14th World Congress on Computational Mechanics (WCCM)
ECCOMAS Congress 2020
Virtual Congress: 11 – 15 January 2021
F. Chinesta, R. Abgrall, O. Allix and M. Kaliske (Eds)

HIGHER-ORDER SPACE-TIME CONTINUOUS GALERKIN METHODS FOR THE WAVE EQUATION

Marco Zank[1]

[1] Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
marco.zank@univie.ac.at

Key words: Wave Equation, Space-Time, Continuous Galerkin, Finite Elements, Higher-Order, CFL Condition

Abstract. We consider a space-time variational formulation of the second-order wave equation, where integration by parts is also applied with respect to the time variable. Conforming tensor-product finite element discretisations with piecewise polynomials of this space-time variational formulation require a CFL condition to ensure stability. To overcome this restriction in the case of piecewise multilinear, continuous ansatz and test functions, a stabilisation is well-known, which leads to an unconditionally stable space-time finite element method. In this work, we generalise this stabilisation idea from the lowest-order case to the higher-order case, i.e. to an arbitrary polynomial degree. We give numerical examples for a one-dimensional spatial domain, where the unconditional stability and optimal convergence rates in space-time norms are illustrated.

1 INTRODUCTION

Standard approaches for the numerical solution of hyperbolic initial-boundary value problems are usually based on semi-discretisations in space and time, where the discretisation in space and time is split accordingly. In contrast to these approaches, space-time methods discretise time-dependent partial differential equations without separating the temporal and spatial directions. In this work, the homogeneous Dirichlet problem for the second-order wave equation,

$$\left.\begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \partial\Omega \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{1}$$

serves as a model problem, where $\Omega = (0,L)$ is an interval for $d = 1$, or Ω is polygonal for $d = 2$, or Ω is polyhedral for $d = 3$, $T > 0$ is a terminal time and f is a given right-hand side. To derive a space-time variational formulation, we define the space-time Sobolev spaces

$$\begin{aligned} H^{1,1}_{0;0,}(Q) &:= L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega)) \subset H^1(Q), \\ H^{1,1}_{0;,0}(Q) &:= L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega)) \subset H^1(Q) \end{aligned}$$

with the Hilbertian norms

$$\|v\|_{H^{1,1}_{0;0,}(Q)} := \|v\|_{H^{1,1}_{0;,0}(Q)} := |v|_{H^1(Q)} := \left(\int_0^T \int_\Omega \left(|\partial_t v(x,t)|^2 + |\nabla_x v(x,t)|^2 \right) \mathrm{d}x\mathrm{d}t \right)^{1/2},$$

where $v \in H^1_{0,}(0,T;L^2(\Omega))$ satisfies $\|v(\cdot,0)\|_{L^2(\Omega)} = 0$ and $w \in H^1_{,0}(0,T;L^2(\Omega))$ fulfils $\|w(\cdot,T)\|_{L^2(\Omega)} = 0$, see [5] for more details. The bilinear form

$$a(\cdot,\cdot)\colon\ H^{1,1}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R},$$

defined by the variational identity

$$a(u,w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)}$$

for $u \in H^{1,1}_{0;0,}(Q)$ and $w \in H^{1,1}_{0;,0}(Q)$, is continuous, i.e. the estimate

$$\forall u \in H^{1,1}_{0;0,}(Q)\colon\ \forall w \in H^{1,1}_{0;,0}(Q)\colon \quad |a(u,w)| \leq |u|_{H^1(Q)}\, |w|_{H^1(Q)}$$

holds true due to the Cauchy-Schwarz inequality. The space-time variational formulation of (1) is to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$\forall w \in H^{1,1}_{0;,0}(Q)\colon \quad a(u,w) = \langle f, w \rangle_{L^2(Q)}, \tag{2}$$

where $f \in L^2(Q)$ is a given right-hand side. Note that the initial condition $u(\cdot,0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot,0) = 0$ is incorporated in a weak sense. The following existence and uniqueness theorem is proven in [1, Theorem 3.2 in Chapter IV], see also [3, 5, 8].

Theorem 1.1 *For $f \in L^2(Q)$, a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation* (2) *exists and the stability estimate*

$$|u|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)}$$

holds true.

