HIGHER-ORDER SPACE-TIME CONTINUOUS GALERKIN METHODS FOR THE WAVE EQUATION

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Abstract. We consider a space-time variational formulation of the second-order wave equation, where integration by parts is also applied with respect to the time variable. Conforming tensor-product finite element discretisations with piecewise polynomials of this space-time variational formulation require a CFL condition to ensure stability. To overcome this restriction in the case of piecewise multilinear, continuous ansatz and test functions, a stabilisation is well-known, which leads to an unconditionally stable space-time finite element method. In this work, we generalise this stabilisation idea from the lowest-order case to the higher-order case, i.e. to an arbitrary polynomial degree. We give numerical examples for a one-dimensional spatial domain, where the unconditional stability and optimal convergence rates in space-time norms are illustrated.

1 INTRODUCTION

Standard approaches for the numerical solution of hyperbolic initial-boundary value problems are usually based on semi-discretisations in space and time, where the discretisation in space and time is split accordingly. In contrast to these approaches, space-time methods discretise time-dependent partial differential equations without separating the temporal and spatial directions. In this work, the homogeneous Dirichlet problem for the second-order wave equation,

$$\left. \begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \partial\Omega \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned} \right\} \quad (1)$$

serves as a model problem, where $\Omega = (0,L)$ is an interval for $d = 1$, or Ω is polygonal for $d = 2$, or Ω is polyhedral for $d = 3$, $T > 0$ is a terminal time and f is a given right-hand side. To derive a space-time variational formulation, we define the space-time Sobolev spaces

$$\begin{aligned} H_{0,0}^{1,1}(Q) &:= L^2(0,T;H_0^1(\Omega)) \cap H_0^1(0,T;L^2(\Omega)) \subset H^1(Q), \\ H_{0,0}^{1,1}(Q) &:= L^2(0,T;H_0^1(\Omega)) \cap H_0^1(0,T;L^2(\Omega)) \subset H^1(Q) \end{aligned}$$

with the Hilbertian norms

$$\|v\|_{H_{0,0}^{1,1}(Q)} := \|v\|_{H_{0,0}^{1,1}(Q)} := |v|_{H^1(Q)} := \left(\int_0^T \int_{\Omega} (|\partial_t v(x,t)|^2 + |\nabla_x v(x,t)|^2) dx dt \right)^{1/2},$$

where $v \in H_{0,0}^1(0, T; L^2(\Omega))$ satisfies $\|v(\cdot, 0)\|_{L^2(\Omega)} = 0$ and $w \in H_{0,0}^1(0, T; L^2(\Omega))$ fulfils $\|w(\cdot, T)\|_{L^2(\Omega)} = 0$, see [5] for more details. The bilinear form

$$a(\cdot, \cdot): H_{0,0}^{1,1}(Q) \times H_{0,0}^{1,1}(Q) \rightarrow \mathbb{R},$$

defined by the variational identity

$$a(u, w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)}$$

for $u \in H_{0,0}^{1,1}(Q)$ and $w \in H_{0,0}^{1,1}(Q)$, is continuous, i.e. the estimate

$$\forall u \in H_{0,0}^{1,1}(Q): \forall w \in H_{0,0}^{1,1}(Q): |a(u, w)| \leq |u|_{H^1(Q)} |w|_{H^1(Q)}$$

holds true due to the Cauchy-Schwarz inequality. The space-time variational formulation of (1) is to find $u \in H_{0,0}^{1,1}(Q)$ such that

$$\forall w \in H_{0,0}^{1,1}(Q): a(u, w) = \langle f, w \rangle_{L^2(Q)}, \quad (2)$$

where $f \in L^2(Q)$ is a given right-hand side. Note that the initial condition $u(\cdot, 0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot, 0) = 0$ is incorporated in a weak sense. The following existence and uniqueness theorem is proven in [1, Theorem 3.2 in Chapter IV], see also [3, 5, 8].

