A NOTE ON A NUMERICAL BENCHMARK TEST: AN AXISYMMETRIC SHELL UNDER RING LOADS

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SUMMARY

In this paper, a well-known numerical benchmark test which is usually solved with displacement control for low values of the load eccentricity is examined for a complete range of eccentricities of the ring load. For a certain range of the eccentricity the response shows either snap-through or snap-back depending on the controlled variable. Thus, in this range of eccentricities, the test can be used to verify implementations of arc-length algorithms using the displacement controlled solution as a reference. Moreover, results are presented for large eccentricities beyond the applicability of displacement-controlled strategies.

Keywords: Nonlinear solvers, Nonlinear Computational Mechanics, Displacement Control, Arc-length methods.

INTRODUCTION

Nonlinear equation solvers in a general sense (i.e. resolution strategies, acceleration techniques, path following methods, etc.) is a subject of much current research.[1–6] Development and analysis of nonlinear techniques requires to use numerical benchmark tests involving both material and geometrical nonlinearities, recall that there are only a few nonlinear mechanics problems with analytical solution.[7] Thus, a collection of benchmark tests in nonlinear computational mechanics has been produced over the past decade, see [1,2,8] among others. These tests are of extreme importance to researchers, because they can check and validate the implementation and performance of new or even classical algorithms. Moreover, it is important to notice that it is difficult to find benchmarks where the same problem can be solved using a displacement controlled technique and an arc-length method to overcome limit points. That is, simple problems that present snap-through and snap-back depending on the controlled variable.

Here, a well-known example: "an axisymmetric shell under ring loads",[2] is reviewed as a possible tool to verify the implementation of arc-length algorithms. This example presents under certain circumstances snap-through or snap-back depending on the controlled variable. Therefore, the easily obtained displacement controlled solution of the snap-through case can be employed, after simple post-processing, as a reference to validate an arc-length implementation which uses as controlling variable the one that presents snap-back. Moreover, the analysis is extended beyond published results and the applicability of displacement-controlled techniques.

PROBLEM STATEMENT

An axisymmetric shell made of an elastic material is clamped at its border, and a ring load is applied to the shell, causing a deflection v of the center point. Figure 1 shows a description of the problem. This problem is classically solved using a displacement-controlled technique [2] on the center point, namely, by forcing increasing values of v.

Figure 2 shows the load-deflection curves presented in Reference 2, each of them associated to a different *eccentricity* of the ring load, $e = r/R_h$. Such a figure clearly shows that this example presents a *snap-through* tendency as the eccentricity increases.

AN ALTERNATIVE TO DISPLACEMENT CONTROL

The behavior of the structure may also be examined by plotting the load versus the displacement of the *loaded* node, ω. This can be done as a simple post-processing of the displacement-controlled solution, i.e. the solution is obtained controlling v but curves P versus ω are plotted. Figure 3 shows the curves load P versus displacement of the loaded node, ω, associated again to eccentricities 0, 0.25 and 0.42. It is important to note that the last one of these curves ($e = 0.42$) shows *snap-back*, which means that this particular problem *cannot be solved using a displacement-controlled procedure on the displacement* ω *of the loaded node*. Therefore, for $e = 0.42$ the center point displacement-controlled solution may be employed as a reference to validate the implementation of arc-length techniques.

Furthermore, as it will be shown in the following examples, the growing complexity of the structure's behavior (increasing number of limit points) is accentuated as the eccentricity of the load increases. In Figure 4 the eccentricity is raised to a value of 0.50, and a qualitatively identical situation to the $e = 0.42$ case is obtained; that is, the $P - v$ curve presents snap-through and may thus be displacement-controlled, whereas the $P - \omega$ curve presents snap-back. Hence, for the $e = 0.42$ and $e = 0.50$ cases, arc-length techniques (restricted, for instance, to the $P - \omega$ plane) can be verified by post-processing the data from a v-displacement-controlled computation. As a matter of fact, Figure 4b shows the reference solution obtained after post-processing the data from a v-displacement-controlled computation, $\square$, and the arc-length results (restricted to the $P - \omega$ plane), $\times$, which as expected coincide with the reference.

FURTHER CONSIDERATIONS

In the previous section the structure shows snap-through or snap-back depending on the controlled variable. However, when the eccentricity of the ring load is increased, for instance up to 0.60, this mixed behavior is not present anymore. Figure 5a shows the $P-v$ response obtained with a displacement-controlled technique, and the behavior seems, at least qualitatively, correct. But if the $P-\omega$ curve is plotted as a post-process of the previous results, see Figure 5b, serious doubts on the validity of the computations obviously appear. In this case, the *correct* response of the structure (i.e. initial *negative* deflections for the $P-v$ curve, and a smooth initial slope for the $P-\omega$ curve) must be determined with an arc-length algorithm, see Figure 6. This solution can never be captured via a displacement-controlled strategy on any of the presented variables. In fact, if in an attempt to capture the correct response the initial displacement increments of v were drastically reduced, the computations would follow an equilibrium path with *negative* values of the load. Thus, an arc-length procedure must be employed. In this particular case, a cylindrical arc-length solution with automatic control of the arc length [1] has been employed to determine Figure 6.

Similar conclusions can be drawn if the eccentricity of the load is raised up to $e = 0.65$. Figure 7 presents the unrealistic results obtained with a v-displacement-controlled strategy. On the other hand, Figure 8 presents the correct behavior obtained with an arc-length method. For even larger values of e, arc-length techniques are definitely necessary, see Figures 9 and 10.

ANALYSIS OF THE RESULTS AND CONCLUSIONS

A well-known spherical shell test has been analyzed from a numerical point of view for a complete range of eccentricities of the ring load, e. Notice that from a physical point of view other concerns not addressed here and apart from the numerical resolution of the nonlinear equations, must be taken into account. Both the displacement of the center point, v, and the displacement under the ring load, ω, have been employed as the controlling variables. Depending on the value of e the following results have been obtained, see Figure 11:

(1) for low values of e, the problem can be solved with displacement control using either v or ω, because the curves $P - v$ and $P - \omega$ do not show snap-back.