Note that the solution operator

$$\mathcal{L}\colon\ L^2(Q) \to H^{1,1}_{0;0,}(Q), \quad \mathcal{L}f := u,$$

of Theorem 1.1 is not an isomorphism, i.e. $\mathcal{L}$ is not surjective, see [4, 5] for more details. In this work, for simplicity, we only consider homogeneous initial conditions, where inhomogeneous initial conditions can be treated analogously as in [1, 7, 8].

A conforming tensor-product space-time discretisation of (2) with piecewise polynomial, continuous ansatz and test functions requires a CFL condition

$$h_t \leq C\, h_x \tag{3}$$

with a constant $C > 0$, depending on the constant of a spatial inverse inequality, where h_t and h_x are the mesh sizes in time and space. For a one-dimensional spatial domain Ω, i.e. $d = 1$, and piecewise multilinear, continuous ansatz and test functions, the CFL condition (3) reads as

$$h_t < h_x$$

for uniform meshes with uniform mesh sizes h_t and h_x, see [3, 5]. To overcome the CFL condition (3), the stabilised space-time finite element method to find $u_h \in \big(V^1_{h_x,0}(\Omega) \otimes S^1_{h_t}(0,T)\big) \cap H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{\alpha=1}^{d} \langle \partial_{x_\alpha} u_h, Q^0_{h_t} \partial_{x_\alpha} w_h \rangle_{L^2(Q)} = \langle f, w_h \rangle_{L^2(Q)} \tag{4}$$

for all $w_h \in \big(V^1_{h_x,0}(\Omega) \otimes S^1_{h_t}(0,T)\big) \cap H^{1,1}_{0;,0}(Q)$ was analysed in [2, 5, 8], where

$$Q^0_{h_t} \colon\ L^2(Q) \to L^2(\Omega) \otimes S^0_{h_t}(0,T) \tag{5}$$

is the extended L^2 projection on the space of the temporal piecewise constant functions and $V^1_{h_x,0}(\Omega) \otimes S^1_{h_t}(0,T)$ is the space of piecewise multilinear, continuous functions, see Section 2 for the notations. The main results for this proposed space-time finite element method (4) are the unconditional stability, i.e. no CFL condition is needed, and the space-time error estimates with

$$h := \max\{h_x, h_t\}, \quad h_x = \max_k h_{x,k}, \quad h_t = \max_\ell h_{t,\ell},$$

which are summarised in the following theorem, where its proof is contained in [2, 5].

Theorem 1.2 *There exists a unique solution* $u_h \in \big(V^1_{h_x,0}(\Omega) \otimes S^1_{h_t}(0,T)\big) \cap H^{1,1}_{0;0,}(Q)$ *of* (4)*, satisfying the* $L^2(Q)$ *stability estimate*

$$\|u_h\|_{L^2(Q)} \le \frac{4}{\pi} T^2 \|f\|_{L^2(Q)}.$$

Further, let the solution u of (1) *and* Ω *be sufficiently regular. Then, the unique solution* $u_h \in \big(V^1_{h_x,0}(\Omega) \otimes S^1_{h_t}(0,T)\big) \cap H^{1,1}_{0;0,}(Q)$ *of* (4) *fulfils the space-time error estimates*

$$\|u - u_h\|_{L^2(Q)} \le Ch^2,$$
$$|u - u_h|_{H^1(Q)} \le Ch,$$

where, for the $H^1(Q)$ *error estimate, a spatial inverse inequality is additionally assumed.*

In this work, we generalise this stabilisation idea from the linear case to the higher-order case. In greater detail, we introduce a new stabilised space-time finite element method of tensor-product type with globally continuous ansatz and test functions, which are piecewise polynomials of an arbitrary polynomial degree p, leading to unconditional stability and optimal convergence rates in the space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$. In other words, the result of Theorem 1.2 is generalised to an arbitrary polynomial degree p. The rest of the paper is organised as follows: In Section 2, notations of the used finite element spaces and L^2 projections are fixed. Section 3 introduces the new space-time finite element method. Numerical examples for a one-dimensional spatial domain and piecewise polynomials of higher-order are presented in Section 4. Finally, we draw some conclusions in Section 5.