Theorem 1.1 For $f \in L^2(Q)$, a unique solution $u \in H_{0,0}^{1,1}(Q)$ of the variational formulation (2) exists and the stability estimate

$$|u|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)}$$

holds true.

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Note that the solution operator

$$\mathcal{L}: L^2(Q) \rightarrow H_{0,0}^{1,1}(Q), \quad \mathcal{L}f := u,$$

of Theorem 1.1 is not an isomorphism, i.e. $\mathcal{L}$ is not surjective, see [4, 5] for more details. In this work, for simplicity, we only consider homogeneous initial conditions, where inhomogeneous initial conditions can be treated analogously as in [1, 7, 8].

A conforming tensor-product space-time discretisation of (2) with piecewise polynomial, continuous ansatz and test functions requires a CFL condition

$$h_t \leq C h_x \quad (3)$$

with a constant $C > 0$, depending on the constant of a spatial inverse inequality, where h_t and h_x are the mesh sizes in time and space. For a one-dimensional spatial domain Ω , i.e. $d = 1$, and piecewise multilinear, continuous ansatz and test functions, the CFL condition (3) reads as

$$h_t < h_x$$

for uniform meshes with uniform mesh sizes h_t and h_x , see [3, 5]. To overcome the CFL condition (3), the stabilised space-time finite element method to find $u_h \in (V_{h_x,0}^1(\Omega) \otimes S_{h_t}^1(0,T)) \cap H_{0,0}^{1,1}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{\alpha=1}^d \langle \partial_{x_\alpha} u_h, Q_{h_t}^0 \partial_{x_\alpha} w_h \rangle_{L^2(Q)} = \langle f, w_h \rangle_{L^2(Q)} \quad (4)$$

for all $w_h \in (V_{h_x,0}^1(\Omega) \otimes S_{h_t}^1(0,T)) \cap H_{0,0}^{1,1}(Q)$ was analysed in [2, 5, 8], where

$$Q_{h_t}^0 : L^2(Q) \rightarrow L^2(\Omega) \otimes S_{h_t}^0(0,T) \quad (5)$$

is the extended L^2 projection on the space of the temporal piecewise constant functions and $V_{h_x,0}^1(\Omega) \otimes S_{h_t}^1(0,T)$ is the space of piecewise multilinear, continuous functions, see Section 2 for the notations. The main results for this proposed space-time finite element method (4) are the unconditional stability, i.e. no CFL condition is needed, and the space-time error estimates with

$$h := \max\{h_x, h_t\}, \quad h_x = \max_k h_{x,k}, \quad h_t = \max_\ell h_{t,\ell},$$

which are summarised in the following theorem, where its proof is contained in [2, 5].

Theorem 1.2 *There exists a unique solution $u_h \in (V_{h_x,0}^1(\Omega) \otimes S_{h_t}^1(0,T)) \cap H_{0,0}^{1,1}(Q)$ of (4), satisfying the $L^2(Q)$ stability estimate*

$$\|u_h\|_{L^2(Q)} \leq \frac{4}{\pi} T^2 \|f\|_{L^2(Q)}.$$

Further, let the solution u of (1) and Ω be sufficiently regular. Then, the unique solution $u_h \in (V_{h_x,0}^1(\Omega) \otimes S_{h_t}^1(0,T)) \cap H_{0,0}^{1,1}(Q)$ of (4) fulfils the space-time error estimates

$$\|u - u_h\|_{L^2(Q)} \leq Ch^2,$$

$$|u - u_h|_{H^1(Q)} \leq Ch$$

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where, for the $H^1(Q)$ error estimate, a spatial inverse inequality is additionally assumed.

In this work, we generalise this stabilisation idea from the linear case to the higher-order case. In greater detail, we introduce a new stabilised space-time finite element method of tensor-product type with globally continuous ansatz and test functions, which are piecewise polynomials of an arbitrary polynomial degree p , leading to unconditional stability and optimal convergence rates in the space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$. In other words, the result of Theorem 1.2 is generalised to an arbitrary polynomial degree p . The rest of the paper is organised as follows: In Section 2, notations of the used finite element spaces and L^2 projections are fixed. Section 3 introduces the new space-time finite element method. Numerical examples for a one-dimensional spatial domain and piecewise polynomials of higher-order are presented in Section 4. Finally, we draw some conclusions in Section 5.

