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Partitioned formulation of internal fluid–structure interaction problems by localized Lagrange multipliers

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Abstract

A partitioned formulation of compressible internal fluid–structure interaction problems is presented by employing a displacement model for both the fluid and structure. Partitioning is effected by a localized version of the method of Lagrange multipliers, which assigns two independent sets of Lagrange multipliers to the structural and fluid interfaces. Two major features of the present formulation include: an interface compliance normalization that helps capture the predominant physics of interaction phenomena when the interfaces are characterized by two radically different rigidities, and a novel transformation of the displacement model into a fluid-pressure model that is suitable for both transient and vibration analyses. The present formulation first solves for the interface Lagrange multipliers, which are subsequently used to solve for the structural displacements and the fluid displacement or pressure by employing two independent analysis modules.

1. Introduction

Formulation, discrete modeling and computer implementation of fluid–structure interaction problems have attracted many researchers' attention in the past. As a result, a variety of fluid–structure interaction analysis approaches exist, ranging from a tightly coupled integrated analysis procedure to two independent fluid and structural analyzers in a staggered manner. For example, for vibration analysis of coupled internal fluid–structure interactions, it has been customary to use integrated equations of motion from which one may obtain the coupled vibratory modes and mode shapes (cf. [11,14]). In most of their formulations, the compressible internal fluid is modeled in terms of fluid pressure while the structure is modeled in terms of displacements. The interaction interfaces are treated by equating the structural normal stress component to the pressure acting on the structure. In the context of partitioned analysis of fluid–structure interaction problems (e.g., [4,7,16,17]), such formulations lead to a tightly coupled set of system equations.

Recently, multi-physics analysis that requires to treat the interactions of more than two fields is emerging as a new challenge as well as new opportunities. This new challenge demands not only modularity of each single-field simulation capability, but also perhaps more importantly the interface phenomena are treated in separate modules as much as possible. This means that a multi-physics-oriented analysis of fluid–structure interaction problems may involve three modular attributes: a fluid analyzer, a structural analyzer, and an interface module that accounts for the fluid–structure interaction phenomena. This has motivated us to

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develop a formulation that leads to modular computational modeling and implementation of compressible, internal fluid–structure interaction problems.

A straightforward way of modeling a compressible internal fluid is to employ fluid displacement as the primary variable. This can be accomplished by invoking an existing finite element structural analysis capability by introducing the fluid constitutive relation and appropriate fluid displacement interpolation schemes. The coupled compressible internal fluid–structure interaction equations can then be constructed by assembling the fluid and structural discrete equations together. There are two difficulties in this monolithic approach. First, the rigidities of the fluid and the structure are typically vastly different, viz., $E/\rho_{\mathrm{f}}c_{\mathrm{f}}^2 \approx 10^5 \sim 10^7$, where E is the structural Young's modulus, ρ_{f} the fluid density, and c_{f} is the speed of propagation of the fluid. Therefore, the solution vector on the interaction interfaces may be dominated by the structural attributes rather than the combined interaction phenomena. This becomes especially problematic for cases that require accurate resolution of high-frequency modes being transmitted across the interaction boundaries. Second, as the volume of the internal fluid and/or frequency resolution of the fluid modes increase, the degrees of freedom for the discrete displacement modeling of fluid may lead to a prohibitively large number of equations. For this reason, the analyst prefers the pressure to the fluid displacement since the number of unknowns can be reduced by a third.

The objective of the present paper is to offer a partitioned formulation of compressible internal fluid–structure interaction problems via a localized version of the method of Lagrange multipliers [20]. Specifically, the interface compatibility condition for the structure with respect to the wetted interface boundary is enforced independently of that for the fluid. The partitioned formulation is carried out first by employing a fluid displacement formulation. The pressure formulation of the fluid is then obtained by means of a *displacement-to-pressure* transformation. A partitioned formulation that leads to a modular computer implementation of internal-fluid–structure interaction analysis involving the structural analyzer, the fluid analyzer, and the interface solver is finally presented.

As the localized treatment of Lagrange multipliers constitutes a central role in the present formulation of fluid–structure interaction problems, we will first review the method of localized Lagrange multipliers. A partitioned formulation of fluid–structure interaction problems and its various specializations will then follow.

2. Basic concepts of a localized version of the method of Lagrange multipliers

Consider a two-spring system partitioned as shown in Fig. 1. For the assembled system shown by Fig. 1(a), the system energy is given by

$$\begin{aligned} &\Pi_{\mathrm{g}}(\mathbf{u}_{\mathrm{g}}) = \mathbf{u}_{\mathrm{g}}^{\mathrm{T}}\left(\frac{1}{2}\mathbf{K}_{\mathrm{g}}\mathbf{u}_{\mathrm{g}} - \mathbf{M}_{\mathrm{g}}\ddot{\mathbf{u}}_{\mathrm{g}} - \mathbf{f}_{\mathrm{g}}\right), \\ &\mathbf{u}_{\mathrm{g}} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \mathbf{K}_{\mathrm{g}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \mathbf{M}_{\mathrm{g}} = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{f}_{\mathrm{g}} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}, \end{aligned} \tag{1}$$

where $\mathbf{K}_{\mathrm{g}}$ is the global or assembled stiffness matrix, $\mathbf{u}_{\mathrm{g}}$ the global nodal displacement, $\mathbf{f}_{\mathrm{g}}$ the force vector acting on the assembled system (a), and the subscript 'g' designates *global* quantities. The stationary value of the preceding functional, namely $\delta\Pi_{\mathrm{g}} = 0$, yields the well-known equilibrium equation

$$\mathbf{M}_{\mathrm{g}}\ddot{\mathbf{u}} + \mathbf{K}_{\mathrm{g}}\mathbf{u}_{\mathrm{g}} = \mathbf{f}_{\mathrm{g}}. \tag{2}$$

We now illustrate two distinct treatments of interface boundary conditions when the assembled structure is partitioned into two or more subdomains. The classical treatment of partition boundaries by the method of Lagrange multipliers is to enforce the interface kinematic conditions. For example, when a node in an assembled structure is partitioned into two substructural nodes, the partition boundary condition is realized by enforcing that the two partition boundary displacements be the same. While the partitioned substructural displacements are 'localized', the Lagrange multiplier is of global nature in that it is common to the two partition boundary nodes. In other words, the classical method of Lagrange multipliers employs the

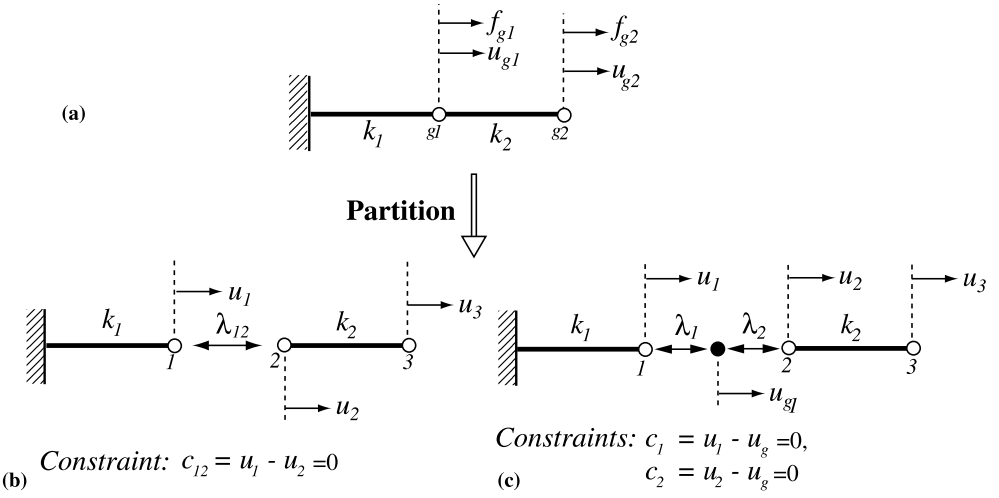

Fig. 1. Partitioning of two springs. (a) Assembled system; (b) partitioned system whose interface is modeled by classical λ-method; (c) partitioned system whose interface is modeled by localized λ-method.

localized displacements in conjunction with the 'globalized' multipliers in constructing the partition boundary constraint condition.

In order to employ localized variables to the maximum degree in constructing the interface boundary constraint conditions, this paper introduces the following consideration. First, instead of requiring the partitioned substructural displacements to be the same, we propose that the two partitioned substructural boundary displacements be equal to a global reference displacement. When this kinematical condition is enforced by Lagrange multipliers, the corresponding Lagrange multipliers become localized variables. Consequently, the partition boundary condition is realized by local substructural displacements, localized Lagrange multipliers, and a global reference displacement. We now illustrate the two methods via a simple example.

2.1. Classical treatment of partitioned boundaries

Let us consider an assembled two-spring system partitioned into two springs as shown in Fig. 1(b). When the classical method of Lagrange multipliers method is used to model the system, the energy functional consists of the energy of the two substructural springs which is augmented with the partition boundary constraint condition

$$\begin{aligned} &\Pi(\mathbf{u}, \boldsymbol{\lambda}_{\mathrm{cl}}) = \mathbf{u}^{\mathrm{T}}\left(\frac{1}{2}\mathbf{K}\mathbf{u} - \mathbf{M}\ddot{\mathbf{u}} - \mathbf{f}\right) + \boldsymbol{\lambda}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{C}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{u}, \\ &\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}, \quad \boldsymbol{\lambda}_{\mathrm{cl}} = \lambda_{12}, \end{aligned} \tag{3}$$

where the interface constraint is given by

$$c_{12} = u_1 - u_2 = \mathbf{C}_{\mathrm{cl}}^{\mathrm{T}}\mathbf{u} = 0 \;\Rightarrow\; \mathbf{C}_{\mathrm{cl}}^{\mathrm{T}} = \left[\, \mathbf{C}_{\mathrm{cl}}^{(1)^{\mathrm{T}}} \quad \mathbf{C}_{\mathrm{cl}}^{(2)^{\mathrm{T}}} \,\right] = [\,[\,1\,] \quad [\,-1 \quad 0\,]\,], \tag{4}$$

in which $\mathbf{u}$ is the partitioned displacement, $\mathbf{K}$ the partitioned stiffness matrix, $\mathbf{f}$ the partitioned force vector acting on each spring, λ_{12} is the interface reaction force to account for the constraint that the displacements u_1 and u_2 are the same, and the superscript (j) denotes springs 1 and 2. Note from Fig. 1(b) that the Lagrange multiplier λ_{12} is common to both the substructures 1 and 2, thus becoming a global variable.