(2) for intermediate values of e, the displacement control is still valid for v, but the arc-length control is needed for ω because the $P - \omega$ curve presents snap-back. Therefore, in the intermediate range, the displacement controlled solution (with v) can be employed to validate the arc-length implementation (with ω).

(3) for large values of e, the arc-length method must be employed with either v or ω, because both $P - v$ and $P - \omega$ curves show complex snap-back behavior.

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FIGURE CAPTIONS

Figure 1: Problem statement.

Figure 2: Load P vs. deflection of central node v; $e = 0$, $e = 0.25$, $e = 0.42$.

Figure 3: Load P vs. deflection of loaded node ω; $e = 0$, $e = 0.25$, $e = 0.42$.

Figure 4: Response for an eccentricity $e = 0.50$: (a) Load P vs. deflection of central node v; (b) Load P vs. deflection of loaded node ω.

Figure 5: Displacement-controlled solution for an eccentricity $e = 0.60$: (a) P versus v; (b) P versus ω.

Figure 6: Arc-length solution for an eccentricity $e = 0.60$: (a) P versus v; (b) P versus ω.

Figure 7: Displacement-controlled solution for an eccentricity $e = 0.65$: (a) P versus v; (b) P versus ω.

Figure 8: Arc-length solution for an eccentricity $e = 0.65$: (a) P versus v; (b) P versus ω.

Figure 9: Response for an eccentricity $e = 0.70$: (a) Load P vs. deflection of central node v; (b) Load P vs. deflection of loaded node ω.

Figure 10: Response for an eccentricity $e = 0.80$: (a) Load P vs. deflection of central node v; (b) Load P vs. deflection of loaded node ω.

Figure 11: Schematic representation of the controlling technique for each variable and the range of eccentricities.

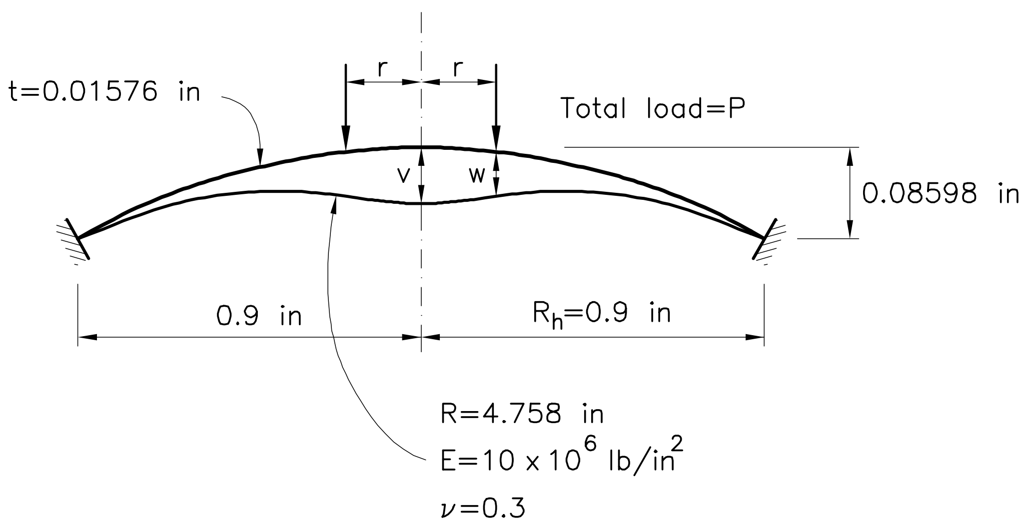
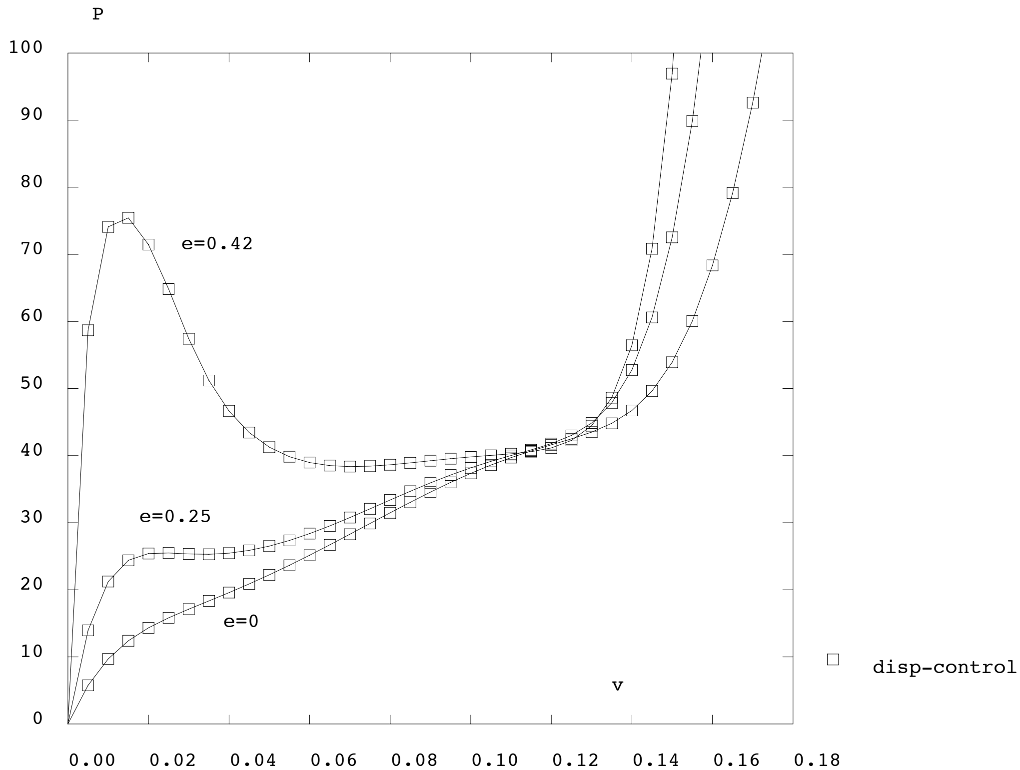

t=0.01576 in
r
r
Total load=P
v
w
0.08598 in
0.9 in
R_h=0.9 in
R=4.758 in
E=10 x 10^6 lb/in^2
ν=0.3

