



Coupled plastic-damaged model

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Abstract

A constitutive model that couples plasticity and damage is presented. The model is thermodynamically consistent and comes from a generalization of classical plasticity theory and isotropic damage theory of Kachanov. Coupling between plasticity and damage is achieved through a simultaneous solution of the plastic and the damage problem. After a description of the model, a numerical algorithm for the integration of the resulting constitutive equations is presented. It is an Euler Backward type of algorithm that is particularly suitable to solve plain stress non-linear problems with a 2D finite element program. The consistent stiffness matrix is also derived. The paper is completed with some application examples that show that the model presented accurately reproduces the behaviour of elastic-plastic-damaged materials.

1. Introduction

The greater part of the materials, in particular geomaterials, presents a non-linear behaviour accompanied by permanent strains and stiffness degradation when they are subjected to mechanical forces.

In geomaterials, permanent strains are caused by microcracking. Plasticity theory can be used as a mathematical framework to treat these permanent strains if it is considered that they resemble plastic stains [1, 2].

Damage or stiffness degradation is related to initiation, growth and interconnection of microcracks and micropores. A great number of approaches have been proposed to simulate this phenomena (see [3, 4]). Among them, continuum damage mechanics has been introduced and widely used to simulate progressive degradation of mechanical properties of materials before the initiation of macro cracking. Kachanov [5] was the first to introduce the concept of *effective stress* to model creep rupture. Later, continuum damage mechanics was extended to model fatigue, creep, creep-fatigue interaction and ductile-plastic damage. Recently, it has been applied to brittle materials like concrete or rock.

Continuum damage mechanics theories are based on the thermodynamics of irreversible process. Only one scalar variable is needed to model isotropic damage while tensorial variables are needed to model damage anisotropy. For simplicity, efficiency and adaptation to different practice applications, isotropic formulations are widely used [3].

There is some experimental evidence that damage is linked to plastic strains [3]. Physical mechanisms of interaction between damage and plastic strains are complicated in nature and cannot be modeled by only one phenomenological approach. Little is known about effects of temperature, strain rate, localization or microstructure in these interactions [4]. A great effort was made by some authors to reproduce coupling of damage and plastic processes. Ju [3] gave an energy-based isotropic elastoplastic

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damage theory that is capable of accommodating non-linear elastic response and general plastic response. The theory can predict degradation in both elastic and plastic responses. This framework was further extended to develop simple energy-based fourth-order anisotropic damage models for brittle materials. Voyiadjis [4] presented a coupled theory of continuum damage mechanics and finite strain plasticity (with small elastic strains). He derived an explicit matrix representation for the damage effect tensor for a general state of deformation and damage. Edlun [6] presented a coupled elastic-plastic damage model for rubber-bond epoxy adhesives. It is a phenomenological model derived from a mathematical framework based on thermodynamic considerations where the micromechanical processes are accounted for by a set of internal variables.

The model presented in this paper is thermodynamically consistent and comes from a generalization of plasticity theory [1, 2] and isotropic damage theory. The plastic model is more general than the one in [2] and the treatment of damage is completely different. Coupling of damage and plastic strains is achieved by solving both problems simultaneously [7]. In this way correct energy dissipation is also assured.

2. Thermodynamic basis

The constitutive model proposed is based on the hypothesis of *uncoupled elasticity* [8, 9]. According to this hypothesis, the total free energy Ψ can be supposed to be formed by two independent parts: an elastic part Ψ^e and a plastic part Ψ^p , corresponding to the elastic and plastic processes, respectively

$$\Psi(\epsilon_{ij}^e; \alpha; \beta) = \Psi^e(\epsilon_{ij}^e; \beta) + \Psi^p(\alpha) \quad (1)$$

where ϵ_{ij}^e is the elastic strains tensor and α and β represent groups of internal variables plastic and non-plastic, respectively.

