0.0009	0.0008	0.0016	0.0008	0.0019	0.0008	0.0013	0.0008	0.0006	0.0008
		$\eta(t;\varpi)$	0.0227	0.1456	0.0241	0.1486	0.0689	0.1477	0.0474	0.1486	0.0291	0.1173	0.068	0.1254	0.0099	0.1126	0.0089	0.1158
	100	a	0.004	0.0311	0.007	0.0313	0.0151	0.0311	0.013	0.0314	0.0156	0.0263	0.0249	0.0269	0.0063	0.0258	0.0081	0.0261
		b	0.0022	0.0216	0.0111	0.022	0.0066	0.0213	0.0067	0.0216	0.0157	0.0119	0.0224	0.0123	0.0091	0.0116	0.0092	0.0117
		$\kappa(t;\varpi)$	0.0028	0.0007	0.0031	0.0007	0.0025	0.0007	0.0004	0.0007	0.0006	0.0006	0.0008	0.0006	0.0003	0.0006	0.0014	0.0006
		$\eta(t;\varpi)$	0.0116	0.1398	0.0342	0.1446	0.0569	0.1398	0.0357	0.1414	0.037	0.1014	0.0737	0.1095	0.0002	0.0965	0.018	0.0995
	130	a	0.0026	0.028	0.0078	0.0283	0.0131	0.028	0.0111	0.0282	0.0123	0.0193	0.0205	0.0197	0.004	0.0191	0.0056	0.0192
		b	0.002	0.0208	0.0067	0.0212	0.0107	0.0206	0.0107	0.0209	0.0081	0.009	0.014	0.0092	0.0022	0.0088	0.0022	0.0089
		$\kappa(t;\varpi)$	0.002	0.0007	0.0023	0.0007	0.0017	0.0007	0.0003	0.0007	0.0001	0.0005	0.0003	0.0005	0.0002	0.0005	0.0017	0.0005
		$\eta(t;\varpi)$	0.0143	0.1243	0.0282	0.1285	0.0566	0.1242	0.0367	0.1255	0.024	0.0715	0.056	0.0764	0.008	0.069	0.0075	0.0706

Table 12: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 1 $\equiv$ $(a = 0.5, b = 1.5)$ for Scheme III