Figure 1



P
100
90
80
70
60
50
40
30
20
10
0
e=0.42
e=0.25
e=0
v
disp-control
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18

Figure 2

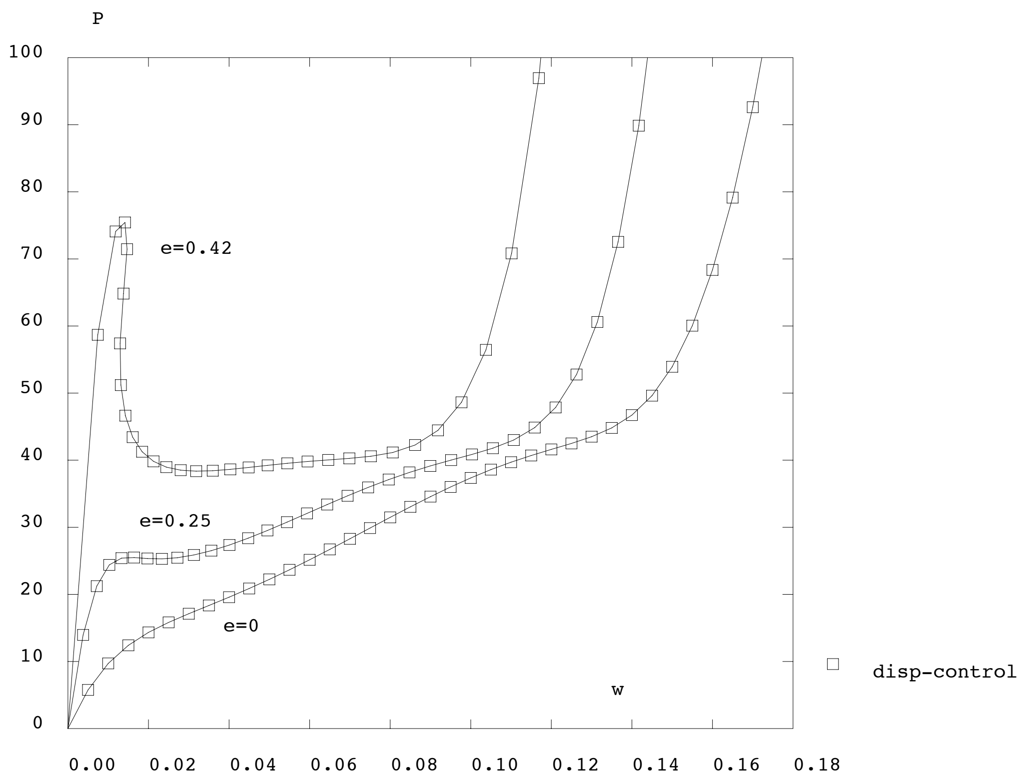
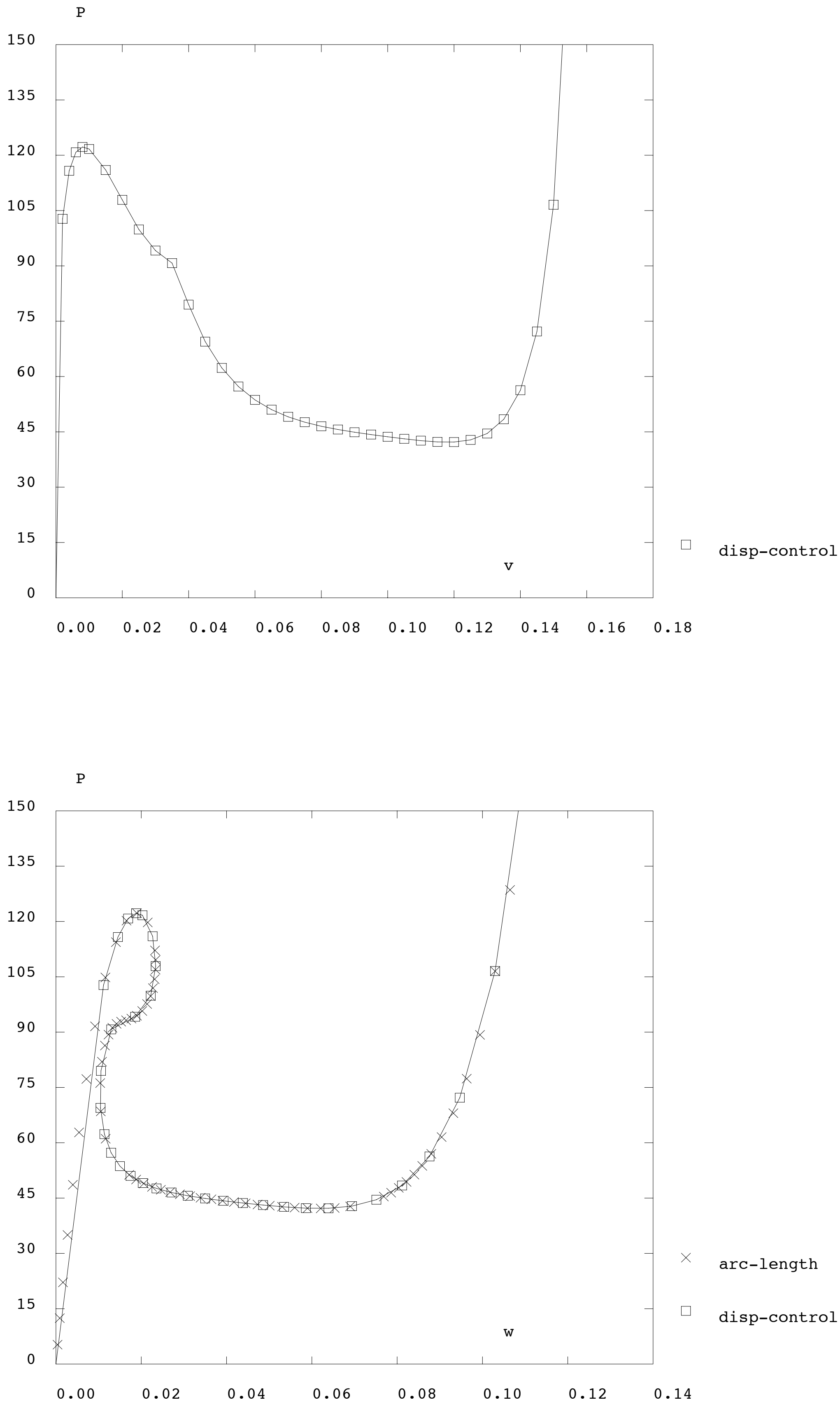