The stationary condition $\delta\Pi(\mathbf{u},\boldsymbol{\lambda}_{\rm cl})=0$ leads to the following partitioned equation set:

$$\begin{aligned}
&\mathbf{S}_{\rm cl}\begin{Bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}_{\rm cl}\end{Bmatrix}=\begin{Bmatrix}\mathbf{f}\\ \mathbf{0}\end{Bmatrix},\\
&\mathbf{S}_{\rm cl}=\begin{bmatrix}\mathbf{K}^{(1)}+\mathbf{M}^{(1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0 & \mathbf{C}_{\rm cl}^{(1)^{\rm T}}\\ 0 & \mathbf{K}^{(2)}+\mathbf{M}^{(2)}\frac{\mathrm{d}^2}{\mathrm{d}t^2} & \mathbf{C}_{\rm cl}^{(2)^{\rm T}}\\ \mathbf{C}_{\rm cl}^{(1)} & \mathbf{C}_{\rm cl}^{(2)} & \mathbf{0}\end{bmatrix}\\
&\quad=\begin{bmatrix}k_1+m_1\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0 & 0 & 1\\ 0 & k_2+m_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} & -k_2 & -1\\ 0 & -k_2 & k_2+m_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0\\ 1 & -1 & 0 & 0\end{bmatrix}.
\end{aligned}\tag{5}$$

Notice that the matrix differential operator $\mathbf{S}_{\rm cl}$ is indefinite. One solution method to bypass the indefiniteness is to eliminate the substructural displacements and solve for the Lagrange multipliers. For example, implicit direct time integration of the preceding equation by the midpoint rule with its step size Δt,

$$\dot{\mathbf{u}}^{n+1/2}=\dot{\mathbf{u}}^{n}+\frac{1}{\beta}\ddot{\mathbf{u}}^{n+1/2},\quad \beta=\frac{2}{\Delta t};\qquad \mathbf{u}^{n+1/2}=\mathbf{u}^{n}+\frac{1}{\beta}\dot{\mathbf{u}}^{n+1/2};\qquad \mathbf{u}^{n+1}=2\mathbf{u}^{n+1/2}-\mathbf{u}^{n};\tag{6}$$

leads to the following equation:

$$\begin{aligned}
&\bar{\mathbf{S}}_{\rm cl}\begin{Bmatrix}\mathbf{u}\\ \boldsymbol{\lambda}\end{Bmatrix}^{n+1/2}=\begin{Bmatrix}\bar{\mathbf{f}}\\ \mathbf{0}\end{Bmatrix}^{n+1/2},\\
&\bar{\mathbf{S}}_{\rm cl}=\begin{bmatrix}\mathbf{K}^{(1)}+\mathbf{M}^{(1)}\beta^2 & 0 & \mathbf{C}_{\rm cl}^{(1)^{\rm T}}\\ 0 & \mathbf{K}^{(2)}+\mathbf{M}^{(2)}\beta^2 & \mathbf{C}_{\rm cl}^{(2)^{\rm T}}\\ \mathbf{C}_{\rm cl}^{(1)} & \mathbf{C}_{\rm cl}^{(2)} & \mathbf{0}\end{bmatrix},\\
&\bar{\mathbf{f}}^{n+1/2}=\mathbf{f}^{n+1/2}+\mathbf{M}(\beta^2\mathbf{u}^{n}+\beta\dot{\mathbf{u}}^{n}).
\end{aligned}\tag{7}$$

Solving for the substructural displacements $\mathbf{u}$ for substructures 1 and 2, and substituting them into the interface compatibility condition, i.e., the last row of (7), we obtain

$$\mathbf{F}_{\rm cl}\boldsymbol{\lambda}_{\rm cl}^{n+1/2}=\mathbf{C}_{\rm cl}^{\rm T}[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\bar{\mathbf{f}}^{n+1/2},\quad \mathbf{F}_{\rm cl}=\{\mathbf{C}_{\rm cl}^{\rm T}[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\mathbf{C}_{\rm cl}\},\tag{8}$$

where $\mathbf{K},\mathbf{M}$ and $\mathbf{C}_{\rm cl}$ are defined in (3) and (4). Specifically, the solution flexibility matrix $\mathbf{F}_{\rm cl}$ for the example problem shown in Fig. 1(b) becomes

$$\mathbf{F}_{\rm cl}=\{\mathbf{C}_{\rm cl}^{\rm T}[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\mathbf{C}_{\rm cl}\}=\left[\frac{1}{(k_1+\beta^2m_1)}+\frac{(k_2+\beta^2m_2)}{\beta^2m_2(2k_2+\beta^2m_2)}\right].\tag{9}$$

First, observe that the dynamic flexibility matrix $\mathbf{F}_{\rm cl}$ is obtained by summing over the interface boundary attributes of both substructures 1 and 2. Second, while the formation of the interface boundary flexibilities as a sum of all the interface substructural attributes is physically consistent, it may cause deleterious effects both on the conditioning of the solution matrix and on software modularity. This can be illustrated by examining a general case of two partitions whose interface flexibility may be expressed as

$$\mathbf{F}_{\rm cl}=[\mathbf{F}_{\rm bb}^{(1)}+\mathbf{F}_{\rm bb}^{(2)}].\tag{10}$$

Observe that, if the interface involves a solid and a plate, the resulting flexibility may experience a high condition number that will adversely affect the solution accuracy as well as efficiency if an iterative procedure is adopted. The loss of modularity, in addition to the non-unique issues associated with the classical

λ-method [20], can be a major concern for coupled-field analysis. This is alleviated by employing a localized version of the method of Lagrange multipliers as discussed below.

2.2. Localized treatment of partitioned boundaries

This section reviews a localized version of the method of Lagrange multipliers that was introduced in the formulation of a variational framework for treating the partition boundary constraints [18,19] and its subsequent formalism [20]. While the classical method of Lagrange multipliers demands the partition boundary displacements to be the same as given by (4), the localized method of Lagrange multipliers requires the partition boundary displacements to be made equal to the global boundary displacement $\mathbf{u}_{\mathrm{g}}$. Specifically, we have from Fig. 1(c)

$$\begin{aligned}
&\left.\begin{Bmatrix} c_1 = u_1 - u_{\mathrm{g}1} = 0 \\ c_2 = u_2 - u_{\mathrm{g}1} = 0 \end{Bmatrix}\right\} \;\Rightarrow\; \mathbf{B}_\ell^{\mathrm{T}}(\mathbf{u} - \mathbf{L}\mathbf{u}_{\mathrm{g}}) = 0 \;\Rightarrow\; \mathbf{B}_\ell^{\mathrm{T}}\mathbf{u} - \mathbf{L}_{\mathrm{b}}\mathbf{u}_{\mathrm{b}} = 0, \\
&\mathbf{B}_\ell^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \;\Rightarrow\; \mathbf{L}_{\mathrm{b}} = \mathbf{B}_\ell^{\mathrm{T}}\mathbf{L} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{aligned} \tag{11}$$

where $\mathbf{B}_\ell$ is the partition-boundary extraction Boolean matrix, $\mathbf{L}$ the well-known finite element assembly operator, and $\mathbf{u}_{\mathrm{b}}$ is the global displacement pertaining to the partition boundary, respectively. For the example problem shown in Fig. 1(c), we have $\mathbf{u}_{\mathrm{b}} = u_{\mathrm{g}1}$.

When (11) is augmented to the partitioned system energy (see Fig. 1(c)), the resulting partitioned system energy functional can be expressed by

$$\Pi_\ell(\mathbf{u}, \mathbf{u}_{\mathrm{b}}, \boldsymbol{\lambda}_\ell) = \mathbf{u}^{\mathrm{T}}\left(\frac{1}{2}\mathbf{K}\mathbf{u} + \mathbf{M}\ddot{\mathbf{u}} - \mathbf{f}\right) + \boldsymbol{\lambda}_\ell^{\mathrm{T}}(\mathbf{B}_\ell^{\mathrm{T}}\mathbf{u} - \mathbf{L}_{\mathrm{b}}\mathbf{u}_{\mathrm{b}}), \quad \boldsymbol{\lambda}_\ell^{\mathrm{T}} = \langle \lambda_1 \ \lambda_2 \rangle. \tag{12}$$

The stationarity of $\delta\Pi_\ell(\mathbf{u}, \mathbf{u}_{\mathrm{b}}, \boldsymbol{\lambda}_\ell)$ yields the following partitioned equations of motion:

$$\begin{aligned}
&\mathbf{S}_\ell \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \\
&\mathbf{S}_\ell = \begin{bmatrix} \mathbf{K}^{(1)} + \mathbf{M}^{(1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0 & \mathbf{B}_1 & 0 \\ 0 & \mathbf{K}^{(2)} + \mathbf{M}^{(2)}\frac{\mathrm{d}^2}{\mathrm{d}t^2} & \mathbf{B}_2 & 0 \\ \mathbf{B}_1^{\mathrm{T}} & \mathbf{B}_2^{\mathrm{T}} & \mathbf{0} & -\mathbf{L}_{\mathrm{b}} \\ 0 & 0 & -\mathbf{L}_{\mathrm{b}}^{\mathrm{T}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} k_1 + m_1\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0 & 0 & 1 & 0 & 0 \\ 0 & k_2 + m_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} & -k_2 & 0 & 1 & 0 \\ 0 & -k_2 & k_2 + m_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}.
\end{aligned} \tag{13}$$

It is noted that the matrix $\mathbf{B}_\ell$ is a localized Boolean matrix and the Lagrange multipliers $\boldsymbol{\lambda}_\ell^{\mathrm{T}} = \langle \lambda_1 \ \lambda_1 \rangle$ are also completely localized. The only global operator is the $\mathbf{L}_{\mathrm{b}}$ matrix.

Remark 2.1. Comparing the two partitioned equations, viz., one augmented with the localized λ-method (13) with the classical λ-method (5), we observe the following.

- The boundary extraction operator $\mathbf{B}_\ell$ is completely localized. Hence, the present formulation does not require any summation of stiffness or other system-dependent attributes across the substructural interfaces.

- The kinematic interface compatibility condition given by the third row of (13),

$$\mathbf{B}_\ell^{\mathrm{T}}\mathbf{u} - \mathbf{L}_{\mathrm{b}}\mathbf{u}_{\mathrm{b}} = 0, \tag{13a}$$

relates the partitioned interface displacement $\mathbf{u}$ to the assembled global displacement $\mathbf{u}_{\mathrm{b}}$ at the partition boundary. Therefore, regardless the number of nodes constrained to a global node, the resulting constraints are unique and rank-sufficient.
- The fourth row of (13) given by

$$-\mathbf{L}^{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{b}} = 0 \;\Rightarrow\; -(\lambda_1 + \lambda_2) = 0 \tag{13b}$$

is the statement that the sum of reaction forces at a node disappears when the partitioned nodes are assembled together, viz., Newton's third law.

Remark 2.2. The preceding formalism possesses the following features:
- The procedure conceptually assumes the availability of the total assembled equilibrium equations.
- The partitioned equilibrium equation for each partition emanates by invoking the finite element *disassembly* process to the system energy functional.
- The resulting time-discretized flexibility matrix $\bar{\mathbf{F}}_{\mathrm{b}}$ forms an uncoupled block diagonal matrix

$$\bar{\mathbf{F}}_\ell = \begin{bmatrix} \bar{\mathbf{F}}^{(1)} & \cdot & \cdots & \cdots \\ \cdot & \bar{\mathbf{F}}^{(2)} & \cdot & \cdots \\ \cdot & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \bar{\mathbf{F}}^{(n)} \end{bmatrix}. \tag{14}$$

Specifically, for the example problem shown in Fig. 1(c), it becomes a (2×2) uncoupled matrix

$$\bar{\mathbf{F}}_\ell = \begin{bmatrix} \frac{1}{(k_1+\beta^2 m_1)} & 0 \\ 0 & \frac{(k_2+\beta^2 m_2)}{D} \end{bmatrix}, \quad D = \det\,|\mathbf{K}^{(2)} + \beta^2\mathbf{M}^{(2)}|. \tag{15}$$

- The resulting Lagrange multipliers possess a localized characteristic. It is this feature that will be exploited in the derivation of fluid–structure interaction problems in order to effect a modular implementation of interaction phenomena.