n	m	Estimate	IP								NIP							
			SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF		SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0081	0.0069	0.0128	0.0071	0.0034	0.0067	0.0107	0.0069	0.0097	0.0131	0.0163	0.0137	0.0032	0.0127	0.0162	0.0132
		b	0.0224	0.0209	0.0308	0.0216	0.014	0.0203	0.0113	0.0205	0.0104	0.0323	0.0211	0.0331	0.0001	0.0318	0.0037	0.0324
		$\kappa(t;\varpi)$	0.0022	0.0014	0.0013	0.0014	0.0031	0.0014	0.0046	0.0014	0.0035	0.0026	0.0022	0.0026	0.0048	0.0027	0.0069	0.0028
		$\eta(t;\varpi)$	0.0136	0.0098	0.0203	0.0104	0.0069	0.0092	0.0115	0.0092	0.0117	0.0174	0.0207	0.0186	0.0029	0.0164	0.0217	0.017
	30	a	0.002	0.0049	0.0017	0.005	0.0057	0.0048	0.017	0.0051	0.0061	0.0104	0.0115	0.0108	0.0007	0.0101	0.0154	0.0105
		b	0.0315	0.0174	0.0392	0.0182	0.0237	0.0168	0.0213	0.0168	0.017	0.0297	0.0268	0.0306	0.0072	0.029	0.0039	0.0295
		$\kappa(t;\varpi)$	0.0023	0.001	0.0031	0.001	0.0016	0.001	0.0004	0.001	0.002	0.0022	0.0009	0.0022	0.0031	0.0023	0.0049	0.0023
		$\eta(t;\varpi)$	0.0028	0.0065	0.0079	0.0068	0.0022	0.0063	0.0164	0.0064	0.0073	0.0124	0.0143	0.013	0.0004	0.0118	0.0191	0.0122
100	20	a	0.0114	0.0065	0.016	0.0067	0.0068	0.0062	0.0068	0.0063	0.0092	0.0129	0.0157	0.0134	0.0027	0.0124	0.0166	0.013
		b	0.0179	0.0182	0.0259	0.0188	0.0099	0.0178	0.0073	0.0179	0.0108	0.0304	0.0211	0.0313	0.0005	0.0299	0.003	0.0305
		$\kappa(t;\varpi)$	0.0038	0.0013	0.0029	0.0012	0.0047	0.0013	0.0061	0.0013	0.0031	0.0026	0.0019	0.0025	0.0044	0.0026	0.0065	0.0027
		$\eta(t;\varpi)$	0.017	0.0094	0.0237	0.0101	0.0105	0.0088	0.0074	0.0087	0.0118	0.0175	0.0209	0.0187	0.0029	0.0166	0.0218	0.0171
	50	a	0.001	0.0029	0.0016	0.003	0.0035	0.0029	0.0113	0.003	0.0048	0.0075	0.0088	0.0077	0.0008	0.0074	0.011	0.0076
		b	0.0201	0.0095	0.026	0.0099	0.0142	0.0092	0.0123	0.0092	0.014	0.0222	0.0223	0.0227	0.0057	0.0218	0.003	0.0221
		$\kappa(t;\varpi)$	0.0013	0.0006	0.0019	0.0006	0.0008	0.0006	0	0.0006	0.0017	0.0017	0.0009	0.0017	0.0025	0.0017	0.0039	0.0018
		$\eta(t;\varpi)$	0.0021	0.0037	0.0055	0.0039	0.0012	0.0036	0.0107	0.0037	0.005	0.0082	0.0099	0.0085	0.0002	0.0079	0.0133	0.0081
	70	a	0.0004	0.0017	0.0016	0.0017	0.0024	0.0017	0.0084	0.0018	0.0047	0.0058	0.0081	0.0059	0.0013	0.0056	0.0086	0.0057
		b	0.0155	0.0073	0.0206	0.0075	0.0104	0.0071	0.0087	0.0071	0.0125	0.02	0.0201	0.0206	0.0048	0.0196	0.0022	0.0198
		$\kappa(t;\varpi)$	0.0008	0.0004	0.0013	0.0004	0.0004	0.0004	0.0002	0.0004	0.0018	0.0014	0.0011	0.0013	0.0025	0.0014	0.0037	0.0014
		$\eta(t;\varpi)$	0.0018	0.0022	0.0043	0.0022	0.0007	0.0021	0.0077	0.0022	0.0042	0.0056	0.0081	0.0058	0.0004	0.0055	0.0103	0.0056
150	50	a	0	0.0027	0.0026	0.0028	0.0025	0.0027	0.0101	0.0028	0.0045	0.0073	0.0085	0.0075	0.0006	0.0072	0.0112	0.0074
		b	0.0183	0.0087	0.024	0.0091	0.0126	0.0085	0.0108	0.0085	0.0142	0.0212	0.0223	0.0218	0.0061	0.0208	0.0034	0.0211
		$\kappa(t;\varpi)$	0.0009	0.0006	0.0014	0.0006	0.0003	0.0006	0.0005	0.0006	0.0015	0.0016	0.0007	0.0016	0.0024	0.0016	0.0037	0.0017
		$\eta(t;\varpi)$	0.0031	0.0036	0.0064	0.0038	0.0002	0.0035	0.0096	0.0035	0.005	0.0081	0.0099	0.0084	0.0002	0.0078	0.0133	0.008
	70	a	0.0002	0.0016	0.0022	0.0016	0.0017	0.0016	0.0075	0.0017	0.0044	0.0056	0.0078	0.0057	0.0011	0.0055	0.0086	0.0055
		b	0.014	0.0067	0.019	0.007	0.0091	0.0065	0.0075	0.0065	0.0125	0.0192	0.02	0.0198	0.005	0.0188	0.0025	0.019
		$\kappa(t;\varpi)$	0.0005	0.0003	0.0009	0.0003	0.0001	0.0003	0.0005	0.0004	0.0017	0.0013	0.001	0.0013	0.0024	0.0013	0.0036	0.0014
		$\eta(t;\varpi)$	0.0024	0.0021	0.0048	0.0021	0	0.002	0.007	0.0021	0.0042	0.0055	0.008	0.0057	0.0004	0.0054	0.0102	0.0055
	100	a	0.0012	0.0012	0.0003	0.0012	0.0027	0.0012	0.0073	0.0012	0.0025	0.0045	0.0052	0.0046	0.0002	0.0044	0.0082	0.0045
		b	0.0137	0.0048	0.0178	0.005	0.0096	0.0047	0.0082	0.0047	0.0129	0.0161	0.0197	0.0167	0.006	0.0157	0.0038	0.0159
		$\kappa(t;\varpi)$	0.0011	0.0003	0.0014	0.0003	0.0008	0.0003	0.0003	0.0003	0.0009	0.0011	0.0003	0.0011	0.0015	0.0011	0.0025	0.0011
		$\eta(t;\varpi)$	0.0007	0.0015	0.0026	0.0016	0.0011	0.0015	0.0064	0.0015	0.0021	0.0041	0.005	0.0042	0.0008	0.004	0.0091	0.0041
	130	a	0.0012	0.0008	0.0025	0.0009	0	0.0008	0.0036	0.0009	0.0039	0.0037	0.0063	0.0038	0.0017	0.0037	0.0052	0.0037
		b	0.0068	0.0036	0.0103	0.0037	0.0034	0.0035	0.0022	0.0035	0.0076	0.0137	0.0137	0.0141	0.0014	0.0135	0.0006	0.0136
		$\kappa(t;\varpi)$	0.0002	0.0002	0	0.0002	0.0005	0.0002	0.0009	0.0002	0.0017	0.0009	0.0012	0.0009	0.0023	0.0009	0.0031	0.0009
		$\eta(t;\varpi)$	0.0023	0.0011	0.0037	0.0011	0.0008	0.0011	0.0033	0.0011	0.0029	0.0031	0.0053	0.0032	0.0005	0.0031	0.0062	0.0031

Table 13: ABs and MSEs for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 2 $\equiv$ $(a = 2.5, b = 2.0)$ for Scheme III