2 PRELIMINARIES

In this section, notations of the used finite element spaces and L^2 projections are stated. For this purpose, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. For a tensor-product ansatz, we consider admissible decompositions

$$\overline{Q} = \overline{\Omega} \times [0,T] = \bigcup_{k=1}^{N_x} \overline{\omega_k} \times \bigcup_{\ell=1}^{N_t} [t_{\ell-1}, t_\ell]$$

with $N := N_x \cdot N_t$ space-time elements, where the time intervals $\tau_\ell := (t_{\ell-1}, t_\ell)$ with mesh sizes $h_{t,\ell} = t_\ell - t_{\ell-1}$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t - 1} < t_{N_t} = T$$

of the time interval $(0,T)$. The maximal and the minimal time mesh sizes are denoted by $h_t := h_{t,\max} := \max_\ell h_{t,\ell}$ and $h_{t,\min} := \min_\ell h_{t,\ell}$, respectively. For the spatial domain Ω, we consider a shape-regular sequence $(\mathcal{T}_\nu)_{\nu \in \mathbb{N}}$ of admissible decompositions

$$\mathcal{T}_\nu := \{\omega_k \subset \mathbb{R}^d \colon k = 1, \ldots, N_x\}$$

of Ω into finite elements $\omega_k \subset \mathbb{R}^d$ with mesh sizes $h_{x,k}$, the maximal mesh size $h_x := h_{x,\max} := \max_k h_{x,k}$ and the minimal mesh size $h_{x,\min} := \min_k h_{x,k}$. The spatial elements ω_k are intervals for $d = 1$, triangles or quadrilaterals for $d = 2$, and tetrahedra or hexahedra for $d = 3$. Next, for a fixed polynomial degree $p \in \mathbb{N}$, we introduce the finite element space

$$Q_h^p(Q) := V_{h_x,0}^p(\Omega) \otimes S_{h_t}^p(0,T)$$

of piecewise polynomial, continuous functions, i.e.

$$V_{h_x,0}^p(\Omega) := V_{h_x}^p(\Omega) \cap H_0^1(\Omega) \subset H_0^1(\Omega), \quad S_{h_t}^p(0,T) \subset H^1(0,T)$$

with $V_{h_x}^p(\Omega) \in \left\{ S_{h_x}^p(\Omega), Q_{h_x}^p(\Omega) \right\}$. Here,

$$S_{h_t}^p(0,T) := \left\{ v_{h_t} \in C[0,T] : \forall \ell \in \{1, \ldots, N_t\} \colon v_{h_t | \overline{\tau_\ell}} \in \mathbb{P}^p(\overline{\tau_\ell}) \right\}$$

denotes the space of piecewise polynomial, continuous functions on intervals, where $\mathbb{P}^p(A)$ is the space of polynomials on a subset $A \subset \mathbb{R}^d$ of global degree at most p. Analogously,

$$S_{h_x}^p(\Omega) := \left\{ v_{h_x} \in C(\overline{\Omega}) : \forall \omega \in \mathcal{T}_\nu \colon v_{h_x | \overline{\omega}} \in \mathbb{P}^p(\overline{\omega}) \right\}$$

is the space of piecewise polynomial, continuous functions on intervals ($d = 1$), triangles ($d = 2$), or tetrahedra ($d = 3$). Moreover,

$$Q_{h_x}^p(\Omega) := \left\{ v_{h_x} \in C(\overline{\Omega}) : \forall \omega \in \mathcal{T}_\nu \colon v_{h_x | \overline{\omega}} \in \mathbb{Q}^p(\overline{\omega}) \right\}$$

is the space of piecewise polynomial, continuous functions on intervals ($d = 1$), quadrilaterals ($d = 2$), or hexahedra ($d = 3$), where $\mathbb{Q}^p(A)$ is the space of polynomials on a subset $A \subset \mathbb{R}^d$ of degree at most p in each variable. The temporal nodal basis functions of $S_{h_t}^p(0,T)$ are denoted by φ_n^p for $n = 0, \ldots, pN_t$, and ψ_j^p, $j = 1, \ldots, M_x$, are the spatial nodal basis functions of $V_{h_x,0}^p(\Omega)$, i.e.

$$S_{h_t}^p(0,T) = \operatorname{span}\{\varphi_n^p\}_{n=0}^{pN_t} \quad \text{and} \quad V_{h_x,0}^p(\Omega) = \operatorname{span}\{\psi_j^p\}_{j=1}^{M_x}.$$

