
# Approximate Reduction in Interval-Valued Ordered Target Information Systems Based on Attribute Weighting

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**INFORMATION**

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# Approximate Reduction in Interval-Valued Ordered Target Information Systems Based on Attribute Weighting

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**ABSTRACT**

Attribute reduction is a critical preprocessing step in interval-valued ordered target information systems. However, conventional methods often inadvertently discard decision critical attributes during dimensionality reduction. This oversight severely degrades subsequent classification and decision making performance. To mitigate this issue, this paper develops a novel attribute weighted reduction framework that preserves essential attributes while compressing redundant features. We first formally introduce interval-valued ordered target information systems with attribute importance, and systematically establish the theoretical foundation of upper and lower approximation reductions. To facilitate efficient computation, we convert interval data into scalar representations by combining interval maxima and radii, constructing a transformed ordered target system that retains both numerical magnitude and inherent uncertainty. On this basis, we design a weighted discernibility matrix to characterize attribute distinguishability under varying importance levels. We further derive rigorous necessary and sufficient conditions for identifying upper and lower approximation reducts, enabling reliable reduction with explicit consideration of attribute weights. The rationality and correctness of the theoretical results are verified through illustrative examples. Extensive experiments on multiple real world datasets demonstrate that the proposed method achieves effective dimensionality reduction. It also consistently retains decision critical attributes.

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## 1 Introduction

Rough set theory, pioneered by Pawlak [1], has established itself as a fundamental paradigm for knowledge discovery, feature selection, and uncertainty reasoning in incomplete or imprecise information systems [2,3]. The theory approximates concepts using upper and lower approximations. This ability, which requires no prior probability distributions, makes it particularly valuable in data mining and decision analysis. To accommodate real world scenarios involving preferenceordered attributes and decision classes, Greco et al. [4] extended classical rough sets to dominance based rough set approaches (DRSA), which replace equivalence relations with dominance relations, enabling the handling of monotonicity constraints in multicriteria decision making.

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Rough set theory has recently been widely incorporated into advanced decision making frameworks. This helps better manage ambiguity and vagueness in multicriteria problems. For example, Kousar and Kausar [5] proposed a fuzzy rough approach for sustainable agritourism, while other studies, such as the bipolar fuzzy VIKOR with $\delta$-covering rough sets and hyperfuzzy based VIKOR–DEMATEL models further improve decision making under uncertainty by capturing complex relationships and higher order fuzziness [6]. Overall, these developments, including shadowed offset models, highlight the growing importance of rough and hybrid rough set methods in modern decision support systems.

Parallel to these developments, the growing complexity of real world data characterized by measurement imprecision, expert judgment ranges, and inherent variability has driven the emergence of interval-valued information systems (IVIS) [7]. Unlike conventional systems where attributes assume precise values, IVIS represent attributes as intervals, offering a more realistic framework for modeling uncertainty in domains including medical diagnosis [8], financial risk assessment [9], and environmental monitoring [10]. When such interval-valued data are coupled with ordered decision semantics, they form interval-valued ordered information systems (IVOIS), which necessitate specialized reduction techniques that preserve both interval structure and dominance relations [11]. In addition, Liu et al. [12] investigated adaptive noise resilient feature extraction for interval-valued data with evolving attributes, addressing dynamic uncertainty in data driven systems. Sultan and Akram [13] applied spherical fuzzy rough numbers to multicriteria decision making problems, showing the applicability of fuzzy rough hybrid frameworks in complex decision environments. The challenge of attribute reduction in IVOIS is compounded by the heterogeneous significance of different attributes in decision processes. Traditional reduction methods whether based on positive region preservation [14], uncertainty measures [15], dependency measures [16], utility theory in incomplete fuzzy decision systems [17] typically assume uniform attribute importance. This assumption is rarely valid in practical applications; for instance, in a clinical decision system for cardiovascular risk prediction, interval-valued measurements of systolic blood pressure (e.g., [120, 140] mmHg) and low density lipoprotein cholesterol (e.g., [100, 130] mg/dL) inherently carry greater diagnostic weight than interval-valued demographic indicators such as age range [18]. Equal weight reduction strategies risk eliminating these semantically critical attributes, thereby degrading classification accuracy, interpretability, and decision robustness [19,20].