n	m	Estimate	IP								NIP							
			SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF		SELF		LLX($\tau=-2$)		LLX($\tau=2$)		MLF	
			AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
50	20	a	0.0058	0.0386	0.0067	0.0388	0.0183	0.0388	0.0159	0.0391	0.0046	0.0426	0.0166	0.0429	0.0075	0.0426	0.0052	0.043
		b	0.0127	0.0291	0.0228	0.0298	0.0027	0.0287	0.0027	0.029	0.0059	0.0379	0.0062	0.0382	0.0179	0.038	0.0181	0.0386
		$\kappa(t;\varpi)$	0.003	0.0012	0.0034	0.0012	0.0026	0.0012	0.0001	0.0012	0.0031	0.0013	0.0035	0.0013	0.0027	0.0013	0.0001	0.0012
		$\eta(t;\varpi)$	0.0052	0.1551	0.0494	0.1609	0.0386	0.1541	0.018	0.1564	0.0341	0.1572	0.0146	0.1597	0.0823	0.1605	0.06	0.1611
	30	a	0.0069	0.0367	0.0053	0.0368	0.0191	0.037	0.0169	0.0373	0.0072	0.0389	0.0188	0.0394	0.0043	0.0387	0.0021	0.0391
		b	0.0221	0.0263	0.0318	0.0272	0.0124	0.0256	0.0125	0.0259	0.0014	0.0369	0.0131	0.0374	0.0102	0.0367	0.0104	0.0373
		$\kappa(t;\varpi)$	0.0033	0.0012	0.0037	0.0012	0.0029	0.0011	0.0004	0.0011	0.0037	0.0012	0.0041	0.0012	0.0033	0.0012	0.0007	0.0011
		$\eta(t;\varpi)$	0.0205	0.1408	0.063	0.1482	0.0216	0.1378	0.0016	0.1407	0.0279	0.1511	0.0201	0.1536	0.0755	0.1539	0.0535	0.1545
100	20	a	0.0058	0.0385	0.0066	0.0387	0.0183	0.0387	0.0159	0.039	0.0045	0.0427	0.0166	0.043	0.0076	0.0426	0.0053	0.0431
		b	0.0107	0.0258	0.0203	0.0264	0.0012	0.0254	0.0011	0.0257	0.0047	0.0358	0.007	0.0362	0.0163	0.0358	0.0165	0.0363
		$\kappa(t;\varpi)$	0.003	0.0012	0.0034	0.0012	0.0027	0.0012	0	0.0012	0.0029	0.0012	0.0033	0.0013	0.0025	0.0012	7.17E-05	0.0012
		$\eta(t;\varpi)$	0.0047	0.152	0.0484	0.1576	0.0387	0.1511	0.0182	0.1533	0.0317	0.1562	0.0166	0.159	0.0796	0.1592	0.0574	0.1599
	50	a	0.0071	0.032	0.0176	0.0325	0.0034	0.0317	0.0014	0.032	0.0072	0.0348	0.0047	0.0349	0.019	0.0351	0.0168	0.0354
		b	0.0173	0.0177	0.0253	0.0183	0.0094	0.0173	0.0094	0.0175	0.0012	0.0293	0.0116	0.0298	0.0092	0.0291	0.0093	0.0295
		$\kappa(t;\varpi)$	0.0025	0.0009	0.0028	0.0009	0.0021	0.0009	4.29E-05	0.0009	0.0035	0.0011	0.0039	0.0011	0.0032	0.001	0.0007	0.001
		$\eta(t;\varpi)$	0.0193	0.1114	0.0576	0.1184	0.019	0.1079	0.0006	0.1104	0.0252	0.1371	0.0207	0.1396	0.0706	0.1394	0.0494	0.1399
	70	a	0.0115	0.0273	0.0213	0.0279	0.0018	0.027	0.0037	0.0272	0.006	0.0321	0.0054	0.0322	0.0174	0.0322	0.0153	0.0324
		b	0.0151	0.0146	0.0224	0.015	0.0077	0.0143	0.0078	0.0144	0.0003	0.027	0.0102	0.0275	0.0095	0.0269	0.0096	0.0272
		$\kappa(t;\varpi)$	0.0013	0.0008	0.0016	0.0008	0.001	0.0008	0.001	0.0007	0.0031	0.0009	0.0034	0.0009	0.0028	0.0009	0.0004	0.0009
		$\eta(t;\varpi)$	0.0262	0.0962	0.0622	0.1029	0.0098	0.0924	0.0076	0.0949	0.0222	0.1314	0.0225	0.1343	0.0665	0.1333	0.0458	0.1339
150	50	a	0.0067	0.0319	0.0172	0.0324	0.0038	0.0317	0.0018	0.032	0.0074	0.0348	0.0045	0.0349	0.0192	0.0352	0.017	0.0354
		b	0.0165	0.0164	0.0242	0.0169	0.0087	0.016	0.0088	0.0161	0.0015	0.0281	0.0118	0.0286	0.0087	0.0279	0.0088	0.0282
		$\kappa(t;\varpi)$	0.0024	0.0009	0.0028	0.0009	0.0021	0.0009	4.17E-05	0.0009	0.0036	0.001	0.004	0.0011	0.0032	0.001	0.0008	0.001
		$\eta(t;\varpi)$	0.0184	0.1099	0.0565	0.1167	0.0198	0.1066	0.0015	0.109	0.0246	0.1362	0.0211	0.1387	0.0699	0.1385	0.0488	0.139
	70	a	0.0115	0.0272	0.0212	0.0278	0.0018	0.0269	0.0037	0.0272	0.0057	0.0322	0.0057	0.0323	0.0171	0.0323	0.015	0.0325
		b	0.0139	0.0135	0.021	0.014	0.0068	0.0133	0.0068	0.0133	0.0005	0.0258	0.0102	0.0263	0.0091	0.0256	0.0092	0.0259
		$\kappa(t;\varpi)$	0.0012	0.0007	0.0015	0.0008	0.0009	0.0007	0.0011	0.0007	0.0031	0.0009	0.0034	0.0009	0.0027	0.0009	0.0004	0.0009
		$\eta(t;\varpi)$	0.0255	0.0949	0.0613	0.1015	0.0104	0.0913	0.0069	0.0937	0.0209	0.1307	0.0236	0.1336	0.065	0.1325	0.0443	0.1331
	100	a	0.0122	0.0218	0.021	0.0222	0.0035	0.0215	0.0052	0.0217	0.0038	0.0295	0.007	0.0297	0.0147	0.0295	0.0126	0.0298
		b	0.0147	0.0102	0.0211	0.0105	0.0083	0.0099	0.0083	0.01	0.0037	0.0231	0.0129	0.0236	0.0055	0.0227	0.0055	0.0229
		$\kappa(t;\varpi)$	0.0008	0.0006	0.0011	0.0006	0.0006	0.0006	0.0012	0.0006	0.0029	0.0008	0.0032	0.0008	0.0025	0.0008	0.0004	0.0008
		$\eta(t;\varpi)$	0.0293	0.0768	0.062	0.0824	0.0035	0.0737	0.0124	0.0756	0.0131	0.1224	0.0291	0.1264	0.0552	0.1225	0.0353	0.1236
	130	a	0.012	0.0177	0.02	0.018	0.004	0.0175	0.0055	0.0176	0.0019	0.0274	0.0084	0.0277	0.0123	0.0274	0.0103	0.0276
		b	0.0075	0.008	0.0132	0.0082	0.0019	0.0079	0.0019	0.0079	0.001	0.0202	0.0077	0.0206	0.0096	0.02	0.0096	0.0202
		$\kappa(t;\varpi)$	0	0.0005	0.0002	0.0005	0.0002	0.0005	0.0018	0.0005	0.002	0.0007	0.0023	0.0007	0.0017	0.0007	0.0003	0.0007
		$\eta(t;\varpi)$	0.0228	0.0622	0.0529	0.0664	0.0072	0.0601	0.0073	0.0614	0.0132	0.1143	0.0274	0.1182	0.0537	0.1142	0.0346	0.1153

