Random Vibration of Systems Subjected to Support Motion

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Random Vibration of Systems Subjected to Support Motion

Lecture notes for the course on

DYNAMIC DESING AND NUMERICAL METHODS

APPLIED TO ENGINES AND TRANSMISSIONS

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CONTENTS

1. Introducción

An action has a dynamic character if it has a rapid time-variation, causing inertia forces in structures. The intensity, the direction or sense, or all these characteristics of dynamic loads are time - dependent. Two different concepts can be used in the definition of the dynamic loads: the *deterministic* concept and the *nondeterministic* or *stochastic* or *random* one. A dynamic load has a deterministic character in the case in which its time variation is completely known at each time instant. On the contrary, a dynamic load is nondeterministic if some parameters of it, or its complete time-history, have been statistically defined(1). The methods of structural analysis have to be in accordance with the concepts used in defining the loads. In the case of a deterministic definition the analysis is performed by means of deterministic methods and, consequently, the dynamic response is described in a deterministic way. When the action is a random process, the methods have to be appropriate for a stochastic treatment of the problem and the structural response is defined through its statistic characteristics.

The dynamic actions defined by using deterministic representations are time-varying functions whose values are known at each time instant. This type of representation is appropriate in an *a posteriori* analysis of structures, that means, in verifying the dynamic behaviour of a structure subjected to a previously defined action. The results calculated in this way are only qualitative ones, due to the fact that they have been obtained under the optimistic assumption that the actions expected in the future will have characteristics similar to the past ones.

Stochastic representations are used in the cases in which the dynamic actions cannot be expressed through time-dependent functions whose values are known at each time instant. In such cases the loads are represented by means of families of possible functions, defined through some probabilistic characteristics. This operation is followed by an evaluation of certain probabilistic parameters of the structural response. Thus, the dynamic actions and, at the same time, the structural response are defined as stochastic processes, that is by means of families of chaotic events having a time evolution(2).

When a structure is subjected to a dynamic load, the actual structural response is the result of filtering the dynamic action through the actual structure and the dynamic analysis requires the previous definition of both the action and the structural characteristics. Nevertheless, as in all branches of Applied Mechanics, in Structural Dynamics the subject of analysis is not an actual structure, but rather a mechanic model of it which, in this case, is a dynamic one. The definition of a dynamic model depends on the type of structure and its objective is to provide not only realistic description of the behaviour of the actual structure, but also a simple relationship between actions and responses. The necessity of a definition of a dynamic model in order to perform the dynamic analysis of a structure demonstrates that the traditional design process is a verification one: starting from a pre-defined structural shape, checks are made to assure that the response comply with some previously stated conditions.

The relationship between action and response is expressed quantitatively by means of a *mathematical model.* The physical characteristics to take into account in defining mathematical models, are the mass (inertia), the damping and the stiffness of the structure. A complete calculation of the dynamic response would require its obtention in every point of the structure, that is, in an infinite number of points and in an infinite number of time instants, a fact that greatly complicates the problem to be solved. In order to simplify the mathematical model of the problem, it is convenient to define dynamic models with a finite number of prescribed points in which the response is to be calculated. One comes to such a result by means of an operation named *spatial discretization*[3–5]. Further simplifications are performed by computing the dynamic response only in a finite number of time instants, by means of a *time discretization.* The definition of both dynamic and mathematical models is dependent not only on the discretization methods used but also on the geometric and mechanical characteristics of the actual structure. Inaccuracies introduced in the dynamic models and in the correspondent mathematical ones, including those affecting the material characteristics, greatly influence the accuracy of the computed response.

An important difference between the effects of a static and a dynamic load on a structure consists in the presence of the inertia forces in the dynamic case, in which the stresses are produced in the structure by the action of both the dynamic force $f(t)$ and the inertia force $f_i(y,t)$. Another difference is the time-evolution of the dynamic response, due to which the analysis has to provide a solution at each time instant, that is a *time history* of the response. The simulation process of the structural response, illustrated in the block scheme of figure 1.1, would introduce physical estimates during the phase of the dynamic modelization, would use an exact definition of the correspondent mathematic model, and, further on, would perform the calculation of the response by suitable numerical procedures.

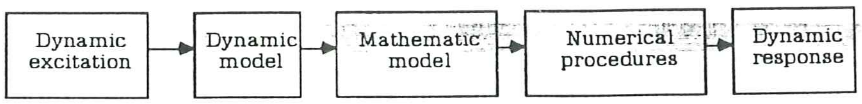

Figure 1.1 Block scheme for the dynamic response computation process.

d'Alembert's principle provides the most direct procedure to write equations of motion[1]. It can be formulated as follows: the dynamic equilibrium of a system is assured if all the forces acting on the system, the inertia ones included, comply with the static equilibrium condition at each instant of time. The inertia forces are expressed according to Newton's second law

$$(F_i)_j(t) = -m_j\ddot{d}_j(t) \qquad (j = 1, 2, \ldots, n)$$

where $\ddot{d}_j(t)$ is the acceleration corresponding to the mass m_j of the system.

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2. Linear systems with a single-degree of freedom

2.1 EQUATIONS OF MOTION

The equation of motion corresponding to the dynamic model of figure 2.1(a) are deduced by using d'Alembert's principle. One considers that the mass is cut off from its connections and that all the forces corresponding to the mentioned connections, the inertia ones included, are applied on the mass [see figure 2.1(b)]. Thus, the equation of equilibrium is written as

$$F_i(t) - F_e(t) - F_a(t) = f(t) \tag{2.1}$$

where $F_i(t)$, $F_e(t)$ and $F_a(t)$ are the inertia, elastic and damping forces respectively, while $f(t)$ is the dynamic force applied on the model.

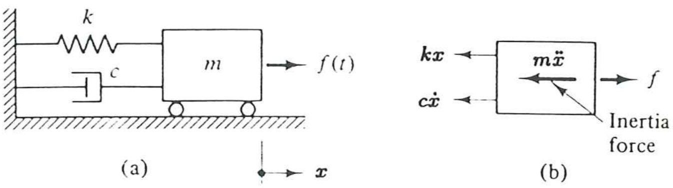

Figure 2.1 Single-degree of freedom model. (a) Dynamic model; (b) forces in dynamic equilibrium.

The elastic force is proportional to the displacement $x(t)$ of the mass m, and to the stiffness k of the model

$$F_e(t) = kx(t) \tag{2.2}$$

The inertia force is generated by the absolute acceleration of the mass m

$$F_i(t) = -m\ddot{x}(t) \tag{2.3}$$

Voigt's hypothesis is admitted for the damping force; according to it, the damping is "viscous", that is proportional to the velocity

$$F_a(t) = c\dot{x}(t) \tag{2.4}$$

With these definitions the equation of equilibrium (2.1) is expressed as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \tag{2.5}$$

For the damped free vibration case, the equation can be written as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{2.6}$$

and for the undamped free vibration case it changes into

$$m\ddot{x}(t) + kx(t) = 0 \tag{2.7}$$

Inertia force can be generated in the model not only by a directly applied dynamic force, but also by the motion of its support points. Consider the case of the system of figure 2.2, which vibrates due to the displacement $x_s(t)$ of its support.

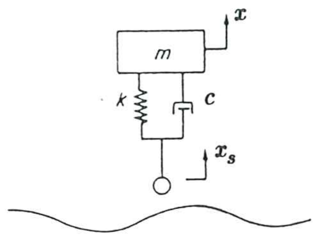

Figure 2.2 Single-degree of freedom model subjected to support excitation.

In this case, the absolute acceleration of the mass is

$$F_i(t) = -m\left[\ddot{x}(t) + \ddot{x}_s(t)\right] \tag{2.8}$$

and the equation of equilibrium (2.1) can be expressed as

$$m\left[\ddot{x}(t) + \ddot{x}_s(t)\right] + c\dot{x}(t) + kx(t) = 0 \tag{2.9}$$

After some transformations it becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -m\ddot{x}_s(t) = f(t) \tag{2.10}$$

where $f(t) = -m\ddot{x}_s(t)$ is the force which acts on the mass m, which has the nature of an inertia force.

2.2 DYNAMIC CHARACTERISTICS OF SINGLE-DEGREE OF FREEDOM MODELS

The dynamic characteristics of a single-degree of freedom model are defined by studying its undamped free vibrations. The model vibrates due to some initial conditions which can be, for example, an initial displacement and an initial velocity and does not suffer the effect of any perturbation during its vibration. Consequently, the model does not dissipate the initially induced energy.

Dividing equation (2.7) by m and using the notation

$$\omega^2 = \frac{k}{m}$$

the following equation is obtained

$$\ddot{x}(t) + \omega^2\, x(t) = 0 \tag{2.11}$$

The quantity ω is the *angular* or *circular natural frequency* of the system, sometimes called simply *frequency* and is expressed in radians per second. This is one of the dynamic characteristics of the system. Another one is the *natural period* T, defined as

$$T = \frac{2\pi}{\omega} \tag{2.12}$$

and measured in seconds. Finally, the *cyclic frequency* f is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{2.13}$$

and is expressed in cycles per second or Hertz.

The general solution of the equation (2.11) can be written in the form

$$x(t) = A\sin(\omega t + \psi) \tag{2.14}$$

where A is the amplitude of the motion and ψ is the phase angle. A and ψ are calculated starting from the initial conditions of the problem. For example, for an initial displacement $x(0) = x_0$ and an initial velocity $\dot{x}(0) = \dot{x}_0$, the resultant values are[1,2]

$$A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2}, \qquad \tan\psi = \frac{\dot{x}_0}{\omega x_0} \tag{2.15}$$

2.3 DAMPING CHARACTERISTICS OF SINGLE-DEGREE OF FREEDOM MODELS

The free damped vibration which will be studied is described by equation (2.6). The damping, characterized by the coefficient c, is proportional to the velocity according to a Kelvin-Voigt model and is called *viscous damping.* The most important reason of the use of such a definition is its simplicity[3,4]. Actually, damping forces in structures are produced by different causes, between which the following can be enumerated[4]:

— Friction between sliding surfaces, which can be dry or lubricated; the damping force is proportional to the force acting normally to the surfaces in contact, according to Coulomb's hypothesis. The mentioned normal force is considered constant and independent from the displacements or velocities.

— Damping due to the vibration of structures in the surrounding medium which, generally, is a fluid.

— Damping due to internal frictions within the structural material, or between structural members, mainly produced by imperfect elasticity. In this case the damping force is proportional to the restoring force, and is called structural damping.

The before mentioned viscous damping is often used to characterize the global damping of a structure; in such cases it is named *equivalent viscous damping.* It can be defined as a force which produces the total energy dissipation in the structure[5].

The equation of motion (2.6) is divided by m, resulting in

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2 x(t) = 0 \tag{2.16}$$

where the notation

$$\frac{c}{m} = 2\beta \tag{2.17}$$

has been introduced. The solution of (2.16) is obtained by starting from the substitution

$$x(t) = e^{rt} \tag{2.18}$$

which provides the characteristic equation

$$r^2 + 2\beta r + \omega^2 = 0$$

whose solutions are

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2} \tag{2.19}$$

The *critical damping* is denoted c_c or β_c and is defined by the equation

$$\beta_c^2 - \omega^2 = 0 \tag{2.20}$$

It results from here

$$\beta_c = \omega \tag{2.21}$$

and by using (2.17), the coefficient c_c is obtained

$$c_c = 2m\beta_c = 2m\omega \tag{2.22}$$

The case in which the damping is greater than the critical one

$$c > c_c \tag{2.23}$$

corresponds to *overdamped systems*. This is not a situation given in the case of normal structures and therefore, it will be not discussed herein. The only observation which will be made is that in such conditions a structure does not oscillate, but rather it returns to its rest position without vibrations.

The typical case given in the dynamic analysis of structures is the undercritical one, defined by

$$c < c_c \tag{2.24}$$

A better definition of this case is achieved by introducing another damping coefficient

$$\zeta = \frac{c}{c_c} \tag{2.25}$$

known as *damping ratio*. The damping ratio is expressed using (2.22)

$$\zeta = \frac{c}{2m\omega} \tag{2.26}$$

and the substitution of (2.17) provides

$$\zeta = \frac{\beta}{\omega} \tag{2.27}$$

In this case the quantity $\beta^2 - \omega^2$ of equation (2.19) is negative and consequently the solutions r_1 and r_2 are complex

$$r_{1,2} = -\beta \pm i\omega\sqrt{1-\zeta^2} \tag{2.28}$$

where $i = \sqrt{-1}$. Defining the damped vibration frequency ω_v as

$$\omega_v = \omega\sqrt{1-\zeta^2} \tag{2.29}$$

the solution (2.28) becomes

$$r_{1,2} = -\zeta\omega \pm i\omega_v \tag{2.30}$$

The general solution of the equation (2.16) can be written in the form

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{2.31}$$

By substituting r_1 and r_2 from (2.19) the final form of the solution is obtained

$$x(t) = Ae^{-\zeta\omega t}\sin(\omega_v t + \psi) \tag{2.32}$$

The coefficients A and ψ are calculated by using the initial conditions of the problem. Obviously, the equation of motion (2.16) can be expressed in the form

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = 0 \tag{2.33}$$

where the definition of the damping ratio (2.27) has been used.

2.4 DYNAMIC RESPONSE OF SINGLE-DEGREE OF FREEDOM SYSTEMS

2.4.1 General considerations

Single-degree of freedom systems are the simplest dynamic models which can be used in analyzing the dynamic behaviour of a structure. From a theoretical point of view, their application in solving practical problems is restricted to a limited class of systems, namely those whose mass is physically concentrated at a given point of the model and which oscilates in one direction. However, their analysis is important, due to the fact that it can be extended to the study of multi-degrees of freedom models by using the superposition concept. Another interesting aspect is that their response can be obtained by expressing the solution of the equation of motion in an explicite form. This solution depends on a reduced number of parameters whose influence can be studied easily.