Recent studies have further explored weighted and interval valued rough set models in decision making systems. Pan et al. [21] proposed an incremental feature selection approach based on weighted dominance based neighborhood rough sets, demonstrating the effectiveness of incorporating attribute importance in dynamic environments. Similarly, Yang et al. [22] developed attribute reduction techniques using generalized weighted neighborhood rough sets in interval set information systems, highlighting the role of weighting mechanisms in interval-based uncertainty modeling. Recent years have witnessed significant advances in weighted rough set models across various extensions. In hesitant fuzzy environments, Xie and Gong [23] introduced weighted similarity relations, while Zhao and Hu [24] developed three-way decision models with weighted loss functions. For fuzzy rough sets, Hu et al. [25] proposed weighted fuzzy neighborhoods, and Wang et al. [26] integrated kernel functions with attribute weighting. However, these approaches predominantly focus on fuzzy or hesitant fuzzy representations and do not directly address the unique structural properties of interval-valued ordered systems, where center radius characteristics, partial order relations, and dominance consistency require specialized mathematical treatment [27,28].

Within the specific domain of interval-valued data processing, several innovative reduction techniques have emerged. Zhang et al. [29] developed $\alpha$-dominance relations for attribute reduction, while Miao et al. [30] employed similarity based approaches. Sun et al. [31] established dominance based rough sets for IVOIS, and Li et al. [9] extended variable precision models to interval-valued contexts. More recently, Yang and Guo [32] proposed multigranulation approaches, and Sang et al. [33] developed incremental reduction Algorithms for dynamic interval-valued data. Although these studies significantly contribute to weighted rough sets, fuzzy rough decision systems, and interval-valued uncertainty modeling, most existing approaches still rely on classical dominance relations or parameter dependent structures. In particular, methods based on $\alpha$-dominance or neighborhood rough sets often introduce threshold sensitivity and may distort the inherent structure of interval-valued data. Moreover, recent weighted rough set approaches do not fully address dominance preserving attribute reduction in interval-valued ordered information systems. This limitation is critical, as it affects the stability and interpretability of decision rules in real world applications. These challenges motivate the development of a center radius based representation, which preserves both the central tendency and dispersion of interval-valued data while enabling a more stable weighted approximate reduction framework.

The concept of approximate reduction has gained traction as a practical alternative to exact reduction. It allows for controlled information loss to achieve more compact feature subsets, particularly in high dimensional or noisy environments [34,35]. When combined with attribute weighting mechanisms, approximate reduction offers a balanced approach that respects both computational efficiency and decision quality. Recent studies have explored weighted approximations in various contexts: Wang et al. [36] investigated weighted neighborhood rough sets, Huang [37] examined weighted fuzzy beta reducts, and Qian et al. [38] developed weighted multigranulation decision theoretic rough sets. However, their application to interval-valued ordered systems remains largely unexplored, creating both a theoretical gap and practical limitation.

Moreover, existing studies rarely address whether consistency is preserved after integrating attribute weighting with interval-valued transformations. In particular, the theoretical conditions are not explicitly discussed in most existing approaches. The conditions concern when dominance relations and decision consistency remain invariant after transformation. This paper addresses these challenges by proposing a novel attribute weighted approximate reduction framework for interval-valued ordered target information systems. Motivated by these limitations, it is evident that existing rough set based reduction methods face three persistent challenges: (i) lack of dominance preserving mechanisms in interval-valued environments, (ii) sensitivity of threshold based models such as $\alpha$-dominance, and (iii) insufficient integration of attribute weighting in approximate reduction processes. Although recent studies have attempted to incorporate weighting strategies and interval-valued rough set extensions, most approaches either rely on rigid dominance relations or fail to simultaneously preserve interval structure and attribute importance. These limitations restrict their applicability in real world decision making systems where uncertainty, imprecision, and heterogeneous attribute significance coexist. Therefore, there is a strong need for a unified framework that integrates attribute weighting with a structure preserving representation of interval-valued data. The center radius approach proposed in this paper addresses this gap. It maintains both the central tendency and dispersion information of intervals while enabling a stable weighted approximate reduction mechanism under ordered decision semantics. Our research makes four key contributions:

(1) Integration of attribute weighting into interval-valued ordered target information systems: Unlike classical dominance based rough set approaches and interval-valued rough set models, which assume uniform attribute importance, our framework explicitly incorporates attribute significance into the dominance relation and reduction process. This allows the method to preserve decision critical attributes that would otherwise be eliminated under equal weight assumptions.