For the stabilisation of the new space-time finite element method, we also need the spaces of piecewise polynomial, discontinuous functions

$$S_{h_t}^{q,\mathrm{disc}}(0,T) := \left\{ v_{h_t} \in L^1(0,T) : \forall \ell \in \{1,\ldots,N_t\} \colon v_{h_t|\tau_\ell} \in \mathbb{P}^q(\tau_\ell) \right\},$$

where $q \in \mathbb{N}_0$ is a fixed polynomial degree. For a given function $v \in L^2(Q)$, the extended L^2 projection $Q_{h_t}^{q,\mathrm{disc}} v \in L^2(\Omega) \otimes S_{h_t}^{q,\mathrm{disc}}(0,T)$ on the space $L^2(\Omega) \otimes S_{h_t}^{q,\mathrm{disc}}(0,T)$ of piecewise polynomial, discontinuous functions with respect to the time variable is defined by

$$\left\langle Q_{h_t}^{q,\mathrm{disc}} v, v_{h_t} \right\rangle_{L^2(Q)} = \langle v, v_{h_t} \rangle_{L^2(Q)}$$

for all $v_{h_t} \in L^2(\Omega) \otimes S_{h_t}^{q,\mathrm{disc}}(0,T)$, satisfying the stability estimate

$$\|Q_{h_t}^{q,\mathrm{disc}} v\|_{L^2(Q)} \leq \|v\|_{L^2(Q)}. \tag{6}$$

Note that $Q_{h_t}^0 = Q_{h_t}^{0,\mathrm{disc}}$ is the extended L^2 projection (5) on the space of the temporal piecewise constant functions $L^2(\Omega) \otimes S_{h_t}^0(0,T) = L^2(\Omega) \otimes S_{h_t}^{0,\mathrm{disc}}(0,T)$. Analogously, for a solely time-dependent function $w \in L^2(0,T)$, we denote $Q_{h_t}^{q,\mathrm{disc}} w \in S_{h_t}^{q,\mathrm{disc}}(0,T)$ as the $L^2(0,T)$ projection on the space $S_{h_t}^{q,\mathrm{disc}}(0,T)$ of piecewise polynomial, discontinuous functions, defined by

$$\left\langle Q_{h_t}^{q,\mathrm{disc}} w, w_{h_t} \right\rangle_{L^2(0,T)} = \langle w, w_{h_t} \rangle_{L^2(0,T)}$$

for all $w_{h_t} \in S_{h_t}^{q,\mathrm{disc}}(0,T)$. We use the same notation $Q_{h_t}^{q,\mathrm{disc}}$ for solely time-dependent functions and functions, which depend on (x,t), since for a function $v \in L^2(Q)$ with $v(x,t) = z(x)w(t)$, $z \in L^2(\Omega)$, $w \in L^2(0,T)$, the equality

$$Q_{h_t}^{q,\mathrm{disc}} v(x,t) = z(x) Q_{h_t}^{q,\mathrm{disc}} w(t), \quad (x,t) \in Q,$$

holds true.

3 NEW STABILISED SPACE-TIME FINITE ELEMENT METHOD

In this section, we introduce a new stabilised space-time finite element method with continuous ansatz and test functions, which are piecewise polynomials of arbitrary polynomial degree $p \in \mathbb{N}$ with respect to the spatial variable and the temporal variable. For this purpose, we fix a polynomial degree $p \in \mathbb{N}$ and we introduce the perturbed bilinear form

$$a_h(\cdot,\cdot) \colon \ Q_h^p(Q) \cap H_{0;0,}^{1,1}(Q) \times Q_h^p(Q) \cap H_{0;,0}^{1,1}(Q) \to \mathbb{R}$$

by defining

$$a_h(u_h, w_h) := -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{\alpha=1}^{d} \langle \partial_{x_\alpha} u_h, Q_{h_t}^{p-1,\mathrm{disc}} \partial_{x_\alpha} w_h \rangle_{L^2(Q)}$$

for $u_h \in Q_h^p(Q) \cap H_{0;0,}^{1,1}(Q)$, $w_h \in Q_h^p(Q) \cap H_{0;,0}^{1,1}(Q)$. Note that the function $\partial_{x_\alpha} w_h$, $\alpha = 1,\ldots,d$, fulfils