2 PRELIMINARIES

In this section, notations of the used finite element spaces and L^2 projections are stated. For this purpose, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or

polyhedral for $d = 3$. For a tensor-product ansatz, we consider admissible decompositions

$$\overline{Q} = \overline{\Omega} \times [0, T] = \bigcup_{k=1}^{N_x} \overline{\omega}_k \times \bigcup_{\ell=1}^{N_t} [t_{\ell-1}, t_\ell]$$

with $N := N_x \cdot N_t$ space-time elements, where the time intervals $\tau_\ell := (t_{\ell-1}, t_\ell)$ with mesh sizes $h_{t,\ell} = t_\ell - t_{\ell-1}$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \dots < t_{N_t-1} < t_{N_t} = T$$

of the time interval $(0, T)$. The maximal and the minimal time mesh sizes are denoted by $h_t := h_{t,\max} := \max_\ell h_{t,\ell}$ and $h_{t,\min} := \min_\ell h_{t,\ell}$, respectively. For the spatial domain Ω , we consider a shape-regular sequence $(\mathcal{T}_v)_{v \in \mathbb{N}}$ of admissible decompositions

$$\mathcal{T}_v := \{\omega_k \subset \mathbb{R}^d : k = 1, \dots, N_x\}$$

of Ω into finite elements $\omega_k \subset \mathbb{R}^d$ with mesh sizes $h_{x,k}$, the maximal mesh size $h_x := h_{x,\max} := \max_k h_{x,k}$ and the minimal mesh size $h_{x,\min} := \min_k h_{x,k}$. The spatial elements ω_k are intervals for $d = 1$, triangles or quadrilaterals for $d = 2$, and tetrahedra or hexahedra for $d = 3$. Next, for a fixed polynomial degree $p \in \mathbb{N}$, we introduce the finite element space

$$Q_h^p(Q) := V_{h_x,0}^p(\Omega) \otimes S_{h_t}^p(0, T)$$

of piecewise polynomial, continuous functions, i.e.

$$V_{h_x,0}^p(\Omega) := V_{h_x}^p(\Omega) \cap H_0^1(\Omega) \subset H_0^1(\Omega), \quad S_{h_t}^p(0, T) \subset H^1(0, T)$$

with $V_{h_x}^p(\Omega) \in \{S_{h_x}^p(\Omega), Q_{h_x}^p(\Omega)\}$. Here,

$$S_{h_t}^p(0, T) := \{v_{h_t} \in C[0, T] : \forall \ell \in \{1, \dots, N_t\} : v_{h_t}|_{\tau_\ell} \in \mathbb{P}^p(\tau_\ell)\}$$

denotes the space of piecewise polynomial, continuous functions on intervals, where $\mathbb{P}^p(A)$ is the space of polynomials on a subset $A \subset \mathbb{R}^d$ of global degree at most p . Analogously,

$$S_{h_x}^p(\Omega) := \{v_{h_x} \in C(\overline{\Omega}) : \forall \omega \in \mathcal{T}_v : v_{h_x}|_{\overline{\omega}} \in \mathbb{P}^p(\overline{\omega})\}$$

is the space of piecewise polynomial, continuous functions on intervals ($d = 1$), triangles ($d = 2$), or tetrahedra ($d = 3$). Moreover,

$$Q_{h_x}^p(\Omega) := \{v_{h_x} \in C(\overline{\Omega}) : \forall \omega \in \mathcal{T}_v : v_{h_x}|_{\overline{\omega}} \in \mathbb{Q}^p(\overline{\omega})\}$$

is the space of piecewise polynomial, continuous functions on intervals ($d = 1$), quadrilaterals ($d = 2$), or hexahedra ($d = 3$), where $\mathbb{Q}^p(A)$ is the space of polynomials on a subset $A \subset \mathbb{R}^d$ of degree at most p in each variable. The temporal nodal basis functions of $S_{h_t}^p(0, T)$ are denoted by ϕ_n^p for $n = 0, \dots, pN_t$, and ψ_j^p , $j = 1, \dots, M_x$, are the spatial nodal basis functions of $V_{h_x,0}^p(\Omega)$, i.e.