(2) Center-radius transformation with dominance preservation: Existing interval-valued methods typically discretize intervals to midpoints or endpoints, which often destroys the inherent dominance structure. Our proposed transformation (Theorem 1) converts interval data into scalar representations while provably preserving the original dominance classes. This is a theoretical guarantee not provided by most existing approaches.

(3) Weighted approximate reduction with dual optimization objectives: We introduce two complementary reduction forms upper approximation reduction (emphasizing compactness) and lower approximation reduction (emphasizing decision reliability). This dual framework is novel in the context of weighted interval-valued ordered systems and provides practitioners with flexible trade offs between efficiency and robustness.

(4) Weighted discernibility matrix with theoretical characterization: We extend the classical discernibility matrix to incorporate attribute weights, and provide rigorous necessary and sufficient conditions (Theorems 4 and 5) for identifying weighted upper and lower approximation reducts. This theoretical foundation distinguishes our work from heuristic or unweighted reduction methods.

Quantitative comparisons further validate the effectiveness of the four contributions in Table 1.

**Table 1:** Quantitative comparison with existing IVOIS methods

| Method | Weighting support | Approximation type | Reduction rate (%) | Accuracy (SVM, %) |
|---|---|---|---|---|
| Qian et al. (2014) [38] | No | Exact | 22.4 | 79.6 |
| Sun et al. (2014) [31] | No | Exact | 26.8 | 81.3 |
| Proposed method (upper) | Yes | Approximate | 38.1 | 87.4 |
| Proposed method (lower) | Yes | Approximate | 20.3 | 89.5 |

By transforming interval data for more accurate dominance modeling and adding attribute weights to better capture feature importance, the results demonstrate that the suggested center radius approach outperforms current IVOIS methods, achieving higher reduction rates and accuracy. This validates its efficacy and uniqueness.

***Related Work***

Table 2 summarizes representative rough set based reduction methods for interval-valued or ordered information systems, highlighting their characteristics and limitations.

Compared to classical $\alpha$-dominance relations, our center radius approach offers two distinct advantages. First, it eliminates the need for a user specified threshold $\alpha$, which is often dataset dependent and difficult to optimize. Second, Theorem 1 guarantees that the original dominance order is preserved under the proposed transformation, a property not generally satisfied by $\alpha$-dominance or midpoint based discretization.

**Table 2:** Comparison of related rough set based reduction methods

| Method | Weighting support | Threshold parameter | Dominance preservation | Approximation type |
|---|---|---|---|---|
| $\alpha$-dominance (Li et al., 2013, [39]) | No | $\alpha$ | Partial | Exact |
| Variable precision (Qian et al., 2014 [40]) | No | $\beta$ | No | Approximate |
| Weighted neighborhood (Yang et al., 2019 [41]) | Yes | $\delta$ | No | Approximate |
| Proposed method | Yes | None | Provable (Theorem 1) | Approximate (dual) |

The remainder of this paper is organized as follows. Section 2 provides necessary preliminaries on interval-valued ordered information systems, dominance relations, and existing reduction approaches. Section 3 presents the proposed weighted approximate reduction model, including the center radius transformation, weighted dominance relations, reduct characterization theorems, and reduction algorithms. Section 4 details experimental settings, datasets, evaluation metrics, and comparative methods. Section 5 analyzes experimental results and discusses practical implications. Finally, Section 6 concludes with summary remarks and suggests future research directions, including dynamic weight adaptation, parallel computation, and integration with deep learning architectures.

## 2 Preliminaries

This section presents the essential concepts and definitions required for formulating interval-valued ordered target information systems.

**Definition 1** (Interval Set of Real Numbers): *Let $\mathbb{R}$ be the set of all real numbers. A set $[a^-, a^+]$ is called an interval set on $\mathbb{R}$, where $a^-, a^+ \in \mathbb{R}$ and $a^- \leq a^+$. Here, $a^-$ and $a^+$ are called the lower endpoint and the upper endpoint of the interval set, respectively. In particular, if $a^- = a^+$, the interval set reduces to a singleton set $\{a^-\}$. Let $\mathcal{I}(\mathbb{R})$ denote the set of all interval sets on $\mathbb{R}$, that is,*

$$\mathcal{I}(\mathbb{R}) = \{\,[a^-, a^+] \mid a^-, a^+ \in \mathbb{R},\ a^- \leq a^+\,\}.$$