Table 14: AIL and CP (in %) for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 1 $\equiv (a = 0.5, b = 1.5)$ for Schemes I, II, and III

n	m	Estimate	Scheme											
			I				II				III			
			IP		NIP		IP		NIP		IP		NIP	
			AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
50	20	a	0.3119	96.3	0.4415	95.6	0.2849	97.1	0.4085	95.4	0.3052	96.4	0.4362	95.7
		b	0.5656	96.6	0.7048	97.5	0.6753	96.6	0.7556	96.3	0.5593	96.8	0.6856	96.1
		$\kappa(t;\varpi)$	0.1413	98.8	0.199	98.6	0.1215	99.4	0.1772	99.3	0.1372	98.6	0.1989	99.3
		$\eta(t;\varpi)$	0.3682	96.2	0.4885	95.7	0.3749	95.8	0.4901	95.9	0.3616	96.1	0.4897	95.9
	30	a	0.2756	96.4	0.3872	96.4	0.2407	96.5	0.3616	96.0	0.2652	96.5	0.3857	96.3
		b	0.5109	96.5	0.654	96.6	0.579	98.3	0.7147	97.4	0.4986	97.0	0.6504	96.8
		$\kappa(t;\varpi)$	0.1283	98.8	0.1792	98.1	0.112	98.2	0.1632	99.0	0.1261	97.8	0.1769	98.5
		$\eta(t;\varpi)$	0.3135	95.9	0.4337	95.8	0.2873	96.5	0.4194	96.3	0.3087	95.7	0.4309	96.0
100	20	a	0.3011	96.7	0.4437	95.4	0.362	96.2	0.4134	95.5	0.2933	96.4	0.4359	96.5
		b	0.5417	97.3	0.6807	98.1	0.6651	97.7	0.7479	96.7	0.5233	97.0	0.6651	96.0
		$\kappa(t;\varpi)$	0.1339	97.3	0.1994	99.5	0.1488	98.8	0.1746	99.4	0.1292	98.6	0.1937	99.2
		$\eta(t;\varpi)$	0.3607	96.0	0.494	95.6	0.4926	95.9	0.5447	95.8	0.3516	96.1	0.4893	96.3
	50	a	0.2096	97.7	0.3349	96.8	0.1657	97.1	0.2793	96.5	0.2044	98.0	0.332	96.5
		b	0.3838	96.6	0.5749	97.6	0.4899	96.4	0.6422	96.6	0.3773	97.6	0.5677	97.5
		$\kappa(t;\varpi)$	0.0979	98.1	0.1602	96.6	0.0793	97.4	0.1278	98.3	0.0964	97.3	0.1586	96.8
		$\eta(t;\varpi)$	0.229	96.9	0.3433	96.8	0.2084	96.4	0.3255	96.8	0.2236	97.0	0.3421	96.2
	70	a	0.1678	97.1	0.2858	97.6	0.1446	95.8	0.2472	96.4	0.1686	97.1	0.283	97.8
		b	0.3215	97.9	0.5424	96.7	0.3812	98.4	0.5941	97.0	0.3196	98.1	0.5365	96.9
		$\kappa(t;\varpi)$	0.0784	98.7	0.1378	97.7	0.07	97.2	0.122	97.9	0.0784	98.6	0.1362	97.4
		$\eta(t;\varpi)$	0.183	96.9	0.2849	96.6	0.1664	96.3	0.2671	97.3	0.1816	97.1	0.2846	96.4
150	50	a	0.204	98.1	0.3326	97.4	0.164	96.7	0.2808	96.0	0.2003	97.9	0.328	96.6
		b	0.3763	96.5	0.5635	97.3	0.5049	96.3	0.6546	96.8	0.3638	97.3	0.5547	97.0
		$\kappa(t;\varpi)$	0.0953	96.6	0.1586	98.0	0.0733	98.1	0.1214	98.2	0.092	96.7	0.1572	95.8
		$\eta(t;\varpi)$	0.226	96.6	0.341	96.4	0.2161	95.8	0.3591	95.9	0.2219	96.6	0.3419	96.8
	70	a	0.1637	97.0	0.2808	96.8	0.1205	96.6	0.2334	97.3	0.1628	96.8	0.2789	97.9
		b	0.3139	97.6	0.5302	97.0	0.4238	97.8	0.6221	96.2	0.3081	98.0	0.5246	96.9
		$\kappa(t;\varpi)$	0.0763	98.4	0.1352	97.8	0.0603	99.2	0.1066	98.6	0.076	99.1	0.1341	97.4
		$\eta(t;\varpi)$	0.1783	96.7	0.2825	96.1	0.1604	97.3	0.2712	95.8	0.1781	96.7	0.2807	96.0
	100	a	0.1367	97.3	0.2526	97.5	0.119	97.8	0.2117	96.7	0.1342	96.3	0.251	97.7
		b	0.259	96.7	0.4965	96.4	0.3253	97.0	0.5524	97.3	0.2589	97.0	0.4913	96.7
		$\kappa(t;\varpi)$	0.0625	97.3	0.1245	97.6	0.0556	97.4	0.1063	97.2	0.0623	98.2	0.1246	97.6
		$\eta(t;\varpi)$	0.1486	96.6	0.2427	97.3	0.1382	96.8	0.217	97.6	0.148	96.6	0.2417	96.6