P
100
90
80
70
60
50
40
30
20
10
0
e=0.42
e=0.25
e=0
w
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
disp-control

Figure 3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135
120
105
90
75
60
45
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15
0
v
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0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
disp-control
P
150
135
120
105
90
75
60
45
30
15
0
w
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
arc-length
disp-control

Figure 4

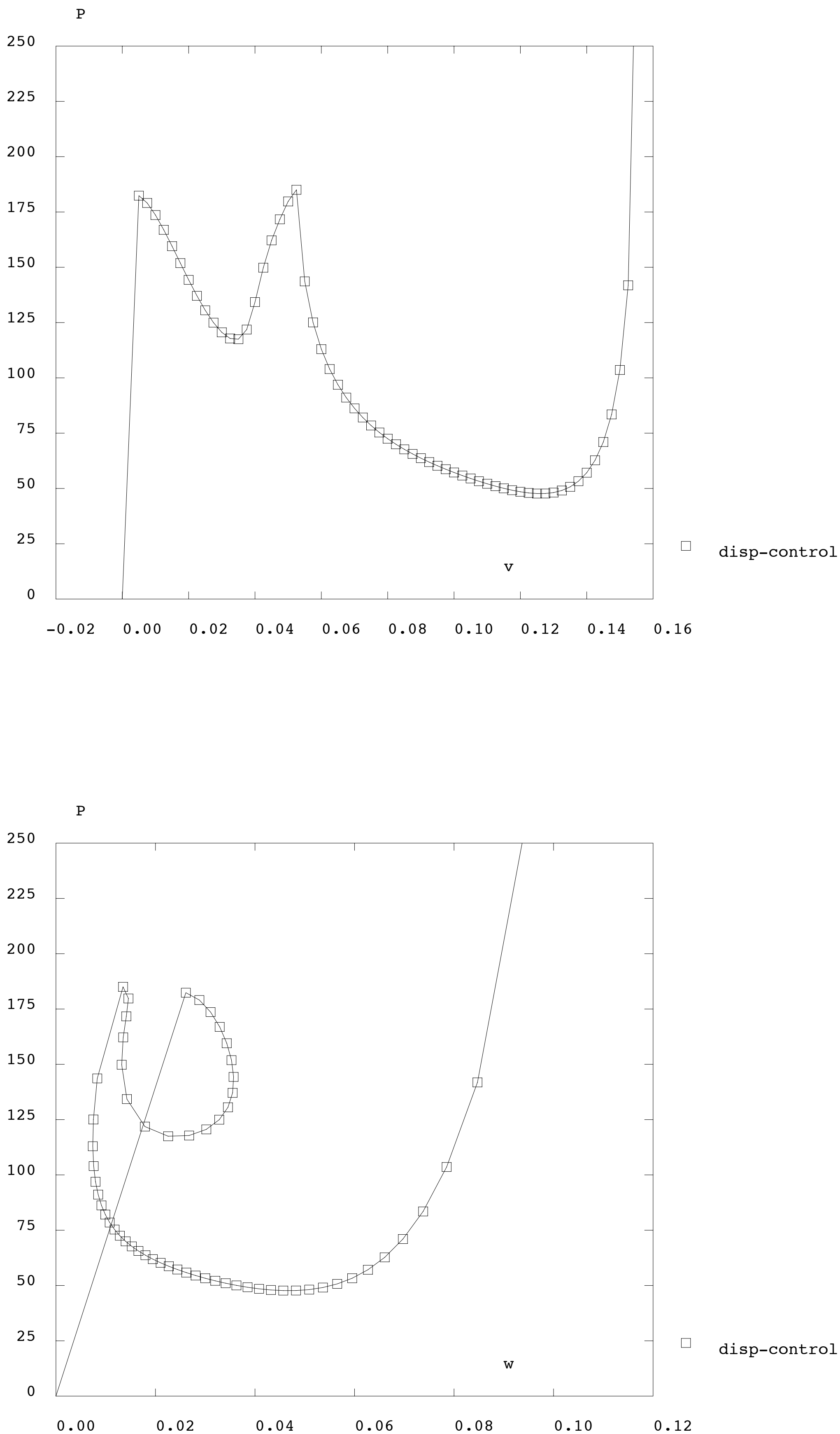

P
250
225
200
175
150
125
100
75
50
25
0
-0.02
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
v
disp-control
P
250
225
200
175
150
125
100
75
50
25
0
0.00
0.02
0.04
0.06
0.08
0.10
0.12
w
disp-control