**Definition 2** (Partial Order of Interval Sets): *For any two interval sets $A = [a^-, a^+]$ and $B = [b^-, b^+] \in I(\mathbb{R})$, their interval order relation $\preccurlyeq$ is defined as follows:*

$$A \preccurlyeq B \iff a^- \leq b^- \text{ and } a^+ \leq b^+.$$

**Definition 3:** *A quintuple $I = (U, AT \cup \{d\}, V, f, g)$ is called an interval-valued target information system, where*

- *$(U, AT, V, f)$ is an interval-valued information system,*
- *$U$ is a finite set of objects,*
- *$AT = \{a_1, a_2, \ldots, a_m\}$ is called the set of conditional attributes,*
- *$\{d\}$ is called the set of target attributes,*

- $V_a$ *is the value domain of conditional attribute* $a$, *i.e.*, $\forall a \in AT, x \in U$, *there exists* $f(x,a) = [a^-(x), a^+(x)]$,
- $f$ *is the relation function between* $U$ *and* $AT$, *denoted as* $f(x,a) = [a^-(x), a^+(x)]$,
- $g$ *is the relation function between* $U$ *and* $\{d\}$, *i.e.*, $g: U \times \{d\} \to V_d$,
- $V_d$ *is the value domain of the decision attribute* $d$.

**Definition 4:** *An interval-valued ordered target information system is defined as the quintuple* $I = (U, AT \cup \{d\}, V, f, g)$. *Within this system, for any subset of conditional attributes* $B \subseteq AT$, *we define a dominance relation as:*

$$R_B^{\geq} = \{(y,x) \in U \times U : a^-(y) \geq a^-(x),\ a^+(y) \geq a^+(x),\ \forall a \in B\},$$

*and for the decision attribute* $d$:

$$R_d^{\geq} = \{(x,y) \in U \times U : g(x) \leq g(y)\}.$$

*The system is referred to as an interval-valued ordered target information system based on the dominance relation. Denote*

$$[x]_B^{\geq} = \{y \in U : (y,x) \in R_B^{\geq}\} = \{y \in U : a^-(y) \geq a^-(x),\ a^+(y) \geq a^+(x), \forall a \in B\},$$

$$[x]_d^{\geq} = \{y \in U : (x,y) \in R_d^{\geq}\} = \{y \in U : g(y) \geq g(x)\}.$$

**Example 1:** Table 3 *shows an interval-valued ordered target information system* $I = (U, A \cup \{d\}, V, f, g)$, *where* $U = \{x_1, x_2, x_3, x_4, x_5\}$ *and* $AT = \{a_1, a_2, a_3\}$.

**Table 3:** An interval-valued ordered target information system

| $U$ | $a_1$ | $a_2$ | $a_3$ | $d$ |
|---|---|---|---|---|
| $x_1$ | [5, 6] | [5, 6] | [5, 6] | 1 |
| $x_2$ | [3, 4] | [4, 6] | [3, 5] | 2 |
| $x_3$ | [5, 6] | [5, 7] | [5, 6] | 1 |
| $x_4$ | [4, 5] | [4, 6] | [4, 6] | 2 |
| $x_5$ | [3, 4] | [3, 5] | [4, 5] | 2 |

According to the definition of the dominance relation, the calculation of the dominance class for each object yields:

$$[x_1]_{AT}^{\succeq} = \{x_1, x_3\}, \quad [x_2]_{AT}^{\succeq} = \{x_1, x_2, x_3, x_4\}, \quad [x_3]_{AT}^{\succeq} = \{x_3\},$$

$$[x_4]_{AT}^{\succeq} = \{x_1, x_3, x_4\}, \quad [x_5]_{AT}^{\succeq} = \{x_1, x_2, x_3, x_4, x_5\};$$

$$[x_1]_d^{\succeq} = [x_3]_d^{\succeq} = \{x_1, x_2, x_3, x_4, x_5\}, \quad [x_2]_d^{\succeq} = [x_4]_d^{\succeq} = [x_5]_d^{\succeq} = \{x_2, x_4, x_5\}.$$

To better study the attribute weighted reduction problem in interval-valued ordered target information systems, the following transformation theorem is proposed.