(Continued)

Table 14 (continued)

n	m	Estimate	Scheme I IP AIL	CP	I NIP AIL	CP	II IP AIL	CP	II NIP AIL	CP	III IP AIL	CP	III NIP AIL	CP
	130	a	0.1103	97.5	0.234	97.3	0.1055	96.7	0.2195	97.0	0.11	97.6	0.234	97.3
		b	0.233	97.0	0.4609	98.0	0.2596	97.3	0.5051	97.5	0.2305	97.7	0.4609	97.9
		$\kappa(t;\varpi)$	0.0526	96.8	0.1173	97.4	0.0504	97.0	0.1113	97.4	0.0524	97.3	0.1164	97.4
		$\eta(t;\varpi)$	0.1223	96.7	0.2148	96.7	0.1204	97.4	0.1986	98.0	0.1223	96.7	0.2145	96.6

Table 15: AIL and CP (in %) for ART–PHL distribution for BEs at a, b, SF, and HRF estimates at Set 2 $\equiv (a = 2.5, b = 2.0)$ for Schemes I, II, and III

n	m	Estimate	Scheme I IP AIL	CP	I NIP AIL	CP	II IP AIL	CP	II NIP AIL	CP	III IP AIL	CP	III NIP AIL	CP
50	20	a	0.8008	96.6	0.7707	97.6	0.837	97.0	0.7778	97.2	0.7986	97.7	0.7703	97.3
		b	0.6851	97.5	0.7292	97.4	0.6734	98.6	0.7002	98.4	0.6741	97.4	0.7214	97.4
		$\kappa(t;\varpi)$	0.1383	96.9	0.1338	96.9	0.1396	97.2	0.1299	97.0	0.1384	96.8	0.1326	97.2
		$\eta(t;\varpi)$	1.5256	97.3	1.5162	97.1	1.5896	96.9	1.5688	97.0	1.5227	96.7	1.5168	97.0
	30	a	0.7614	97.4	0.7584	96.8	0.7902	97.8	0.7497	96.5	0.7631	97.9	0.7558	97.0
		b	0.6367	96.6	0.7138	96.7	0.6503	97.7	0.7037	96.4	0.6291	96.4	0.7076	96.6
		$\kappa(t;\varpi)$	0.1317	97.2	0.1309	97.0	0.133	97.8	0.1255	97.8	0.1315	97.0	0.1305	97.0
		$\eta(t;\varpi)$	1.4491	97.6	1.4923	96.9	1.5444	98.0	1.5483	96.8	1.4516	97.7	1.4988	96.4
100	20	a	0.8004	96.8	0.7710	97.3	0.8476	97.2	0.779	97.3	0.7917	96.6	0.7689	97.3
		b	0.6638	97.7	0.7164	97.4	0.6184	96.3	0.6363	97.0	0.6418	97.2	0.6994	97.2
		$\kappa(t;\varpi)$	0.1367	96.9	0.1335	97.3	0.1375	97.8	0.1252	96.9	0.1367	97.1	0.1323	97.2
		$\eta(t;\varpi)$	1.5202	96.8	1.5020	97.0	1.6012	97.0	1.5389	96.9	1.515	96.8	1.5174	96.8
	50	a	0.6958	96.3	0.7267	97.4	0.7402	96.5	0.7305	98.1	0.6932	96.4	0.7242	97.3
		b	0.5330	98.0	0.6705	97.4	0.554	98.1	0.6285	97.5	0.5237	98.1	0.6667	96.4
		$\kappa(t;\varpi)$	0.1134	97.9	0.1234	97.6	0.1154	96.9	0.1137	97.2	0.1132	97.7	0.1226	97.0
		$\eta(t;\varpi)$	1.2831	97.3	1.4472	96.4	1.4423	97.1	1.5311	98.3	1.2814	97.2	1.4453	96.3
	70	a	0.6453	98.1	0.6861	97.5	0.6811	96.9	0.6996	97.9	0.6469	98.1	0.6832	97.6
		b	0.4835	97.9	0.6403	97.1	0.5124	97.7	0.6152	97.2	0.4754	97.8	0.6394	97.1
		$\kappa(t;\varpi)$	0.1076	97.6	0.1131	97.6	0.1105	97.0	0.1071	96.9	0.1071	97.4	0.113	97.6
		$\eta(t;\varpi)$	1.1942	97.5	1.3700	97.1	1.3281	97.1	1.4815	96.2	1.1973	97.4	1.3632	96.8

(Continued)