Figure 5

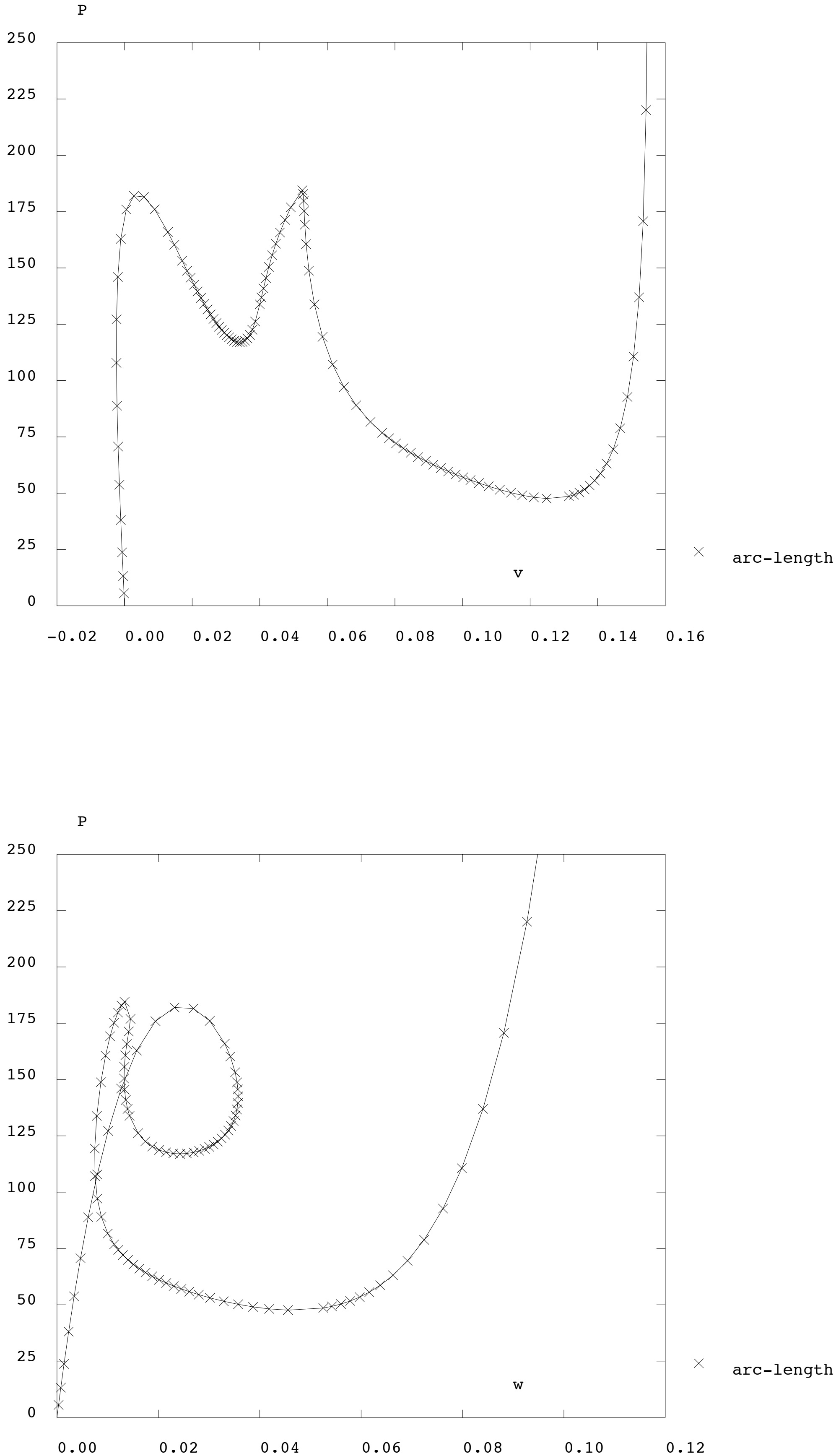
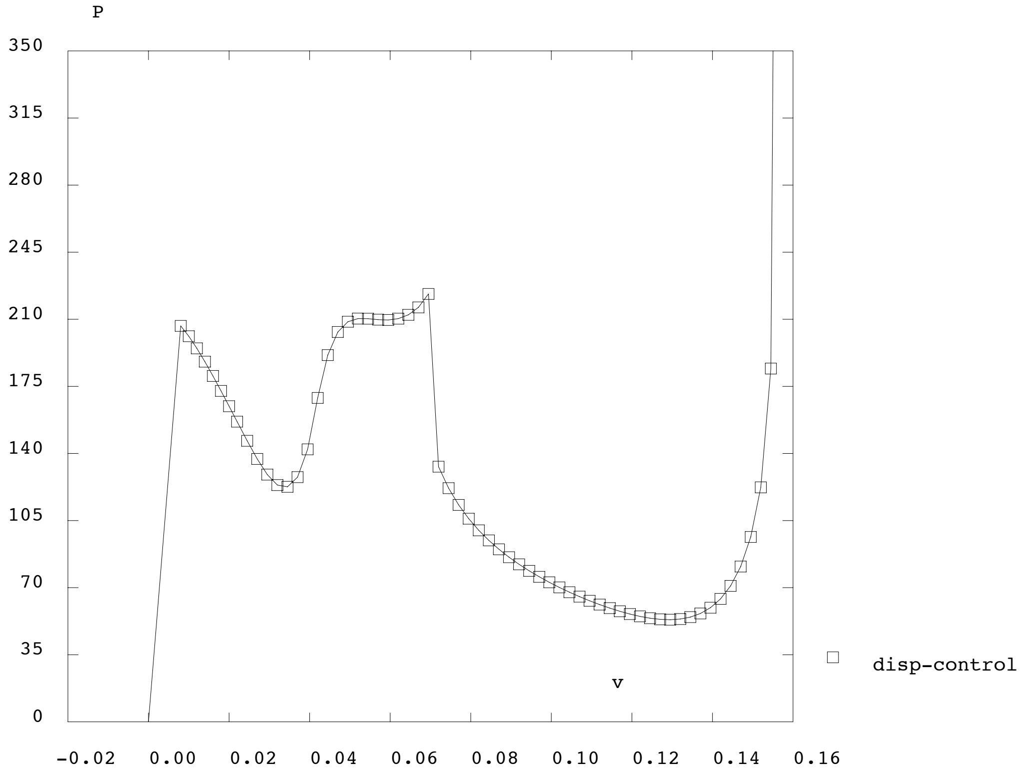

P
250
225
200
175
150
125
100
75
50
25
0
v
-0.02 0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16
arc-length
P
250
225
200
175
150
125
100
75
50
25
0
w
0.00 0.02 0.04 0.06 0.08 0.10 0.12
arc-length