$$S_{h_t}^p(0, T) = \text{span}\{\phi_n^p\}_{n=0}^{pN_t} \quad \text{and} \quad V_{h_x,0}^p(\Omega) = \text{span}\{\psi_j^p\}_{j=1}^{M_x}.$$

For the stabilisation of the new space-time finite element method, we also need the spaces of piecewise polynomial, discontinuous functions

$$S_{h_t}^{q,\text{disc}}(0, T) := \{v_{h_t} \in L^1(0, T) : \forall \ell \in \{1, \dots, N_t\} : v_{h_t}|_{\tau_\ell} \in \mathbb{P}^q(\tau_\ell)\},$$

where $q \in \mathbb{N}_0$ is a fixed polynomial degree. For a given function $v \in L^2(Q)$, the extended L^2 projection $Q_{h_t}^{q,\text{disc}} v \in L^2(\Omega) \otimes S_{h_t}^{q,\text{disc}}(0, T)$ on the space $L^2(\Omega) \otimes S_{h_t}^{q,\text{disc}}(0, T)$ of piecewise polynomial, discontinuous functions with respect to the time variable is defined by

$$\langle Q_{h_t}^{q,\text{disc}} v, v_{h_t} \rangle_{L^2(Q)} = \langle v, v_{h_t} \rangle_{L^2(Q)}$$

for all $v_{h_t} \in L^2(\Omega) \otimes S_{h_t}^{q,\text{disc}}(0, T)$, satisfying the stability estimate

$$\|Q_{h_t}^{q,\text{disc}} v\|_{L^2(Q)} \leq \|v\|_{L^2(Q)}. \quad (6)$$

Note that $Q_{h_t}^0 = Q_{h_t}^{0,\text{disc}}$ is the extended L^2 projection (5) on the space of the temporal piecewise constant functions $L^2(\Omega) \otimes S_{h_t}^0(0, T) = L^2(\Omega) \otimes S_{h_t}^{0,\text{disc}}(0, T)$. Analogously, for a solely time-dependent function $w \in L^2(0, T)$, we denote $Q_{h_t}^{q,\text{disc}} w \in S_{h_t}^{q,\text{disc}}(0, T)$ as the $L^2(0, T)$ projection on the space $S_{h_t}^{q,\text{disc}}(0, T)$ of piecewise polynomial, discontinuous functions, defined by

$$\langle Q_{h_t}^{q,\text{disc}} w, w_{h_t} \rangle_{L^2(0, T)} = \langle w, w_{h_t} \rangle_{L^2(0, T)}$$

for all $w_{h_t} \in S_{h_t}^{q,\text{disc}}(0, T)$. We use the same notation $Q_{h_t}^{q,\text{disc}}$ for solely time-dependent functions and functions, which depend on (x, t) , since for a function $v \in L^2(Q)$ with $v(x, t) = z(x)w(t)$, $z \in L^2(\Omega)$, $w \in L^2(0, T)$, the equality

$$Q_{h_t}^{q,\text{disc}} v(x, t) = z(x) Q_{h_t}^{q,\text{disc}} w(t), \quad (x, t) \in Q,$$

holds true.