**Theorem 1:** *Let $I = (U, AT \cup \{d\}, V, f, g)$ be an interval-valued ordered target information system and denote*

$$\lambda_a = \max_{x \in U}\left[a^+(x) - a^-(x)\right], \quad F(x, a) = a^+(x) - \frac{a^+(x) - a^-(x)}{\lambda_a}.$$

*Then, for any $B \subseteq AT$, the following equivalence holds:*

$$x_j \in [x_i]_B^{\succeq} \quad \Longleftrightarrow \quad F(x_j, a) \geq F(x_i, a) \quad \text{for all } a \in B.$$

**Proof:** "⇒" Assume $x_j \in [x_i]_B^{\succeq}$. By definition of the dominance relation, we have $a^-(x_j) \geq a^-(x_i)$ and $a^+(x_j) \geq a^+(x_i)$ for all $a \in B$. Then

$$F(x_j, a) - F(x_i, a) = \frac{(\lambda_a - 1)[a^+(x_j) - a^+(x_i)] + [a^-(x_j) - a^-(x_i)]}{\lambda_a} \geq 0,$$

hence $F(x_j, a) \geq F(x_i, a)$.

"⇐" We prove the contrapositive. Suppose $x_j \notin [x_i]_B^{\succeq}$. Since all objects have comparable interval values on each attribute, there exists some attribute $a \in B$ such that either (i) $a^-(x_j) < a^-(x_i)$ or (ii) $a^+(x_j) < a^+(x_i)$. For case (i), since $a^-(x_j) - a^-(x_i) < 0$ and all other terms are bounded, we have

$$F(x_j, a) - F(x_i, a) = \frac{(\lambda_a - 1)[a^+(x_j) - a^+(x_i)] + (a^-(x_j) - a^-(x_i))}{\lambda_a} < 0,$$

because the negative contribution from the lower endpoint dominates. Case (ii) is analogous. Therefore, if $x_j \notin [x_i]_B^{\succeq}$, there exists $a \in B$ such that $F(x_j, a) < F(x_i, a)$. Hence, if $F(x_j, a) \geq F(x_i, a)$ for all $a \in B$, we must have $x_j \in [x_i]_B^{\succeq}$. □

To numerically validate Theorem 1, consider objects $x_2$ and $x_4$ from Table 3. Their interval values for attribute $a_1$ are $[3, 4]$ and $[4, 5]$, respectively. Compute $\lambda_{a_1} = \max_{x \in U}(a_1^+(x) - a_1^-(x)) = 1$. Then $F(x_2, a_1) = 4 - (4 - 3)/1 = 3$, and $F(x_4, a_1) = 5 - (5 - 4)/1 = 4$. Clearly $F(x_4) > F(x_2)$, and indeed $a_1^-(x_4) = 4 > 3 = a_1^-(x_2)$ and $a_1^+(x_4) = 5 > 4 = a_1^+(x_2)$, so $x_4$ dominates $x_2$. For a non-dominating pair, take $x_1$ ($[5, 6]$) and $x_4$ ($[4, 5]$). We have $F(x_1) = 6 - 1 = 5$, $F(x_4) = 4$, so $F(x_1) > F(x_4)$, but $x_1$ does not dominate $x_4$ because $a_1^-(x_1) = 5 > 4 = a_1^-(x_4)$ fails (the lower endpoint is larger, which is actually a reversal). This illustrates that the transformation preserves the direction of dominance without creating false relationships.

This theorem shows that the dominance classes obtained from the interval-valued data remain unchanged after transforming the attribute values into single numerical values via $F$. For convenience, the new information system $(U, AT \cup \{d\}, F, g)$ will be called the transformed target information system of the original interval-valued ordered target information system $I$.

**Definition 5:** *Let $(U, AT \cup \{d\}, F, g)$ be a transformed target information system. For any $x \in U$ and $a_i \in AT$, define the weighted score of $x$ over all attributes $(a_1, a_2, \ldots, a_m)$ as $S(x) = \sum_{i=1}^{m} \omega_i F_i(x)$, where $\omega_i$ denotes the weight of attribute $a_i$ satisfying $\sum_{i=1}^{m} \omega_i = 1$ and $\omega_i > 0$, and $F_i(x)$ denotes the value of $x$ on attribute $a_i$ as defined in Theorem 1.*

In practice, weights can be determined in various ways depending on the available information. When decision classes are known, supervised methods such as information gain ratio or correlation with the decision attribute can be used. When decision classes are unavailable, unsupervised methods such as entropy based or variance based weights may be applied. The choice of weighting scheme is application dependent, and the proposed framework remains valid for any positive normalized weight vector. Using the weighted scores, one can compute and compare the evaluations of objects under the condition attributes. Therefore, the following dominance relation can be considered in the transformed target information system.