Table 15 (continued)

n	m	Estimate	Scheme I				Scheme II				Scheme III			
			IP		NIP		IP		NIP		IP		NIP	
			AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP	AIL	CP
150	50	a	0.6938	97.6	0.7245	97.4	0.7637	96.8	0.7516	98.3	0.6915	96.2	0.7254	97.6
		b	0.5214	98.1	0.6614	96.4	0.5214	97.5	0.561	96.5	0.5084	97.7	0.6473	97.0
		$\kappa(t;\varpi)$	0.1120	97.9	0.1229	97.5	0.1169	97.0	0.1136	97.6	0.1119	97.9	0.1218	97.5
		$\eta(t;\varpi)$	1.2936	97.7	1.4307	96.5	1.5122	96.6	1.5527	98.5	1.284	97.4	1.4342	96.3
	70	a	0.6448	97.1	0.6827	97.5	0.7186	96.5	0.7064	97.6	0.6421	96.5	0.6865	97.4
		b	0.4650	97.8	0.6350	96.9	0.4872	97.7	0.5536	96.5	0.4584	98.4	0.631	97.1
		$\kappa(t;\varpi)$	0.1060	97.4	0.1123	97.7	0.1131	96.6	0.1057	98.2	0.1056	97.5	0.1119	97.6
		$\eta(t;\varpi)$	1.1896	97.4	1.3675	96.8	1.44	96.6	1.4926	97.3	1.1792	97.4	1.3728	96.6
	100	a	0.5724	97.4	0.6577	96.3	0.6242	97.8	0.6822	96.5	0.5697	97.5	0.6574	96.4
		b	0.3884	98.2	0.6084	96.1	0.4247	98.9	0.5806	96.2	0.385	98.3	0.6008	97.4
		$\kappa(t;\varpi)$	0.0954	97.5	0.1089	98.1	0.0975	97.6	0.1014	98.1	0.095	97.5	0.109	97.1
		$\eta(t;\varpi)$	1.0656	96.7	1.3557	96.5	1.2145	97.0	1.4337	96.4	1.0586	96.8	1.359	96.1
	130	a	0.5045	97.1	0.6254	97.0	0.5323	97.4	0.6338	97.0	0.5045	97.1	0.6254	97.0
		b	0.3525	97.9	0.5735	98.1	0.3767	95.8	0.581	97.7	0.3521	96.0	0.5689	98.0
		$\kappa(t;\varpi)$	0.0828	98.1	0.1039	96.6	0.0851	98.3	0.1013	96.6	0.0834	98.3	0.1039	97.4
		$\eta(t;\varpi)$	0.9590	97.2	1.2843	97.4	1.0346	97.3	1.3494	97.3	0.9602	97.2	1.2843	97.5

Overall, the simulation results show that the suggested ART-PHL estimators have robust performance in finite samples. It is true that accuracy increases with the increase in sample sizes and the reduction in censoring. However, the increase in variability observed in heavy censoring is consistent with the behavior of likelihood-based estimators. This does not weaken the model's robustness. In comparison to other related models, the ART-PHL technique has better adaptability in scenarios that have complex-shaped hazard rates. Even though it is difficult to develop a precise theoretical basis for asymptotic approximations, the results provide useful guidelines. In scenarios in which the number of observed failures $m \geq 20$, or $m/n \geq 0.3$, asymptotic techniques, such as the normal approximation to the MLE distribution, can be useful for developing.

Although the ML technique has greater computational advantages in terms of direct numerical optimization, the Bayesian technique using the MCMC algorithm is still a viable option in terms of its performance in comparison to the intensive requirement. In this study, the Bayesian technique involved 10,000 iterations for convergence after a burn-in period of 2000 iterations. The Bayesian technique can be said to be highly viable in terms of its performance using current computing power and its ability to produce results with lower standard errors and smaller credible intervals. It can be noted that the computational cost of the MCMC algorithm depends on the number of iterations and not the sample size.

The reason behind the superior performance of Scheme III, in which the removals are equally spaced, is the capacity of this scheme to cover the failure times throughout the entire spectrum of the lifetime. This is in contrast to the front-loading and back-loading of the failure times in the other schemes. The coverage of the failure times is important in the context of the ART-PHL distribution, in which the hazard rate has complex and non-monotone behavior, with the possibility of J-shaped and reversed J-shaped behavior. This makes the bias of the other schemes in the estimation of the reliability characteristics of the distribution less accurate.

8 Concluding Remarks

This study introduces the ART-PHL distribution, a versatile extension of the PHL model. The ART–PHL hazard rate effectively accommodates decreasing, J-shaped, and inverted J-shaped failure patterns. We derive its key statistical properties and evaluate its performance under PT-IIC. Parameter estimation is conducted via ML and Bayesian frameworks, the latter utilizing various loss functions under IP and NIP. Given the analytical intractability of the posterior distributions, we employ the M-H using MCMC algorithm. Monte Carlo simulations demonstrate that Bayesian point and interval estimates generally outperform frequentist methods, maintaining coverage probabilities near 95%. Sensitivity analyses and convergence diagnostics, including trace and autocorrelation plots confirm the stability and robustness of the Bayesian framework.

While the ART–PHL distribution offers significant advantages in modeling non-monotonic hazards and stability under censoring, it reduces to the standard PHL model when strictly monotonic. Furthermore, high censoring rates introduce estimation variability across all models. This study is currently limited to univariate lifetime data and a single engineering application. Future work includes extending the ART-PHL distribution to multivariate and regression settings, exploring alternative censoring schemes, and developing Bayesian hierarchical versions for small samples.

Acknowledgement: The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R913), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Funding Statement: The authors express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2026R913), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Author Contributions: The authors confirm contribution to the paper as follows: Conceptualization, Amal S. Hassan, Maysaa Elmahi Abd Elwahab, Mohammed Elgarhy, Haifa Alqahtani, Rokaya Elmorsy Mohamed; methodology, Amal S. Hassan, Rokaya Elmorsy Mohamed; software, Mohammed Elgarhy; validation, Rokaya Elmorsy Mohamed; formal analysis, Rokaya Elmorsy Mohamed; investigation, Amal S. Hassan; resources, Maysaa Elmahi Abd Elwahab; data curation, Amal S. Hassan; writing—original draft preparation, Amal S. Hassan, Maysaa Elmahi Abd Elwahab, Mohammed Elgarhy, Haifa Alqahtani, Rokaya Elmorsy Mohamed; writing—review and editing, Amal S. Hassan, Mohammed Elgarhy, Rokaya Elmorsy Mohamed; project administration, Mohammed Elgarhy. All authors reviewed and approved the final version of the manuscript.