3 NEW STABILISED SPACE-TIME FINITE ELEMENT METHOD

In this section, we introduce a new stabilised space-time finite element method with continuous ansatz and test functions, which are piecewise polynomials of arbitrary polynomial degree $p \in \mathbb{N}$ with respect to the spatial variable and the temporal variable. For this purpose, we fix a polynomial degree $p \in \mathbb{N}$ and we introduce the perturbed bilinear form

$$a_h(\cdot, \cdot) : Q_h^p(Q) \cap H_{0;0}^{1,1}(Q) \times Q_h^p(Q) \cap H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$$

by defining

$$a_h(u_h, w_h) := -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{\alpha=1}^d \langle \partial_{x_\alpha} u_h, Q_{h_t}^{p-1,\text{disc}} \partial_{x_\alpha} w_h \rangle_{L^2(Q)}$$

for $u_h \in Q_h^p(Q) \cap H_{0;0}^{1,1}(Q)$, $w_h \in Q_h^p(Q) \cap H_{0;0}^{1,1}(Q)$. Note that the function $\partial_{x_\alpha} w_h$, $\alpha = 1, \dots, d$, fulfils

$$\partial_{x_\alpha} w_h \in L^2(\Omega) \otimes S_{h_t}^p(0, T),$$

i.e. $\partial_{x_\alpha} w_h$ is still a piecewise polynomial of degree p with respect to the temporal variable. The perturbed bilinear form $a_h(\cdot, \cdot)$ is continuous since the Cauchy-Schwarz inequality and the $L^2(Q)$ stability (6) of $\mathcal{Q}_h^{p-1, \text{disc}}$ yield

$$|a_h(u_h, w_h)| \leq |u_h|_{H^1(Q)} |w_h|_{H^1(Q)}$$

for all $u_h \in \mathcal{Q}_h^p(Q) \cap H_{0;0}^{1,1}(Q)$, $w_h \in \mathcal{Q}_h^p(Q) \cap H_{0;0}^{1,1}(Q)$. The perturbed variational formulation, corresponding to (2), is to find $u_h \in \mathcal{Q}_h^p(Q) \cap H_{0;0}^{1,1}(Q)$ such that

$$\forall w_h \in \mathcal{Q}_h^p(Q) \cap H_{0;0}^{1,1}(Q): a_h(u_h, w_h) = \langle f, w_h \rangle_{L^2(Q)}. \quad (7)$$

This perturbed variational formulation (7) coincides with the perturbed variational formulation (4) for $p = 1$. In other words, the new perturbed variational formulation (7) is a generalisation of the perturbed variational formulation (4) from $p = 1$ to arbitrary $p \in \mathbb{N}$. The numerical analysis, i.e. an analogous result as Theorem 1.2, of the perturbed variational formulation (7) is far beyond the scope of this contribution, we refer to [6].

The discrete variational formulation (7) is equivalent to the linear system

$$K_h \underline{u} = \underline{f} \quad (8)$$

with the system matrix

$$K_h := -A_{h_t} \otimes M_{h_x} + \tilde{M}_{h_t} \otimes A_{h_x} \in \mathbb{R}^{M_x \cdot pN_t \times M_x \cdot pN_t},$$

where $M_{h_x}, A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ are the mass and stiffness matrix with respect to the spatial variable, which are given by

$$\begin{aligned} M_{h_x}[i, j] &= \langle \Psi_j^p, \Psi_i^p \rangle_{L^2(\Omega)}, & i, j &= 1, \dots, M_x, \\ A_{h_x}[i, j] &= \langle \nabla_x \Psi_j^p, \nabla_x \Psi_i^p \rangle_{L^2(\Omega)}, & i, j &= 1, \dots, M_x, \end{aligned}$$

and $\tilde{M}_{h_t}, A_{h_t} \in \mathbb{R}^{pN_t \times pN_t}$ are the perturbed mass and stiffness matrix with respect to temporal variable, which are defined by

$$\begin{aligned} \tilde{M}_{h_t}[n, m] &= \langle \Phi_m^p, \mathcal{Q}_{h_t}^{p-1, \text{disc}} \Phi_n^p \rangle_{L^2(0, T)}, & n &= 0, \dots, pN_t - 1, m = 1, \dots, pN_t, \\ A_{h_t}[n, m] &= \langle \partial_t \Phi_m^p, \partial_t \Phi_n^p \rangle_{L^2(0, T)}, & n &= 0, \dots, pN_t - 1, m = 1, \dots, pN_t. \end{aligned}$$