3. Variational formulation of elasto-internal acoustic problem by primitive variables

3.1. Motivation

One of the most widely adopted formulations of elasto-internal acoustic problems is a family of the so-called (u,p)-formulations (Everstine [3], Felippa [5], Felippa et al. [7], Geers and Felippa [8], Geers and Zhang [9], Morand and Ohayon [11], Ohayon [12] and Sandberg and Göransson [25] among others). In order to achieve a symmetric formulation, they introduced a pressure potential with a proviso that the associated nonphysical zero-mode be projected out a priori. The resulting formulation, while achieving symmetry, introduces a highly dense mass matrix (for more details, see Section 8.4 of [11]). A closer examination of their formulation reveals that the pointwise enforcement of the interface condition between the fluid and structural surface dictated by

$$\sigma_{ij}n_j^{\mathrm{S}} = -pn_i^{\mathrm{S}} = pn_i \tag{16}$$

is responsible for the subsequent tight coupling between the fluid and structural governing equations. In (16), σ_{ij} is the stress tensor in the structure, p the fluid pressure, and n_i is the unit normal to the interface surface.

In the present formulation to be described below, we bypass the above pointwise Neumann boundary condition by treating both the fluid and structure domains as a single continuum. A justification for this integrated continuum modeling of fluid and structure is that we substitute the pointwise interface condition by its weak constraint, viz., the sum of the discrete nodal is forced to vanish at the fluid–structure interface. Specifically, we replace the pointwise interface normal stress condition (16) by a constraint functional in terms of a set of localized Lagrange multipliers and the associated kinematical relations

$$\begin{aligned} &\pi_\ell(\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{s},\mathbf{u}_\mathrm{b},\boldsymbol{\lambda}_\ell) = \boldsymbol{\lambda}_\mathrm{f}^\mathrm{T}\mathscr{S}_\mathrm{f}^\mathrm{T}(\mathbf{u}_\mathrm{f}-\mathbf{u}_\mathrm{b}) + \boldsymbol{\lambda}_\mathrm{s}^\mathrm{T}\mathscr{S}_\mathrm{s}^\mathrm{T}(\mathbf{u}_\mathrm{f}-\mathbf{u}_\mathrm{b}),\\ &\quad\Downarrow\\ &\mathscr{S}_\mathrm{f}^\mathrm{T}(\mathbf{u}_\mathrm{f}-\mathbf{u}_\mathrm{b}) = 0, \qquad \mathscr{S}_\mathrm{s}^\mathrm{T}(\mathbf{u}_\mathrm{s}-\mathbf{u}_\mathrm{b}) = 0, \quad \mathscr{S}_\mathrm{f}\boldsymbol{\lambda}_\mathrm{f} + \mathscr{S}_\mathrm{s}\boldsymbol{\lambda}_\mathrm{s} = 0, \end{aligned} \tag{17}$$

where $\boldsymbol{\lambda}_\mathrm{f}$ and $\boldsymbol{\lambda}_\mathrm{s}$ are the Lagrange multipliers representing a set of generalized interacting forces on the fluid and structural interfaces, and $\mathscr{S}_\mathrm{f}$ and $\mathscr{S}_\mathrm{s}$ are the rigidity regularization operators (or filters) that scale the interface kinematic constraints commensurate with the relative rigidities of the structure and fluid.

3.2. Interface regularization

Observe that $\boldsymbol{\lambda}_\mathrm{f}$ and $\boldsymbol{\lambda}_\mathrm{s}$ are independent, which offers the possibility of modeling them to reflect the predominant physics of each field. For example, the structure is in general characterized by much higher frequencies than the fluid. These distinct frequency characteristics can be exploited in approximating the two Lagrange multipliers. To this end, we will specialize the so-called flexibility normalization scheme [20] that has proved to be effective for contact enforcement and parallel computations of partitioned structural equations of motion [21].

In order to arrive at a rational choice of the rigidity regularization operators $(\mathscr{S}_\mathrm{s},\mathscr{S}_\mathrm{f})$, we first obtain a least-square solution of the reference interface displacement (or velocity) $\mathbf{u}_\mathrm{b}$ from (17):

$$\mathbf{u}_\mathrm{b} = \frac{\mathscr{S}_\mathrm{f}^2\mathbf{u}_\mathrm{f} + \mathscr{S}_\mathrm{s}^2\mathbf{u}_\mathrm{s}}{\mathscr{S}_\mathrm{f}^2 + \mathscr{S}_\mathrm{s}^2}. \tag{18}$$

Observe that the preceding relation is analogous to the well-known momentum conservation relation when two rigid bodies (m_1, m_2) are in contact,

$$\dot{\mathbf{u}} = \frac{m_1\dot{\mathbf{u}}_1 + m_2\dot{\mathbf{u}}_2}{m_1+m_2}, \tag{19}$$

which suggests that, when the interacting two bodies are elastic, a dimensional analysis suggests that the rigidity regularization operators $(\mathscr{S}_\mathrm{s},\mathscr{S}_\mathrm{f})$ should be of the form

$$\mathscr{S}_\mathrm{s} = (c\Delta t^2 k_\mathrm{s} + m_\mathrm{s})^{1/2}, \qquad \mathscr{S}_\mathrm{f} = (c\Delta t^2 k_\mathrm{f} + m_\mathrm{f})^{1/2}, \tag{20}$$

where $(k_\mathrm{s}, k_\mathrm{f})$ are the characteristic stiffness of the structure and fluid, Δt is characteristic time constant, and c is a constant.

As a simple generalization of the above characteristic regularization parameters to the discrete matrix system, we have chosen the following:

$$\mathscr{S}_\mathrm{s} = (\mathrm{diag}\,(\delta^2\mathbf{K}_\mathrm{s} + \mathbf{M}_\mathrm{s}))^{1/2}, \qquad \mathscr{S}_\mathrm{f} = (\mathrm{diag}\,(\delta^2\mathbf{K}_\mathrm{f} + \mathbf{M}_\mathrm{f}))^{1/2}, \quad \delta = \beta\Delta t \ \ (\beta \leqslant 1), \tag{21}$$

where $(\mathbf{K}_\mathrm{s},\mathbf{K}_\mathrm{f})$ are the partitioned structural and fluid stiffness matrices at the interface boundary, and $(\mathbf{M}_\mathrm{s},\mathbf{M}_\mathrm{f})$ are the partitioned structural and fluid inertia matrices at the interface boundary, respectively.

Remark 3.1. It should be noted that such rigidity regularization is not available when one employs the classical $\boldsymbol{\lambda}$-method.

3.3. Variational formulation

The variational formulation of a small-amplitude elasto-internal acoustic problem in terms of displacement variables can be expressed as

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_s}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\,\mathrm{d}\Omega_s+\frac{1}{2}\rho_f c_f^2\int_{\Omega_f}(\nabla\mathbf{u}_f)^2\,\mathrm{d}\Omega_f+\int_{\Omega_s}\mathbf{u}_s^{\mathrm{T}}(\mathbf{f}_s-\rho_s\ddot{\mathbf{u}}_s)\,\mathrm{d}\Omega_s+\int_{\Omega_f}\mathbf{u}_f^{\mathrm{T}}(\mathbf{f}_f-\rho_f\ddot{\mathbf{u}}_f)\,\mathrm{d}\Omega_f\\&\quad=\int_{\Gamma_s}\mathbf{u}_s^{\mathrm{T}}\mathbf{T}_s\,\mathrm{d}\Gamma+\int_{\Gamma_f}\mathbf{u}_f^{\mathrm{T}}\mathbf{T}_f\,\mathrm{d}\Gamma,\qquad p=-\rho_f c_f^2\nabla\mathbf{u}_f,\\&\quad\text{with the irrotationality condition: }\mathbf{curl}\,\mathbf{u}_f=0\text{ in }\Omega_f,\end{aligned}\tag{22}$$

where (ρ_s,ρ_f) are the density of the structure and fluid; c_f is the sound speed of the fluid; $(\mathbf{u}_s,\mathbf{u}_f)$ are the structural and fluid displacement vectors; $(\boldsymbol{\sigma},\boldsymbol{\epsilon})$ are the stress and strain vectors of the structure; $(\mathbf{f}_s,\mathbf{f}_f)$ are the body forces of the structure and fluid; and (Ω_s,Ω_f) designate the interior structure and fluid domains; (Γ_s,Γ_f) represent the physical boundaries of the structure and fluid; and the superscript dots $(\dot{\ })$ designate time differentiation.

The finite element discretization of (22) (see, for example, Section 8.2-4 of [11]) can be expressed as

$$\begin{aligned}&\delta\Pi(\mathbf{u}_g)=\delta\mathbf{u}_g^{\mathrm{T}}(\mathbf{K}_g\mathbf{u}_g+\mathbf{M}_g\ddot{\mathbf{u}}_g-\mathbf{F}_g),\\&\mathbf{curl}\,\mathbf{u}_f\approx\mathscr{C}_f^{\mathrm{T}}\mathbf{u}_f=0,\\&\mathbf{K}_g=\begin{bmatrix}\mathbf{L}_s^{\mathrm{T}} & \mathbf{L}_f^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{K}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_f\end{bmatrix}\begin{bmatrix}\mathbf{L}_s\\ \mathbf{L}_f\end{bmatrix},\\&\mathbf{M}_g=\begin{bmatrix}\mathbf{L}_s^{\mathrm{T}} & \mathbf{L}_f\end{bmatrix}\begin{bmatrix}\mathbf{M}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_f\end{bmatrix}\begin{bmatrix}\mathbf{L}_s^{\mathrm{T}}\\ \mathbf{L}_f\end{bmatrix},\\&\begin{Bmatrix}\mathbf{u}_s\\ \mathbf{u}_f\end{Bmatrix}=\begin{bmatrix}\mathbf{L}_s\\ \mathbf{L}_f\end{bmatrix}\mathbf{u}_g,\\&\mathbf{f}_g=\begin{bmatrix}\mathbf{L}_s^{\mathrm{T}} & \mathbf{L}_f^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{f}_{\gamma s}+\mathbf{t}_s\\ \mathbf{f}_{\gamma f}+\mathbf{t}_f\end{Bmatrix},\end{aligned}\tag{23}$$

where $\mathbf{L}$ denotes a Boolean matrix that assembles elements to a global matrix, and various matrix and vectorial quantities are given by

$$\begin{aligned}&\delta\int_{\Omega_s}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\,\mathrm{d}\Omega\approx\delta\mathbf{u}_s^{\mathrm{T}}\mathbf{K}_s\mathbf{u}_s,\\&\rho_f c_f^2\delta\int_{\Omega_f}(\nabla\mathbf{u}_f)^2\,\mathrm{d}\Omega\approx\delta\mathbf{u}_f^{\mathrm{T}}\mathbf{K}_f\mathbf{u}_f,\\&\int_{\Omega_s}\rho_s\delta\mathbf{u}_s^{\mathrm{T}}\ddot{\mathbf{u}}_s\,\mathrm{d}\Omega\approx\delta\mathbf{u}_s^{\mathrm{T}}\mathbf{M}_s\ddot{\mathbf{u}}_s,\\&\int_{\Omega_f}\rho_f\delta\mathbf{u}_f^{\mathrm{T}}\ddot{\mathbf{u}}_f\,\mathrm{d}\Omega\approx\delta\mathbf{u}_f^{\mathrm{T}}\mathbf{M}_f\ddot{\mathbf{u}}_f,\\&\delta\int_{\Omega_s}\mathbf{u}_s^{\mathrm{T}}\mathbf{f}_s\,\mathrm{d}\Omega\approx\delta\mathbf{u}_s^{\mathrm{T}}\mathbf{f}_{\gamma s},\\&\delta\int_{\Omega_f}\mathbf{u}_f^{\mathrm{T}}\mathbf{f}_f\,\mathrm{d}\Omega\approx\delta\mathbf{u}_f^{\mathrm{T}}\mathbf{f}_{\gamma f},\\&\delta\int_{\Gamma_s}\mathbf{u}_s^{\mathrm{T}}\mathbf{T}_s\,\mathrm{d}\Gamma\approx\delta\mathbf{u}_s^{\mathrm{T}}\mathbf{t}_s,\\&\delta\int_{\Gamma_f}\mathbf{u}_f^{\mathrm{T}}\mathbf{T}_f\,\mathrm{d}\Gamma\approx\delta\mathbf{u}_f^{\mathrm{T}}\mathbf{t}_f.\end{aligned}\tag{24}$$

It should be noted that several investigators [1,2,10,22] proposed a similar fluid-displacement formulation.