Availability of Data and Materials: The data that support the findings of this study are available from the Corresponding Author.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alzaatreh A, Famoye F, Lee C. A new method for generating families of continuous distributions. Metron. 2013;71(1):63–79. doi:10.1007/s40300-013-0007-y.
2. Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. J Data Sci. 2014;12(1):53–68. doi:10.6339/jds.201401_12(1).0004.
3. Orji GO, Etaga HO, Almetwally EM, Igbokwe CP, Aguwa OC, Obulezi OJ. A new odd reparameterized exponential transformed-X family of distributions with applications to public health data. Innovat Statist Probab. 2025;1(1):88–118. doi:10.64389/isp.2025.0110.
4. Chipepa F, Oluyede B, Mukubate B. The Topp-Leone Marshall-Olkin-G family of distributions with applications. Int J Stat Probab. 2020;9(4):15–28. doi:10.5539/ijsp.v9n4p15.
5. Cordeiro GM, Alizadeh M, Diniz Marinho PR. The type I half-logistic family of distributions. J Stat Comput Simul. 2016;86(4):707–28. doi:10.1080/00949655.2015.1031233.
6. Nofal ZM, Afify AZ, Yousof HM, Cordeiro GM. The generalized transmuted-G family of distributions. Commun Stat Theory Methods. 2016;46(8):4119–36. doi:10.1080/03610926.2015.1078478.
7. Zhao W, Khosa SK, Ahmad Z, Aslam M, Afify AZ. Type-I heavy tailed family with applications in medicine, engineering and insurance. PLoS One. 2020;15(8):e0237462. doi:10.1371/journal.pone.0237462.
8. Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution. Commun Stat Theory Methods. 2017;46(13):6543–57. doi:10.1080/03610926.2015.1130839.
9. Cordeiro GM, Ortega EM, Popovic BV, Pescim RR. The Lomax generator of distributions: properties, minification process and regression model. Appl Math Comput. 2014;247:465–86.
10. Sule I, Doguwa S, Isah A, Jibril H. The Topp Leone Kumaraswamy-G family of distributions with applications to cancer disease data. J Biomed Eng. 2020;6:40–51. doi:10.18502/jbe.v6i1.4758.
11. Eghwerido JT, Ibidoja OJ, Karokatose GB. The exponential alpha power-G family of distributions: properties and applications. J Stat Manag Syst. 2021;25(2):393–422.
12. Kavya P, Manoharan M. Some parsimonious models for lifetimes and applications. J Stat Comput Simul. 2021;91(18):3693–708. doi:10.1080/00949655.2021.1946064.
13. Elsherpieny EA, Assar SM, Helal T. Odd Kumaraswamy Fréchet generated family of distributions with applications. J Xidian Univ. 2021;15(5):498–511. doi:10.37896/jxu15.5/056.
14. Atchade MN, Agbahide AA, Otodji T, Bogninou MJ, Moussa Djibril A. A new shifted Lomax-X family of distributions: properties and applications to actuarial and financial data. Comput J Math Stat Sci. 2025;4(1):41–71.
15. Alkhairy I, Nagy M, Muse AH, Hussam E. The Arctan-X family of distributions: properties, simulation, and applications to actuarial sciences. Complexity. 2021;2021:4689010.
16. Balakrishnan N. Handbook of the logistic distribution. Boca Raton, FL, USA: CRC Press; 1991.
17. Torabi H, Bagheri FL. Estimation of parameters for an extended generalized half-logistic distribution based on complete and censored data. J Iran Stat Soc. 2010;9(2):171–95. doi:10.1016/j.matcom.2017.11.005.
18. Krishnarani SD. On a power transformation of half-logistic distribution. J Probab Stat. 2016;2016(5):1–10. doi:10.1155/2016/2084236.
19. Oliveira J, Santos J, Xavier C, Trindade D, Cordeiro GM. The McDonald half-logistic distribution: theory and practice. Commun Stat Theory Methods. 2016;45(7):2005–22.
20. Rao BS, Nagendram S, Rosaiah K. Exponential half-logistic additive failure rate model. Int J Sci Res Publ. 2013;3(5):1–10. doi:10.15623/ijret.2015.0410038.
21. Samuel AF, Kehinde OA. A study on transmuted half logistic distribution: properties and application. Int J Stat Distrib Appl. 2019;5(3):54–9. doi:10.11648/j.ijsd.20190503.12.
22. Mustpha MH, Jakperik D, Nasiru S. Tangent exponentiated odd log-logistic Weibull quantile regression with applications to complete and censored data. Egypt Stat J. 2025;69(1):173–205. doi:10.21608/esju.2025.359062.1070.