For small strain and thermally stable problems, the elastic part of the free energy is written as quadratic function as follows [10]

$$\Psi^e(\epsilon_{ij}^e; \beta) = \frac{1}{2m_0} [\epsilon_{ij}^e C_{ijkl}^s(\beta) \epsilon_{kl}^e] \quad (2)$$

where m_0 is the density of the material and $C_{ijkl}^s(\beta)$ the secant constitutive tensor affected by the evolution of non-plastic internal variables which can be written as [10]

$$C_{ijkl}^s(\beta) = f(\beta) C_{ijkl}^0 \quad (3)$$

where C_{ijkl}^0 is the initial secant constitutive tensor of the virgin material and $f(\beta)$ a tensorial transformation function from an undamaged equivalent space to the real damaged space. The most simple form for the transformation function is the one coinciding with isotropic damage theory of Kachanov [5]

$$f(\beta) = (1 - d) \quad (4)$$

where $\beta = d$ is the *inner damage variable* and is such that

$$\begin{aligned} d &= 0 && \text{for the undamaged virgin material} \\ d &= 1 && \text{for the completely damaged material} \end{aligned}$$

For this particular choice, the elastic part of the free energy can be written as

$$\Psi^e = (1 - d) \Psi^0 = (1 - d) \frac{1}{2m_0} [\epsilon_{ij}^e C_{ijkl}^0 \epsilon_{kl}^e] \quad (5)$$

where Ψ^0 represents the elastic free energy of the undamaged material.

Inequality of Clausius–Duhem [11] can be written in terms of free energy as follows

$$\Xi = m_0(-\dot{\Psi} - \eta\dot{\theta}) + \sigma_{ij}\dot{\epsilon}_{ij} - \frac{1}{\theta} q_i \frac{\partial\theta}{\partial x_i} \geq 0 \quad (9)$$

where η is the entropy, θ a measure of temperature, q the heat flow and x_i a set of spatial coordinates.

Substituting the form proposed for the free energy (Eq. (1)) in Eq. (2) and splitting the total strain tensor in an elastic and a plastic part, the following inequality is obtained

$$\left(\sigma_{ij} - m_0 \frac{\partial\Psi}{\partial\epsilon_{ij}^c}\right)\dot{\epsilon}_{ij}^c - m_0 \left(\frac{\partial\Psi}{\partial\theta} + \eta\right)\dot{\theta} + \sigma_{ij}\dot{\epsilon}_{ij}^p - m_0 \frac{\partial\Psi}{\partial\alpha_i} \dot{\alpha}_i - m_0 \frac{\partial\Psi}{\partial d} \dot{d} - \frac{1}{\theta} q_i \frac{\partial\theta}{\partial x_i} \geq 0 \quad (7)$$

The fulfillment of inequality of Clausius–Planck for a given thermodynamic state is guaranteed if the following equations, known as Coleman [12] relations, are satisfied

$$\sigma_{ij} = m_0 \frac{\partial\Psi^c}{\partial\epsilon_{ij}^c} \quad \text{and} \quad \eta = -\frac{\partial\Psi}{\partial\theta} \quad (8)$$

On the other hand, for uncoupled thermomechanical problems, the inequalities of Clausius–Planck must be independently satisfied

(a) Mechanical dissipation

$$\begin{aligned} \Xi_m^p: \text{Plastic dissipation} \quad \Xi_m^d: \text{Damage dissipation} \\ \Xi_m = \sigma_{ij}\dot{\epsilon}_{ij}^p - m_0 \frac{\partial\Psi^p}{\partial\alpha_i} \dot{\alpha}_i - m_0 \frac{\partial\Psi^c}{\partial d} \dot{d} \geq 0 \end{aligned} \quad (9)$$

(b) Thermal dissipation

$$\Xi_\theta = \frac{1}{\theta} q_i \frac{\partial\theta}{\partial x_i} \leq 0$$

Mechanical dissipation must satisfy first inequality of Clausius–Planck and can be decomposed in two parts: one part due to the plastic process Ξ_m^p and the other due to the damage process Ξ_m^d .