2.4.2 Transfer function of a dynamic system

The Fourier transforms of the dynamic excitation $f(t)$ or of the support acceleration $\ddot{x}_s(t)$ are defined as

$$F(\theta) = \int_{-\infty}^{\infty} f(t)\, e^{-i\theta t}\, dt \tag{2.34a}$$

$$-mA(\theta) = -m \int_{-\infty}^{\infty} \ddot{x}_s(t)\, e^{-i\theta t}\, dt \tag{2.34b}$$

while the Fourier transform of the response $x(t)$ is

$$X(\theta) = \int_{-\infty}^{\infty} x(t)\, e^{-i\theta t}\, dt \tag{2.35}$$

In equations (2.34) and (2.35), θ is the frequency of the excitation and $A(\theta)$ is the Fourier transform of the support acceleration $\ddot{x}_s(t)$. Since in dynamic problems the excitation signals $f(t)$ or $\ddot{x}_s(t)$ and the response ones $x(t)$ are always finite, continuous and bounded, the integrals of the Fourier transforms (2.34) and (2.35) always exist and can be evaluated[1,6–8]. The same statement can be made on the corresponding inverse Fourier transforms defined by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta)\, e^{i\theta t}\, d\theta \tag{2.36}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\theta)\, e^{i\theta t}\, d\theta \tag{2.37}$$

The *transfer function* $H(\theta)$ of the system, formulated in the complex frequency domain, is defined as

$$H(\theta) = \frac{X(\theta)}{F(\theta)} \tag{2.38}$$

The complex frequency response is thus expressed as

$$X(\theta) = H(\theta)\, F(\theta) \tag{2.39}$$

The convolution theorem is used now, which states that the inverse transform of the product of two Fourier transforms is equal to the convolution integral of the inverse of their transforms. By applying this theorem to equation (2.39), the following equation is obtained[9,10]:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)\, e^{i\theta t}\, d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\theta)\, F(\theta)\, e^{i\theta t}\, d\theta$$

$$= \int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau)\, f(\tau)\, d\tau \qquad (2.40)$$

where $h(t)$ is the inverse Fourier transform of the transfer function $H(\theta)$ of the system

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\theta)\, e^{i\theta t}\, d\theta \qquad (2.41)$$

Taking into account that $h(t)$ is meaningless for $t < 0$ and that the excitation $f(t)$ is a finite signal, distinct from zero only for $t > 0$, equation (2.40) can be rewritten as

$$x(t) = \int_0^t h(\tau)\, f(t-\tau)\, d\tau = \int_0^t h(t-\tau)\, f(\tau)\, d\tau \qquad (2.42)$$

2.4.3 Response to unit impulse

Consider a single-degree of freedom system subjected to a unit impulse defined by means of a delta Dirac function

$$\delta(t-t_0) = 0 \quad for\ t \neq t_0 \quad (t > 0) \qquad (2.43a)$$

$$\int_0^{\infty} \delta(t-t_0)\, dt = 1 \qquad (2.43b)$$

This function has the property that

$$\int_0^{\infty} \delta(t-t_0)\, q(t) dt = q(t_0) \qquad (2.44)$$

where $q(t)$ is any time function.

The Fourier transform of the unit impulse is

$$F(\theta) = \int_{-\infty}^{\infty} \delta(t)\, e^{-i\theta t}\, dt \qquad (2.45)$$

and, according to (2.44)

$$F(\theta) = 1 \qquad (2.46)$$

Consequently, the complex frequency response of the system subjected to a unit impulse, which will be denoted by $X^P(\theta)$, is

$$X^P(\theta) = H(\theta)F(\theta) = H(\theta) \qquad (2.47)$$

It can be stated from here that the Fourier transform of the response to a unit Dirac impulse is equal to the transfer function of the system. The time domain response of the system $x^P(t)$ produced by a unit impulse is equal to the inverse Fourier transform of the complex impulse response of the system

$$x^P(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^P(\theta)\, e^{i\theta t}\, d\theta \qquad (2.48)$$

By using (2.47), the *unit-impulse response* is expressed as

$$x^p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\theta)\, e^{i\theta t}\, d\theta \;=\; h(t) \tag{2.49}$$

where the transfer function of the system

$$H(\theta) \;=\; \int_{-\infty}^{\infty} h(t)\, e^{-i\theta t}\, dt \tag{2.50}$$

is the Fourier transform of the unit-impulse response. Equation (2.49) states that the inverse transform $h(t)$ of the transfer function is equal to the unit impulse response of the system.

2.4.4 Response to a general excitation

Consider that the system is now subjected to a general dynamic load $f(t)$ or to a support acceleration $\ddot{x}_s(t)$. The displacement response of the system is obtained by applying Fourier transforms to the terms of equation (2.1). Considering zero initial conditions, the following linear algebric equation with complex coefficients is obtained

$$[-m\theta^2 + i\,\theta c + k]\; X(\theta) = -m\; A(\theta) = F(\theta) \tag{2.51}$$

where $A(\theta)$ and $F(\theta)$ are the Fourier transforms of $f(t)$ and $x_s(t)$, respectively and $X(\theta)$ is the Fourier transform of the displacement response. Using (2.51), the complex frequency response can be expressed in the following form:

$$X(\theta) = \frac{-mA(\theta)}{-m\theta^2 + i\theta c + k} = \frac{F(\theta)}{-m\theta^2 + i\theta c + k} \tag{2.52}$$

By comparing (2.39) with (2.52), it results that, in the general dynamic case, the complex frequency domain transfer function of the single-degree of freedom system is

$$H(\theta) = \frac{1}{-m\theta^2 + i\theta c + k}$$

which can be transformed into

$$H(\theta) = \frac{1}{m(-\theta^2 + i2\zeta\omega\theta + \omega^2)} \tag{2.53a}$$

where the well known expressions

$$c = 2\zeta\omega m$$

and

$$\omega^2 = \frac{k}{m}$$

have been used. In the support excitation case, the complex frequency domain transfer function of the single-degree of freedom model will change according to (2.52), becoming

$$H(\theta) = -\frac{1}{(-\theta^2 + i2\zeta\omega\theta + \omega^2)} \tag{2.53b}$$

More transformations can be performed on the expression (2.53a) of the transfer function, which allows to express it in the form

$$H(\theta) = \frac{1}{m\;[(i\theta + \zeta\omega)^2 + \omega^2(1-\zeta^2)]} \tag{2.54}$$

By using now the frequency of the damped vibration

$$\omega_v = \omega\,\sqrt{1-\zeta^2}$$

the equation (2.54) becomes

$$H(\theta) = \frac{1}{m}\frac{1}{[(i\theta + \zeta\omega)^2 + \omega_v^2]} \qquad (2.55a)$$

Obviously, in the support excitation case the transfer function has to be written according to (2.53b)

$$H(\theta) = -\frac{1}{[(i\theta + \zeta\omega)^2 + \omega_v^2]} \qquad (2.55b)$$

Remember that the transfer function $H(\theta)$ is the Fourier transform of the unit impulse response of the system. The inverse Fourier transform $h(t)$ of $H(\theta)$ is given by the equation (2.41). The integral of this equation can be solved analytically. For example, if $H(\theta)$ is expressed by means of (2.55a), the unit impulse response of the system is

$$h(t) = \frac{1}{m\omega_v} e^{-\zeta\omega t}\sin(\omega_v t) \qquad (2.56)$$

By using now the equation (2.42), the time domain response of the system is formulated in the follwing final form:

$$x(t) = \frac{1}{m\omega_v}\int_0^t f(\tau)\, e^{-\zeta\omega(t-\tau)}\, \sin[\omega_v(t-\tau)]\, d\tau \qquad (2.57)$$

In the case of a support excitation, equation (2.57) transforms into

$$x(t) = -\frac{1}{\omega_v}\int_0^t \ddot{x}_s(\tau)\, e^{-\zeta\omega(t-\tau)}\, \sin[\omega_v(t-\tau)]\, d\tau \qquad (2.58)$$

The integrals in (2.57) and (2.58) are known as Duhamel's integrals, which have analytical solution only for some particular cases of functions describing the excitation. In a general case it has to be solved numerically.

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3. Linear multi-degree of freedom systems

3.1 EQUATIONS OF MOTION

3.1.1 General formulation

Consider a multi-degree of freedom model obtained by means of the concentrated mass method. The corresponding equations of motion can be written by expressing the dynamic equilibrium of each mass of the model, according to d'Alembert's principle. This operation requires to cut off the connections of each mass m_r, $r = 1, 2, \ldots, i, \ldots, n$, to introduce all the corresponding forces, the inertia ones included and to express the dynamic equilibrium of the mass. One obtains

$$F_{i_r}(t) - F_{e_r}(t) - F_{a_r}(t) = f_r \qquad (r = 1, 2, \ldots, n) \tag{3.1}$$

Obviously, the entire dynamic model is in equilibrium if all its concentrated masses are in equilibrium, that is, that all (3.1) type equations are similar to (2.4). By writing these n equations in matrix form, it results

$$\boldsymbol{F}_i(t) - \boldsymbol{F}_e(t) - \boldsymbol{F}_a(t) = \boldsymbol{F}(t) \tag{3.2}$$

The equations (2.2), (2.3) and (2.4), which define the elastic, inertia and damping forces of a single-degree of freedom model, turn in this case into the following matrix relations(1–3):

$$\begin{aligned} \boldsymbol{F}_e(t) &= \boldsymbol{K}\boldsymbol{X}(t) \\ \boldsymbol{F}_i(t) &= -\boldsymbol{M}\left[\ddot{\boldsymbol{X}}(t)\right] \\ \boldsymbol{F}_a(t) &= \boldsymbol{C}\dot{\boldsymbol{X}}(t) \end{aligned} \tag{3.3}$$

$\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and stiffness matrices, respectively. The substitution of equations (3.3) in (3.2) provides

$$\boldsymbol{M}\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}\dot{\boldsymbol{X}}(t) + \boldsymbol{K}\boldsymbol{X}(t) = F(t) \tag{3.4}$$

If the same model is subjected to a support displacement $x_s(t)$, the equation of motion becomes

$$\boldsymbol{F}_i(t) - \boldsymbol{F}_e(t) - \boldsymbol{F}_a(t) = \boldsymbol{0} \tag{3.5}$$

where now

$$\begin{aligned} \boldsymbol{F}_e(t) &= \boldsymbol{K}\boldsymbol{X}(t) \\ \boldsymbol{F}_i(t) &= -\boldsymbol{M}\left[\ddot{\boldsymbol{X}}(t) + \boldsymbol{J}\ddot{x}_s(t)\right] \\ \boldsymbol{F}_a(t) &= \boldsymbol{C}\dot{\boldsymbol{X}}(t) \end{aligned} \tag{3.6}$$

and the vector $\boldsymbol{J}$ defines the rigid body displacements of the model according to each degree of freedom. By substituting the equations (3.6) in (3.5), the following equations of motion are obtained:

$$\boldsymbol{M}\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}\dot{\boldsymbol{X}}(t) + \boldsymbol{K}\boldsymbol{X}(t) = -\boldsymbol{M}\boldsymbol{J}\ddot{x}_s(t) \tag{3.7}$$

3.1.2 Kelvin's model

The simplest multi-degree of freedom model which can be used in the dynamic modelling of structures is constituted by a series of mass-spring-damper systems which are schematically pictured in figure 3.1(a). It is based on the hypothesis that the only possible displacement of the interconnected rigid bodies is the horizontal one. The dynamic forces $f_r(t)$ act according to the degrees of freedom.

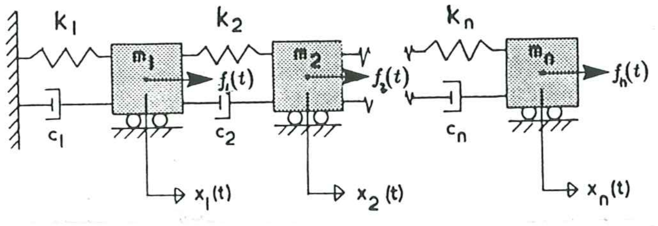

Figure 3.1 Multi-degree of freedom model. (a) Dynamic model; (b) force equilibrium.

In this particular case the stiffness matrix has the following particular form:

$$K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 & & & \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 & & \mathbf{0} & \\ 0 & -k_3 & k_3+k_4 & -k_4 & 0 & & & \\ & & & 0 & -k_r & k_r+k_{r+1} & -k_{r+1} & 0 \\ & \mathbf{0} & & & & & & \\ & & & & & & & k_n \end{bmatrix} \quad (3.8)$$

where k_r is the stiffness of each spring. The mass matrix $\boldsymbol{M}$ of the same model is diagonal and the damping matrix $\boldsymbol{C}$ can be considered, for instance, of the same type.

3.1.3 3D beam model

In the more general case of a three-dimensional beam structure subjected to a spatial support motions, six degrees of freedom have to be considered for each node of the model. With this aim, the the three components of the support acceleration, according to both horizontal axes x_s and y_s and to the vertical one, z_s, have to be considered. Thus, the equations of motion (3.7) can be completed to cover these possibilities, resulting in

$$\boldsymbol{M}\ddot{\boldsymbol{D}}(t)+\boldsymbol{C}\dot{\boldsymbol{D}}(t)+\boldsymbol{K}\boldsymbol{D}(t)=-\boldsymbol{M}\left[\boldsymbol{J}_x\ddot{x}_s(t)+\boldsymbol{J}_y\ddot{y}_s(t)+\boldsymbol{J}_z\ddot{z}_s(t)\right] \quad (3.9)$$

The vector of the unknown displacements $\boldsymbol{D}(t)$ has six elements for each degree of freedom r: three translations (x_r, y_r, z_r) and three rotations $(\varphi_{x_r}, \varphi_{y_r}, \varphi_{z_r})$, being of the form

$$[\boldsymbol{D}(t)]^{\mathrm{T}}=[\ x_1\ y_1\ z_1\ \varphi_{x1}\ \varphi_{y1}\ \varphi_{z1}\ldots x_r\ y_r\ z_r\ \varphi_{x_r}\ \varphi_{y_r}\ \varphi_{z_r}\ldots \\ \ldots x_n\ y_n\ z_n\ \varphi_{xn}\ \varphi_{yn}\ \varphi_{zn}\ ] \quad (3.10)$$

Other notations that have been introduced in the system of differential equation (3.9) are

$$\begin{aligned} \boldsymbol{J}_x^{\mathrm{T}} &= [1\,0\,0\,0\,0\,0\ \ldots\ldots\ 1\,0\,0\,0\,0\,0\ \ldots\ldots\ 1\,0\,0\,0\,0\,0] \\ \boldsymbol{J}_y^{\mathrm{T}} &= [0\,1\,0\,0\,0\,0\ \ldots\ldots\ 0\,1\,0\,0\,0\,0\ \ldots\ldots\ 0\,1\,0\,0\,0\,0] \\ \boldsymbol{J}_z^{\mathrm{T}} &= [0\,0\,1\,0\,0\,0\ \ldots\ldots\ 0\,0\,1\,0\,0\,0\ \ldots\ldots\ 0\,0\,1\,0\,0\,0] \end{aligned} \quad (3.11)$$

The formulation given for the seismic force vector assures that the acceleration components produce nonzero elements only for the translational degrees of freedom. One

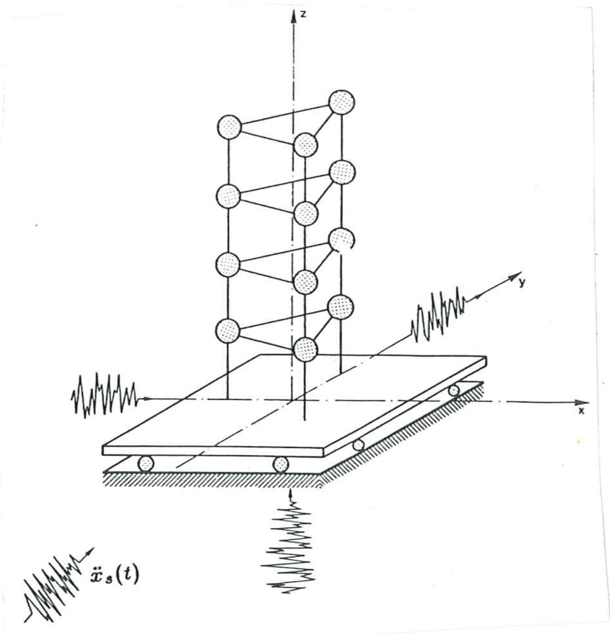

Figure 3.2 Dynamic three-dimensional model.

must remark that the matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ of (3.9) are expanded, so as to be concordant with the vector $\boldsymbol{D}(t)$. Obviously, it would by possible to include in the equations of motion the effect of the action of dynamic loads in an arbitrary direction respecting the structure. In such a case, the dynamic forces should be decomposed according to both horizontal axes x and y and to the vertical one, z, the components $f_x(t)$, $f_y(t)$ and $f_z(t)$ being thus obtained. The dynamic models used in the analysis may include only some of these degrees of freedom, depending on the actual characteristics of the structure studied.