Figure 6


P
350
315
280
245
210
175
140
105
70
35
0
v
disp-control
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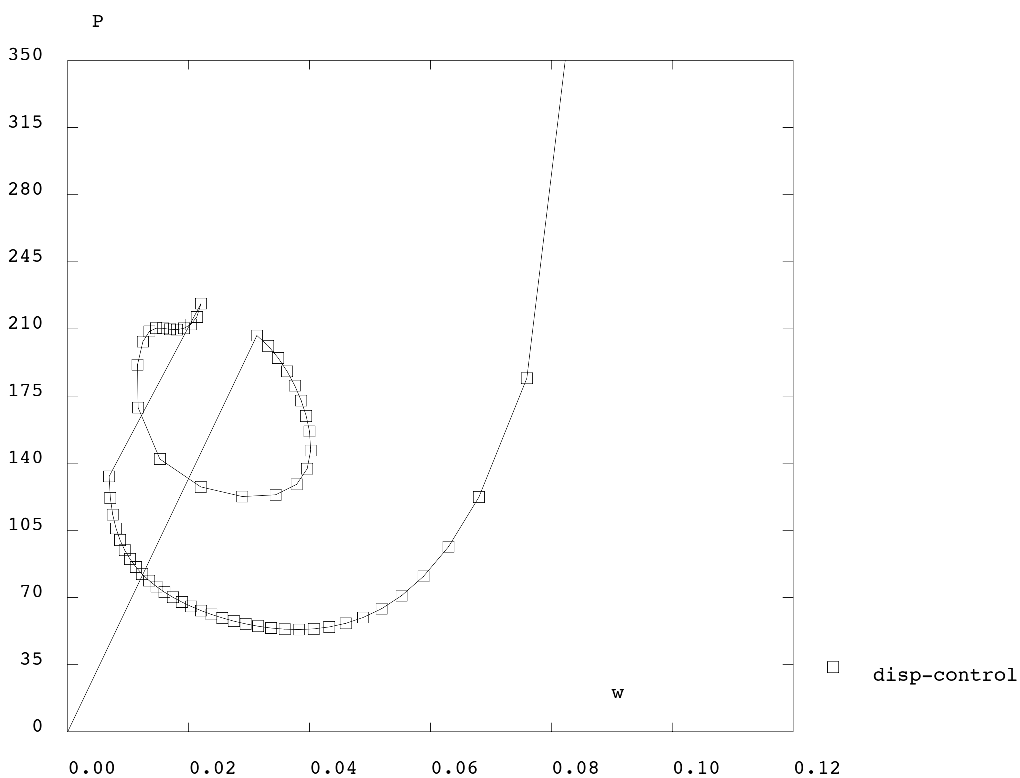
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16


P
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315
280
245
210
175
140
105
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35
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w
disp-control
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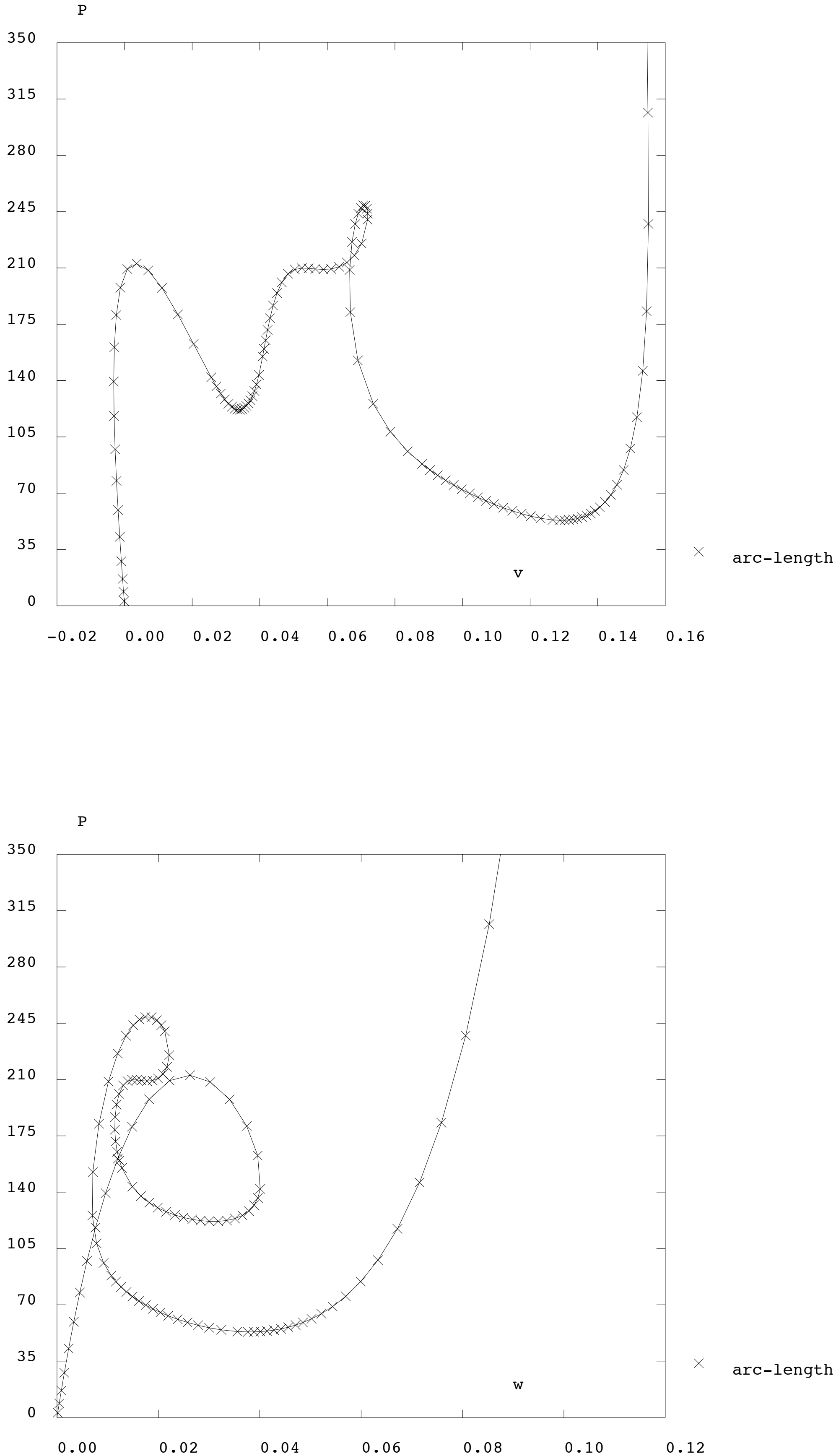
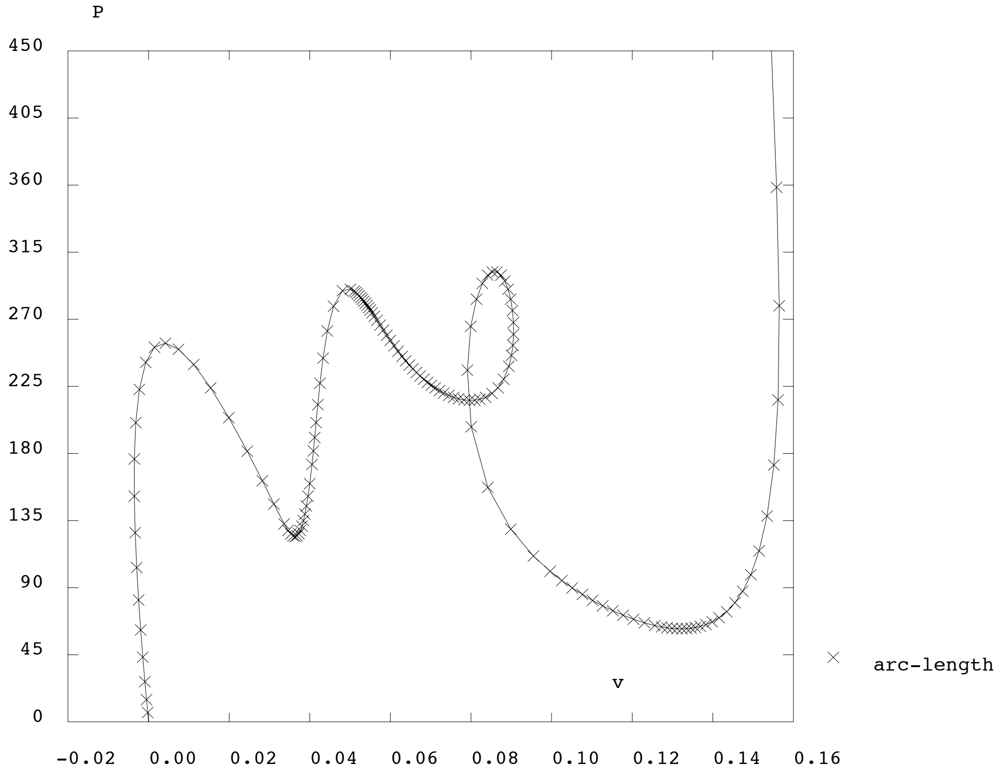
0.04
0.06
0.08
0.10
0.12