If we augment $\delta\Pi(\mathbf{u}_g)$ with the irrotational condition, we have the following modified functional:

$$\delta\Pi(\mathbf{u}_g,\boldsymbol{\mu}) = \delta\mathbf{u}_g^T(\mathbf{K}_g\mathbf{u}_g + \mathbf{M}_g\ddot{\mathbf{u}}_g - \mathbf{f}_g) + \delta\boldsymbol{\mu}^T\mathscr{C}_f^T\mathbf{u}_f + \delta\mathbf{u}_f^T\mathscr{C}_f\boldsymbol{\mu}. \tag{25}$$

Hence, the governing discrete equations of motion subject to the irrotational constraint lead to the following equation:

$$\begin{bmatrix} \mathbf{K}_g + \mathbf{M}_g\frac{d^2}{dt^2} & \mathscr{C}_g \\ \mathscr{C}_g^T & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{u}_g \\ \boldsymbol{\mu} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_g \\ \mathbf{0} \end{Bmatrix}, \quad \mathscr{C}_g^T = \begin{bmatrix} \mathbf{0} & \mathscr{C}_f^T \end{bmatrix}\begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix}. \tag{26}$$

It should noted that, for each fluid element,$\mathscr{C}_f$ becomes $(3 \times n_f)$ where n_f is the total elemental degrees of freedom. Care must be exercised in constructing the discrete irrotational operator in an analogous manner to the incompressibility condition, which has been extensively studied (see, e.g., [23] and references therein). Alternatively, one can employ the fluid displacement interpolation basis that satisfies the irrotational condition a priori [24].

3.4. Admissible rigid body modes in two-dimensional irrotational flow

The Raviart–Thomas two-dimensional irrotational flow element is touted as an innovative approach to model irrotational flow. In connection with the present fluid displacement approach, we offer the following observation.

The shape functions employed in [24] are given by

$$\begin{aligned} u &= a + cx, \\ v &= b + cy. \end{aligned} \tag{27}$$

It can be easily shown that this interpolation automatically satisfies the irrotational constraint

$$\mathbf{curl}\,(\mathbf{u}) = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{28}$$

The bulk strain given by

$$\epsilon = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = c \tag{29}$$

indicates that the pressure will be constant in each element.

The fluid displacement (or velocity) given by (27) possesses two zero-energy modes, vertical (y-direction) and horizontal (x-direction) motions. However, when the gravity effect is absent as is the case in internal acoustics, the fluid displacement must possess the following third zero-energy mode shown in Fig. 2.

Hence, we conclude that, when one employs the Raviart–Thomas element to model two-dimensional irrotational acoustics problems, a fix must be introduced to accommodate the filling mode, by either pairing two or more elements such that the superelement possesses the filling mode or some other ways.

4. Localized partitioning of fluid–structure interaction equations

A key aspect in the localized treatment of the partitioned-boundary condition as discussed in Sections 2.2 and 3.1 is to relate the partitioned-boundary displacements to the assembled displacement as formulated in (11) and (17). When the partition boundary constitutes the fluid–structure interface, this can be expressed as

$$\begin{Bmatrix} \mathbf{u}_{fb} \\ \mathbf{u}_{sb} \end{Bmatrix} - \mathbf{L}_b\mathbf{u}_b = 0 \;\Rightarrow\; \mathbf{B}_\ell^T\mathbf{u} - \mathbf{L}_b\mathbf{u}_b = 0, \quad \mathbf{u} = \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \end{Bmatrix}, \; \mathbf{L}_b = \mathbf{B}_\ell^T\mathbf{L} = \begin{bmatrix} \mathbf{L}_{bs} \\ \mathbf{L}_{bf} \end{bmatrix}, \; \mathbf{B}_\ell = \begin{bmatrix} \mathbf{B}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f \end{bmatrix}. \tag{30}$$

Hence, if the fluid domain is partitioned from the structural domain as shown in Fig. 3, the corresponding energy functional becomes

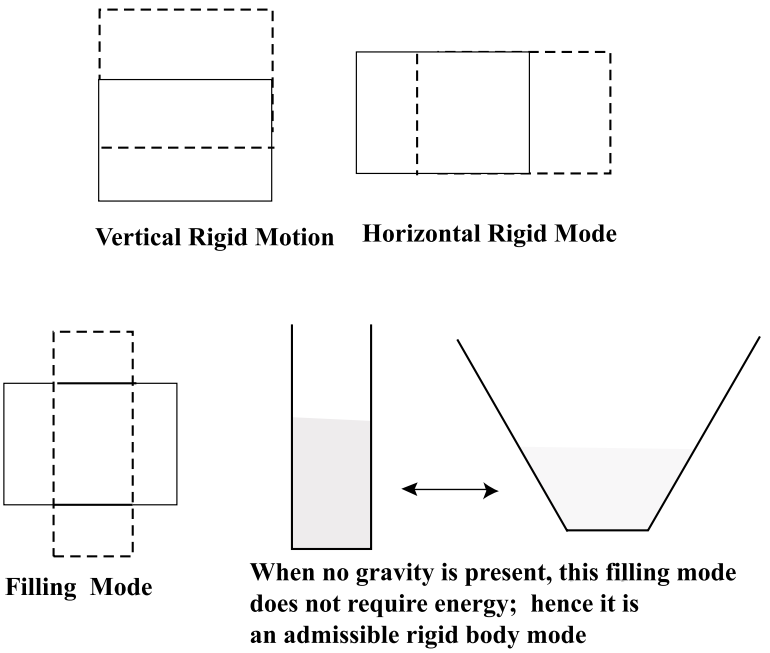

Fig. 2. Three admissible rigid body modes.

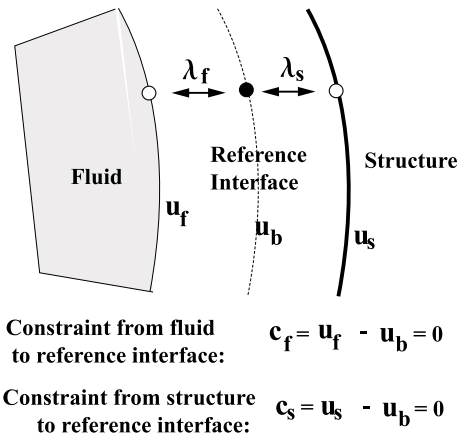

Fig. 3. Fluid–structure interface descriptions.

$$\Pi(\mathbf{u},\boldsymbol{\lambda}_\ell,\mathbf{u}_\mathrm{b},\boldsymbol{\mu}) = \mathbf{u}^\mathrm{T}\left(\frac{1}{2}\mathbf{K}\mathbf{u}+\mathbf{M}\ddot{\mathbf{u}}-\mathbf{F}\right)+\boldsymbol{\mu}^\mathrm{T}\mathscr{C}_\mathrm{f}^\mathrm{T}\mathbf{u}_\mathrm{f}+\boldsymbol{\lambda}_\ell^\mathrm{T}\mathscr{S}^\mathrm{T}(\mathbf{B}^\mathrm{T}\mathbf{u}-\mathbf{L}_\mathrm{b}\mathbf{u}_\mathrm{g}),\quad \mathscr{S}=\begin{bmatrix}\mathscr{S}_\mathrm{s} & 0\\ 0 & \mathscr{S}_\mathrm{f}\end{bmatrix}, \tag{31}$$

where $\mathscr{S}$ is the rigidity regularization operator discussed in (17)–(21).

The variation of $\Pi(\mathbf{u},\boldsymbol{\lambda}_\ell,\mathbf{u}_\mathrm{b},\boldsymbol{\mu})$ in terms of the six partitioned variables leads to

$$\begin{aligned}\delta\Pi(\mathbf{u}_\mathrm{s},\mathbf{u}_\mathrm{f},\boldsymbol{\lambda}_\mathrm{s},\boldsymbol{\lambda}_\mathrm{f},\mathbf{u}_\mathrm{b},\boldsymbol{\mu}) &= \delta\mathbf{u}_\mathrm{s}^\mathrm{T}(\mathbf{K}_\mathrm{s}\mathbf{u}_\mathrm{s}+\mathbf{M}_\mathrm{s}\ddot{\mathbf{u}}_\mathrm{s}-\mathbf{f}_\mathrm{s}+\mathscr{B}_\mathrm{s}\boldsymbol{\lambda}_\mathrm{s})+\delta\boldsymbol{\mu}^\mathrm{T}\mathscr{C}_\mathrm{f}^\mathrm{T}\mathbf{u}_\mathrm{f}\\ &\quad+\delta\mathbf{u}_\mathrm{f}^\mathrm{T}(\mathbf{K}_\mathrm{f}\mathbf{u}_\mathrm{f}+\mathbf{M}_\mathrm{f}\ddot{\mathbf{u}}_\mathrm{f}-\mathbf{f}_\mathrm{f}+\mathscr{C}_\mathrm{f}\boldsymbol{\mu}+\mathscr{B}_\mathrm{f}\boldsymbol{\lambda}_\mathrm{f})\\ &\quad+\delta\boldsymbol{\lambda}_\ell^\mathrm{T}\left(\begin{bmatrix}\mathscr{B}_\mathrm{s}^\mathrm{T} & \mathbf{0}\\ \mathbf{0} & \mathscr{B}_\mathrm{f}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\mathbf{u}_\mathrm{s}\\ \mathbf{u}_\mathrm{f}\end{Bmatrix}-\begin{bmatrix}\mathscr{L}_\mathrm{bs}\\ \mathscr{L}_\mathrm{bf}\end{bmatrix}\mathbf{u}_\mathrm{b}\right)-\delta\mathbf{u}_\mathrm{b}^\mathrm{T}\begin{bmatrix}\mathscr{L}_\mathrm{bs}^\mathrm{T} & \mathscr{L}_\mathrm{bf}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\lambda}_\mathrm{s}\\ \boldsymbol{\lambda}_\mathrm{f}\end{Bmatrix},\end{aligned} \tag{32}$$

$$\boldsymbol{\lambda}_\ell=\begin{Bmatrix}\boldsymbol{\lambda}_\mathrm{s}\\ \boldsymbol{\lambda}_\mathrm{f}\end{Bmatrix},\quad \begin{Bmatrix}\mathscr{B}_\mathrm{s}\\ \mathscr{B}_\mathrm{f}\end{Bmatrix}=\begin{Bmatrix}\mathbf{B}_\mathrm{s}\mathscr{S}_\mathrm{s}\\ \mathbf{B}_\mathrm{f}\mathscr{S}_\mathrm{f}\end{Bmatrix},\quad \begin{Bmatrix}\mathscr{L}_\mathrm{bs}\\ \mathscr{L}_\mathrm{bf}\end{Bmatrix}=\begin{Bmatrix}\mathscr{S}_\mathrm{s}^\mathrm{T}\mathbf{L}_\mathrm{bs}\\ \mathscr{S}_\mathrm{f}^\mathrm{T}\mathbf{L}_\mathrm{bf}\end{Bmatrix}.$$