3. Plastic process

The plastic process is described by a generalization of classical plasticity theory that takes into account many aspects of geomaterials behaviour.

Elastic threshold is defined by a yield function:

$$F(\sigma_{ij}; \alpha_k) = f(\sigma_{ij}) - K(\sigma_{ij}; \alpha_k) \leq 0 \quad (10)$$

where $f(\sigma_{ij})$ is the equivalent tension defined in the damaged space. $K(\sigma_{ij}; \alpha_k)$ is the equivalent yielding threshold and $\dot{\alpha}_k$ a set of internal variables defined as follows

$$\alpha_k = \{\kappa^p; \varphi; K\} \quad (11)$$

where κ^p is the *plastic damage variable* [1, 2] and φ the angle of internal friction.

The following rules are used for the evolution of plastic strains ϵ_{ij}^p and plastic internal variables α_k

$$\dot{\epsilon}_{ij}^p = \dot{\lambda} \frac{\partial G(\sigma_{mn}; \alpha_k)}{\partial \sigma_{ij}} \quad (12)$$

$$\dot{\alpha}_k = \dot{\lambda} H_k(\sigma_{ij}; \alpha_m) = \dot{\lambda} (h_k)_{ij} \frac{\partial G(\sigma_{mn}; \alpha_l)}{\partial \sigma_{ij}} \quad (13)$$

where $\dot{\lambda}$ is the plastic consistency factor. G the potential function and $(h_k)_{ij}$ a tensor to be defined for each plastic internal variable [1, 2].

The plastic damage variable κ^p is obtained normalizing energy dissipated by the plastic process to unity [7] and varies from 0, for the virgin material, to 1, when the maximum energy is plastically dissipated.

$$\kappa^p = \left[\frac{r}{g_t^{*p}} + \frac{(1-r)}{g_c^{*p}} \right] \sigma_{ij} \dot{\epsilon}_{ij}^p \quad (14)$$

where

$$r = \frac{\sum_{i=1}^3 \langle \sigma_i \rangle}{\sum_{i=1}^3 |\sigma_i|} \quad \langle \sigma_i \rangle = \frac{1}{2} [\sigma_i + |\sigma_i|], \quad (15)$$

σ_i : principal stresses,

$$g_t^{*p} = \frac{\sum_{i=1}^3 |\sigma_i| R^{up}}{f(\sigma_{ij})} g_t^p \quad g_c^{*p} = \frac{\sum_{i=1}^3 |\sigma_i|}{f(\sigma_{ij})} g_c^p, \quad (16)$$

R^{up} is the relation between the yielding threshold in uniaxial compression and that corresponding to uniaxial tension; g_t^p and g_c^p are the maximum energy densities dissipated by the plastic process in uniaxial tension and compression processes, respectively. In the case of a thermodynamic process with no damage dissipation, they can be evaluated as follows

$$g_t^p = \frac{G_f}{l_c} \quad \text{and} \quad g_c^p = \frac{G_c}{l_c} \quad (17)$$

where G_f and G_c are the fracture and crushing energies, respectively, and l_c is an external parameter that depends on the characteristic size of the finite element mesh that is introduced in order to obtain objectivity of the solid response respect to the mesh size [13].

The following evolution equation is proposed for the equivalent yielding threshold [7]:

$$K(\sigma_{ij}, \kappa^p) = r\sigma_t(\kappa^p) + (1+r)\sigma_c(\kappa^p) \quad (18)$$

where $\sigma_t(\kappa^p)$ and $\sigma_c(\kappa^p)$ represent the evolution of the yielding threshold in uniaxial tension and compression tests, respectively.