Figure 7

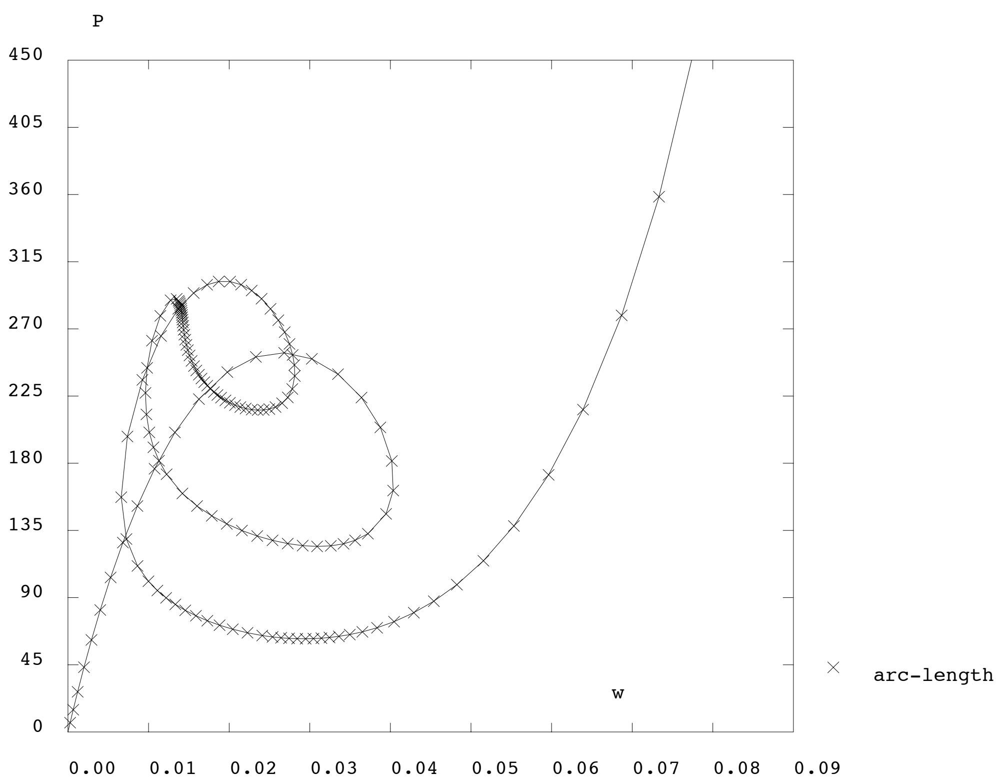


P
350
315
280
245
210
175
140
105
70
35
0
-0.02
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
v
arc-length
P
350
315
280
245
210
175
140
105
70
35
0
0.00
0.02
0.04
0.06
0.08
0.10
0.12
w
arc-length