It should be emphasized that the Lagrange multipliers $\boldsymbol{\lambda}_{\mathrm{f}}$ at the fluid interface nodes and those at the structure interface nodes $\boldsymbol{\lambda}_{\mathrm{s}}$ are independently defined. Setting the variational expression $\delta\Pi(\mathbf{u}_{\mathrm{s}}, \mathbf{u}_{\mathrm{f}}, \boldsymbol{\lambda}_{\mathrm{s}}, \boldsymbol{\lambda}_{\mathrm{f}}, \mathbf{u}_{\mathrm{b}}, \boldsymbol{\mu}) = 0$ we obtain the following partitioned equations of motion:

$$\begin{matrix} \text{Structural equilibrium :} \\ \text{Fluid equilibrium :} \\ \text{Structural interface constraint :} \\ \text{Fluid interface constraint :} \\ \text{Fluid irrotational condition :} \\ \text{Newton's third law at interfaces :} \end{matrix} \begin{bmatrix} \mathbf{K}_{\mathrm{s}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{M}_{\mathrm{s}} & 0 & \mathscr{B}_{\mathrm{s}} & 0 & 0 & 0 \\ 0 & \mathbf{K}_{\mathrm{f}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{M}_{\mathrm{f}} & 0 & \mathscr{B}_{\mathrm{f}} & \mathscr{C}_{\mathrm{f}} & 0 \\ \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bf}} \\ 0 & \mathscr{C}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & -\mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{f}} \\ \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \boldsymbol{\mu} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathbf{f}_{\mathrm{s}} \\ \mathbf{f}_{\mathrm{f}} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{33}$$

The above partitioned equation for compressible internal fluid–structure interaction problems is perhaps most general in that several special formulations can be derived from it. Three specializations will be subsequently discussed.

5. Partitioned transient analysis of compressible internal fluid–structure interactions

Time discretization of the partitioned compressible internal fluid–structure interaction Eq. (33) can be carried out by integrating the structural and fluid displacement equations. To this end, we employ the midpoint integration rule given by

$$\mathbf{u}^{n+1/2} = \mathbf{u}^{n} + \frac{1}{2}\Delta t\dot{\mathbf{u}}^{n} + \left(\frac{1}{2}\Delta t\right)^{2}\ddot{\mathbf{u}}^{n+1/2}, \qquad \dot{\mathbf{u}}^{n+1/2} = \dot{\mathbf{u}}^{n} + \frac{1}{2}\Delta t\ddot{\mathbf{u}}^{n+1/2}, \qquad \mathbf{u}^{n+1} = 2\mathbf{u}^{n+1/2} - \mathbf{u}^{n}, \tag{34}$$

where Δt is the time step size.

Integrating the left-hand side of (33) by (34) yields

$$\begin{bmatrix} \mathbf{K}_{\mathrm{s}} + \beta^2\mathbf{M}_{\mathrm{s}} & 0 & \mathscr{B}_{\mathrm{s}} & 0 & 0 & 0 \\ 0 & \mathbf{K}_{\mathrm{f}} + \beta^2\mathbf{M}_{\mathrm{f}} & 0 & \mathscr{B}_{\mathrm{f}} & \mathscr{C}_{\mathrm{f}} & 0 \\ \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bf}} \\ 0 & \mathscr{C}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & -\mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{f}} \\ \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \boldsymbol{\mu} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathbf{g}_{\mathrm{s}} \\ \mathbf{g}_{\mathrm{f}} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{n+1/2}, \tag{35}$$

$$\mathbf{g}_{\mathrm{s}}^{n+1/2} = \mathbf{f}_{\mathrm{s}}^{n+1/2} + \mathbf{M}_{\mathrm{s}}(\beta^2\mathbf{u}_{\mathrm{s}}^{n} + \beta\dot{\mathbf{u}}_{\mathrm{s}}^{n}), \quad \beta = \frac{1}{(1/2)\Delta t}, \quad \mathbf{g}_{\mathrm{f}}^{n+1/2} = \mathbf{f}_{\mathrm{f}}^{n+1/2} + \mathbf{M}_{\mathrm{f}}(\beta^2\mathbf{u}_{\mathrm{f}}^{n} + \beta\dot{\mathbf{u}}_{\mathrm{f}}^{n}).$$

For an efficient solution of the above time-discretized equation, we first solve for $\mathbf{u}_{\mathrm{s}}^{n+1/2}$ and $\mathbf{u}_{\mathrm{f}}^{n+1/2}$ to obtain

$$\begin{aligned} \mathbf{u}_{\mathrm{s}}^{n+1/2} &= \mathbf{F}_{\mathrm{s}}(\mathbf{g}_{\mathrm{s}}^{n+1/2} - \mathscr{B}_{\mathrm{s}}\boldsymbol{\lambda}_{\mathrm{s}}^{n+1/2}), \quad \mathbf{F}_{\mathrm{s}} = (\mathbf{K}_{\mathrm{s}} + \beta^2\mathbf{M}_{\mathrm{s}})^{-1}, \\ \mathbf{u}_{\mathrm{f}}^{n+1/2} &= \mathbf{F}_{\mathrm{f}}(\mathbf{g}_{\mathrm{f}}^{n+1/2} - \mathscr{B}_{\mathrm{f}}\boldsymbol{\lambda}_{\mathrm{f}}^{n+1/2} - \mathscr{C}_{\mathrm{f}}\boldsymbol{\mu}_{\mathrm{f}}^{n+1/2}), \quad \mathbf{F}_{\mathrm{f}} = (\mathbf{K}_{\mathrm{f}} + \beta^2\mathbf{M}_{\mathrm{f}})^{-1}. \end{aligned} \tag{36}$$

Substituting these into the remainder of (35), we obtain the following equation in terms of the interface variables $(\boldsymbol{\lambda}_{\mathrm{f}}, \boldsymbol{\lambda}_{\mathrm{s}}, \mathbf{u}_{\mathrm{b}})$ and the irrotationality multipliers $\boldsymbol{\mu}$,

$$\begin{bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathscr{B}_{\mathrm{s}} & 0 & 0 & \mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{B}_{\mathrm{f}} & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{C}_{\mathrm{f}} & \mathscr{L}_{\mathrm{bf}} \\ 0 & \mathscr{C}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{B}_{\mathrm{f}} & \mathscr{C}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{C}_{\mathrm{f}} & 0 \\ \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \boldsymbol{\mu} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathbf{g}_{\mathrm{s}} \\ \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathbf{g}_{\mathrm{f}} \\ \mathscr{C}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathbf{g}_{\mathrm{f}} \\ 0 \end{Bmatrix}^{n+1/2}. \tag{37}$$

Remark 5.1.

- If the shape functions for the fluid displacement satisfies the irrotational constraint a priori (see, for example, [24]), the preceding equation becomes

$$\begin{bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathscr{B}_{\mathrm{s}} & 0 & \mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{B}_{\mathrm{f}} & \mathscr{L}_{\mathrm{bf}} \\ \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & \mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathbf{g}_{\mathrm{s}} \\ \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathbf{g}_{\mathrm{f}} \\ 0 \end{Bmatrix}^{n+1/2}. \tag{38}$$

  Thus, the independently assigned localized Lagrange multipliers give rise to uncoupled system dynamic flexibility matrices for the fluid and structure. This leads to a modular implementation for the transient analysis of compressible internal fluid–structure interaction problems. An efficient parallel solution procedure for solving the preceding equation is presented in [21].
- When the fluid displacement shape functions do not satisfy the irrotational constraint, one can express (37) as

$$\begin{bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathscr{B}_{\mathrm{s}} & 0 & \mathscr{L}_{\mathrm{bs}} \\ 0 & \bar{\mathscr{B}}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\bar{\mathscr{B}}_{\mathrm{f}} & \bar{\mathscr{L}}_{\mathrm{bf}} \\ \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & \bar{\mathscr{L}}_{\mathrm{bf}}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{s}} \\ \bar{\boldsymbol{\lambda}}_{\mathrm{f}} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathbf{g}_{\mathrm{s}} \\ \bar{\mathscr{B}}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathbf{g}_{\mathrm{f}} \\ 0 \end{Bmatrix}^{n+1/2}, \tag{39}$$

$$\bar{\boldsymbol{\lambda}}_{\mathrm{f}} = \begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{f}} \\ \boldsymbol{\mu} \end{Bmatrix}, \quad \bar{\mathscr{B}}_{\mathrm{f}} = \begin{bmatrix} \mathscr{B}_{\mathrm{f}} \\ \mathscr{C}_{\mathrm{f}} \end{bmatrix}, \quad \bar{\mathscr{L}}_{\mathrm{bf}} = \begin{bmatrix} \mathscr{L}_{\mathrm{bf}} \\ 0 \end{bmatrix}.$$

  Clearly, even when the irrotationality constraint is imposed, it does not couple with the structural flexibility matrix, including the fluid–structure boundary structural attributes.

6. Vibration analysis

A vibration analysis can be effected by replacing the time-differentiation operator $(\mathrm{d}^2/\mathrm{d}t^2)$ in (33) by $(-\omega^2)$:

$$\begin{bmatrix} \mathbf{K}_{\mathrm{s}} - \omega^2\mathbf{M}_{\mathrm{s}} & 0 & \mathscr{B}_{\mathrm{s}} & 0 & 0 & 0 \\ 0 & \mathbf{K}_{\mathrm{f}} - \omega^2\mathbf{M}_{\mathrm{f}} & 0 & \mathscr{B}_{\mathrm{f}} & \mathscr{C}_{\mathrm{f}} & 0 \\ \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bf}} \\ 0 & \mathscr{C}_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & -\mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{f}} \\ \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \boldsymbol{\mu} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{\mathrm{s}} \\ \mathbf{f}_{\mathrm{f}} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{40}$$

Efficient solutions of the preceding partitioned eigenanalysis on both sequential and parallel computers presently constitute a challenging computational mechanics research task.