Loading/unloading conditions are derived from the Kuhn–Tucker relations formulated for problems with unilateral restrictions:

$$\begin{aligned} \text{(a)} \quad & \dot{\lambda} \geq 0 \\ \text{(b)} \quad & F \leq 0 \\ \text{(c)} \quad & \dot{\lambda} F = 0 \end{aligned} \quad (19)$$

4. Damage process

The damage threshold is described by a damage function in the following way [7]

$$G^D = \bar{\sigma}(\sigma_{ij}) - f_c(\sigma_{ij}, \kappa^d) \leq 0 \quad (20)$$

where $\bar{\sigma}(\sigma_{ij})$ is the equivalent tension defined in the damaged space, $f_c(\sigma_{ij}, \kappa^d)$ is the equivalent damage threshold and κ^d is the degradation variable.

The equivalent tension $\bar{\sigma}(\sigma_{ij})$ can be evaluated using known yielding functions (Tresca, Von-Mises, Mohr–Coulomb or Drucker–Prager) or any function specially developed for damage.

The degradation variable varies from 0, for the virgin material, to 1, for the completely damaged material and is obtained normalizing energy dissipated by damage to unity [7].

$$\kappa^d = \left[\frac{r}{g_t^{*d}} + \frac{(1-r)}{g_c^{*d}} \right] m_0 \Psi^{0,d} \quad (21)$$

where

$$g_t^{*d} = \frac{\sum_1^3 |\sigma_i| R^{0,d}}{\bar{\sigma}} g_t^d \quad g_c^{*d} = \frac{\sum_1^3 |\sigma_i|}{\bar{\sigma}} g_c^d \quad (22)$$

$R^{0,d}$ is the relation between the damage threshold in uniaxial compression and that corresponding to uniaxial tension and g_t^d and g_c^d are maximum energy densities dissipated by damage in uniaxial tension and compression processes, respectively.

The following evolution equation is proposed for the equivalent damage threshold:

$$f_c(\sigma_{ij}, \kappa^d) = r\sigma_i(\kappa^d) + (1-r)\sigma_c(\kappa^d) \quad (23)$$

where $\sigma_i(\kappa^d)$ and $\sigma_c(\kappa^d)$ represent the evolution of the damage threshold in uniaxial tension and compression tests, respectively.

The loading/unloading conditions are derived from the Kuhn–Tucker relations and are analogous to the ones corresponding to the plastic process:

$$\begin{aligned} (a) \quad & \dot{d} \geq 0 \\ (b) \quad & G^D \leq 0 \\ (c) \quad & dG^D = 0 \end{aligned} \quad (24)$$

5. Coupled plastic-damaged response

Evolution of permanent strains and damage is obtained from the simultaneous solution of the following equations called the *consistency conditions* of the problem [7]

$$\begin{aligned} \dot{F} &= 0 \\ \dot{G}^D &= 0 \end{aligned} \quad (25)$$

Eqs. (25) are two linear equations in $\dot{\lambda}$ and \dot{d} that can be easily solved.

The secant constitutive law can be written as follows

$$\sigma_{ij} = m_0 \frac{\partial \Psi^c}{\partial \epsilon_{ij}^c} = C_{ijkl}^c \epsilon_{kl}^c = (1-d) C_{ijkl}^0 (\epsilon_{kl} - \epsilon_{kl}^p) \quad (26)$$

$$\sigma_{ij} = (1-d) \sigma_{ij}^0 \quad (27)$$

where σ_{ij}^0 is the stress tensor of a fictitious undamaged solid. Eq. (27) can be interpreted as a transformation between the real damaged space and a fictitious undamaged space.