An alternative formulation can be given, by writing the equations of motion as

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{C}\dot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = -\boldsymbol{M}\boldsymbol{J}\ddot{x}_s(t) \tag{3.12}$$

$\boldsymbol{J}$ is, in this case, a vector which performs the decomposition of $\ddot{x}_s(t)$ according to the directions x, y and z and has values distinct from zero only for the translational degrees of freedom of the model. Generally, the nonzero elements of $\boldsymbol{J}$ are direction cosine functions.

The free damped vibration of the dynamic model is described by

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{C}\dot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = \boldsymbol{0} \tag{3.13}$$

and when the damping is neglected, the following system of equations:

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = \boldsymbol{0} \tag{3.14}$$

describes the vibration of the model. In the equations of motion (3.10), (3.12), (3.13) and (3.14), the stiffness matrix $\boldsymbol{K}$ is expressed exactly as in the static case[(4)], while the mass matrix $\boldsymbol{M}$ is normally diagonal. If rotations are considered in vector $\boldsymbol{D}(t)$, it is necessary to consider in the mass matrix $\boldsymbol{M}$ elements corresponding to the rotational inertia. The influence of these elements on the solution of the problem is generally small, being therefore substituted in many cases by zero[(5–7)].

3.2 MODES OF VIBRATION

3.2.1 Spectral and modal matrices

The dynamic characteristics of concentrated mass multi-degree of freedom models are defined now, analyzing the undamped free vibration. For any of the studied cases, the motion of the system is governed by equation (3.14). The system of differential equations (3.12) is verified by a particular solution of the following type:

$$\boldsymbol{D}(t) = \boldsymbol{A}e^{i\omega t} \tag{3.15}$$

or by the similar one

$$\boldsymbol{D}(t) = \boldsymbol{A}\sin(\omega t + \psi) \tag{3.16}$$

where the vector $\boldsymbol{A}$ contains the amplitudes of the vibration, ω is the natural frequency of the model and ψ is the phase angle. The substitution of (3.15) or (3.16) in the equations (3.14) gives

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\,\boldsymbol{A} = \boldsymbol{0} \tag{3.17}$$

This algebraic system of homogeneous linear equations constitutes an *eigenvalues problem*. The system has solutions $\boldsymbol{A}$ distinct from the trivial ones (with the physical meaning that the system vibrates), only if the determinant of the matrix coefficients is equal to zero

$$|\boldsymbol{K} - \omega^2\boldsymbol{M}| = \boldsymbol{0} \tag{3.18}$$

Equation (3.18) can be developed in the following polynomial form:

$$\omega^{2n} + \alpha_1\omega^{2n-2} + \alpha_2\omega^{2n-4} + \ldots + \alpha_{n-1}\omega^2 + \alpha_n = 0 \tag{3.19}$$

and is called *characteristic equation*. The matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ are real, and symmetric. $\boldsymbol{K}$ is also positive definite. $\boldsymbol{M}$ is at least semipositive definite. In the case in which $\boldsymbol{M}$ is also positive definite, the characteristic equation provides n positive solutions ω_i^2, and consequently n real values ω_i. If $\boldsymbol{M}$ is only semipositive definite, the number of finite solutions ω_i^2 is smaller than n. The n eigenvalues ω_i are the *natural frequencies* of the multi-degree-of-freedom model. The frequencies ω_i can be arranged in a sequence in the diagonal matrix $\boldsymbol{\Omega}$ which is called *spectral matrix*. The lowest frequency ω_1 is known as *fundamental frequency*. A similar spectral matrix $\boldsymbol{\Omega}^2$ can be defined, which has as elements the values ω_i^2. The natural periods of the model are defined by

$$T_i = \frac{2\pi}{\omega_i} \qquad i = 1, 2, \ldots, n \tag{3.20}$$

where T_1 is the fundamental period. The amplitude vector $\boldsymbol{A}_i$ corresponding to the frequency ω_i satisfies identically the equations (3.17). Therefore, the vector $\boldsymbol{A}_i$, known as *eigenvector*, can be obtained by expressing all the elements of $\boldsymbol{A}_i$ in terms of any one of them. For example, the elements of $\boldsymbol{A}_i$ can be divided by $\boldsymbol{A}_{i1}$, resulting thus

$$\boldsymbol{\varphi}_i = \frac{\boldsymbol{A}_i}{A_{i1}} \tag{3.21}$$

where all the *normalized eigenvectors* $\boldsymbol{\varphi}_i$, $i = 1, 2, 3, \ldots, n$, have the first element equal to the unit. This operation is called normalization. Another possibility to normalize the eigenvectors is based on the use of the relation

$$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{M}\boldsymbol{A}_i = M_i^* \tag{3.22}$$

which permits the application of the following scaling formula:

$$\boldsymbol{\varphi}_i = \boldsymbol{A}_i \times (M_i^*)^{-\frac{1}{2}} \qquad i = 1, 2, \ldots, n \tag{3.23}$$

This normalization procedure assures the fulfilment of the condition

$$\boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\varphi}_i = 1 \tag{3.24}$$

where the eigenvectors $\boldsymbol{\varphi}_i$ can be organized in the matrix $\boldsymbol{\Phi}$

$$\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2 \quad \cdots \quad \boldsymbol{\varphi}_i \quad \cdots \quad \boldsymbol{\varphi}_n] \tag{3.25}$$

The eigenvectors $\boldsymbol{\varphi}_i$ does not express the amplitudes of the vibration, which are indeterminate; actually, they describe the *shape* of the system during its vibration, corresponding to each one of the eigenvalues. Therefore, in the structural analysis, the eigenvectors are called sometimes *natural shapes of vibration*. An eigenvalue ω_i together with the correspondent eigenvector $\boldsymbol{\varphi}_i$ constitute *the natural mode of vibration* i, and the matrix $\boldsymbol{\Phi}$ is called *modal matrix*.

3.2.2 Orthogonality conditions

The orthogonality condition of two eigenvectors is defined by

$$\boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{\varphi}_j = 0 \qquad i \neq j \tag{3.26}$$

Similar orthogonality conditions with respect to the mass matrix $\boldsymbol{M}$ and to the stiffness matrix $\boldsymbol{K}$ can be now introduced, by means of the expressions(2,3)

$$\boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\varphi}_j = 0 \qquad i \neq j \tag{3.27}$$

$$\boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{K} \boldsymbol{\varphi}_j = 0 \qquad i \neq j \tag{3.28}$$

The normality condition (3.24), together with the orthogonality one (3.27), can be expressed by an unique *orthonormality condition* respecting the mass matrix

$$\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\Phi} = \boldsymbol{I} \tag{3.29}$$

where $\boldsymbol{\Phi}$ is the modal matrix and $\boldsymbol{I}$ the identity one. In a similar way, the orthogonality condition respecting the stiffness matrix (3.28) is expressed as

$$\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{\Phi} = \boldsymbol{K}^* \tag{3.30}$$

where

$$K_i^* = \boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{K} \boldsymbol{\varphi}_i \tag{3.31}$$

$\boldsymbol{K}^*$ is thus a diagonal matrix. In the case of a modal matrix for which the normalization condition (3.29) has been not used, the following orthogonality condition respecting the mass matrix can be written:

$$\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \boldsymbol{\Phi} = \boldsymbol{M}^* \tag{3.32}$$

where $\boldsymbol{M}^*$ is again diagonal matrix.

3.3 CLASSICAL DAMPING

It has been explained previously how the mass and stiffness matrices of a multi-degree of freedom model can be defined. In the definition of the damping matrix, some simplifying hypotheses have to be made in order to permit a reasonable numerical description of the damping properties of the structure. One of these hypotheses makes the assumption that a homogeneous energy loss mechanism exists throughout the entire structure(8). In such a case, a damping matrix which fulfils the orthogonality condition respecting the modal matrix can be developed. This condition defines a *classical* damping[9], which is considered to be proportional to the mass matrix, to the stiffness matrix or is obtained as a linear combination of both. Thus, the damping can be specified for each mode of vibration of the structure, by using the damping ratio ζ[10].

When the damping matrix $\boldsymbol{C}$ is proportional to the mass matrix

$$\boldsymbol{C} = \alpha_1 \boldsymbol{M} \tag{3.33}$$

the following orthogonality condition holds:

$$\boldsymbol{\varphi}_i^{\mathrm{T}} \boldsymbol{C} \boldsymbol{\varphi}_i = 0 \qquad i \neq j \tag{3.34}$$

and a diagonal *generalized damping* matrix $\boldsymbol{C}^*$ is thus obtained

$$\boldsymbol{C}^* = \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{C} \, \boldsymbol{\Phi} \tag{3.35}$$

By substituting (3.33) in (3.35), the generalized damping matrix results in

$$\boldsymbol{C}^* = \alpha_1 \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \, \boldsymbol{\Phi} = \alpha_1 \boldsymbol{M}^* \tag{3.36}$$

If the definition of the damping ratio of a single-degree of freedom model is now applied to the generalized damping coefficient $\boldsymbol{C}^*$, the damping ratio ζ_i is expressed for each mode of vibration by means of

$$\zeta_i = \frac{C_i^*}{2 M_i^* \omega_i} \qquad i = 1, 2, \ldots, n$$

In the case in which the damping matrix $\boldsymbol{C}$ is proportional to the stiffness matrix $\boldsymbol{K}$

$$\boldsymbol{C} = \alpha_2 \boldsymbol{K}$$

a similar orthogonality condition is obtained

$$\boldsymbol{C}^* = \alpha_2 \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{M} \, \boldsymbol{\Phi} = \alpha_2 \boldsymbol{K}^*$$

where $\boldsymbol{C}^*$ is diagonal. The combination of these two cases

$$\boldsymbol{C} = \alpha_1 \boldsymbol{M} + \alpha_2 \boldsymbol{K} \tag{3.37}$$

defines Rayleigh's damping.

3.4 NON-CLASSICAL DAMPING

The case of the structural models with proportional damping has to be considered as a particular one, which is based on the assumption of the homogeneity throughout the entire model of the energy loss mechanism. This hypothesis permits to develop orthogonal damping matrices which simplifies the procedures to solve the equations of motion of the structure.

If there are important differences between the energy loss mechanisms of the distinct parts of a structure, the correspondent damping is non-classical[8]. Such a damping is characteristic, for example, for models composed of structures laying on a soft soil, or for structures coupled with fluid media. In such cases the damping matrix is not orthogonal, with the consequent increase of the numerical difficulties in solving the equations of motion[11].

3.5 MODAL ANALYSIS

3.5.1 Uncoupling of the equations of motion

According to the equations (3.4), (3.7) or (3.12) the system of dynamic equations corresponding to a linear structure with n degrees of freedom is

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{C}\dot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = \boldsymbol{F}(t) \tag{3.38}$$

The free vibrations of the corresponding system without damping are governed by the equation

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = \boldsymbol{0} \tag{3.39}$$

There are n eigenvalues and n eigenvectors associated to equation (3.39). The eigenvalues are the squares of the eigenfrequencies ω, while the corresponding eigenvectors are the mode shapes $\boldsymbol{\varphi}$ of the system. They are obtained as solutions of the following homogeneous system of algebraic equations

$$(-\omega^2\, \boldsymbol{M} + \boldsymbol{K})\, \boldsymbol{\varphi} = \boldsymbol{0} \tag{3.40}$$

The mode shapes $\boldsymbol{\varphi}$ are orthonormal respecting the mass and stiffnes matrices(12). As the modal matrix $\boldsymbol{\Phi} = [\, \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_i, \ldots, \boldsymbol{\varphi}_n \,]$ constitutes a complet set of n orthonormal eigenvectors, the solution of the system of equations (3.38) can be expressed in the following form:

$$\boldsymbol{D} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i\, y_i\,(t) \tag{3.41}$$

where $y_i(t)$, $i = 1, 2, \ldots, n$ are scalar functions of time to be determinated, denominated generalized coordinates. By substituting (3.41) in (3.38), the equation of motion expressed in this new system of coordinates results in

$$\boldsymbol{M}\sum_{i=1}^{n} \boldsymbol{\varphi}_i \ddot{y}_i(t) + \boldsymbol{C}\sum_{i=1}^{n} \boldsymbol{\varphi}_i \dot{y}_i(t) + \boldsymbol{K}\sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i(t) = \boldsymbol{F}(t) \tag{3.42}$$

If (3.42) is now premultiplied by the transpose of any eigenvector $\boldsymbol{\varphi}_j$, it becomes

$$\boldsymbol{\varphi}_j^{\mathrm{T}}\boldsymbol{M}\sum_{i=1}^{n} \boldsymbol{\varphi}_i \ddot{y}_i(t) + \boldsymbol{\varphi}_j^{\mathrm{T}}\boldsymbol{C}\sum_{i=1}^{n} \boldsymbol{\varphi}_i \dot{y}_i(t) + \boldsymbol{\varphi}_j^{\mathrm{T}}\boldsymbol{K}\sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i(t) = \boldsymbol{\varphi}_j^{\mathrm{T}}\boldsymbol{F}(t) \tag{3.43}$$

The orthogonality condition of the modal matrix $\boldsymbol{\Phi}$, prvides

$$\boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{M} \sum_{i=1}^{n} \boldsymbol{\varphi}_i = \boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{M}\, \boldsymbol{\varphi}_j = M_j^* \tag{3.44a}$$

$$\boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{K} \sum_{i=1}^{n} \boldsymbol{\varphi}_i = \boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{K}\, \boldsymbol{\varphi}_j = K_j^* \tag{3.44b}$$

Particular types of damping matrices are usually considered in the dynamic analysis of structures. For example, if the damping is a classical one, such as the Rayleigh damping, the modal matrix is orthonormal respecting the damping matrix. In this case, the following equation, similar to (3.44a) and (3.44b) can be written:

$$\boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{C} \sum_{i=1}^{n} \boldsymbol{\varphi}_i = \boldsymbol{\varphi}_j^{\mathrm{T}}\, \boldsymbol{C}\, \boldsymbol{\varphi}_j = C_j^* \tag{3.44c}$$

Using the three equations (3.44), the equation (3.43) can be written in a form similar to that of the corresponding to a single-degree of freedom model

$$M_j^* \ddot{y}_j(t) + C_j^* \dot{y}_j(t) + K_j^* y_j(t) = \boldsymbol{\varphi}_j^{\mathrm{T}} \boldsymbol{MJ}\, \ddot{x}_s(t), \qquad j = 1, 2, \ldots, n \tag{3.45}$$

As it can be seen, the system of differential equations (3.38) whose solution is $\boldsymbol{D}$, has been reduced to n independent differential equations whose solutions are y_j, $j = 1, 2, \ldots, n$. Equation (3.45) is now writen in the following usual form:

$$\ddot{y}_j(t) + 2\,\zeta_j\,\omega_j\,\dot{y}_j(t) + \omega_j^2 y_j(t) = -\frac{\boldsymbol{\varphi}_j^{\mathrm{T}} \boldsymbol{MJ}}{\boldsymbol{\varphi}_j^{\mathrm{T}} \boldsymbol{M}\, \boldsymbol{\varphi}_j} \ddot{x}_s(t), \qquad j = 1, 2, \ldots, n \tag{3.46}$$

where ω_j is the eigenfrequency associated to the mode shape $\boldsymbol{\varphi}_j$. The equation (3.46) can be solved by using any of the procedures developed for single-degree of freedom models.