7. $(\mathbf{p}_{\mathrm{f}}, \boldsymbol{\alpha}_{\mathrm{f}}, \mathbf{u}_{\mathrm{s}}, \boldsymbol{\lambda}_{\ell})$ formulation

In practice, some of the widely used fluid–structure formulations (e.g., [8,11,13–15,26]) involve pressure $(\mathbf{p}_{\mathrm{f}})$ as the primary variable for the fluid domain. This section describes a novel transformation procedure

for obtaining the governing equations for the fluid in terms of the fluid pressure from the basic displacement formulation (31). The motivations for such a formulation are reduction of the size of the fluid variable and an automatic satisfaction of the irrotationality condition.

The starting point for deriving such a formulation is that the discrete pressure in an element (e) can be obtained in terms of the elemental discrete fluid displacement $\mathbf{u}_{\mathrm{f}}^{(e)}$ as

$$p = -\rho_{\mathrm{f}} c^2 \nabla \mathbf{u}_{\mathrm{f}} \quad \Rightarrow \quad \mathbf{p}^{(e)} = \mathscr{D}_{\mathrm{f}}^{(e)} \mathbf{u}_{\mathrm{f}}^{(e)}. \tag{41}$$

If this equation is augmented with the irrotational constraints for each fluid element, the resulting equation becomes

$$\left\{ \begin{array}{c} \mathbf{p}^{(e)} \\ \mathbf{0} \end{array} \right\} = \left[\begin{array}{c} \mathscr{D}_{\mathrm{f}}^{(e)} \\ \mathscr{C}_{\mathrm{f}}^{(e)} \end{array} \right] \mathbf{u}_{\mathrm{f}}^{(e)}. \tag{42}$$

Second, note that the elemental fluid displacement $\mathbf{u}_{\mathrm{f}}^{(e)}$ can be decomposed into two parts, that is, deformation $\mathbf{d}_{\mathrm{f}}^{(e)}$ and zero-energy motion $\mathbf{r}_{\mathrm{f}}^{(e)}$ (see, e.g., [18,19]),

$$\mathbf{u}_{\mathrm{f}}^{(e)} = \mathbf{d}_{\mathrm{f}}^{(e)} + \mathbf{r}_{\mathrm{f}}^{(e)}, \tag{43}$$

where the zero-energy part can be represented by

$$\mathbf{r}_{\mathrm{f}}^{(e)} = \mathbf{R}_{\mathrm{f}}^{(e)} \boldsymbol{\alpha}_{\mathrm{f}}^{(e)}, \tag{44}$$

where $\mathbf{R}_{\mathrm{f}}^{(e)}$ is the elemental zero-energy modes that depend only on the geometry of the element under consideration, and $\boldsymbol{\alpha}^{(e)}$ is the associated generalized coordinates.

Third, the deformation vector $\mathbf{d}_{\mathrm{f}}^{(e)}$ can be expressed in terms of the pressure. To this end, we substitute (43) into the displacement-to-pressure relation (41)

$$\mathbf{p}^{(e)} = \mathscr{D}_{\mathrm{f}}^{(e)} (\mathbf{d}_{\mathrm{f}}^{(e)} + \mathbf{r}_{\mathrm{f}}^{(e)}) = \mathscr{D}_{\mathrm{f}}^{(e)} (\mathbf{d}_{\mathrm{f}}^{(e)} + \mathbf{R}_{\mathrm{f}}^{(e)} \boldsymbol{\alpha}_{\mathrm{f}}^{(e)}) = \mathscr{D}_{\mathrm{f}}^{(e)} \mathbf{d}_{\mathrm{f}}^{(e)} \tag{45}$$

since the zero-energy modes $\mathbf{R}_{\mathrm{f}}^{(e)}$ do not incur any pressure change.

Similarly, the irrotational constraint given by the second row of (42) reduces to

$$\mathscr{C}_{\mathrm{f}}^{(e)} \mathbf{d}_{\mathrm{f}}^{(e)} = 0. \tag{46}$$

Therefore, the elemental fluid deformation $\mathbf{d}_{\mathrm{f}}^{(e)}$ can be obtained from (45) and (46) as

$$\mathbf{d}_{\mathrm{f}}^{(e)} = \mathbf{D}_p^{(e)} \mathbf{p}^{(e)}, \quad \mathbf{D}_p^{(e)} = \bar{\mathscr{D}}_{\mathrm{f}}^{(e)} \left(\bar{\mathscr{D}}_{\mathrm{f}}^{(e)^{\mathrm{T}}} \bar{\mathscr{D}}_{\mathrm{f}}^{(e)} \right)^{-1}, \quad \bar{\mathscr{D}}_{\mathrm{f}}^{(e)} = \left\{ \begin{array}{c} \mathscr{D}_{\mathrm{f}}^{(e)} \\ \mathscr{C}_{\mathrm{f}}^{(e)} \end{array} \right\}. \tag{47}$$

Substituting (47) and (44) into (43), we obtain the following *pressure-to-displacement* relation:

$$\mathbf{u}_{\mathrm{f}}^{(e)} = \mathbf{D}_p^{(e)} \mathbf{p}^{(e)} + \mathbf{R}_{\mathrm{f}}^{(e)} \boldsymbol{\alpha}_{\mathrm{f}}^{(e)}. \tag{48}$$

Obviously a desirable discretization is one that engenders no spurious mode. Hence, if there is no spurious mode in the discretization, the size of $\boldsymbol{\alpha}_{\mathrm{f}}^{(e)}$ is at most six. Note also that the size of the pressure vector is substantially smaller than that of the fluid displacement vector.

To obtain the total fluid displacement $\mathbf{u}_{\mathrm{f}}$ from the above equation, we invoke the assembly relation

$$\begin{aligned} &\mathbf{u}_{\mathrm{f}}^{(e)} = \mathbf{L}^{(e)} \mathbf{u}_{\mathrm{f}}, \\ &\quad \Downarrow \\ &\mathbf{u}_{\mathrm{f}} = \mathscr{L}_{\mathrm{f}} \mathbf{u}_{\mathrm{f}}^{(e)}, \quad \mathscr{L}_{\mathrm{f}} = \left(\mathbf{L}^{(e)^{\mathrm{T}}} \mathbf{L}^{(e)} \right)^{-1} \mathbf{L}^{(e)^{\mathrm{T}}} \end{aligned} \tag{49}$$

so that, by substituting (48) into the above equation, the fluid displacement $\mathbf{u}_\mathrm{f}$ becomes

$$\mathbf{u}_\mathrm{f} = \mathbf{D}_p\mathbf{p} + \mathscr{L}_\mathrm{f}\mathbf{R}_\mathrm{f}^{(e)}\boldsymbol{\alpha}_\mathrm{f}^{(e)}, \quad \mathbf{D}_p = \mathscr{L}_\mathrm{f}\mathbf{D}_p^{(e)}. \tag{50}$$

If $\mathbf{R}_\mathrm{f}$ and $\boldsymbol{\alpha}_\mathrm{f}$ are the rigid-body modes and their generalized coordinates for the assembled fluid partition, we have the following identity,

$$\mathscr{L}_\mathrm{f}\mathbf{R}_\mathrm{f}^{(e)}\boldsymbol{\alpha}_\mathrm{f}^{(e)} = \mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f}, \tag{51}$$

since the sum of the elemental rigid-body displacements must be equal to the assembled rigid-body displacements. Substituting the above relation into (50) we obtain

$$\mathbf{u}_\mathrm{f} = \mathbf{D}_p\mathbf{p} + \mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f}. \tag{52}$$

Therefore, the total degrees of freedom for the fluid domain in terms of the pressure are reduced to $(1/3)(n^3+6)$ from n^3 for an $(n\times n\times n)$ cube.

When (52) is substituted into (32), the variational functional becomes

$$\begin{aligned}\delta\Pi(\mathbf{u}_\mathrm{s},\mathbf{p},\boldsymbol{\alpha}_\mathrm{f},\boldsymbol{\lambda}_\mathrm{s},\boldsymbol{\lambda}_\mathrm{f},\mathbf{u}_\mathrm{b},\boldsymbol{\mu}) = {}& \delta\mathbf{u}_\mathrm{s}^\mathrm{T}(\mathbf{K}_\mathrm{s}\mathbf{u}_\mathrm{s}+\mathbf{M}_\mathrm{s}\ddot{\mathbf{u}}_\mathrm{s}-\mathbf{f}_\mathrm{s}+\mathscr{B}_\mathrm{s}\boldsymbol{\lambda}_\mathrm{s}) + \delta\boldsymbol{\mu}^\mathrm{T}\mathscr{C}_\mathrm{f}^\mathrm{T}(\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})\\ &+\delta(\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})^\mathrm{T}[\mathbf{K}_\mathrm{f}(\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})+\mathbf{M}_\mathrm{f}(\mathbf{D}_p\ddot{\mathbf{p}}+\mathbf{R}_\mathrm{f}\ddot{\boldsymbol{\alpha}}_\mathrm{f})-\mathbf{f}_\mathrm{f}+\mathscr{C}_\mathrm{f}\boldsymbol{\mu}+\mathscr{B}_\mathrm{f}\boldsymbol{\lambda}_\mathrm{f}]\\ &+\delta\boldsymbol{\lambda}_\ell^\mathrm{T}\left(\begin{bmatrix}\mathscr{B}_\mathrm{s}^\mathrm{T} & \mathbf{0}\\ \mathbf{0} & \mathscr{B}_\mathrm{f}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\mathbf{u}_\mathrm{s}\\ (\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})\end{Bmatrix}-\begin{bmatrix}\mathscr{L}_\mathrm{bs}\\ \mathscr{L}_\mathrm{bf}\end{bmatrix}\mathbf{u}_\mathrm{b}\right)\\ &-\delta\mathbf{u}_\mathrm{b}^\mathrm{T}\begin{bmatrix}\mathscr{L}_\mathrm{bs}^\mathrm{T} & \mathscr{L}_\mathrm{bf}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\lambda}_\mathrm{s}\\ \boldsymbol{\lambda}_\mathrm{f}\end{Bmatrix}.\end{aligned} \tag{53}$$