Tangent constitutive law can be obtained from Eqs. (25) and (26) and results

$$\dot{\sigma}_{ij} = C_{ijkl}^t \dot{\epsilon}_{kl} \quad (28)$$

where

$$C_{ijkl}^t = C_{ijkl}^c - \frac{C_{ijrs}^c \frac{\partial G}{\partial \sigma_{rs}} \frac{\partial F}{\partial \sigma_{mn}} C_{mnkl}^c}{-\frac{\partial F}{\partial \alpha_p} (h_p)_{iu} \frac{\partial G}{\partial \sigma_{iu}} + \frac{\partial F}{\partial \sigma_{mn}} C_{mnr s}^c \frac{\partial G}{\partial \sigma_{rs}}} \quad (29)$$

and

$$C_{ijkl}^e = C_{ijkl}^s - \frac{\sigma_{ij}^0 \left[\frac{\partial \bar{\sigma}}{\partial \sigma_{mn}} \left(-\frac{\partial F}{\partial \alpha_p} (h_p)_{mn} \frac{\partial G}{\partial \sigma_{mn}} + \frac{\partial F}{\partial \sigma_{mn}} C_{mnpq}^s \frac{\partial G}{\partial \sigma_{rs}} \right) - \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{mn}} C_{mnpq}^s \frac{\partial G}{\partial \sigma_{rs}} \right) \frac{\partial F}{\partial \sigma_{tu}} \right] C_{ijkl}^s}{\left(-\frac{\partial F}{\partial \alpha_p} (h_p)_{mn} \frac{\partial G}{\partial \sigma_{mn}} + \frac{\partial F}{\partial \sigma_{mn}} C_{mnpq}^s \frac{\partial G}{\partial \sigma_{rs}} \right) \left(\frac{\partial f_c}{\partial \kappa^q} h_{k,q} + \frac{\partial \bar{\sigma}}{\partial \sigma_{tu}} \sigma_{tu}^0 \right) - \left(\frac{\partial F}{\partial \sigma_{mn}} C_{mnpq}^s \frac{\partial G}{\partial \sigma_{rs}} \right) \frac{\partial F}{\partial \sigma_{tu}} \sigma_{tu}^0} \quad (30)$$

6. Algorithm for the numerical implementation of the plastic-damaged model

Capacity of a constitutive model to predict solid response not only depends on the model itself but also on the numerical integration of the model. The precision with which constitutive equations are integrated has a direct impact on the accuracy of the results [14, 15].

For the proposed model, yielding and damage equations must be integrated simultaneously. An Euler-Backward algorithm [7, 16, 17] is presented for this purpose. Solution of plane stress problems with this type of methods is straightforward. Between two equilibrium configurations n and $n-1$ the variables of the problem are updated as follows

$$(\epsilon_{ij}^p)_n = (\epsilon_{ij}^p)_{n-1} + \Delta \lambda \left(\frac{\partial G}{\partial \sigma_{ij}} \right)_n \quad (31)$$

$$(\alpha_i)_n = (\alpha_i)_{n-1} + \Delta \lambda (H_i)_n \quad (32)$$

$$d_n = d_{n-1} + \Delta d_n \quad (33)$$

$$(\sigma_{ij})_n = (1 - d_n) C_{ijkl}^0 [(\epsilon_{kl})_n - (\epsilon_{kl}^p)_n] \quad (34)$$

Replacing this equation in Eqs. (10) and (20), corresponding to yielding and damage conditions, the following non-linear system of equations is obtained

$$\begin{aligned} H^p(\Delta \lambda_n, \Delta d_n) &= F[(\sigma_{ij})_n; (\alpha_k)_n] = 0 \\ H^d(\Delta \lambda_n, \Delta d_n) &= G^D[(\sigma_{ij})_n; (k^d)_n] = 0 \end{aligned} \quad (35)$$

This system can be solved, for example, by the Newton–Raphson method

$$\begin{Bmatrix} \Delta \lambda_n \\ \Delta d_n \end{Bmatrix}^k = \begin{Bmatrix} \Delta \lambda_n \\ \Delta d_n \end{Bmatrix}^{k-1} - \begin{bmatrix} \left(\frac{\partial H^p}{\partial \Delta \lambda} \right)_n & \left(\frac{\partial H^p}{\partial \Delta d} \right)_n \\ \left(\frac{\partial H^d}{\partial \Delta \lambda} \right)_n & \left(\frac{\partial H^d}{\partial \Delta d} \right)_n \end{bmatrix}^{k-1} \begin{Bmatrix} H^p(\Delta \lambda_n, \Delta d_n) \\ H^d(\Delta \lambda_n, \Delta d_n) \end{Bmatrix}^{k-1} \quad (36)$$

The algorithm is summarized in Table 1.