$$\partial_{x_\alpha} w_h \in L^2(\Omega) \otimes S_{h_t}^p(0,T),$$

i.e. $\partial_{x_\alpha} w_h$ is still a piecewise polynomial of degree p with respect to the temporal variable. The perturbed bilinear form $a_h(\cdot,\cdot)$ is continuous since the Cauchy-Schwarz inequality and the $L^2(Q)$ stability (6) of $Q_{h_t}^{p-1,\mathrm{disc}}$ yield

$$|a_h(u_h,w_h)| \le |u_h|_{H^1(Q)}\,|w_h|_{H^1(Q)}$$

for all $u_h \in Q_h^p(Q)\cap H^{1,1}_{0;0,}(Q)$, $w_h \in Q_h^p(Q)\cap H^{1,1}_{0;,0}(Q)$. The perturbed variational formulation, corresponding to (2), is to find $u_h \in Q_h^p(Q)\cap H^{1,1}_{0;0,}(Q)$ such that

$$\forall w_h \in Q_h^p(Q)\cap H^{1,1}_{0;,0}(Q):\; a_h(u_h,w_h) = \langle f, w_h\rangle_{L^2(Q)}. \tag{7}$$

This perturbed variational formulation (7) coincides with the perturbed variational formulation (4) for $p=1$. In other words, the new perturbed variational formulation (7) is a generalisation of the perturbed variational formulation (4) from $p=1$ to arbitrary $p\in\mathbb{N}$. The numerical analysis, i.e. an analogous result as Theorem 1.2, of the perturbed variational formulation (7) is far beyond the scope of this contribution, we refer to [6].

The discrete variational formulation (7) is equivalent to the linear system

$$K_h \underline{u} = \underline{f} \tag{8}$$

with the system matrix

$$K_h := -A_{h_t}\otimes M_{h_x} + \widetilde{M}_{h_t}\otimes A_{h_x} \in \mathbb{R}^{M_x\cdot pN_t\times M_x\cdot pN_t},$$

where $M_{h_x}, A_{h_x} \in \mathbb{R}^{M_x\times M_x}$ are the mass and stiffness matrix with respect to the spatial variable, which are given by

$$\begin{aligned} M_{h_x}[i,j] &= \langle \psi_j^p, \psi_i^p\rangle_{L^2(\Omega)}, && i,j=1,\ldots,M_x,\\ A_{h_x}[i,j] &= \langle \nabla_x\psi_j^p, \nabla_x\psi_i^p\rangle_{L^2(\Omega)}, && i,j=1,\ldots,M_x, \end{aligned}$$

and $\widetilde{M}_{h_t}, A_{h_t} \in \mathbb{R}^{pN_t\times pN_t}$ are the perturbed mass and stiffness matrix with respect to temporal variable, which are defined by

$$\begin{aligned} \widetilde{M}_{h_t}[n,m] &= \langle \varphi_m^p, Q_{h_t}^{p-1,\mathrm{disc}}\varphi_n^p\rangle_{L^2(0,T)}, && n=0,\ldots,pN_t-1,\, m=1,\ldots,pN_t,\\ A_{h_t}[n,m] &= \langle \partial_t\varphi_m^p, \partial_t\varphi_n^p\rangle_{L^2(0,T)}, && n=0,\ldots,pN_t-1,\, m=1,\ldots,pN_t. \end{aligned}$$

Here, the nodal basis function φ_0^p corresponds to the vertex $t_0=0$ and the nodal basis function $\varphi_{pN_t}^p$ corresponds to the vertex $t_{N_t}=T$. As the $L^2(0,T)$ projection $Q_{h_t}^{p-1,\mathrm{disc}}$ can be computed locally, i.e. on each temporal element τ_ℓ for $\ell=1,\ldots,N_t$, the assembling of the perturbed mass matrix $\widetilde{M}_{h_t}$ can be realised, as for the classical mass matrix, via local matrices.