By making use of the relations

$$\begin{aligned}&\mathscr{C}_\mathrm{f}^\mathrm{T}\mathbf{D}_p = 0 \quad \Leftarrow \quad \mathrm{div}\cdot(\mathbf{curl}\,\mathbf{u}_\mathrm{f}) = 0,\\ &\mathscr{C}_\mathrm{f}^\mathrm{T}\mathbf{R}_\mathrm{f} = 0 \text{ since the irrotational modes are orthogonal to rigid modes,}\\ &\mathbf{K}_\mathrm{f}\mathbf{R}_\mathrm{f} = 0 \text{ as } \mathbf{R}_\mathrm{f} \text{ is a null space of } \mathbf{K}_\mathrm{f},\end{aligned} \tag{54}$$

the foregoing variational functional (53) reduces to

$$\begin{aligned}\delta\Pi(\mathbf{u}_\mathrm{s},\mathbf{p},\boldsymbol{\alpha}_\mathrm{f},\boldsymbol{\lambda}_\mathrm{s},\boldsymbol{\lambda}_\mathrm{f},\mathbf{u}_\mathrm{b},\boldsymbol{\mu}) = {}& \delta\mathbf{u}_\mathrm{s}^\mathrm{T}(\mathbf{K}_\mathrm{s}\mathbf{u}_\mathrm{s}+\mathbf{M}_\mathrm{s}\ddot{\mathbf{u}}_\mathrm{s}-\mathbf{f}_\mathrm{s}+\mathscr{B}_\mathrm{s}\boldsymbol{\lambda}_\mathrm{s})\\ &+\delta(\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})^\mathrm{T}[\mathbf{K}_\mathrm{f}(\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})+\mathbf{M}_\mathrm{f}(\mathbf{D}_p\ddot{\mathbf{p}}+\mathbf{R}_\mathrm{f}\ddot{\boldsymbol{\alpha}}_\mathrm{f})-\mathbf{f}_\mathrm{f}+\mathscr{B}_\mathrm{f}\boldsymbol{\lambda}_\mathrm{f}]\\ &+\delta\boldsymbol{\lambda}_\ell^\mathrm{T}\left(\begin{bmatrix}\mathscr{B}_\mathrm{s}^\mathrm{T} & \mathbf{0}\\ \mathbf{0} & \mathscr{B}_\mathrm{f}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\mathbf{u}_\mathrm{s}\\ (\mathbf{D}_p\mathbf{p}+\mathbf{R}_\mathrm{f}\boldsymbol{\alpha}_\mathrm{f})\end{Bmatrix}-\begin{bmatrix}\mathscr{L}_\mathrm{bs}\\ \mathscr{L}_\mathrm{bf}\end{bmatrix}\mathbf{u}_\mathrm{b}\right)\\ &-\delta\mathbf{u}_\mathrm{b}^\mathrm{T}\begin{bmatrix}\mathscr{L}_\mathrm{bs}^\mathrm{T} & \mathscr{L}_\mathrm{bf}^\mathrm{T}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\lambda}_\mathrm{s}\\ \boldsymbol{\lambda}_\mathrm{f}\end{Bmatrix}.\end{aligned} \tag{55}$$

The stationarity of (55) yields the following partitioned system equation:

$$\begin{bmatrix}\mathbf{K}_\mathrm{s}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{M}_\mathrm{s} & 0 & 0 & \mathscr{B}_\mathrm{s} & 0 & 0\\ 0 & \mathbf{K}_p+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{M}_p & 0 & 0 & \mathbf{D}_p^\mathrm{T}\mathscr{B}_\mathrm{f} & 0\\ 0 & 0 & \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{M}_{\alpha\alpha} & 0 & \mathbf{R}_\mathrm{f}^\mathrm{T}\mathscr{B}_\mathrm{f} & 0\\ \mathscr{B}_\mathrm{s}^\mathrm{T} & 0 & 0 & 0 & 0 & -\mathscr{L}_\mathrm{bs}\\ 0 & \mathscr{B}_\mathrm{f}^\mathrm{T}\mathbf{D}_p & \mathscr{B}_\mathrm{f}^\mathrm{T}\mathbf{R}_\mathrm{f} & 0 & 0 & -\mathscr{L}_\mathrm{bf}\\ 0 & 0 & -\mathscr{L}_\mathrm{bs}^\mathrm{T} & -\mathscr{L}_\mathrm{bf}^\mathrm{T} & 0 & 0\end{bmatrix}\begin{Bmatrix}\mathbf{u}_\mathrm{s}\\ \mathbf{p}\\ \boldsymbol{\alpha}_\mathrm{f}\\ \boldsymbol{\lambda}_\mathrm{s}\\ \boldsymbol{\lambda}_\mathrm{f}\\ \mathbf{u}_\mathrm{b}\end{Bmatrix} = \begin{Bmatrix}\mathbf{f}_\mathrm{s}\\ \mathbf{D}_p^\mathrm{T}\mathbf{f}_\mathrm{f}\\ \mathscr{R}_\mathrm{f}^\mathrm{T}\mathbf{f}_\mathrm{f}\\ 0\\ 0\\ 0\end{Bmatrix}, \tag{56}$$

$$\mathbf{K}_p = \mathbf{D}_p^\mathrm{T}\mathbf{K}_\mathrm{f}\mathbf{D}_p, \quad \mathbf{M}_p = \mathbf{D}_p^\mathrm{T}\mathbf{M}_\mathrm{f}\mathbf{D}_p, \quad \mathbf{M}_{\alpha\alpha} = \mathbf{R}_\mathrm{f}^\mathrm{T}\mathbf{M}_\mathrm{f}\mathbf{R}_\mathrm{f},$$

where we have utilized the relation $\mathbf{M}_{p\alpha} = \mathbf{D}_p^\mathrm{T}\mathbf{M}_\mathrm{f}\mathbf{R}_\mathrm{f} = 0$ since the pressure modes are orthogonal to the rigid-body modes.

The preceding partitioned equation, while equivalent to some of the existing fluid pressure and structural displacement formulations [6,11], possesses one distinctly additional variable, that is, the fluid rigid-body variable $\boldsymbol{\alpha}_{\mathrm{f}}$. When the fluid container is grounded, of course, there will be no fluid rigid-body motions, so that $\mathbf{R}_{\mathrm{f}} = 0$. However, when the container is in a free–free state as in a rocket in motion, the present formulation (56) will automatically account for the necessary rigid-body motion.

We will now present two applications of this formulation: vibration analysis and transient analysis.

8. Vibration analysis by $(\mathbf{u}_{\mathrm{s}}, \mathbf{p}_{\mathrm{f}}, \boldsymbol{\lambda}_{\mathrm{s}}, \mathbf{u}_{\mathrm{b}})$ formulation

Of several possible further eliminations of the system variables for vibration analysis, we present the case of eliminating $\boldsymbol{\alpha}_{\mathrm{f}}$ and $\boldsymbol{\lambda}_{\mathrm{f}}$ below. First, from the third row of (56) we obtain

$$\omega^2 \boldsymbol{\alpha}_{\mathrm{f}} = -\mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} (\mathbf{f}_{\mathrm{f}} - \mathscr{B}_{\mathrm{f}} \boldsymbol{\lambda}_{\mathrm{f}}). \tag{57}$$

Substituting this into the fifth equation of (56), we obtain

$$\begin{aligned} &\omega^2 \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{D}_p \mathbf{p}_{\mathrm{f}} - \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{R}_{\mathrm{f}} \mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} (\mathbf{f}_{\mathrm{f}} - \mathscr{B}_{\mathrm{f}} \boldsymbol{\lambda}_{\mathrm{f}}) - \omega^2 \mathscr{L}_{\mathrm{bf}} \mathbf{u}_{\mathrm{b}} = 0, \\ &\qquad \Downarrow \\ &\boldsymbol{\lambda}_{\mathrm{f}} = \mathbf{M}_{\mathrm{b}} [\mathbf{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{R}_{\mathrm{f}} \mathbf{M}_{\alpha}^{-1} \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} \mathbf{f}_{\mathrm{f}} - \omega^2 \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{D}_p \mathbf{p}_{\mathrm{f}} + \omega^2 \mathscr{L}_{\mathrm{bf}} \mathbf{u}_{\mathrm{b}}], \\ &\mathbf{M}_{\mathrm{b}} = [\mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{R}_{\mathrm{f}} \mathbf{M}_{\alpha\alpha}^{-1} \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} \mathscr{B}_{\mathrm{f}}]^{-1}. \end{aligned} \tag{58}$$

Eq. (56) can now be reduced to read, with $\mathbf{f}_{\mathrm{f}} = 0$ for simplicity, as

$$\begin{bmatrix} \mathbf{K}_{\mathrm{s}} - \omega^2 \mathbf{M}_{\mathrm{s}} & 0 & \mathscr{B}_{\mathrm{s}} & 0 \\ 0 & \mathbf{K}_p - \omega^2 \hat{\mathbf{M}}_p & 0 & \omega^2 \hat{\mathscr{L}}_{\mathrm{bf}} \\ \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} & 0 & 0 & -\mathscr{L}_{\mathrm{bs}} \\ 0 & \omega^2 \hat{\mathscr{L}}_{\mathrm{bf}}^{\mathrm{T}} & -\mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & -\omega^2 \mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} \mathbf{M}_{\mathrm{b}} \mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{p}_{\mathrm{f}} \\ \boldsymbol{\lambda}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{\mathrm{s}} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{59}$$

$$\hat{\mathbf{M}}_p = \mathbf{M}_p - \mathbf{D}_p^{\mathrm{T}} \mathscr{B}_{\mathrm{f}} \mathbf{M}_{\mathrm{b}} \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{D}_p, \quad \hat{\mathscr{L}}_{\mathrm{bf}} = \mathbf{D}_p^{\mathrm{T}} \mathscr{B}_{\mathrm{f}} \mathbf{M}_{\mathrm{b}} \mathscr{L}_{\mathrm{bf}}.$$

It should be noted that a further reduction is possible, for example, the elimination of the interface boundary displacement $\mathbf{u}_{\mathrm{b}}$. Further work along this line may offer new insight as well as new formulations.

9. Transient fluid–structure interaction analysis by three solver modules

We now present a main result of the present paper, viz., analysis of coupled internal acoustics and structural response by three analyzer modules: the structural analyzer, the fluid analyzer and the interface solver. This is illustrated in Fig. 4. To this end we first time-discretize (56) by the midpoint rule (39) to obtain

$$\begin{bmatrix} \mathbf{K}_{\mathrm{s}} + \beta^2 \mathbf{M}_{\mathrm{s}} & 0 & 0 & \mathscr{B}_{\mathrm{s}} & 0 & 0 \\ 0 & \mathbf{K}_p + \beta^2 \mathbf{M}_p & 0 & 0 & \mathbf{D}_p^{\mathrm{T}} \mathscr{B}_{\mathrm{f}} & 0 \\ 0 & 0 & \beta^2 \mathbf{M}_{\alpha\alpha} & 0 & \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} \mathscr{B}_{\mathrm{f}} & 0 \\ \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -\mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{D}_p & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}} \mathbf{R}_{\mathrm{f}} & 0 & 0 & -\mathscr{L}_{\mathrm{bf}} \\ 0 & 0 & -\mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & -\mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{p} \\ \boldsymbol{\alpha}_{\mathrm{f}} \\ \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathbf{g}_u \\ \mathbf{g}_p \\ \mathbf{g}_\alpha \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{n+1/2}, \tag{60}$$

where we have invoked the mass-orthogonality between the pressure and rigid-body modes and the right-hand vectors are given by

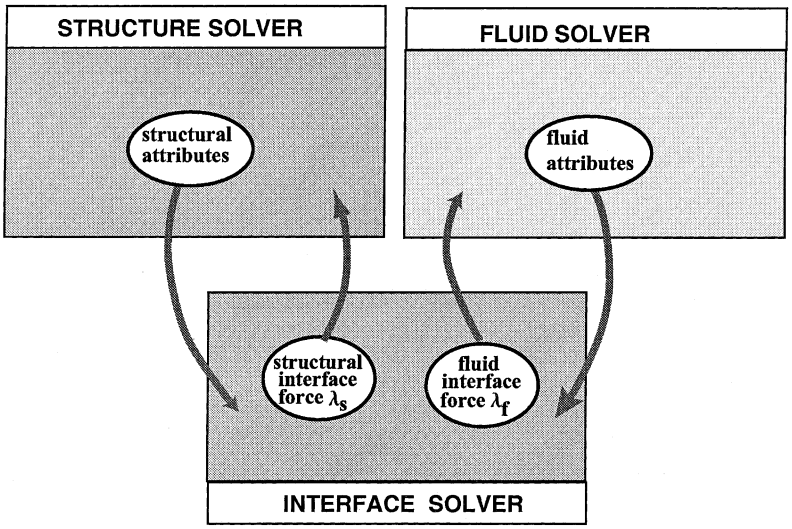

Fig. 4. Interaction analysis by three modular analyzers.