Some recent publications [16–19] pointed out the advantage of using a consistent stiffness matrix when solving elastoplastic problems. It has been proved that the quadratic rate of convergence of an incremental solution based on a Newton–Raphson procedure can only be ensured if the tangent modulus is derived in a way consistent with the constitutive integration algorithm.

The consistent tangent modulus corresponding to the plastic-damaged model and the algorithm proposed is developed below.

Differentiating Eq. (34) and taking into account Eqs. (31)–(33), it results

$$\begin{aligned} (d\sigma_{ij})_n &= -d(d_n) C_{ijkl}^0 [(\epsilon_{kl})_n - (\epsilon_{kl}^p)_{n-1} - \Delta \lambda_n \left(\frac{\partial G}{\partial \sigma_{kl}} \right)_n] \\ &+ (C_{ijkl}^s)_n (d\epsilon_{ij})_n - d\lambda_n (C_{ijkl}^s)_n \left(\frac{\partial G}{\partial \sigma_{kl}} \right)_n - \Delta \lambda_n (C_{ijkl}^s)_n \left(\frac{\partial^2 G}{\partial \sigma_{kl} \partial \sigma_{pq}} \right)_n (d\sigma_{pq})_n \end{aligned} \quad (37)$$

Table 1

Algorithm for the numerical integration of constitutive equations

-
- (1) Initialization: $k = 0$; $\Delta\lambda_n^0 = 0$; $\Delta d_n^0 = 0$
- (2) Plastic strain updating: $(\epsilon_{ij})_n^k = (\epsilon_{ij})_{n-1} + \Delta\lambda_n^k \left(\frac{\partial G}{\partial \sigma_{rs}} \right)_n^{k-1}$
- (3) Damage variable updating: $d_n^k = d_{n-1} + \Delta d_n^k$
- (4) Non-damaged stress updating: $(\sigma_{ij})_n^k = C_{ijkl}^0 [(\epsilon_{rs})_n - (\epsilon_{rs}^p)_n^k]$
- (5) Damaged stress updating: $(\sigma_{ij})_n^k = (1 - d_n^k)(\sigma_{ij})_n^k$
- (6) Updating of the other internal variables: $(\alpha_i)_n^k = (\alpha_i)_{n-1} + \Delta\lambda_n^k (H_i)_n^k$
- $$(\Delta\kappa^a)_n^k = \left[\frac{r}{R_1^a} + \frac{(1-r)}{R_2^a} \right]_n^k m_n (\Psi^a)_n^k \Delta d_n^k; \quad (\kappa^a)_n^k = (\kappa^a)_{n-1} + (\Delta\kappa^a)_n^k$$
- (7) Yielding and damage conditions: If $(H^p)_n^k < 0$ and $(H^d)_n^k < 0$ goto (13)
- (8) If $(H^p)_n^k \geq 0$ goto (10)
- (9) $(\partial H^p / \partial \Delta d)_n^k = 0$ goto (12)
- (10) If $(H^d)_n^k \geq 0$ goto (12)
- (11) $(\partial H^d / \partial \Delta \lambda)_n^k = 0$
- (12) $\Delta\lambda_n^{k+1} = \Delta\lambda_n^k + \frac{\langle H^p(\Delta\lambda_n^k, \Delta d_n^k) \rangle (\partial H^p / \partial \Delta d)_n^k - \langle H^d(\Delta\lambda_n^k, \Delta d_n^k) \rangle (\partial H^p / \partial \Delta \lambda)_n^k}{(\partial H^p / \partial \Delta \lambda)_n^k (\partial H^d / \partial \Delta d)_n^k - (\partial H^d / \partial \Delta \lambda)_n^k (\partial H^p / \partial \Delta d)_n^k}$
- $$\Delta d_n^{k+1} = \Delta d_n^k + \frac{\langle H^d(\Delta\lambda_n^k, \Delta d_n^k) \rangle (\partial H^d / \partial \Delta \lambda)_n^k - \langle H^p(\Delta\lambda_n^k, \Delta d_n^k) \rangle (\partial H^d / \partial \Delta d)_n^k}{(\partial H^p / \partial \Delta \lambda)_n^k (\partial H^d / \partial \Delta d)_n^k - (\partial H^d / \partial \Delta \lambda)_n^k (\partial H^p / \partial \Delta d)_n^k}$$
- $k = k + 1$ Goto (2)
- (13) $d_n = d_n^k$; $(\sigma_{ij})_n = (\sigma_{ij})_n^k$; $(\kappa^a)_n = (\kappa^a)_n^k$; $(f_i)_n = (f_i)_n^k$; $(\epsilon_{ij}^p)_n = (\epsilon_{ij}^p)_n^k$; $(\alpha_i)_n = (\alpha_i)_n^k$
- (14) END
-