4 NUMERICAL EXAMPLES

In this section, numerical examples for the new space-time finite element method (7) are given. For this purpose, we consider the hyperbolic initial-boundary value problem (1) in the one-dimensional spatial domain $\Omega := (0,1)$ with the terminal time $T=10$, i.e. the rectangular space-time domain

$$Q := \Omega\times(0,T) := (0,1)\times(0,10). \tag{9}$$

As exact solutions, we choose

$$u_1(x,t) = t^2 \sin(10\pi x) \sin(t\,x), \tag{10}$$

$$u_2(x,t) = t^2 (T-t)^{9/5} \sqrt{t+x^2+1} \sin(\pi x) \tag{11}$$

for $(x,t) \in Q$. The spatial domain $\Omega = (0,1)$ is decomposed into nonuniform elements with the vertices

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1, \tag{12}$$

whereas the temporal domain $(0,T) = (0,10)$ is decomposed into nonuniform elements with the vertices

$$t_0 = 0, \quad t_1 = T/8, \quad t_2 = T/4, \quad t_3 = T. \tag{13}$$

We apply a uniform refinement strategy for the meshes (12), (13), which do not fulfil the CFL condition (3) at least for piecewise multilinear, continuous functions, i.e. $p = 1$. Additionally, we choose $p = 1$ for Table 1, $p = 2$ for Table 2, and $p = 6$ for Table 3, where the number of the degrees of freedom is denoted by

$$\text{dof} = M_x \cdot p \cdot N_t.$$

The global linear system (8) is solved by a direct solver, where the appearing integrals to compute the related right-hand side are calculated by using high-order quadrature rules.

In the case of piecewise multilinear, continuous functions, i.e. $p = 1$, the numerical results for the smooth solution u_1 in (10) are given in Table 1, where we observe unconditional stability, quadratic convergence in $\|\cdot\|_{L^2(Q)}$ and linear convergence in $|\cdot|_{H^1(Q)}$, as predicted by Theorem 1.2.

Table 1: Numerical results of the Galerkin finite element discretisation (7) for $p = 1$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\lvert u_1 - u_{1,h}\rvert_{H^1(Q)}$	eoc
3	0.7500	0.2500	7.5000	1.2500	9.4e+01	-	2.2e+03	-
18	0.3750	0.1250	3.7500	0.6250	8.7e+01	0.1	2.2e+03	0.0
84	0.1875	0.0625	1.8750	0.3125	7.7e+01	0.2	2.0e+03	0.1
360	0.0938	0.0312	0.9375	0.1562	4.5e+01	0.8	1.7e+03	0.3
1488	0.0469	0.0156	0.4688	0.0781	1.3e+01	1.8	9.3e+02	0.8
6048	0.0234	0.0078	0.2344	0.0391	3.5e+00	1.9	4.9e+02	0.9
24384	0.0117	0.0039	0.1172	0.0195	8.8e-01	2.0	2.5e+02	1.0
97920	0.0059	0.0020	0.0586	0.0098	2.2e-01	2.0	1.2e+02	1.0
392448	0.0029	0.0010	0.0293	0.0049	5.6e-02	2.0	6.1e+01	1.0
1571328	0.0015	0.0005	0.0146	0.0024	1.4e-02	2.0	3.1e+01	1.0
6288384	0.0007	0.0002	0.0073	0.0012	3.5e-03	2.0	1.5e+01	1.0
25159680	0.0004	0.0001	0.0037	0.0006	8.7e-04	2.0	7.7e+00	1.0

For $p = 2$ and $p = 6$, the results for the smooth solution u_1 in (10) are stated in Table 2 and Table 3, respectively, where we illustrate that the new space-time finite element method (7) is unconditionally stable and the convergence rates with respect to the space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$ are as expected. Moreover, a comparison of Table 1, Table 2 and Table 3 show that a polynomial degree $p > 1$ is advisable since the numbers of the degrees of freedom are much lower for $p > 1$ than for $p = 1$ when a fixed accuracy is desired. For example, we need dof $= 25159680$ degrees of freedom for $p = 1$, dof $= 392448$ degrees of freedom for $p = 2$ and dof $= 13680$ degrees of freedom for $p = 6$ to receive the error in $|\cdot|_{H^1(Q)}$ within a comparable range.