An important aspect concerning the modal analysis has to be pointed out: due to approximations owning to the discretization process of the structure, as well as to approximations associated with the numerical calculations, the errors wich appear in the computation of the frequencies increase with the increase of the order of the modes of vibration. Consequently, the frequencies corresponding to the first modes of vibration are more exact than those corresponding to the higher modes. Moreover, due to the fact that the lower modes contain smaller elastic energy of deformation, their contribution to the structural response is the most important. And this is so to such an extent that the higher modes only contribute with perturbations and errors to the correct response of the structure. That is why equation (3.41) is normally written in the following form:

$$\boldsymbol{D} = \sum_{i=1}^{q} \boldsymbol{\varphi}_i\, y_i\,(t) \tag{3.47}$$

where usually $q << n$ and, in any case, $q < n$. Therefore, the number of equations of the type (3.46) to be solved is considerably smaller than n, a fact that simplifies the numerical process remarkably.

The trasformation of equation (3.41) into (3.47) is very advantageous, given the fact that computers spend most of the calculation time required by the computation of the structural response in evaluating the eigenfrequencies and eigenvectors. Thus, if the calculation of the response includes only a few modes of vibration, corresponding to the lower frequencies, much of the computer time can be saved. Therefore, a procedure capable to calculate only the first desired modes of vibration is necessary to be used to take advantage of the modal analysis. Such a procedure, the Determinant Search Method, is described in Appendix 1.

3.5.2 Computation of the dynamic response

The dynamic response of structures is described through the *time history* of the displacements, bending moments, shear forces, stresses, etc., corresponding to each degree of freedom of the structural model. This description normally provides an excess of information on the structural response, due to the fact that for design purposes the knowledge of the maximum response is sufficient. Such an excess of information obviously implies longer calculation time and a larger amount of computer memory storage.

The computation of the time history of the response starts from the numerical evaluation of the solution $y_i(t)$ of the equations (3.46), for all the modes of vibration considered in the analysis. This evaluation can be made using known numerical procedures for single-degree of freedom models. The displacement response history $\boldsymbol{D}(t)$ is then obtained by applying the modal superposition according to equation (3.47). The calculation of the time history of the displacements, bending moments, shear forces, stresses, etc., is performed by means of well-known static equations, applied at each time instant by starting from the values $\boldsymbol{D}(t)$ previously established.

3.6 FREQUENCY DOMAIN ANALYSIS

Consider a multi-degree dynamic model whose behaviour is governed by an equation of the type (3.38). The excitation signal, $f(t)$, as well as the response signals $\ddot{\boldsymbol{D}}(t)$, $\dot{\boldsymbol{D}}(t)$ and $\boldsymbol{D}(t)$ are considered finite, continuous and bounded. In these conditions their direct and inverse Fourier transforms always exist and can be evaluated. Moreover, the dynamic excitation $f(t)$ is a finite signal and distinct from zero only for $0 < t < t_f$, and the same statement will be made on the dynamic response. Thus, the Fourier transform of the excitation can be written as

$$F(\theta) \;=\; = \int_0^{t_f} f(t)\, e^{-i\theta t}\, dt \tag{3.48}$$

and the Fourier transforms of the response signals are

$$\overline{\boldsymbol{D}}(\theta) \;=\; \int_0^{t_f} \boldsymbol{D}(t)\, e^{-i\theta t}\, dt \tag{3.49}$$

$$\overline{\dot{\boldsymbol{D}}}(\theta) \;=\; \int_0^{t_f} \dot{\boldsymbol{D}}(t)\, e^{-i\theta t}\, dt \;=\; i\theta\overline{\boldsymbol{D}}(\theta) \tag{3.50}$$

$$\overline{\ddot{\boldsymbol{D}}}(\theta) \;=\; \int_0^{t_f} \ddot{\boldsymbol{D}}(t)\, e^{-i\theta t}\, dt \;=\; -\theta^2\overline{\boldsymbol{D}}(\theta) \tag{3.51}$$

where θ is the frequency of the excitation. Using these equations, the system of differential equations (3.38) transforms into the following system of algebraic equations with complex coefficients

$$(-\theta^2\boldsymbol{M} + i\theta\boldsymbol{C} + \boldsymbol{K})\overline{\boldsymbol{D}}(\theta) \;=\; -\boldsymbol{MJ}F(\theta) \tag{3.52}$$

The time domain response of the system is obtained by taking the inverse Fourier transform of the complex frequency response

$$\boldsymbol{D}(t) \;=\; \frac{1}{2\pi}\int_0^{t} \overline{\boldsymbol{D}}(\theta)\, e^{i\theta t}\, d\theta \tag{3.53}$$

$$\dot{\boldsymbol{D}}(t) \;=\; \frac{1}{2\pi}\int_0^{t} (i\theta\overline{\boldsymbol{D}}(\theta))\, e^{i\theta t}\, d\theta \tag{3.54}$$

$$\ddot{\boldsymbol{D}}(t) \;=\; \frac{1}{2\pi}\int_0^{t} (-\theta^2\overline{\boldsymbol{D}}(\theta))\, e^{i\theta t}\, d\theta \tag{3.55}$$

Both the direct and inverse Fourier transform have to be evaluated numerically, by means of the discret Fourier transform, which is usually computed by means of the Fast Fourier Transform (FFT) algorithm[13–15].

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4. Random vibrations

4.1 SINGLE-DEGREE OF FREEDOM MODELS SUBJECTED TO STATIONARY DYNAMIC ACTION

4.1.1 Probabilistic input-output relations

Consider the linear single-degree of freedom model of figure 4.1, subjected to a dynamic action $f(t)$, which can represent both an applied dynamic force or the force produced on the model by a support displacement $x_s(t)$. The force $f(t)$ is modelled as an ergodic, normal, zero mean random process, characterized through its power spectral density $S_f(\theta)$.

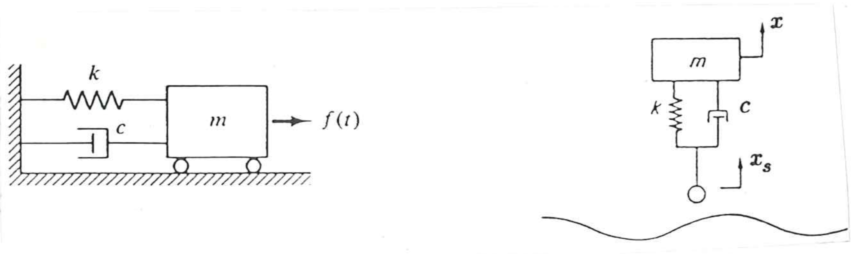

Figure 4.1 Single-degree of freedom models subjected to a random action.

A sample function $f(t)$ of the stochastic process is expressed by means of the inverse Fourier transform

$$f(t) \;=\; \int_{-\infty}^{+\infty} F(\theta) e^{i\theta t}\, \mathrm{d}\theta \tag{4.1}$$

where

$$F(\theta) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t) e^{-i\theta t}\, \mathrm{d}t \tag{4.2}$$

is the direct Fourier transform of $f(t)$.

Consider now the equation of motion of the model of figure 4.1, subjected to the sample function $f(t)$ of the action

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \tag{4.3}$$

As the analyzed system is linear, the response $x(t)$ is also a stochastic ergodic process. In a frequency domain analysis, the Fourier transform of the response, $X(\theta)$, is

$$X(\theta) = H(\theta)F(\theta) \tag{4.4}$$

where $H(\theta)$ is the frequency response function of the model expressed by

$$H(\theta) = \frac{1}{k\left(-\frac{\theta^2}{\omega^2} + 2i\zeta\frac{\theta}{\omega} + 1\right)} \tag{4.5}$$

θ is the frequency of the excitation, ω is the natural frequency of the system and ζ is the damping ratio.

Each term of equation (4.4) is now multiplied by its complex conjugate and divided by $2\pi T_0$, resulting thus

$$\frac{1}{2\pi T_0}\, X(\theta)X^*(\theta) = \frac{1}{2\pi T_0}\, H(\theta)H^*(\theta)F(\theta)F^*(\theta) \tag{4.6}$$

where $X^*(\theta)$, $H^*(\theta)$ and $F^*(\theta)$ are the complex conjugate values of $X(\theta)$, $H(\theta)$ and $F(\theta)$, respectively. Thus, (4.6) can be written in the following form:

$$\frac{1}{2\pi T_0}\, |X(\theta)|^2 = \frac{1}{2\pi T_0}\, |H(\theta)|^2\, |F(\theta)|^2 \tag{4.7}$$

Estimates for the power spectral density of the excitation random process $S_f(\theta)$ and of the response random process $S_x(\theta)$ can be now written[1-3]

$$S_f(\theta) = \lim_{T_0 \to \infty} \frac{1}{2\pi T_0} |F(\theta)|^2 \tag{4.8}$$

$$S_x(\theta) = \lim_{T_0 \to \infty} \frac{1}{2\pi T_0}\, |X(\theta)|^2 \tag{4.9}$$

At limit, for $T_0 \to \infty$ and by using (4.8) and (4.9), equation (4.7) becomes[2]

$$S_x(\theta) = |H(\theta)|^2\, S_f(\theta) \tag{4.10}$$

where

$$|H(\theta)| = \frac{1}{k\sqrt{\left(1 - \frac{\theta^2}{\omega^2}\right)^2 + \left(2\zeta\frac{\theta}{\omega}\right)^2}} \tag{4.11}$$

Thus, the filter which performs the transformation of the power spectral density of the action $S_f(\theta)$ into the power spectral density of the displacement response of the system is

$$|H(\theta)|^2 = \frac{\omega^{-4}}{m^2\left[\left(1 - \frac{\theta^2}{\omega^2}\right)^2 + \left(2\zeta\frac{\theta}{\omega}\right)\right]^2} \tag{4.12}$$

Equations similar to (4.10), but relating the power spectral densities of the processes describing the velocity and acceleration responses to the power spectral density of the action can be obtained[3]

$$S_{\dot{x}}(\theta) = \theta^2\, |H(\theta)|^2 S_f(\theta) \tag{4.13}$$

$$S_{\ddot{x}}(\theta) = \theta^4\, |H(\theta)|^2 S_f(\theta) \tag{4.14}$$

The variance of the displacement, velocity and acceleration response of the structure is defined by[3]

$$\sigma_x^2 = \int_{-\infty}^{+\infty} S_x(\theta)\, d\theta \tag{4.15}$$

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{+\infty} S_{\dot{x}}(\theta)\,\mathrm{d}\theta \tag{4.16}$$

$$\sigma_{\ddot{x}}^2 = \int_{-\infty}^{+\infty} S_{\ddot{x}}(\theta)\,\mathrm{d}\theta \tag{4.17}$$

By using now in (4.15) the expression of $S_x(\theta)$ given by (4.10), the variance of the displacement response becomes

$$\sigma_x^2 = \int_{-\infty}^{+\infty} |H(\theta)|^2 S_f(\theta)\,\mathrm{d}\theta \tag{4.18}$$

Similarly, the substitution of (4.13) and (4.14) in (4.16) and (4.17) respectively, provides

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{+\infty} \theta^2 |H(\theta)|^2 S_f(\theta)\,\mathrm{d}\theta \tag{4.19}$$

$$\sigma_{\ddot{x}}^2 = \int_{-\infty}^{+\infty} \theta^4 |H(\theta)|^2 S_f(\theta)\,\mathrm{d}\theta \tag{4.20}$$

4.1.2 Structural response to a white noise

In the case in which $f(t)$ is described by an ideal white noise, $S_f(\theta)$ is constant and equal to S_0. It follows from here that

$$\sigma_x^2 = S_0 \int_{-\infty}^{+\infty} |H(\theta)|^2\,\mathrm{d}\theta \tag{4.21}$$

By using the expression of $H(\theta)|^2$ given by (4.12), the variance of the displacement response becomes

$$\sigma_x^2 = \frac{\omega^{-4}}{m^2} S_0 \int_{-\infty}^{+\infty} \frac{1}{\left(1 - \frac{\theta^2}{\omega^2}\right)^2 + \left(2\zeta\frac{\theta}{\omega}\right)^2}\mathrm{d}\theta \tag{4.22}$$

and solving the integral, the last equation turns into[1-3]

$$\sigma_x^2 = \frac{\pi S_0}{2\zeta\omega^3 m^2} \tag{4.23}$$

By making the same substitution in (4.19) and (4.20), the variances of the velocity and acceleration responses, become

$$\sigma_{\dot{x}}^2 = \frac{\pi S_0}{2\zeta\omega m^2} \tag{4.24}$$

$$\sigma_{\ddot{x}}^2 = \frac{\pi S_0 \omega}{2\zeta m^2} \tag{4.25}$$

The interpretation of these results in term of values used in structural analysis will be given in a further section.