Figure 8

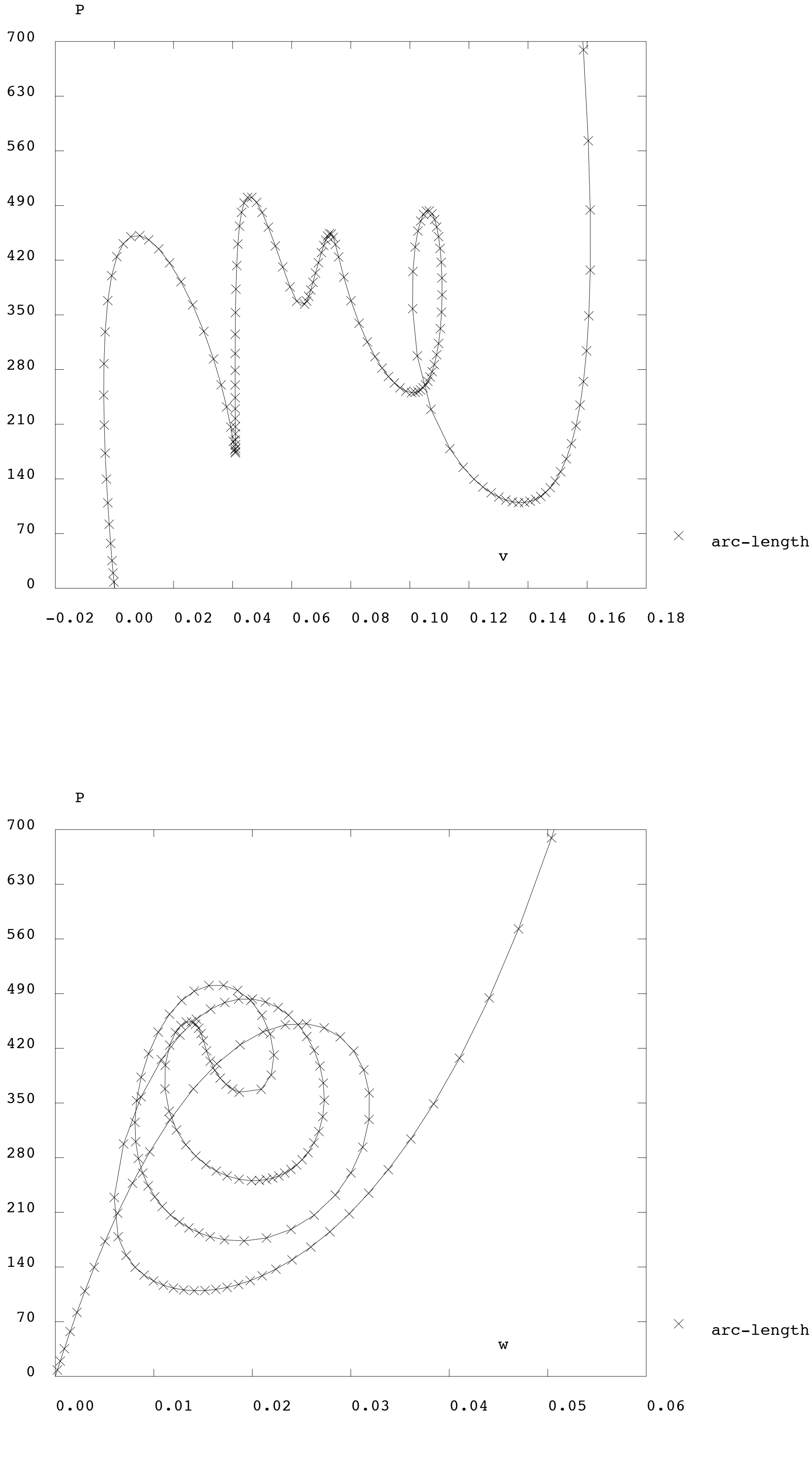


P
450
405
360
315
270
225
180
135
90
45
0
v
-0.02
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
arc-length



P
450
405
360
315
270
225
180
135
90
45
0
w
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
arc-length

Figure 9


P
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630
560
490
420
350
280
210
140
70
0
-0.02 0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16 0.18
v
arc-length
P
700
630
560
490
420
350
280
210
140
70
0
0.00 0.01 0.02 0.03 0.04 0.05 0.06
w
arc-length

Figure 10

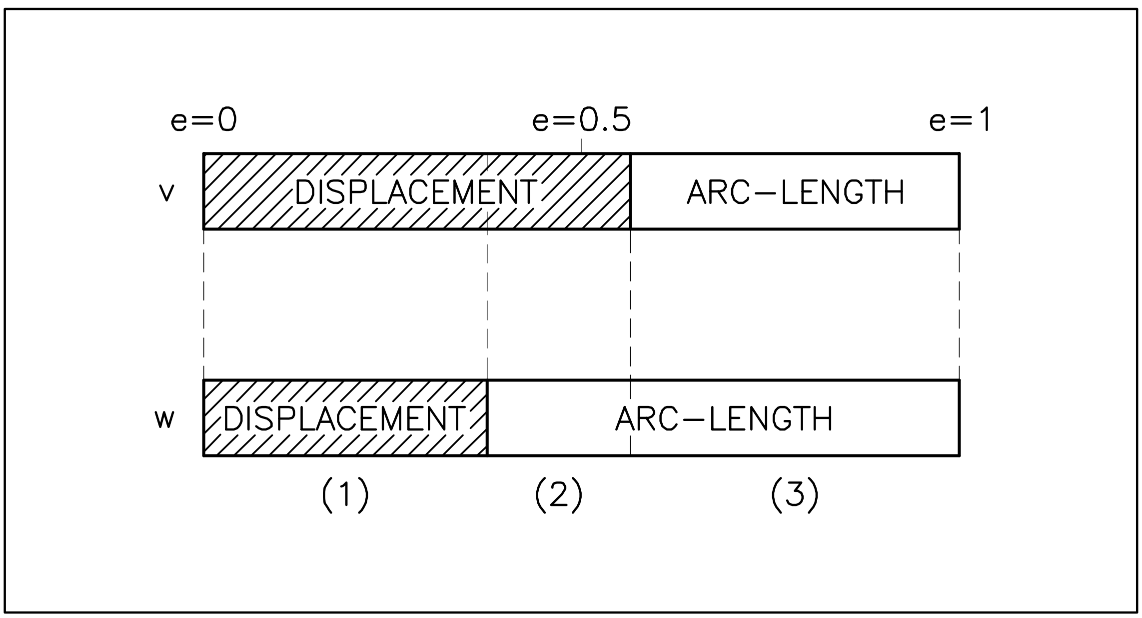
e=0
e=0.5
e=1
v
DISPLACEMENT
ARC-LENGTH
w
DISPLACEMENT
ARC-LENGTH
(1)
(2)
(3)

Figure 11