Table 2: Numerical results of the Galerkin finite element discretisation (7) for $p = 2$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\lvert u_1 - u_{1,h}\rvert_{H^1(Q)}$	eoc
18	0.7500	0.2500	7.5000	1.2500	4.4e+03	-	1.4e+04	-
84	0.3750	0.1250	3.7500	0.6250	7.8e+01	5.8	2.1e+03	2.8
360	0.1875	0.0625	1.8750	0.3125	4.6e+01	0.8	1.7e+03	0.3
1488	0.0938	0.0312	0.9375	0.1562	1.2e+01	2.0	7.5e+02	1.2
6048	0.0469	0.0156	0.4688	0.0781	2.6e+00	2.2	2.4e+02	1.7
24384	0.0234	0.0078	0.2344	0.0391	2.2e-01	3.6	5.7e+01	2.1
97920	0.0117	0.0039	0.1172	0.0195	2.6e-02	3.1	1.4e+01	2.0
392448	0.0059	0.0020	0.0586	0.0098	3.2e-03	3.0	3.6e+00	2.0
1571328	0.0029	0.0010	0.0293	0.0049	4.0e-04	3.0	9.0e-01	2.0
6288384	0.0015	0.0005	0.0146	0.0024	5.1e-05	3.0	2.2e-01	2.0
25159680	0.0007	0.0002	0.0073	0.0012	6.3e-06	3.0	5.6e-02	2.0

Table 3: Numerical results of the Galerkin finite element discretisation (7) for $p = 6$ for the space-time cylinder (9) for the smooth function u_1 in (10) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\lvert u_1 - u_{1,h}\rvert_{H^1(Q)}$	eoc
198	0.7500	0.2500	7.5000	1.2500	5.2e+01	-	2.0e+03	-
828	0.3750	0.1250	3.7500	0.6250	3.0e+01	0.8	1.3e+03	0.6
3384	0.1875	0.0625	1.8750	0.3125	9.0e-01	5.0	8.6e+01	3.9
13680	0.0938	0.0312	0.9375	0.1562	8.9e-03	6.7	1.7e+00	5.6
55008	0.0469	0.0156	0.4688	0.0781	8.0e-05	6.8	3.1e-02	5.8
220608	0.0234	0.0078	0.2344	0.0391	6.4e-07	7.0	4.9e-04	6.0
883584	0.0117	0.0039	0.1172	0.0195	5.0e-09	7.0	7.7e-06	6.0

For the singular solution u_2 in (11), the related results are given in Table 4 for $p = 1$, Table 5 for $p = 2$ and Table 6 for $p = 6$, where we observe for $p > 1$ a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$. These convergence rates correspond to the reduced Sobolev regularity $u_2 \in H^{23/10-\varepsilon}(Q)$, $\varepsilon > 0$.

5 CONCLUSIONS

In this work, we introduced new stabilised higher-order space-time continuous Galerkin methods for the wave equation with globally continuous ansatz and test functions, which are piecewise polynomials of arbitrary polynomial degree. These methods are based on a space-time variational formulation, using also integration by parts with respect to the time variable, and its discretisation of tensor-product type with the help of a certain stabilisation. Thus, we generalised the well-known stabilisation idea from the lowest-order case to the higher-order case, i.e. to an arbitrary polynomial degree. We gave numerical

Table 4: Numerical results of the Galerkin finite element discretisation (7) for $p = 1$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
3	0.7500	0.2500	7.5000	1.2500	1.1e+03	-	4.4e+03	-
18	0.3750	0.1250	3.7500	0.6250	7.2e+02	0.6	2.9e+03	0.6
84	0.1875	0.0625	1.8750	0.3125	3.1e+02	1.2	1.4e+03	1.0
360	0.0938	0.0312	0.9375	0.1562	8.7e+01	1.8	5.6e+02	1.4
1488	0.0469	0.0156	0.4688	0.0781	2.4e+01	1.9	2.5e+02	1.2
6048	0.0234	0.0078	0.2344	0.0391	6.5e+00	1.9	1.1e+02	1.1
24384	0.0117	0.0039	0.1172	0.0195	1.6e+00	2.0	5.6e+01	1.0
97920	0.0059	0.0020	0.0586	0.0098	4.1e-01	2.0	2.8e+01	1.0
392448	0.0029	0.0010	0.0293	0.0049	1.0e-01	2.0	1.4e+01	1.0
1571328	0.0015	0.0005	0.0146	0.0024	2.6e-02	2.0	7.0e+00	1.0
6288384	0.0007	0.0002	0.0073	0.0012	6.5e-03	2.0	3.5e+00	1.0
25159680	0.0004	0.0001	0.0037	0.0006	1.6e-03	2.0	1.7e+00	1.0