4.1.3 Structural response to a filtered white noise

Consider that the dynamic force acting on the system is a white noise process with a power spectral density S_0, filtered through a model with a frequency response function $|H_g(\theta)|^2$. The power spectral density resulting after the filtering operation is $S_g^f(\theta)$

$$S_g^f(\theta) = |H_g(\theta)|^2 S_0 \qquad \theta \in (-\infty, +\infty) \tag{4.26}$$

Using (4.10), (4.13) and (4.14), the power spectral densities of the displacement, velocity and acceleration responses of the single-degree of freedom model subjected to the filtered white noise process are given by the following equations:

$$S_x(\theta) = |H(\theta)|^2 \, |H_g(\theta)|^2 \, S_0 \tag{4.27}$$

$$S_{\dot{x}}(\theta) = \theta^2 \, |H(\theta)|^2 |H_g(\theta)|^2 \, S_0 \tag{4.28}$$

$$S_{\ddot{x}}(\theta) = \theta^4 \, |H(\theta)|^2 \, |H_g(\theta)|^2 \, S_0 \tag{4.29}$$

If equations (4.15), (4.16) and (4.17) are now applied, the correspondent variances are obtained

$$\sigma_x^2 = S_0 \int_{-\infty}^{+\infty} |H(\theta)|^2 \, |H_g(\theta)|^2 d\theta \tag{4.30}$$

$$\sigma_{\dot{x}}^2 = S_0 \int_{-\infty}^{+\infty} \theta^2 |H(\theta)|^2 \, |H_g(\theta)|^2 d\theta \tag{4.31}$$

$$\sigma_{\ddot{x}}^2 = S_0 \int_{-\infty}^{+\infty} \theta^4 |H(\theta)|^2 \, |H_g(\theta)|^2 d\theta \tag{4.32}$$

The integrals of these equations can be generally solved only by using numerical procedures.

4.2 Models of random dynamic excitations

4.2.1 Seismic ground acceleration

As a first example, the case of an earthquake ground motion is considered. The seismic acceleration in the epicenter is defined as a white noise process[4,5] having a sample function $a(t)$ and the power spectral density S_0. The equation of motion of a single degree of freedom model subjected to $a(t)$ is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -ma(t) \tag{4.33}$$

The power spectral density resulting after filtering the white noise process through the soil layers is $S_a^f(\theta)$. Kanai and Tajimi[6] have introduced a single-degree of freedom model for the soil layer, whose characteristics are its natural frequency θ_g and its damping ratio ζ_g. The model has the following frequency response function:

$$|H_{1g}(\theta)|^2 = \frac{1 + 4\zeta_{g1}\left(\frac{\theta}{\theta_{g1}}\right)^2}{\left[1 - \left(\frac{\theta}{\theta_{g1}}\right)^2\right]^2 + 4\zeta_{g1}^2\left(\frac{\theta}{\theta_{g1}}\right)^2} \tag{4.34}$$

For firm soil conditions Kanai has recommended $\theta_{g1} = 15.6$ rad/s and $\zeta_{g1} = 0.6$.

By using (4.34) the power spectral density of the filtered process, $a^f(t)$ can be expressed as

$$S_a^f(\theta) = |H_{1g}(\theta)|^2 \, S_0 \qquad \theta \in (-\infty, +\infty) \tag{4.35}$$

This filter reduces the high frequency components and amplifies those components which have the frequency in the neighbourhood of $\theta_g = 4\pi$. Nevertheless, this filter produce at the same time singularities for very low frequencies and consequently, unbounded variance of the ground velocities and displacements[1]. A plot of $S_a^f(\theta)$ is given in figure 4.2.

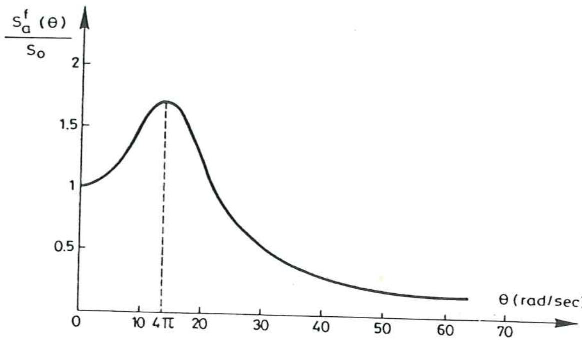

Figure 4.2 Plot of the power spectral density function $S_a^f(\theta)$.

To avoid the mentioned problem, the process $a^f(t)$ can be filtered another time, through a filter which reduces the components with very low frequency. The following transfer function can be used for the second filter[(2)]:

$$|H_{2g}(\theta)|^2 = \frac{\left(\frac{\theta}{\theta_{g2}}\right)^2}{\left[1-\left(\frac{\theta}{\theta_{g2}}\right)^2\right]^2 + 4\zeta_{g2}^2\left(\frac{\theta}{\theta_{g2}}\right)^2} \tag{4.36}$$

The process $a^s(t)$ resulted by passing $a^f(t)$ through this second filter has a power spectral density $S_a^s(\theta)$ given by

$$S_a^s(\theta) = |H_{2g}(\theta)|^2 S_a^f \tag{4.37}$$

By using now (4.35) the following equation is obtained:

$$S_a^s(\theta) = |H_g(\theta)|^2 \; S_o \tag{4.38}$$

where

$$|H_g(\theta)|^2 = |H_{1g}(\theta)|^2 \; |H_{2g}(\theta)|^2 \tag{4.39}$$

Figure 4.3 shows a plot of $S_a^s(\theta)$.

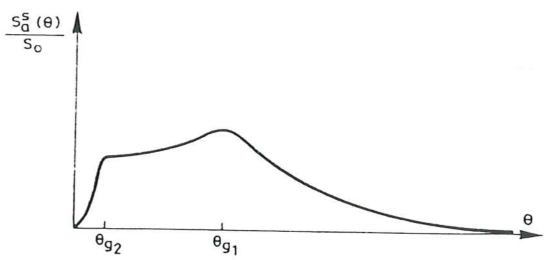

Figure 4.3 Plot of the power spectral density function $S_a^s(\theta)$.

The power spectral densities of the displacement, velocity and acceleration responses are now given by the equations (4.27), (4.28) and (4.29), with $|H_g(\theta)|^2 = |H_f(\theta)|^2$. Similarly, the corresponding variances are expressed by (4.30), (4.31) and (4.32), with the inclusion of the same modification.

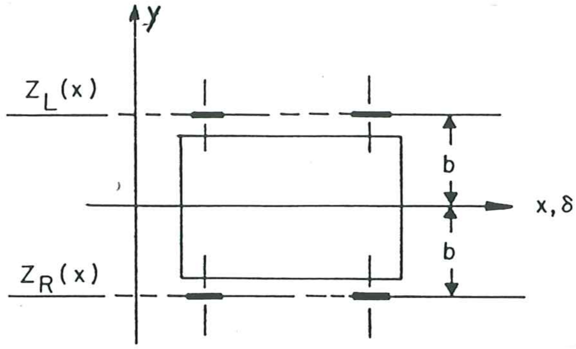

Figure 4.4 A four wheel vehicle moving on a road[8].

4.2.2 Roughness of road surface

A second example considers the case of a stochastic model of the roughness of a road surface[7–9]. A scheme of a four-wheel vehicle moving on a road can be seen in figure 4.4.

The displacements $Z(x, y)$ describing the roughness of the road induce in the wheels excitations which are described through their statistical properties. In order to obtain the dynamic response of the vehicle, the cross-spectral densities or the cross-correlation functions between the displacements applied on the four wheels have to be estimated. The statistics of the road surface roughness can be evaluated starting from measurements. As an example, the spectral densities obtained in this way are shown in figure 4.5. The spectral density of the road surface roughness is usually plotted against the wave number n, which is the reciprocal of the wave length, and not respecting to the frequency. The values given in figure 4.5 can be simulated by means of the following equation[8]:

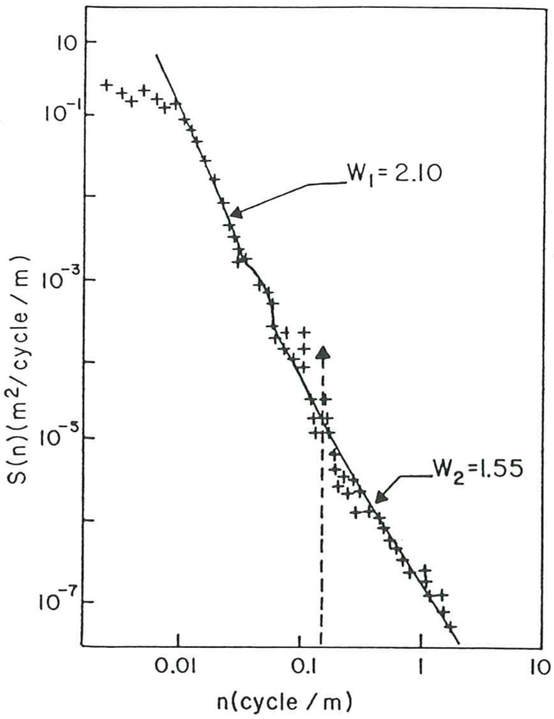

Figure 4.5 Example of the spectral density of a typical road surface roughness[8,10].

$$S(n) = S(n_0)\left(\frac{n}{n_0}\right)^{w_1}, \qquad n \leq n_0$$
$$= S(n_0)\left(\frac{n}{n_0}\right)^{w_2}, \qquad n \geq n_0 \tag{4.40}$$

where $n_0 = 1/(2\pi)$ cycle/m and $S(n_0)$ is the roughness coefficient. Some numerical values of $S(n_0)$, whose units are 10^{-6} m^3/cycle, can be seen in table 4.1 for different classes of roads.

Table 4.1 Values of the roughness coefficient[9].

Road class	Quality	Range of roughness coefficient
motorways	very good good	2–8 8–32
principal roads	very good good average poor	2–8 8–32 32–128 128–512
minor roads	average poor very poor	32–128 128–512 512–2048

Consider that the displacement $Z(x, y)$ induced by the road surface on a wheel is a three-dimensional, zero-mean, stationary Gaussian process. Suppose, also, that the rear wheels of the vehicle follow the same profile as the front wheels. In such a case, only the auto-correlation functions of the profiles of two parallel tracks $Z_L(x)$ and $Z_R(x)$ and their corresponding cross-correlation functions have to be determined. The auto-correlation functions are defined by

$$R_{LL}(\delta) = E\{Z_L(x)Z_L(x+\delta)\}$$
$$R_{RR}(\delta) = E\{Z_R(x)Z_R(x+\delta)\} \tag{4.41}$$

while the cross-correlation function between the profiles are

$$R_{LR}(\delta) = E\{Z_L(x)Z_R(x+\delta)\}$$
$$R_{RL}(\delta) = E\{Z_R(x)Z_L(x+\delta)\} \tag{4.42}$$

Using now the additional assumption that the road surface profiles are isotropic, which means that the process $Z(x, y)$ has circular symmetry, the following equations can be written:

$$R_{LL}(\delta) = R_{RR}(\delta) = R(\delta) \tag{4.43}$$

$$R_{RL}(\delta) = R_{LR}(\delta) = R\sqrt{\delta^2 + 4b^2} \tag{4.44}$$

where $R(\delta)$ is the profile autocorrelation function in any direction and b is defined in figure 4.4 as one half of the track width. Thus, the auto-spectral densities of the road profiles are

$$S_{LL}(n) = S_{RR}(n) = S(n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(\delta)\,\mathrm{e}^{-i2\pi n\delta}\,\mathrm{d}\delta \tag{4.45}$$

while their cross-spectral densities can be expressed as

$$S_{LR}(n) = S_{RL}(n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(\sqrt{\delta^2 + 4b^2})\, e^{-i2\pi n\delta}\, d\delta \tag{4.46}$$

The hypotheses which have been made are supported by results obtaind from measurements as those shown in figure 4.6, where the spectral densities of two parallel track profiles are plotted for three different roads. As it can be seen that the auto-spectral densitie of the road profile is almost independent of the track.

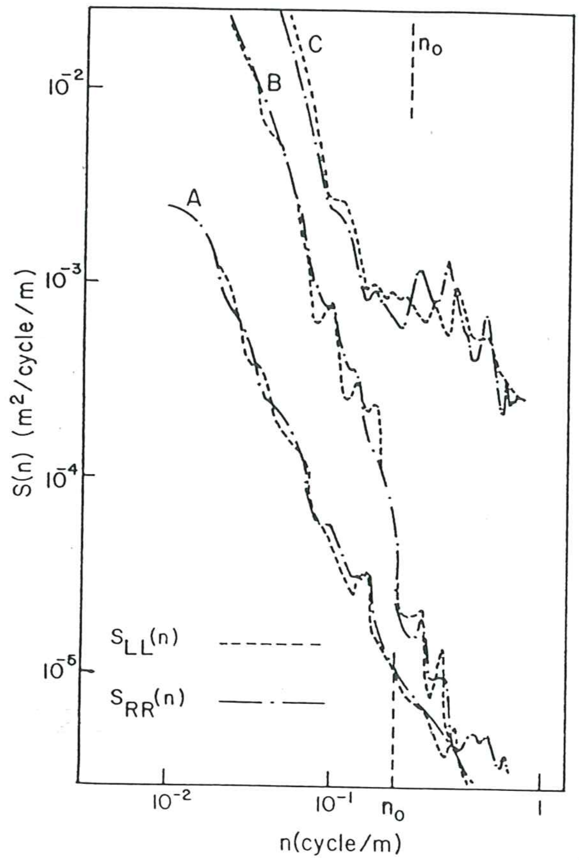

Figure 4.6 Spectral densities of two parallel track profiles corresponding to three typical roads(8): A – motorway, B – minor road, C – paved road.

The interdependence between the road profiles corresponding to two parallel tracks can be analyzed by using the following coherency function(8,10):

$$g^2(n) = \frac{|S_{LR}(n)|^2}{S_{LL}(n)S_{RR}(n)} \tag{4.47}$$

where $0 \leq g^2(n) \leq 1$. If the function $g^2(n) \to 1$, it means that the two profiles are highly dependent at wave number n. Figure 4.7 showes a comparison between the measured coherency functions of the same three typical motor roads and the computed ones. It can be observed a high dependence between two parallel track profiles at low wave numbers, i.e. long wave length.