Eq. (37) can be written as

$$(d\sigma_{ij})_n = (C_{ijkl}^{e*})_n (d\epsilon_{kl})_n - (C_{ijkl}^{e*})_n d\lambda_n \left(\frac{\partial G}{\partial \sigma_{kl}} \right)_n \quad (38)$$

where

$$(C_{ijkl}^{e*})_n = \left[\delta_{ij} \delta_{pq} + \Delta\lambda_n (C_{ijrs}^s)_n \left(\frac{\partial^2 G}{\partial \sigma_{rs} \partial \sigma_{pq}} \right)_n \right]^{-1} C_{pqkl}^e \quad (39)$$

$$(C_{ijkl}^{s*})_n = \left[\delta_{ij} \delta_{pq} + \Delta\lambda_n (C_{ijrs}^s)_n \left(\frac{\partial^2 G}{\partial \sigma_{rs} \partial \sigma_{pq}} \right)_n \right]^{-1} C_{pqkl}^s \quad (40)$$

The consistent tangent modulus results

$$(\hat{C}_{ijkl})_n = (C_{ijkl}^{e*})_n - \frac{(C_{ijpq}^{s*})_n \left(\frac{\partial G}{\partial \sigma_{pq}} \right)_n \left(\frac{\partial F}{\partial \sigma_{rs}} \right)_n (C_{rskl}^{e*})_n}{-\left(\frac{\partial F}{\partial \sigma_{rs}} \right)_n [(h_p)_{/w}]_n \left(\frac{\partial G}{\partial \sigma_{rs}} \right)_n + \left(\frac{\partial F}{\partial \sigma_{pq}} \right)_n (C_{pqrs}^{s*})_n \left(\frac{\partial G}{\partial \sigma_{rs}} \right)_n} \quad (41)$$

7. Application examples

7.1. Concrete under biaxial compression

In this example the behavior of concrete under biaxial compression [20] is studied. Concrete was modeled as an elasto-plastic-damaged material. Mechanical properties of concrete are summarized in Table 2. finite element mesh and loading conditions are represented in Fig. 1. The problem was solved

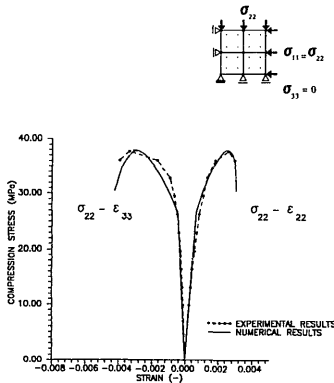


Fig. 1. Concrete under cyclic biaxial compression.