Here, the nodal basis function Φ_0^p corresponds to the vertex $t_0 = 0$ and the nodal basis function $\Phi_{pN_t}^p$ corresponds to the vertex $t_{N_t} = T$. As the $L^2(0, T)$ projection $\mathcal{Q}_{h_t}^{p-1, \text{disc}}$ can be computed locally, i.e. on each temporal element τ_ℓ for $\ell = 1, \dots, N_t$, the assembling of the perturbed mass matrix $\tilde{M}_{h_t}$ can be realised, as for the classical mass matrix, via local matrices.

4 NUMERICAL EXAMPLES

In this section, numerical examples for the new space-time finite element method (7) are given. For this purpose, we consider the hyperbolic initial-boundary value problem (1) in the one-dimensional spatial domain $\Omega := (0, 1)$ with the terminal time $T = 10$, i.e. the rectangular space-time domain

$$Q := \Omega \times (0, T) := (0, 1) \times (0, 10). \quad (9)$$

As exact solutions, we choose

$$u_1(x, t) = t^2 \sin(10\pi x) \sin(tx), \quad (10)$$

$$u_2(x, t) = t^2(T - t)^{9/5} \sqrt{t + x^2 + 1} \sin(\pi x) \quad (11)$$

for $(x, t) \in Q$. The spatial domain $\Omega = (0, 1)$ is decomposed into nonuniform elements with the vertices

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1, \quad (12)$$

whereas the temporal domain $(0, T) = (0, 10)$ is decomposed into nonuniform elements with the vertices

$$t_0 = 0, \quad t_1 = T/8, \quad t_2 = T/4, \quad t_3 = T. \quad (13)$$

We apply a uniform refinement strategy for the meshes (12), (13), which do not fulfil the CFL condition (3) at least for piecewise multilinear, continuous functions, i.e. $p = 1$. Additionally, we choose $p = 1$ for Table 1, $p = 2$ for Table 2, and $p = 6$ for Table 3, where the number of the degrees of freedom is denoted by

$$\text{dof} = M_x \cdot p \cdot N_t.$$

The global linear system (8) is solved by a direct solver, where the appearing integrals to compute the related right-hand side are calculated by using high-order quadrature rules.

In the case of piecewise multilinear, continuous functions, i.e. $p = 1$, the numerical results for the smooth solution u_1 in (10) are given in Table 1, where we observe unconditional stability, quadratic convergence in $\|\cdot\|_{L^2(Q)}$ and linear convergence in $|\cdot|_{H^1(Q)}$, as predicted by Theorem 1.2.

Table 1: Numerical results of the Galerkin finite element discretisation (7) for $p = 1$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
31	0.7500	0.2500	7.5000	1.2500	9.4e+01		2.2e+03	
18	0.3750	0.1250	3.7500	0.6250	8.7e+01	0.1	2.2e+03	0.0
84	0.1875	0.0625	1.8750	0.3125	7.7e+01	0.2	2.0e+03	0.1
360	0.0938	0.0312	0.9375	0.1562	4.5e+01	0.8	1.7e+03	0.3
1488	0.0469	0.0156	0.4688	0.0781	1.3e+01	1.8	9.3e+02	0.8
6048	0.0234	0.0078	0.2344	0.0391	3.5e+00	1.9	4.9e+02	0.9
24384	0.0117	0.0039	0.1172	0.0195	8.8e-01	2.0	2.5e+02	1.0
97920	0.0059	0.0020	0.0586	0.0098	2.2e-01	2.0	1.2e+02	1.0
392448	0.0029	0.0010	0.0293	0.0049	5.6e-02	2.0	6.1e+01	1.0
1571328	0.0015	0.0005	0.0146	0.0024	1.4e-02	2.0	3.1e+01	1.0
6288384	0.0007	0.0002	0.0073	0.0012	3.5e-03	2.0	1.5e+01	1.0
25159680	0.0004	0.0001	0.0037	0.0006	8.7e-04	2.0	7.7e+00	1.0