$$\begin{aligned}
\mathbf{g}_u^{n+1/2} &= \mathbf{f}_{\mathrm{s}}^{n+1/2} + \mathbf{M}_{\mathrm{s}}(\beta^2\mathbf{u}_{\mathrm{s}}^n + \beta\dot{\mathbf{u}}_{\mathrm{s}}^n), \quad \beta = \frac{1}{(1/2)\Delta t},\\
\mathbf{g}_p^{n+1/2} &= \mathbf{D}_p^{\mathrm{T}}\mathbf{f}_{\mathrm{f}}^{n+1/2} + \mathbf{M}_p(\beta^2\mathbf{p}_{\mathrm{f}}^n + \beta\dot{\mathbf{p}}_{\mathrm{f}}^n),\\
\mathbf{g}_\alpha^{n+1/2} &= \mathbf{R}_{\mathrm{f}}^{\mathrm{T}}\mathbf{f}_{\mathrm{f}}^{n+1/2} + \mathbf{M}_\alpha(\beta^2\boldsymbol{\alpha}_{\mathrm{f}}^n + \beta\dot{\boldsymbol{\alpha}}_{\mathrm{f}}^n).
\end{aligned} \tag{61}$$

If both the structural degrees of freedom n_{s} and the fluid pressure degrees of freedom n_p are substantially larger than the interface unknown size $n_{\mathrm{b}} = n_{\lambda_{\mathrm{s}}} = n_{\lambda_{\mathrm{f}}}$, which is usually the case in practice, it is beneficial to employ three separate analysis modules: the structural analyzer (the u_{s}-module), the fluid analyzer (the p-module) and the interface analyzer (the $\boldsymbol{\lambda}$-module). To this end, we solve for $\mathbf{u}_{\mathrm{s}}$ and $\langle \mathbf{p}_{\mathrm{f}}\ \boldsymbol{\alpha}_{\mathrm{f}} \rangle$ to obtain

$$\begin{aligned}
\mathbf{u}_{\mathrm{s}}^{n+1/2} &= \mathbf{F}_{\mathrm{s}}(\mathbf{g}_u - \mathscr{B}_{\mathrm{s}}\boldsymbol{\lambda}_{\mathrm{s}})^{n+1/2}, \quad \mathbf{F}_{\mathrm{s}} = (\mathbf{K}_{\mathrm{s}} + \beta^2\mathbf{M}_{\mathrm{s}})^{-1},\\
\mathbf{p}_{\mathrm{f}}^{n+1/2} &= \mathbf{F}_p(\mathbf{g}_p - \mathbf{D}_p^{\mathrm{T}}\mathscr{B}_{\mathrm{f}}\boldsymbol{\lambda}_{\mathrm{f}})^{n+1/2}, \quad \mathbf{F}_p = (\mathbf{K}_p + \beta^2\mathbf{M}_p)^{-1},\\
\boldsymbol{\alpha}_{\mathrm{f}}^{n+1/2} &= \mathbf{F}_\alpha(\mathbf{g}_\alpha - \mathbf{R}_{\mathrm{f}}^{\mathrm{T}}\mathscr{B}_{\mathrm{f}}\boldsymbol{\lambda}_{\mathrm{f}})^{n+1/2}, \quad \mathbf{F}_\alpha = (\beta^2\mathbf{M}_\alpha)^{-1}.
\end{aligned} \tag{62}$$

Substituting these into the fourth and fifth rows of (60) yields the desired interface equation

$$\begin{aligned}
&\begin{bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathscr{B}_{\mathrm{s}} & 0 & \mathscr{L}_{\mathrm{bs}} \\ 0 & \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}\mathbf{F}_{\mathrm{f}}\mathscr{B}_{\mathrm{f}} & \mathscr{L}_{\mathrm{bf}} \\ \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & \mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \\ \mathbf{u}_{\mathrm{b}} \end{Bmatrix}^{n+1/2} = \begin{Bmatrix} \mathbf{g}_{\lambda_{\mathrm{s}}} \\ \mathbf{g}_{\lambda_{\mathrm{f}}} \\ 0 \end{Bmatrix}^{n+1/2}, \quad \mathbf{F}_{\mathrm{f}} = [\,\mathbf{D}_p \quad \mathbf{R}_{\mathrm{f}}\,] \begin{bmatrix} \mathbf{F}_p & 0 \\ 0 & \mathbf{F}_\alpha \end{bmatrix} \begin{bmatrix} \mathbf{D}_p^{\mathrm{T}} \\ \mathbf{R}_{\mathrm{f}}^{\mathrm{T}} \end{bmatrix},\\
&\mathbf{g}_{\lambda_{\mathrm{s}}} = \mathscr{B}_{\mathrm{s}}^{\mathrm{T}}\mathbf{F}_{\mathrm{s}}\mathbf{g}_{\mathrm{s}}, \quad \mathbf{g}_{\lambda_{\mathrm{f}}} = \mathscr{B}_{\mathrm{f}}^{\mathrm{T}}[\,\mathbf{D}_p \quad \mathbf{R}_{\mathrm{f}}\,] \begin{bmatrix} \mathbf{F}_p & 0 \\ 0 & \mathbf{F}_\alpha \end{bmatrix} \begin{Bmatrix} \mathbf{g}_p \\ \mathbf{g}_\alpha \end{Bmatrix}.
\end{aligned} \tag{63}$$

It should be emphasized that the preceding interface Eq. (63) is the same as (40) that was derived by using an irrotationality-embedded displacement formulation. However, the origins of the two equations are distinctly different in that the coupled fluid equations are in pressure and rigid-body variables as given by (50). We now summarize the sequence of computations.

Step 1: Solve for $\langle \boldsymbol{\lambda}_{\mathrm{s}}\ \boldsymbol{\lambda}_{\mathrm{f}} \rangle$ by using (63).

Step 2: Either in parallel or in sequential mode, solve for $\mathbf{u}_{\mathrm{s}}$ and $\langle \mathbf{p}_{\mathrm{f}}\ \boldsymbol{\alpha}_{\mathrm{f}} \rangle$ by using (62).

Step 3: Update the variables and generate the necessary time step-dependent vectors and repeat Steps 1 and 2 until the simulation time is covered.

Remark 9.1. It should be remarked that the interface equation by the classical $\boldsymbol{\lambda}$-method can be obtained by the following relation:

$$\mathbf{C}_{\mathrm{cl}} = \mathrm{null}\left(\begin{bmatrix} \mathscr{L}_{\mathrm{bs}} \\ \mathscr{L}_{\mathrm{bf}} \end{bmatrix} \right). \tag{64}$$

Pre-multiplying (63) by $\mathbf{C}_{\mathrm{cl}}^{\mathrm{T}}$ and substituting

$$\begin{Bmatrix} \boldsymbol{\lambda}_{\mathrm{s}} \\ \boldsymbol{\lambda}_{\mathrm{f}} \end{Bmatrix} = \mathbf{C}_{\mathrm{cl}} \boldsymbol{\lambda}_{\mathrm{cl}}, \tag{65}$$

we obtain

$$\left\{ \mathbf{C}_{\mathrm{cl}}^{\mathrm{T}} \begin{bmatrix} \mathscr{B}_{\mathrm{s}}^{\mathrm{T}} \mathbf{F}_{\mathrm{s}} \mathscr{B}_{\mathrm{s}} & 0 \\ 0 & \mathscr{B}_{\mathrm{f}} \mathbf{F}_{\mathrm{f}} \mathscr{B}_{\mathrm{f}} \end{bmatrix} \mathbf{C}_{\mathrm{cl}} \right\} \boldsymbol{\lambda}_{\mathrm{cl}}^{n+1/2} = \mathbf{C}_{\mathrm{cl}}^{\mathrm{T}} \begin{Bmatrix} \mathbf{g}_{\boldsymbol{\lambda}_{\mathrm{s}}} \\ \mathbf{g}_{\boldsymbol{\lambda}_{\mathrm{f}}} \end{Bmatrix}^{n+1/2}, \tag{66}$$

since

$$\begin{bmatrix} \mathscr{L}_{\mathrm{bs}}^{\mathrm{T}} & \mathscr{L}_{\mathrm{bf}}^{\mathrm{T}} \end{bmatrix} \mathbf{C}_{\mathrm{cl}} = 0.$$

Hence, the use of the classical $\boldsymbol{\lambda}$-method tightly couples the interface equation as seen by (66). On the other hand, the interface Eq. (63) derived by utilizing the localized Lagrange multipliers preserves the modularity of the flexibility matrices.

10. Discussions

The present paper has presented a partitioned formulation of vibro-acoustic interaction problems by employing a localized version of the method of Lagrange multipliers. To preserve a general nature of the present formulation, both the structure and fluid are modeled in terms of their displacements.

In order to illustrate the present localized version of the method of Lagrange multipliers [20], a two-spring model is used to illustrate the basic features of both the present localized $\boldsymbol{\lambda}$-method and the classical $\boldsymbol{\lambda}$-method. It is demonstrated that the classical Lagrange multipliers couple directly from one interface node i to its connecting node j as characterized by its corresponding multiplier $\boldsymbol{\lambda}_{ij}$. In addition, for multipoints with more than two nodes constrained to a global node, the constraints are not uniquely constructed. On the other hand, the present version of the localized $\boldsymbol{\lambda}$-method offers a unique way of constructing constraints for multipoints, localizes the multipliers as characterized by a single subscript $\boldsymbol{\lambda}_i$ for node i, and renders a direct physical interpretation as each of them becomes the physical nodal force.

A variational formulation of elasto-internal acoustic problems is introduced to model small-amplitude fluid motions and a linear structural system in terms of their displacements. The total system is then discretized as one entity. Partition of the fluid domain and the structural domain is realized employing the localized localized $\boldsymbol{\lambda}$-method. It is shown that the partitioned structural and fluid equations are coupled through the partition boundary global displacement vector that are common both to the fluid and structure, and through the sum of the localized Lagrange multipliers via Newton's third law. The partitioned compressible internal fluid–structure interaction equations are then specialized to: implicit–implicit transient analysis, vibration analysis using substructuring methods.

A novel procedure of transforming the fluid displacement variable to the pressure variable is introduced via a *pressure-to-displacement* relation (52) that satisfies both the correct rigid-body modes in the fluid and the irrotationality condition. The transformed partitioned formulation is then specialized to vibration analysis using (p, u, λ)-formulation and transient analysis of coupled fluid–structure interactions. It is shown that the present pressure–displacement formulation correctly models the presence of zero-energy modes in the fluid without which incorrect vibration characterization would result.

Additional formulations can be derived from the present variational formulation, which are presently under investigation and will be reported in the future.

So far we have not utilized a continuum-based variational principle that incorporates the interface condition in continuum form. A recent work along this line for the formulation of partitioned structural equations of motion [19] indicates that the continuum-based interface conditions can accommodate non-collocated meshes and interface traction discontinuities. Extensions of this variational principle to coupled-field problems are under study and will be presented in the future.

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