Table 5: Numerical results of the Galerkin finite element discretisation (7) for $p = 2$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
18	0.7500	0.2500	7.5000	1.2500	5.8e+02	-	1.9e+03	-
84	0.3750	0.1250	3.7500	0.6250	2.0e+02	1.6	7.4e+02	1.4
360	0.1875	0.0625	1.8750	0.3125	3.2e+01	2.6	1.7e+02	2.1
1488	0.0938	0.0312	0.9375	0.1562	2.4e+00	3.7	2.7e+01	2.6
6048	0.0469	0.0156	0.4688	0.0781	3.9e-01	2.6	6.6e+00	2.0
24384	0.0234	0.0078	0.2344	0.0391	6.4e-02	2.6	2.0e+00	1.7
97920	0.0117	0.0039	0.1172	0.0195	1.1e-02	2.5	6.7e-01	1.6
392448	0.0059	0.0020	0.0586	0.0098	2.1e-03	2.4	2.4e-01	1.5
1571328	0.0029	0.0010	0.0293	0.0049	4.0e-04	2.4	9.3e-02	1.4
6288384	0.0015	0.0005	0.0146	0.0024	7.9e-05	2.3	3.7e-02	1.3
25159680	0.0007	0.0002	0.0073	0.0012	1.6e-05	2.3	1.5e-02	1.3

Table 6: Numerical results of the Galerkin finite element discretisation (7) for $p = 6$ for the space-time cylinder (9) for the singular function u_2 in (11) for a uniform refinement strategy with the starting meshes (12), (13).

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
198	0.7500	0.2500	7.5000	1.2500	2.7e+00	-	1.6e+01	-
828	0.3750	0.1250	3.7500	0.6250	6.2e-01	2.1	3.5e+00	2.2
3384	0.1875	0.0625	1.8750	0.3125	8.2e-02	2.9	8.8e-01	2.0
13680	0.0938	0.0312	0.9375	0.1562	1.5e-02	2.4	3.3e-01	1.4
55008	0.0469	0.0156	0.4688	0.0781	3.0e-03	2.3	1.3e-01	1.3
220608	0.0234	0.0078	0.2344	0.0391	6.1e-04	2.3	5.3e-02	1.3
883584	0.0117	0.0039	0.1172	0.0195	1.2e-04	2.3	2.1e-02	1.3

examples, where the unconditional stability, i.e. no CFL condition is required, and optimal convergence rates in space-time norms were illustrated.

REFERENCES

[1] LADYZHENSKAYA, O. A. *The boundary value problems of mathematical physics*, vol. 49 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1985.

[2] STEINBACH, O., AND ZANK, M. A stabilized space–time finite element method for the wave equation. In *Advanced Finite Element Methods with Applications. Selected papers from the 30th Chemnitz FEM Symposium 2017, (T. Apel, U. Langer, A. Meyer, O. Steinbach eds.), Lecture Notes in Computational Science and Engineering*. Springer, 2019, pp. 315–342.

[3] STEINBACH, O., AND ZANK, M. Coercive space-time finite element methods for initial boundary value problems. *Electron. Trans. Numer. Anal. 52* (2020), 154–194.

[4] STEINBACH, O., AND ZANK, M. A generalized inf-sup stable variational formulation for the wave equation. [math.NA] 2101.06293, arXiv.org, 2021.

[5] ZANK, M. *Inf–sup stable space–time methods for time–dependent partial differential equations*. volume 36 of Monographic Series TU Graz: Computation in Engineering and Science. Feb 2020.

[6] ZANK, M. Stability and error estimates in space-time norms of a stabilized space-time finite element method for the wave equation for an arbitrary polynomial degree. *In preparation* (2021).

[7] ZANK, M. The Newmark method and a space-time fem for the second–order wave equation. In *Numerical Mathematics and Advanced Applications ENUMATH 2019, (F. J. Vermolen, C. Vuik eds.), Lecture Notes in Computational Science and Engineering*. Cham: Springer, 2021, accepted.

[8] ZLOTNIK, A. A. Convergence rate estimates of finite-element methods for second-order hyperbolic equations. In *Numerical methods and applications*. CRC, Boca Raton, FL, 1994, pp. 155–220.