23. Seo JI, Kang SB. Notes on the exponentiated half logistic distribution. Appl Math Model. 2015;39(21):6491–500. doi:10.1016/j.apm.2015.01.039.
24. Hassan AS, Alsadat N, Elgarhy M, Sapkota LP, Balogun OS, Gemeay AM. A novel asymmetric form of the power half-logistic distribution with statistical inference and real data analysis. Electron Res Arch. 2025;33(2):791–825. doi:10.3934/era.2025036.
25. Ragab I, Elgarhy M. Type II half logistic Ailamujia distribution with numerical illustrations to medical data. Comput J Math Stat Sci. 2025;4(2):379–406.
26. Genc M, Ozbilen O. Sine unit exponentiated half-logistic distribution: theory, estimation, and applications in reliability modeling. Mathematics. 2025;13:1871.
27. Hussam E, Aldallal RA, Sapkota SP, Gemea AM. A modified half-logistic distribution with regression analysis. Eng Rep. 2025;7:e70487.
28. Hassan AS, Abdalla GSS, Elgarhy M, Mendy JT, Almetwally EM. Efficient inference of $P(W < Z)$ using ranked set sampling: the inverse power half-logistic model. J Stat Appl Probab. 2025;14(5):761–85.
29. Chipepa F, Abdelwahab MM, Charumbira WF, Oluyede B, Dingalo N, Ghorbal AB, et al. A new cosine Topp-Leone exponentiated half logistic-G family of distributions with applications. Mathematics. 2026;14(3):472. doi:10.3390/math14030472.
30. Wang X, Gui W. Bayesian estimation of entropy for Burr Type XII distribution under progressive Type-II censored data. Mathematics. 2021;9(4):313. doi:10.3390/math9040313.
31. Abdelall Y, Abdullah A. Bayesian inference on residual Tsallis entropy of the inverse Weibull model: a progressive Type I censoring approach. Egypt Stat J. 2024;68(2):104–28. doi:10.21608/esju.2024.315059.1041.
32. Ahmadini A, El-Saeed A, Almetwally E, Elgarhy M. Classical and bayesian inference of engineering and disability data: using the kavya manoharan power chris-jerry distribution under hybrid censoring. Rev Int Métod Numér Cálc Diseño Ing. 2025;41:80.
33. Salah R, Abuelamayem O. Partially accelerated dependent competing risks model under adaptive type-II progressive censoring based on copulas. Egypt Statist J. 2025;69(1):103–26. doi:10.21608/esju.2025.349560.1061.
34. Balakrishnan N, Aggarwala R. Progressive censoring: theory, methods, and applications. Cham, Switzerland: Springer; 2000.
35. Hassan AS, Abd Elghaffar AM. E-Bayesian and hierarchical Bayesian stress-strength reliability modeling under partially accelerated life testing. J Stat Theory Pract. 2026;20(1):21. doi:10.1007/s42519-025-00532-5.
36. Hassan AS, Moheb S, El-Arishy SM. Reliability analysis under extreme conditions for the Burr XII distribution utilizing upper record values. Hacettepe J Math Stat. 2025;54(6):2399–425.
37. Elgarhy M. Garhy distribution with different estimation methods and applications to engineering and medical data. Rev Int Métodos Numér Cálc Diseño Ing. 2026;42(3):90. doi:10.23967/j.rimni.2026.078324.
38. Hassan AS, Alballa T, Alshawarbeh E, Alsultan R, Nassr SG, Mohamed RE. Efficient entropy estimation for inverted exponentiated Pareto distribution using ranked set sampling: a comparative study. Alex Eng J. 2026;137(3):242–69. doi:10.1016/j.aej.2026.01.018.
39. Hassan AS, Seyam EA, Nassr SG, Mohamed RE. Statistical inference of weighted extropy under outlier influence: an MCMC approach with data applications. Sci Afr. 2025;30:e02919. doi:10.1016/j.sciaf.2025.e02919.
40. Rényi A. On measures of entropy and information. In: Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Oakland, CA, USA: University of California Press; 1960. p. 547–61.
41. Havrda J, Charvat FS. Quantification method of classification processes: concept of structural-entropy. Kybernetika. 1967;3:30–5.
42. Tsallis C. Possible generalization of Boltzmann-Gibbs statistics. J Stat Phys. 1988;52(1–2):479–87. doi:10.1007/bf01016429.

43. Arimoto S. Information-theoretical considerations on estimation problems. Inf Control. 1971;19(3):181–94. doi:10.1016/s0019-9958(71)90065-9.
44. Vander Wiel SA, Meeker WQ. Accuracy of approximate confidence bounds using censored Weibull regression data from accelerated life tests. IEEE Trans Reliab. 1990;39(3):346–51. doi:10.1109/24.103016.
45. Cohen AC. Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. Technometrics. 1965;7(4):579–88. doi:10.1080/00401706.1965.10490300.
46. Lawless JF. Statistical models and methods for lifetime data. Hoboken, NJ, USA: John Wiley & Sons, Inc.; 1982.
47. Greene WH. Econometric analysis. 4th Ed. Englewood Cliffs, NJ, USA: Prentice Hall; 2000.
48. Metropolis N, Rosenbluth AW, Rosenbluth MN, Teller AH, Teller E. Equations of state calculations by fast computing machines. J Chem Phys. 1953;21(6):1087–92. doi:10.1063/1.1699114.
49. Hastings WK. Monte Carlo sampling methods using Markov chains and their applications. Biometrika. 1970;57(1):97–109. doi:10.1093/biomet/57.1.97.
50. Dixit UJ, Nooghabi MJ. Estimation of parameters of gamma distribution in the presence of outliers in right censored samples. Aligar J Statist. 2011;31(4):17–29. doi:10.1016/j.stamet.2011.01.011.
51. Muhammad M, Liu L. A new extension of the generalized half logistic distribution with applications to real data. Entropy. 2019;21(4):339. doi:10.3390/e21040339.
52. Muhammad M. Generalized half-logistic Poisson distributions. Commun Stat Appl Methods. 2017;24(4):353–65.
53. Balakrishnan N, Sandhu RA. A simple simulational algorithm for generating progressive Type-II censored samples. Am Stat. 1995;49(2):229–30. doi:10.1080/00031305.1995.10476150.
54. Dey S, Dey T, Luckett DJ. Statistical inference for the generalized inverted exponential distribution based on upper record values. Math Comput Simul. 2016;120(11):64–78. doi:10.1016/j.matcom.2015.06.012.