For $p = 2$ and $p = 6$, the results for the smooth solution u_1 in (10) are stated in Table 2 and Table 3, respectively, where we illustrate that the new space-time finite element method (7) is unconditionally stable and the convergence rates with respect to the space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$ are as expected. Moreover, a comparison of Table 1, Table 2 and Table 3 show that a polynomial degree $p > 1$ is advisable

since the numbers of the degrees of freedom are much lower for $p > 1$ than for $p = 1$ when a fixed accuracy is desired. For example, we need $\text{dof} = 25159680$ degrees of freedom for $p = 1$, $\text{dof} = 392448$ degrees of freedom for $p = 2$ and $\text{dof} = 13680$ degrees of freedom for $p = 6$ to receive the error in $|\cdot|_{H^1(Q)}$ within a comparable range.

Table 2: Numerical results of the Galerkin finite element discretisation (7) for $p = 2$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
18	0.7500	0.2500	7.5000	1.2500	4.4e+03	-	1.4e+04	-
84	0.3750	0.1250	3.7500	0.6250	7.8e+01	5.8	2.1e+03	2.8
360	0.1875	0.0625	1.8750	0.3125	4.6e+01	0.8	1.7e+03	0.3
1488	0.0938	0.0312	0.9375	0.1562	1.2e+01	2.0	7.5e+02	1.2
6048	0.0469	0.0156	0.4688	0.0781	2.6e+00	2.2	2.4e+02	1.7
24384	0.0234	0.0078	0.2344	0.0391	2.2e-01	3.6	5.7e+01	2.1
97920	0.0117	0.0039	0.1172	0.0195	2.6e-02	3.1	1.4e+01	2.0
392448	0.0059	0.0020	0.0586	0.0098	3.2e-03	3.0	3.6e+00	2.0
1571328	0.0029	0.0010	0.0293	0.0049	4.0e-04	3.0	9.0e-01	2.0
6288384	0.0015	0.0005	0.0146	0.0024	5.1e-05	3.0	2.2e-01	2.0
25159680	0.0007	0.0002	0.0073	0.0012	6.3e-06	3.0	5.6e-02	2.0

Table 3: Numerical results of the Galerkin finite element discretisation (7) for $p = 6$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
198	0.7500	0.2500	7.5000	1.2500	5.2e+01	-	2.0e+03	-
828	0.3750	0.1250	3.7500	0.6250	3.0e+01	0.8	1.3e+03	0.6
3384	0.1875	0.0625	1.8750	0.3125	9.0e-01	5.0	8.6e+01	3.9
13536	0.0938	0.0312	0.9375	0.1562	8.0e-01	6.7	1.6e+01	5.6
55008	0.0469	0.0156	0.4688	0.0781	8.0e-05	6.8	3.1e-02	5.8
220608	0.0234	0.0078	0.2344	0.0391	6.4e-07	7.0	4.9e-04	6.0
883584	0.0117	0.0039	0.1172	0.0195	5.0e-09	7.0	7.7e-06	6.0

For the singular solution u_2 in (11), the related results are given in Table 4 for $p = 1$, Table 5 for $p = 2$ and Table 6 for $p = 6$, where we observe for $p > 1$ a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$. These convergence rates correspond to the reduced Sobolev regularity $u_2 \in H^{23/10-\varepsilon}(Q)$, $\varepsilon > 0$.