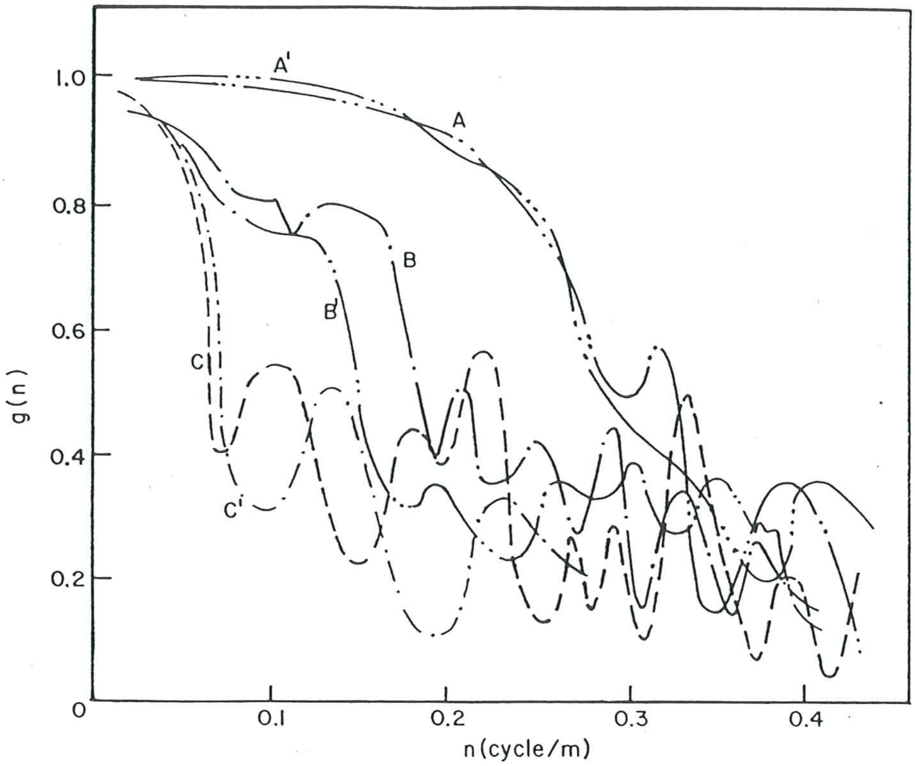

Figure 4.7 Measured and computed coherency functions $g(n)$ for: A – motorway, B – minor road, C – paved road[8].

4.3 SINGLE-DEGREE OF FREEDOM SYSTENS SUBJECTED TO A NONSTATIONARY DYNAMIC ACTION

4.3.1 Nonstationary excitations

A nonstationary random processes can be generated in a simplified way by transforming a stationary process $f(t)$ in a nonstationary one $\tilde{f}(t)$, through its multiplication by a deterministic time functions $I(t)$[11–13]

$$\tilde{f}(t) = f(t)I(t) \tag{4.48}$$

The shape of the intensity functions depends on the concrete problem to be solved. For instance, intensity functions like the following proposed by Oto[14] are usualy considered:

$$I(t) = \frac{t}{t_0}\,\mathrm{e}^{1-\frac{t}{t_0}} \tag{4.49}$$

The equation of motion for a single-degree of freedom model subjected to a nonstationary action $\tilde{f}(t)$ is

$$m\ddot{\tilde{x}}(t) + c\dot{\tilde{x}}(t) + k\tilde{x}(t) \ = \tilde{f}(t) \tag{4.50}$$

where the response $\tilde{x}(t)$ is a nonstationary random process. By using Oto's intensity function (4.49) in (4.48), by substituting the obtained expression of $\tilde{f}(t)$ in (4.50) and finally by dividing this equation by m, it results

$$\ddot{\tilde{x}}(t) + 2\zeta\omega\dot{\tilde{x}}(t) + \omega^2\tilde{x}(t) = \frac{t}{t_0}\,\mathrm{e}^{1-\frac{t}{t_0}}\,\frac{1}{m}f(t) \tag{4.51}$$

This is the equation of motion of the model subjected to a ground acceleration defined in a nondeterministic way.

4.3.2 Solution of the equation of motion

The deduction of the solution of equation (4.51) starts from expressing the stationary process $f(t)$ by means of

$$f(t) = \int_{-\infty}^{+\infty} |F(\theta)|\,\mathrm{e}^{i\left[\theta t+\varphi(\theta)\right]}\,\mathrm{d}\theta \tag{4.52}$$

where $F(\theta)$ is the Fourier transform of $f(t)$

$$F(\theta) = \frac{1}{m}\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\,\mathrm{e}^{-i\theta t}\,\mathrm{d}t \tag{4.53}$$

and where the phase angles $\varphi(\theta)$ are random variables. By substituting (4.52) in (4.51), this turns into

$$\ddot{\tilde{x}}(t) + 2\zeta\omega\dot{\tilde{x}}(t) + \omega^2\tilde{x}(t) = \frac{t}{t_0}\,\mathrm{e}^{1-\frac{t}{t_0}}\int_{-\infty}^{+\infty} |F(\theta)|\,\mathrm{e}^{i\left[\theta t+\varphi(\theta)\right]}\,\mathrm{d}\theta \tag{4.54}$$

The solution of (4.54) will be obtained by starting from[14]

$$\tilde{x}(t) = \int_{-\infty}^{+\infty} |F(\theta)|\,\xi(\theta,t)\mathrm{d}\theta \tag{4.55}$$

The substitution of (4.55) in (4.54) provides the equation

$$\ddot{\xi}(\theta,t) + 2\zeta\omega\dot{\xi}(\theta,t) + \omega^2\xi(\theta,t) = \frac{t}{t_0}\,\mathrm{e}^{1-\frac{t}{t_0}}\,\mathrm{e}^{i\left[\theta t+\varphi(\theta)\right]} \tag{4.56}$$

which has the initial conditions

$$\begin{aligned} \xi(\theta,0) &= 0 \\ \dot{\xi}(\theta,0) &= 0 \end{aligned} \tag{4.57}$$

The solution $\xi(\theta,t)$ of (4.56) is obtained as a sum of the solution of the homogeneous equation $\xi_0(\theta,t)$ with a particular solution of the nonhomogeneous equation $\xi^*(\theta,t)$

$$\xi(\theta,t) = \xi_0(\theta,t) + \xi^*(\theta,t) \tag{4.58}$$

The *general solution* $\xi_0(\theta,t)$ is of the form

$$\xi_0(\theta,t) = e^{-\zeta\omega t}\left[C_1(\theta)\sin\,\omega_v t + C_2(\theta)\cos\,\omega_v t\right] \tag{4.59}$$

where $\omega_v = \sqrt{1-\zeta^2}$ and the constants $C_1(\theta)$ and $C_2(\theta)$ are calculated from the initial conditions (4.57), resulting

$$C_1(\theta) = -\frac{1}{\omega_v}\left[\dot{\xi}^*(\theta,0) + \zeta\omega\xi^*(\theta,0)\right] \tag{4.60}$$

$$C_2(\theta) = -\xi^*(\theta,0) \tag{4.61}$$

The *particular solution* $\xi^*(\theta,t)$ is expressed in the following form

$$\xi^*(\theta,t) = \mathrm{e}^{1-\frac{t}{t_0}}\,\varsigma(\theta,t) \tag{4.62a}$$

being its derivatives

$$\dot{\xi}^*(\theta,t) = \mathrm{e}^{1-\frac{t}{t_0}}\left[\dot{\varsigma}(\theta,t) - \frac{1}{t_0}\varsigma(\theta,t)\right] \tag{4.62b}$$

$$\ddot{\xi}^*(\theta,t) = \mathrm{e}^{1-\frac{t}{t_0}}\left[\ddot{\varsigma}(\theta,t) - \frac{2}{t_0}\dot{\varsigma}(\theta,t) + \frac{1}{t_0^2}\varsigma(\theta,t)\right] \tag{4.62c}$$

By substituting $\xi^*(\theta,t)$ and its derivatives in (4.56), this becomes

$$\ddot{\varsigma}(\theta,t) + 2\Big(\zeta\omega - \frac{1}{t_0}\Big)\dot{\varsigma}(t) + \omega^2\Big(\zeta\omega - \frac{1}{t_0}\Big)^2\varsigma(t) = \frac{t}{t_0}\mathrm{e}^{i[\theta t+\varphi(\theta)]} \tag{4.63}$$

whose solution is

$$\varsigma(\theta,t) = \mathrm{e}^{i[\theta t+\varphi(\theta)]}\left[\gamma_1(\theta)\frac{t}{t_0} + \gamma_2(\theta)\right] \tag{4.64}$$

It has to be remarked that γ_1 and γ_2 are coefficients independent of time.

By introducing (4.64) in (4.63), the following equation is obtained:

$$\frac{t}{t_0}\Big[\alpha_1(\theta)\,\gamma_1(\theta) - 1\Big] + \Big[\alpha_2(\theta)\,\gamma_1(\theta) + \alpha_1(\theta)\,\gamma_2(\theta)\Big] = 0 \tag{4.65}$$

where

$$\alpha_1(\theta) = \omega_v^2 - \theta^2 + \Big(\zeta\omega - \frac{1}{t_0}\Big)^2 + 2i\theta\Big(\zeta\omega - \frac{1}{t_0}\Big) \tag{4.66}$$

$$\alpha_2(\theta) = \frac{2}{t_0}\Big[i\theta + \Big(\zeta\omega - \frac{1}{t_0}\Big)\Big] \tag{4.67}$$

$\gamma_1(\theta)$ and $\gamma_2(\theta)$ are obtained from (4.65) by identifying with zero both terms of the left side member

$$\gamma_1(\theta) = \frac{1}{\alpha_1(\theta)} \tag{4.68}$$

$$\gamma_2(\theta) = \frac{\alpha_2(\theta)}{\alpha_1(\theta)}\,\gamma_1(\theta) \tag{4.69}$$

These equations can be written in the following form:

$$\gamma_1(\theta) = \Re\Big[\gamma_1(\theta)\Big] + i\,\Im\Big[\gamma_1(\theta)\Big] \tag{4.70}$$

$$\gamma_2(\theta) = \Re\Big[\gamma_2(\theta)\Big] + i\,\Im\Big[\gamma_1(\theta)\Big] \tag{4.71}$$

where the real part, denoted $\Re$, and the imaginary part, denoted $\Im$, have been separated. The equations (4.62) provides now the particular solution of (4.56), solution which is written by substituting $\varsigma(\theta,t)$ from (4.64) and $\gamma_1(\theta)$ and $\gamma_2(\theta)$ from (4.70) and (4.71). Finally it results

$$\begin{aligned}\xi^*(\theta,t) = \mathrm{e}^{1-\frac{t}{t_0}}\Bigg\{&\Big[\frac{t}{t_0}\,\Re(\gamma_1(\theta)) + \Re(\gamma_2(\theta))\Big]\cos\Big[\theta t + \varphi(\theta)\Big] - \\ &\Big[\frac{t}{t_0}\Im(\gamma_1(\theta)) + \Im(\gamma_2(\theta))\Big]\sin\Big[\theta t + \varphi(\theta)\Big]\Bigg\}\end{aligned} \tag{4.72}$$

The general solution of (4.56) can be now rewritten by substituting in (4.60) and (4.61) the expressions of $\xi^*(\theta,0)$ and $\dot{\xi}^*(\theta,0)$ obtained from (4.72)

$$\xi^*(\theta,0) = \theta \left\{ \Re\left[\gamma_2(\theta)\right]\cos(\varphi(\theta)) - \Im\left[\gamma_2(\theta)\right]\sin(\varphi(\theta)) \right\}$$

$$\begin{aligned}\dot{\xi}^*(\theta,0) =\theta \Bigg\{ & \frac{1}{t_0}\Big[\Re(\gamma_1(\theta)) - \Re(\gamma_2(\theta))\Big]\cos\Big[\varphi(\theta)\Big] - \\ & \frac{1}{t_0}\Big[\Im(\gamma_1(\theta)) - \Im(\gamma_2(\theta))\Big]\sin\Big[\varphi(\theta)\Big] - \\ & \theta\Re(\gamma_2(\theta))\sin\Big[\varphi(\theta)\Big] - \theta\Im(\gamma_2(\theta))\cos\Big[\varphi(\theta)\Big]\Bigg\}\end{aligned}$$

The solution of (4.56) will be now written by using in (4.58) the general solution (4.59) and the particular solution (4.72). It can be expressed in the form

$$\xi(\theta,t) \;=\; \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_1\sin[\varphi(\theta)] + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_2\cos[\varphi(\theta)] \tag{4.73}$$

where the following matrix notations have been introduced

$$\boldsymbol{A} = \begin{Bmatrix} \mathrm{e}^{1-\zeta\omega t} \ \sin\omega_v t \\ \mathrm{e}^{1-\zeta\omega t} \ \cos\omega_v t \\ \frac{t}{t_0}\ \mathrm{e}^{1-\frac{t}{t_0}} \ \sin\theta t \\ \frac{t}{t_0}\ \mathrm{e}^{1-\frac{t}{t_0}} \ \cos\theta t \\ \mathrm{e}^{1-\frac{t}{t_0}} \ \sin\theta t \\ \mathrm{e}^{1-\frac{t}{t_0}} \ \cos\theta t \end{Bmatrix}$$

$$\begin{aligned}\boldsymbol{Z}_1^{\mathrm{T}} =\Big\{ & q_1(\theta) \quad \Im[\gamma_2(\theta)] \quad -\Re[\gamma_1(\theta)] \\ & -\Im[\gamma_1(\theta)] \quad -\Re[\gamma_2(\theta)] \quad -\Im[\gamma_2(\theta)]\Big\}\end{aligned}$$

$$\begin{aligned}\boldsymbol{Z}_2^{\mathrm{T}} =\Big\{ & q_2(\theta) \quad -\Re[\gamma_2(\theta)] \quad -\Im[\gamma_1(\theta)] \quad \Re[\gamma_1(\theta)] \\ & -\Im[\gamma_2(\theta)] \quad \Re[\gamma_2(\theta)]\Big\}\end{aligned}$$

The elements $q_1(\theta)$ and $q_2(\theta)$ in these last two equations are

$$q_1 = \frac{1}{\omega_v}\left\{ \frac{1}{t_0}\Im[\gamma_1(\theta)] + \left(\zeta\omega - \frac{1}{t_0}\right)\Im[\gamma_2(\theta)] + \theta\,\Re[\gamma_2(\theta)]\right\}$$

$$q_2 = \frac{1}{\omega_v}\left\{ -\frac{1}{t_0}\Re[\gamma_1(\theta)] - \left(\zeta\omega - \frac{1}{t_0}\right)\Re[\gamma_2(\theta)] + \theta\,\Im[\gamma_2(\theta)]\right\}$$

By using trigonometric transformations, the solution (4.73) becomes

$$\xi(\theta,t) = G_{\tilde{x}}(\theta,t)\sin\left[\psi(\theta,t) + \varphi(\theta)\right]$$

where

$$G_{\tilde{x}}(\theta,t) = \sqrt{(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_1)^2 + (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_2)^2} \tag{4.74}$$

$$\psi(\theta,t) = \arctan\frac{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_2}{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}_1} \tag{4.75}$$

The function $G_{\tilde{x}}(\theta,t)$ can be finally expressed as

$$G_{\tilde{x}}(\theta,t) = \sqrt{\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Z}\,\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{A}} \tag{4.76}$$

where the notation

$$\boldsymbol{Z} = [\boldsymbol{Z}_1 \quad \boldsymbol{Z}_2] \tag{4.77}$$

has been introduced.