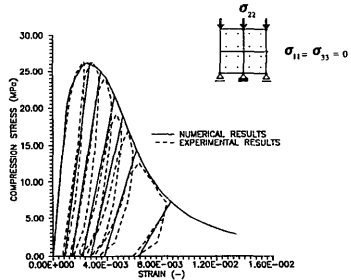


Fig. 2. Concrete under uniaxial compression.

with a 2D finite element program and the algorithm previously described. A consistent tangent modulus was used and no more than two iterations were needed for each loading step.

Stress–strain curves corresponding to the loading plane and the direction normal to the plane are represented in Fig. 1. It can be seen that a good agreement between numerical and experimental results is achieved with the elasto-plastic-damaged model.

7.2. Concrete under cyclic uniaxial compression

This example was used to prove the ability of the model and the algorithm proposed to reproduce cyclic behavior of an elasto-plastic-damage material like concrete. The behaviour of concrete under cyclic uniaxial compression [21] was studied.

Table 2
Mechanical properties of concrete

| |
|--|
| $E = 30\,000$ MPa |
| $\nu = 0.24$ |
| $\sigma_{cu} = 22.9$ MPa |
| $\sigma_{peak} = 32.8$ MPa |
| $\frac{\sigma_{cu}}{\sigma_{cr}} = 10$ |
| $\kappa_{peak}^p = 0.38$ |
| $G_p^* = 0.08$ N/mm |
| $G_p^d = 8.0$ N/mm |
| Lubliner–Oller Yielding Criteria [1] ($\alpha = 0.12; \gamma = 3.0; p = 0.0$) |
| Mohr–Coulomb Potential Function: $\psi = 15^\circ$ |
| $G_p^d = 0.08$ N/mm |
| $G_p^d = 8.0$ N/mm |
| Lubliner–Oller Damage Criteria [1] ($\alpha = 0.12; \gamma = 3.0; p = 0.0$) |

Table 3
Mechanical properties of concrete

| |
|--|
| $E = 19324.4$ MPa |
| $\nu = 0.24$ |
| $\sigma_{cu} = 22.0$ MPa |
| $\sigma_{peak} = 26.5$ MPa |
| $\frac{\sigma_{cu}}{\sigma_{cr}} = 10$ |
| $\kappa_{peak}^p = 0.12$ |
| $G_p^* = 0.08$ N/mm |
| $G_p^d = 30.0$ N/mm |
| Lubliner–Oller Yielding Criteria [1] ($\alpha = 0.12; \gamma = 3.0; p = 0.0$) |
| Associated flaw |
| $\kappa_{peak}^d = 0.12$ |
| $G_p^* = 0.08$ N/mm |
| $G_p^d = 10.0$ N/mm |
| Lubliner–Oller Damage Criteria [1] ($\alpha = 0.12; \gamma = 3.0; p = 0.0$) |

The finite element mesh and loading conditions are represented in Fig. 2. Concrete was modeled as an elasto-plastic-damaged material with the mechanical properties summarized in Table 3.

Experimental and numerical stress–strain curves are represented in Fig. 2. It can be seen that the model accurately reproduces the behavior of concrete. Unloading branches have almost the same average modulus. Obviously, the model is not able to reproduce hysteresis loops appearing in experimental results because linear elastic behaviour was assumed for unloading cases.

8. Concluding remarks

The constitutive model presented, simultaneously solves the problem of evolution of permanent strains and stiffness degradation. Yielding and damage consistent conditions are both satisfied in each loading step. In this way, using hardening variables related to energy dissipation in each problem, correct energy dissipation of the whole problem is achieved.

The model is simple and presents a total analogy with other elasto-plastic models used to reproduce the behaviour of frictional materials.

The algorithm presented for the numerical implementation of the model proposed is a generalization of Euler-Backward algorithms commonly used in plasticity to the case of plasticity coupled with damage. It is particularly suitable for plane stress problems. The consistent tangent modulus developed preserves quadratic rate of convergence of Newton–Raphson method.

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