5 CONCLUSIONS

In this work, we introduced new stabilised higher-order space-time continuous Galerkin methods for the wave equation with globally continuous ansatz and test functions, which are piecewise polynomials of arbitrary polynomial degree. These methods are based on a space-time variational formulation, using also integration by parts with respect to the time variable, and its discretisation of tensor-product type with the help of a certain stabilisation. Thus, we generalised the well-known stabilisation idea from the lowest-order case to the higher-order case, i.e. to an arbitrary polynomial degree. We gave numerical

Table 4: Numerical results of the Galerkin finite element discretisation (7) for $p = 1$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
3	0.7500	0.2500	7.5000	1.2500	1.1e+03	-	4.4e+03	-
18	0.3750	0.1250	3.7500	0.6250	7.2e+02	0.6	2.9e+03	0.6
84	0.1875	0.0625	1.8750	0.3125	3.1e+02	1.2	1.4e+03	1.0
360	0.0938	0.0312	0.9375	0.1562	8.7e+01	1.8	5.6e+02	1.4
1488	0.0469	0.0156	0.4688	0.0781	2.4e+01	1.9	2.5e+02	1.2
6048	0.0234	0.0078	0.2344	0.0391	6.5e+00	1.9	1.1e+02	1.1
24384	0.0117	0.0039	0.1172	0.0195	1.6e+00	2.0	5.6e+01	1.0
97920	0.0059	0.0020	0.0586	0.0098	4.1e-01	2.0	2.8e+01	1.0
392448	0.0029	0.0010	0.0293	0.0049	1.0e-01	2.0	1.4e+01	1.0
1571328	0.0015	0.0005	0.0146	0.0024	2.6e-02	2.0	7.0e+00	1.0
6288384	0.0007	0.0002	0.0073	0.0012	6.5e-03	2.0	3.5e+00	1.0
25159680	0.0004	0.0001	0.0037	0.0006	1.6e-03	2.0	1.7e+00	1.0

Table 5: Numerical results of the Galerkin finite element discretisation (7) for $p = 2$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
18	0.7500	0.2500	7.5000	1.2500	5.8e+02	-	1.9e+03	-
84	0.3750	0.1250	3.7500	0.6250	2.0e+02	1.6	7.4e+02	1.4
360	0.1875	0.0625	1.8750	0.3125	3.2e+01	2.6	1.7e+02	2.1
1488	0.0938	0.0312	0.9375	0.1562	2.4e+00	3.7	2.7e+01	2.6
6048	0.0469	0.0156	0.4688	0.0781	3.9e-01	2.6	6.6e+00	2.0
24384	0.0234	0.0078	0.2344	0.0391	6.4e-02	2.6	2.0e+00	1.7
97920	0.0117	0.0039	0.1172	0.0195	1.1e-02	2.5	6.7e-01	1.6
392448	0.0059	0.0020	0.0586	0.0098	2.1e-03	2.4	2.4e-01	1.5
1571328	0.0029	0.0010	0.0293	0.0049	4.0e-04	2.4	9.3e-02	1.4
6288384	0.0015	0.0005	0.0146	0.0024	7.9e-05	2.3	3.7e-02	1.3
25159680	0.0007	0.0002	0.0073	0.0012	1.6e-05	2.3	1.5e-02	1.3

Table 6: Numerical results of the Galerkin finite element discretisation (7) for $p = 6$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
198	0.7500	0.2500	7.5000	1.2500	2.7e+00	-	1.6e+01	-
828	0.3750	0.1250	3.7500	0.6250	6.2e-01	2.1	3.5e+00	2.2
3384	0.1875	0.0625	1.8750	0.3125	8.2e-02	2.9	8.8e-01	2.0
13680	0.0938	0.0312	0.9375	0.1562	1.5e-02	2.4	3.3e-01	1.4
55008	0.0469	0.0156	0.4688	0.0781	3.0e-03	2.3	1.3e-01	1.3
220608	0.0234	0.0078	0.2344	0.0391	6.1e-04	2.3	5.3e-02	1.3
883584	0.0117	0.0039	0.1172	0.0195	1.2e-04	2.3	2.1e-02	1.3

examples, where the unconditional stability, i.e. no CFL condition is required, and optimal convergence rates in space-time norms were illustrated.

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