Once obtained $\xi(\theta, t)$, the solution of the equation (4.54) is written by using (4.55)

$$\tilde{x}(t) = \int_{-\infty}^{+\infty} |A(\theta)|\; G_{\tilde{x}}(\theta, t) \sin\,[\psi(\theta, t) + \varphi(\theta)] d\theta \tag{4.78}$$

It is obvious from (4.68) and (4.69) that for $\alpha_1(\theta) = 0$ the solution (4.78) can be not calculated because it would imply divisions by zero. It can be seen from equation (4.66) that this case occurs if

$$\omega_v^2 - \theta^2 + \left(\zeta\omega - \frac{1}{t_0}\right)^2 + 2i\theta\left(\zeta\omega - \frac{1}{t_0}\right) = 0 \tag{4.79}$$

resulting from here

$$\omega_v = \theta \tag{4.80}$$

$$\zeta\omega = \frac{1}{t_0} \tag{4.81}$$

In this particular situation, the vectors $\boldsymbol{A}$, $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ have to be calculated by means of the following relations(14):

$$\boldsymbol{A} = e^{1-\frac{t}{t_0}} \begin{Bmatrix} \sin\,\theta t \\ \cos\,\theta t \\ \frac{t}{t_0} \sin\,\theta t \\ \frac{t}{t_0} \cos\,\theta t \\ \frac{t^2}{t_0} \sin\,\theta t \\ \frac{t^2}{t_0} \cos\,\theta t \end{Bmatrix} \tag{4.82}$$

$$\boldsymbol{Z}_1^{\mathrm{T}} = \left[\; 0 \quad 0 \quad -\frac{1}{4\theta^2} \quad 0 \quad 0 \quad \frac{t_0}{4\theta} \;\right] \tag{4.83}$$

$$\boldsymbol{Z}_2^{\mathrm{T}} = \left[\; -\frac{1}{4t_0\theta^3} \quad 0 \quad 0 \quad \frac{1}{4\theta^2} \quad \frac{t_0}{4\theta} \quad 0 \;\right] \tag{4.84}$$

The velocity response process of the analyzed system can be deduced in a similar way and expressed as(14,15)

$$\dot{\tilde{x}}(t) = \int_{-\infty}^{+\infty} |A(\theta)|\; G_{\dot{\tilde{x}}}(\theta, t) \cos\,[\psi(\theta, t) + \varphi(\theta)]\, d\theta \tag{4.85}$$

where

$$G_{\dot{\tilde{x}}}(\theta, t) = \sqrt{\boldsymbol{A}^{\mathrm{T}} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{Z}\, \boldsymbol{Z}^{\mathrm{T}} \boldsymbol{B}\, \boldsymbol{A}} \tag{4.86}$$

The matrix $\boldsymbol{B}$ is defined by the time derivation of the functions contained in the vector $\boldsymbol{A}$,

$$\dot{\boldsymbol{A}} = \boldsymbol{B}\, \boldsymbol{A} \tag{4.87}$$

and has the form

$$\boldsymbol{B} = \begin{bmatrix} -\zeta\omega & \omega_v & 0 & 0 & 0 & 0 \\ -\omega_v & -\zeta\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{t_0} & \theta & \frac{1}{t_0} & 0 \\ 0 & 0 & -\theta & -\frac{1}{t_0} & 0 & \frac{1}{t_0} \\ 0 & 0 & 0 & 0 & -\frac{1}{t_0} & \theta \\ 0 & 0 & 0 & 0 & -\theta & -\frac{1}{t_0} \end{bmatrix} \tag{4.88}$$

In the case in which $\alpha_1(\theta) = 0$, the matrix $\boldsymbol{B}$ turns into

$$\boldsymbol{B} = \begin{bmatrix} -\frac{1}{t_0} & \theta & 0 & 0 & 0 & 0 \\ -\theta & -\frac{1}{t_0} & 0 & 0 & 0 & 0 \\ \frac{1}{t_0} & 0 & -\frac{1}{t_0} & \theta & 0 & 0 \\ 0 & \frac{1}{t_0} & -\theta & -\frac{1}{t_0} & 0 & 0 \\ 0 & 0 & \frac{2}{t_0} & 0 & -\frac{1}{t_0} & \theta \\ 0 & 0 & 0 & \frac{2}{t_0} & -\theta & -\frac{1}{t_0} \end{bmatrix} \tag{4.89}$$

The statistic characteristics of the nonstationary processes $\tilde{x}(t)$, defined by equation (4.78) and $\dot{\tilde{x}}(t)$, defined by equation (4.85), are time dependent. The hypothesis that these processes are Gaussian and with zero mean is made now.

4.3.3 Statistics of the response process

For the stationary case corresponding to equation (4.3), the power spectral density of the response is expressed by (4.10) in function of the power spectral density of the process $f(t)$

$$S_x(\theta) = m^2 |H(\theta)|^2 S_f(\theta) \tag{4.90}$$

An evolutionary power spectral density of the response process $\tilde{x}(t)$, denoted $S_{\tilde{x}}(\theta, t)$ will be defined now. Such a spectral density has been introduced by Priestley(16), Damrath(14) and others, and is expressed as

$$S_{\tilde{x}}(\theta, t) \ = |G_{\tilde{x}}(\theta, t)\ |^2 S_x(\theta) \tag{4.91}$$

or, by using (4.90)

$$S_{\tilde{x}}(\theta, t) = m^2 |G_{\tilde{x}}(\theta, t)|^2 |H(\theta)|^2 S_f(\theta) \tag{4.92}$$

$|G_{\tilde{x}}(\theta, t)|^2$ can be interpreted as complex transfer function of the nonstationary displacement response. It can be seen from (4.90) that for a stationary process $|G_{\tilde{x}}(\theta, t)|^2 = 1$.

A similar expression can be found for the power spectral density of the velocity response process $S_{\dot{\tilde{x}}}(\theta, t)$

$$S_{\dot{\tilde{x}}}(\theta, t) = m^2 |G_{\dot{\tilde{x}}}(\theta, t)|^2 |H(\theta)|^2 S_f(\theta) \tag{4.93}$$

where $|G_{\dot{\tilde{x}}}(\theta, t)|^2$ is the transfer function of the nonstationary velocity response process. By comparing (4.93) whith (4.13) it results that for a stationary process $S_{\dot{\tilde{x}}}(\theta, t) = \theta^2$.

In the case of a nonstationary process with evolutionary power spectral density, the variance of the process is also a function of time. The variance of the displacement response process $[\sigma_{\tilde{x}}(t)]^2$ and of the velocity response process $[\sigma_{\dot{\tilde{x}}}(t)]^2$ can be calculated by means of expressions similar to (4.15) and (4.16), such as it is demonstrated in references (16–19)

$$\left[\sigma_{\tilde{x}}(t)\right]^2 = \int_{-\infty}^{+\infty} S_{\tilde{x}}(\theta, t)\ \mathrm{d}\theta \tag{4.94}$$

$$\left[\sigma_{\dot{x}}(t)\right]^2 = \int_{-\infty}^{+\infty} S_{\dot{x}}(\theta, t)\, \mathrm{d}\theta \tag{4.95}$$

The integrals in (4.94) and (4.95) have to be solved numerically.

4.4 EXTREME MEAN RESPONSE OF SINGLE-DEGREE OF FREEDOM MODELS

4.4.1 Formulation of the problem

The *extreme mean value* x_e of a structural response process $x(t)$ is defined as the response expected only one time during a given finite time interval t_e. The problem of the computation of x_e can be solved under some restrictive hypotheses. For example the hypothesis that the process is a narrow band one is considered herein(14).

The number of cases in which the response overpass the positive or negative value x_e during the time interval t_e is denoted $N(\pm x_e)$, while $n(\pm x_e)$ is the number of overpasses on time unit. Thus

$$N(\pm x_e) = n(\pm x_e)\, t_e = 1 \tag{4.96}$$

It can be demonstrated that the following equation holds

$$n(+x_e) = \int_0^{\infty} p\,(x_e, \dot{x})\, \dot{x}\, \mathrm{d}\dot{x} \tag{4.97}$$

which express the probability that $x(t) \in [x_e, x_e + dx]$ and at the same time $\dot{x}(t) \in [\dot{x}, \dot{x} + \mathrm{d}\dot{x}]$. The second order Gaussian probability density of the variables x and $\dot{x}$ can be written according to the equation (A8.10) as

$$p\,(x, \dot{x}) = \frac{1}{2\pi\sigma_x\, \sigma_{\dot{x}}}\; \mathrm{e}^{-\frac{1}{2}\left[\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{x}}}\right)^2\right]} \tag{4.98}$$

where the hypothesis has been made that $x(t)$ and $\dot{x}(t)$ have zero mean. For the extreme mean value x_e the probability density becomes

$$p\,(x_e, \dot{x}) = \frac{1}{2\pi\sigma_x\, \sigma_{\dot{x}}}\; \mathrm{e}^{-\frac{1}{2}\left[\left(\frac{x_e}{\sigma_x}\right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{x}}}\right)^2\right]} \tag{4.99}$$

The substitution of (4.99) in (4.97) leads to(14)

$$n(+x_e) = \frac{1}{2\pi}\, \frac{\sigma_{\dot{x}}}{\sigma_x}\; \mathrm{e}^{-\frac{1}{2}\left(\frac{x_e}{\sigma_x}\right)^2} \tag{4.100}$$

4.4.2 Stationary process

If the process $x(t)$ is stationary, the relation

$$\frac{\sigma_{\dot{x}}}{\sigma_x} = \theta_m \tag{4.101}$$

can be used to define the mean frequency θ_m of the process(14). The substitution of (4.101) in (4.100) gives

$$n(+x_e) = \frac{\theta_m}{2\pi}\, \mathrm{e}^{-\frac{1}{2}\left(\frac{x_e}{\sigma_x}\right)^2} \tag{4.102}$$

By taking into account that

$$n(\pm x_e) = 2\; n(+x_e) \tag{4.103}$$

the substitution of (4.102) into (4.103) results in

$$n(\pm x_e) = t_e \, \frac{\theta_m}{\pi} \, e^{-\frac{1}{2}\left(\frac{\tilde{x}_e}{\sigma_x}\right)^2} \tag{4.104}$$

The expression (4.96) permits to establish the following equation which defines x_e:

$$t_e \, \frac{\theta_m}{\pi} \, e^{-\frac{1}{2}\left(\frac{x_e}{\sigma_x}\right)^2} = 1 \tag{4.105}$$

whose solution is

$$x_e = \sigma_x \sqrt{2 \ln\left(t_e \, \frac{\theta_m}{\pi}\right)} \tag{4.106}$$

4.4.3 Nonstationary process

In the case in which the response process is nonstationary $\tilde{x}(t)$, the variances $\sigma_{\dot{\tilde{x}}}$ and $\sigma_{\tilde{x}}$ are only approximately proportional. But including in this case, Damrath[14] has defined a mean frequency of the process, by means of the equation

$$\theta_m = \frac{\sigma_{\dot{\tilde{x}}}}{\sigma_{\tilde{x}}} \tag{4.107}$$

The process can be treated as stationary in an infinitesimal time interval dt. Thus, according to (4.104), the expected number of cases in which $\tilde{x}(t)$ overpass a given value $\pm\tilde{x}_e$ in the time interval dt is

$$n(\pm\tilde{x}_e) = \frac{\theta_m}{\pi} \, e^{-\frac{1}{2}\left(\frac{\tilde{x}_e}{\sigma_{\tilde{x}}}\right)^2} \, dt \tag{4.108}$$

By taking into account that in this case

$$N(\pm\tilde{x}_e) = \int_0^\infty n(\pm\tilde{x}_e)\, dt \tag{4.109}$$

the equation which defines $\tilde{x}_e$

$$N(\pm\tilde{x}_e) = 1$$

can be expressed in the following form:

$$\frac{\theta_m}{\pi} \int_0^\infty e^{-\frac{1}{2}\left(\frac{\tilde{x}_e}{\sigma_{\tilde{x}}}\right)^2} \, dt = 1 \tag{4.110}$$

The equation (4.110) can be solved for x_e only numerically, but it can be simplified under the hypothesis that the following equation holds:

$$\sigma_{\tilde{x}}(t) = \sigma_{\tilde{x}m} \, \frac{t}{t_{\tilde{x}m}} \, e^{1-\frac{t}{t_{\tilde{x}m}}} \tag{4.111}$$

where $\sigma_{\tilde{x}m}$ is the maximum standard deviation and $t_{\tilde{x}m}$ is the time instant at wich the maximum standard deviation occurs. By using now the substitution

$$z = \frac{t}{t_{\tilde{x}m}} \tag{4.112}$$

the equation (4.111) takes the form

$$\sigma_{\tilde{x}}(z) = \sigma_{\tilde{x}m} \; z \; e^{1-z} \tag{4.113}$$

and (4.110) can be expressed as

$$\frac{\theta_m}{\pi} \int_0^\infty e^{-\frac{1}{2}\left(\frac{\tilde{x}_e}{\sigma_{\tilde{x}m}\, z\, e^{1-z}}\right)^2} t_{\tilde{x}_m}\, dz = 1 \tag{4.114}$$

The time instant $t_{\tilde{x}_m}$ is calculated by means of the formula[14]

$$t_{\tilde{x}_m} = \frac{1}{2}\,(t_{G_m} + t_0) \tag{4.115}$$

where t_0 and t_{G_m} are respectively the time instants at which the intensity function $I(t)$ and the function $G_{\tilde{x}}(\omega_v, t)$ take a maximum value. t_{G_m} is obtained by means of a step by step procedure. Thus, the maximum standard deviation of $\tilde{x}$ and $\dot{\tilde{x}}$ can be calculated by means of

$$\sigma_{\tilde{x}_m} = \sigma_{\tilde{x}}\,(t_{\tilde{x}_m}) \tag{4.116}$$

$$\sigma_{\dot{\tilde{x}}_m} = \sigma_{\dot{\tilde{x}}}\,(t_{xm}) \tag{4.117}$$

and θ_m is given by

$$\theta_m = \frac{\sigma_{\dot{\tilde{x}}}}{\sigma_{\tilde{x}}} \tag{4.118}$$

The equation (4.114) is now expressed in the form

$$\int_0^\infty e^{-\frac{1}{2}\left(\frac{\tilde{x}_e}{\sigma_{\tilde{x}m}}\, \frac{1}{z\, e^{1-z}}\right)^2} dz = \frac{\pi}{\theta_m\, t_{\tilde{x}_m}} \tag{4.119}$$

and the value

$$\alpha = \frac{\tilde{x}_e}{\sigma_{\tilde{x}_m}} \tag{4.120}$$

is obtained numerically. Finally

$$\tilde{x}_e = \alpha\, \sigma_{\tilde{x}_m} \tag{4.121}$$

is obtained as a solution of the problem.

4.5 MULTI-DEGREE OF FREEDOM MODEL SUBJECTED TO STOCHASTIC DYANMIC ACTIONS

4.5.1 Modal uncoupling

Consider a linear multi-degree of freedom system subjected to a motion of its support x_s modelled as a random process. According to the formulations given in Chapter 3, the system of equations of motion is

$$\boldsymbol{M}\ddot{\boldsymbol{D}}(t) + \boldsymbol{C}\dot{\boldsymbol{D}}(t) + \boldsymbol{K}\boldsymbol{D}(t) = -\boldsymbol{M}\boldsymbol{J}\ddot{x}_s(t) \tag{4.122}$$

Due to the linearity of the model the elements of the vectors $\boldsymbol{D}(t)$, $\dot{\boldsymbol{D}}(t)$ and $\ddot{\boldsymbol{D}}(t)$ are also stochastic processes of the same type as $f(t)$. The system of equations of motion (4.122) is uncoupled by using the standard procedure developed in Chapter 6. The usual transformation

$$\boldsymbol{D}(t) = \boldsymbol{\Phi}\, \boldsymbol{y}(t) \tag{4.123}$$

is applied, which leads to the following n independent equations

$$\ddot{y}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i{}^2 y_i(t) = -\frac{1}{M_i^*}\,\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}\,\boldsymbol{J}\ddot{x}_s(t) \tag{4.124}$$

in which i takes values from 1 to n and $y_i(t)$ is also a random process. The generalized mass M_i^* is defined by

$$M_i^* = \boldsymbol{\varphi}_i^{\mathrm{T}}\,\boldsymbol{M}\,\boldsymbol{\varphi}_i$$

and the coefficient Q_i has the form

$$Q_i = \frac{\boldsymbol{\varphi}_i^{\mathrm{T}}\,\boldsymbol{M}\,\boldsymbol{J}}{\boldsymbol{\varphi}_i^{\mathrm{T}}\,\boldsymbol{M}\,\boldsymbol{\varphi}_i}$$

The vector of the displacement response corresponding to the mode of vibration i is denoted $\boldsymbol{D}_i(t)$ and is given by

$$\boldsymbol{D}_i(t) = \boldsymbol{\varphi}_i^{\mathrm{T}}\,y_i(t) \tag{4.125}$$

where $\boldsymbol{\varphi}_i$ is the column i of the modal matrix $\boldsymbol{\Phi}$. The displacement according to the degree of freedom r and the mode of vibration i is

$$d_{ri} = \varphi_{ri}\,y_i(t) \tag{4.126}$$

The equation (4.124) is now rewritten in the form

$$\ddot{y}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i{}^2 y_i(t) = -Q_i\,\ddot{x}_s(t) \tag{4.127}$$

whose solution in the frequency domain is

$$\overline{y}_i(\theta) = -Q_i\,H_i(\theta)\,X_s(\theta) \tag{4.128}$$

where $X_s(\theta)$ is the Fourier transform of $\ddot{x}_s(t)$ and $H_i(\theta)$ is

$$H_i(\theta) = \frac{1}{k_i\left(-\frac{\theta^2}{\omega_i^2} + 2i\zeta_i\,\frac{\theta}{\omega_i} + 1\right)} \tag{4.129}$$

The frequency domain displacement response $\bar{d}_{ri}(\theta)$ is expressed by using (4.125) in the following form:

$$\bar{d}_{ri}(\theta) = \varphi_{ri}\,\overline{y}_i(\theta) \tag{4.130}$$

and by using now (4.128), it turns into

$$\bar{d}_{ri}(\theta) = -\varphi_{ri}\,Q_i\,H_i(\theta)\,X_s(\theta) \tag{4.131}$$

By using the mode shape coefficients Q^f_{ri} defined by

$$Q^f_{ri} = \varphi_{ri}\,Q_i \tag{4.132}$$

the solution (4.131) becomes

$$\bar{d}_{ri}(\theta) = -Q^f_{ri}\,H_i(\theta)\,X_s(\theta) \tag{4.133}$$

Input-output statistical relations will be now given for both stationary and nonstationary ground acceleration processes.

4.5.2 Stationary ground acceleration

The power spectral densities of the displacement response can be organized in the following matrix form

$$\boldsymbol{S}_D(\theta) = \begin{bmatrix} S_{d_{11}} & S_{d_{12}} & \cdots & S_{d_{1i}} & \cdots & S_{d_{1n}} \\ S_{d_{21}} & S_{d_{22}} & \cdots & S_{d_{2i}} & \cdots & S_{d_{2n}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{d_{r1}} & S_{d_{r2}} & \cdots & S_{d_{ri}} & \cdots & S_{d_{rn}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{d_{n1}} & S_{d_{n2}} & \cdots & S_{d_{ni}} & \cdots & S_{d_{nn}} \end{bmatrix} \qquad (4.134)$$

which is calculated by means of the following expression:

$$\boldsymbol{S}_D(\theta) = \boldsymbol{Q}^f \boldsymbol{H}^2(\theta)\, S_{\ddot{x}_s}(\theta) \qquad (4.135)$$

$S_{\ddot{x}_s}(\theta)$ is the power spectral density of the random support motion. The other notations used in equation (4.135) have the following meaning:

$$\boldsymbol{Q}^f = \begin{bmatrix} Q^f_{11} & Q^f_{12} & \cdots & Q^f_{1i} & \cdots & Q^f_{1n} \\ Q^f_{21} & Q^f_{22} & \cdots & Q^f_{2i} & \cdots & Q^f_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Q^f_{r1} & Q^f_{r2} & \cdots & Q^f_{ri} & \cdots & Q^f_{rn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ Q^f_{n1} & Q^f_{n2} & \cdots & Q^f_{ni} & \cdots & Q^f_{nn} \end{bmatrix} \qquad (4.136)$$

$$\boldsymbol{H}^2(\theta) = \begin{bmatrix} |H_1(\theta)|^2 & & & & & \\ & |H_2(\theta)|^2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & |H_i(\theta)|^2 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & |H_n(\theta)|^2 \end{bmatrix} \qquad (4.137)$$

The elements of the diagonal matrix $\boldsymbol{H}^2(\theta)$ are of the form

$$|H_i(\theta)|^2 = \frac{1}{m_i^2\,\omega_i^4\left[\left(1-\frac{\theta^2}{\omega_i^2}\right)^2+\left(2\zeta_i\frac{\theta}{\omega_i}\right)^2\right]} \qquad (4.138)$$

The matrices of the variance of the displacement, velocity and acceleration responses are finally given by means of equations similar to (4.18), (4.19) and (4.20), respectively

$$\boldsymbol{\sigma}_D^2 = \boldsymbol{Q}^f \int_{-\infty}^{+\infty} \boldsymbol{H}^2(\theta) S_{\ddot{x}_s}(\theta)\, \mathrm{d}\theta \tag{4.139}$$

$$\boldsymbol{\sigma}_{\dot{D}}^2 = \boldsymbol{Q}^f \int_{-\infty}^{+\infty} \theta^2 \boldsymbol{H}^2(\theta) S_{\ddot{x}_s}(\theta)\, \mathrm{d}\theta \tag{4.140}$$

$$\boldsymbol{\sigma}_{\ddot{D}}^2 = \boldsymbol{Q}^f \int_{-\infty}^{+\infty} \theta^4 \boldsymbol{H}^2(\theta) S_{\ddot{x}_s}(\theta)\, \mathrm{d}\theta \tag{4.141}$$

The matrix $\boldsymbol{\sigma}_D^2$ has as elements the variances of the displacement response according to each degree of freedom r and each mode of vibration i

$$\boldsymbol{\sigma}_D^2 = \begin{bmatrix} \sigma_{d_{11}}^2 & \sigma_{d_{12}}^2 & \cdots & \sigma_{d_{1i}}^2 & \cdots & \sigma_{d_{1n}}^2 \\ \sigma_{d_{21}}^2 & \sigma_{d_{22}}^2 & \cdots & \sigma_{d_{2i}}^2 & \cdots & \sigma_{d_{2n}}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{d_{r1}}^2 & \sigma_{d_{r2}}^2 & \cdots & \sigma_{d_{ri}}^2 & \cdots & \sigma_{d_{rn}}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{d_{n1}}^2 & \sigma_{d_{n2}}^2 & \cdots & \sigma_{d_{ni}}^2 & \cdots & \sigma_{d_{nn}}^2 \end{bmatrix} \tag{4.142}$$

The matrices $\boldsymbol{\sigma}_{\dot{D}}^2$ and $\boldsymbol{\sigma}_{\ddot{D}}^2$ are defined similarly.

The equation (4.139) can be particularized for the calculation of the response of structures subjected to ground accelerations defined as white noise with a power spectral density $S_{\ddot{x}_s}(\theta) = S_0$, resulting thus for the variance of the displacement response the expression

$$\boldsymbol{\sigma}_D^2 = \frac{\Pi S_0}{2} \boldsymbol{Q}^f \begin{bmatrix} \frac{1}{\zeta_1 m_1^2 \omega_1^3} & & & & & \\ & \frac{1}{\zeta_2 m_2^2 \omega_2^3} & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & \frac{1}{\zeta_i m_i^2 \omega_i^3} & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & \frac{1}{\zeta_n m_n^2 \omega_n^3} \end{bmatrix} \tag{4.143}$$

Similar equations can be written for the variance of the velocity response (4.140), by substituting the diagonal elements of (4.143) by

$$\frac{1}{\zeta_i m_i^2 \omega_i} \qquad i = 1,\ 2, \ldots, n$$

and for the variance of the acceleration response (4.141), by substituting the elements of the principal diagonal of (4.143) by

$$\frac{\omega_i}{\zeta_i m_i^2} \qquad i = 1,\ 2, \ldots, n$$

For a filtered white noise process, expressions similar to (4.35), (4.36) and (4.37) can be derived.

The usual systems of structural engineering are lightly damped systems, and the response processes corresponding to two different modes y_i and y_j are almost statistically independent. In this condition the total variance of the response process corresponding to a given degree of freedom r can be calculated by means of

$$\sigma_r^2 = \sigma_{r1}^2 + \sigma_{r2}^2 + \ldots \sigma_{ri}^2 + \ldots$$

where a sufficient number of modes of vibration have to be introduced.

4.5.3 Nonstationary support motion

If the support motion is defined as a nonstationary process $\tilde{x}_s(t)$ defined by the equations (4.48) and (4.49), the system of equations of motion to be solved becomes

$$\boldsymbol{M}\ddot{\tilde{\boldsymbol{D}}}(t) + \boldsymbol{C}\dot{\tilde{\boldsymbol{D}}}(t) + \boldsymbol{K}\tilde{\boldsymbol{D}}(t) = -\boldsymbol{M}\boldsymbol{J}\ \frac{t}{t_0}\ e^{1-\frac{t}{t_0}}\ \ddot{\tilde{x}}_s(t) \tag{4.144}$$

where $\ddot{\tilde{x}}_s(t)$ is a stationary process and $\tilde{\boldsymbol{D}}(t)$ is the the nonstationary displacement response vector. Matrices of the variance of the response of the same type as (4.142) are introduced[(1)]

$$\boldsymbol{\sigma}^2_{\tilde{D}} = \boldsymbol{Q}^f \int_{-\infty}^{+\infty} \boldsymbol{G}^2_{\tilde{x}}(\theta, t)\ \boldsymbol{H}^2(\theta) S_{\ddot{x}_s}(\theta)\, \mathrm{d}\theta \tag{4.145}$$

$$\boldsymbol{\sigma}^2_{\dot{\tilde{D}}} = \boldsymbol{Q}^f \int_{-\infty}^{+\infty} \boldsymbol{G}^2_{\dot{\tilde{x}}}(\theta, t)\ \boldsymbol{H}^2(\theta) S_{\ddot{x}_s}(\theta)\, \mathrm{d}\theta \tag{4.146}$$

where $\boldsymbol{G}^2_{\tilde{x}}(\theta, t)$ and $\boldsymbol{G}^2_{\dot{\tilde{x}}}(\theta, t)$ are expressed as

$$\boldsymbol{G}^2_{\tilde{x}}(\theta,t) = \begin{bmatrix} |G_{\tilde{x}_1}(\theta,t)|^2 & & & & & \\ & |G_{\tilde{x}_2}(\theta,t)|^2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & |G_{\tilde{x}_i}(\theta,t)|^2 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & |G_{\tilde{x}_i}(\theta,t)|^2 \end{bmatrix} \tag{4.147}$$

$$\boldsymbol{G}^2_{\dot{\tilde{x}}}(\theta,t) = \begin{bmatrix} |G_{\dot{\tilde{x}}_1}(\theta,t)|^2 & & & & & \\ & |G_{\dot{\tilde{x}}_2}(\theta,t)|^2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & |G_{\dot{\tilde{x}}_i}(\theta,t)|^2 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & |G_{\dot{\tilde{x}}_i}(\theta,t)|^2 \end{bmatrix} \tag{4.148}$$

The elements $|G^2_{\tilde{x}}(\theta,t)|^2$ and $|G^2_{\dot{\tilde{x}}}(\theta,t)|^2$ of the matrices $\boldsymbol{G}^2_{\tilde{x}}(\theta,t)$ and $\boldsymbol{G}^2_{\dot{\tilde{x}}}(\theta,t)$, respectively, are obtained by particularizing (4.76) and (4.86) for each mode of vibration.

4.6 EXTREME MEAN RESPONSE OF MULTI-DEGREE OF FREEDOM MODELS

The same procedures developed for the computation of the extreme mean response of single-degree of freedom models, can be extended to multi-degree of freedom models, for both stationary and nonstationary processes.

The extreme mean values of a stationary response process can be obtained for each mode of vibration and for each degree of freedom of the model and can be organized in a matrix $\boldsymbol{X}_e$ of the form

$$\boldsymbol{X}_e = \begin{bmatrix} \tilde{x}_{e11} & \tilde{x}_{e12} & \cdots & \tilde{x}_{e1i} & \cdots & \tilde{x}_{e1n} \\ \tilde{x}_{e21} & \tilde{x}_{e22} & \cdots & \tilde{x}_{e2i} & \cdots & \tilde{x}_{e2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{x}_{er1} & \tilde{x}_{er2} & \cdots & \tilde{x}_{eri} & \cdots & \tilde{x}_{ern} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{x}_{en1} & \tilde{x}_{en2} & \cdots & \tilde{x}_{eni} & \cdots & \tilde{x}_{enn} \end{bmatrix} \tag{4.149}$$

The values x_{e_r} representing the total extreme mean values corresponding to any degree of freedom r of the model, can be calculated by the well known root mean square approximation

$$\tilde{x}_{e_r} = \sqrt{\sum_{i=1}^{n} x_{eri}^2} \tag{4.150}$$

In the case in which the action is modelled as a nonstationary process, a matrix $\tilde{\boldsymbol{X}}_e$, similar to $\boldsymbol{X}_e$ defined equation (4.149), is obtained. It contains the modal extreme mean values of the nonstationary response and an equation similar to (4.150) permits the calculation of the total extreme values of the structural response.

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