**Definition 6:** *Let $(U, AT \cup \{d\}, F, g)$ be a transformed target information system. For any $B \subseteq AT$ and $X \subseteq U$, define the dominance relations as:*

$$R_B^{\leq} = \{(x_i, x_j) \in U \times U : S_B(x_i) \leq S_B(x_j)\},$$

*and for the decision attribute $d$:*

$$R_d^{\leq} = \{(x_i, x_j) \in U \times U : g(x_i) \leq g(x_j)\}.$$

*Denote:* $[x_i]_B^{\leq} = \{x_j \in U : (x_i, x_j) \in R_B^{\leq}\} = \{x_j \in U : S_B(x_i) \leq S_B(x_j)\}$,

$[x_i]_d^{\leq} = \{x_j \in U : (x_i, x_j) \in R_d^{\leq}\} = \{x_j \in U : g(x_i) \leq g(x_j)\}$.

*The lower approximation and upper approximation of $X$ are respectively defined as*

$$\underline{R_B^{\leq}}(X) = \{x_i \in U : [x_i]_B^{\leq} \subseteq X\}, \overline{R_B^{\leq}}(X) = \{x_i \in U : [x_i]_B^{\leq} \cap X \neq \emptyset\}.$$

***System Consistency after Weight Transformation***

Let $(U, AT \cup \{d\}, F, g)$ be a transformed target information system. The system is said to be *consistent* if $R_{AT}^{\leq} \subseteq R_d^{\leq}$, i.e., for any $x_i, x_j \in U$, $S(x_i) \leq S(x_j)$ implies $g(x_i) \leq g(x_j)$. The following proposition provides a sufficient condition for consistency preservation.

**proposition 1:** *If the original interval-valued system is consistent with respect to the dominance relation defined on intervals, and the weight vector $\omega = (\omega_1, \ldots, \omega_m)$ satisfies $\omega_i > 0$ for all $i$, then there exists a threshold $\epsilon > 0$ such that the weighted transformed system remains consistent provided that for any pair $(x_i, x_j)$ with $S(x_i) = S(x_j)$, the decision values satisfy $g(x_i) = g(x_j)$.*

In practice, when using information gain ratio weights, the weighted transformed system typically preserves consistency because the weight assignment is monotonic with respect to discriminative power. For inconsistent original systems, our approximate reduction framework is explicitly designed to handle inconsistency through the upper and lower approximation formulations.

**Example 2:** *Apply attribute weights $\omega_1 = 0.5$, $\omega_2 = 0.3$, $\omega_3 = 0.2$ to Table 3 and compute the classification.*

Table 4 shows the transformed target information system derived from the interval-valued data in Table 3.

$$S(x_1) = 5.5, \quad S(x_2) = 4.05, \quad S(x_3) = 5.65, \quad S(x_4) = 4.75, \quad S(x_5) = 3.85.$$

By comparing the weighted scores, the dominance class for each object is obtained:

$$[x_1]_{AT}^{\leq} = \{x_1, x_3\}, [x_2]_{AT}^{\leq} = \{x_1, x_2, x_3, x_4\}, [x_3]_{AT}^{\leq} = \{x_3\},$$

$$[x_4]_{AT}^{\leq} = \{x_1, x_3, x_4\}, [x_5]_{AT}^{\leq} = \{x_1, x_2, x_3, x_4, x_5\}.$$

Since $D_1 = D_3 = \{x_1, x_2, x_3, x_4, x_5\}$ and $D_2 = D_4 = D_5 = \{x_2, x_4, x_5\}$, by the definitions of lower and upper approximations we have:

$$\underline{R^{\leq}_{AT}}(D_1) = \underline{R^{\leq}_{AT}}(D_3) = \{x_1, x_2, x_3, x_4, x_5\}, \quad \underline{R^{\leq}_{AT}}(D_2) = \underline{R^{\leq}_{AT}}(D_4) = \underline{R^{\leq}_{AT}}(D_5) = \emptyset,$$

$$\overline{R^{\leq}_{AT}}(D_1) = \overline{R^{\leq}_{AT}}(D_3) = \{x_1, x_2, x_3, x_4, x_5\}, \quad \overline{R^{\leq}_{AT}}(D_2) = \overline{R^{\leq}_{AT}}(D_4) = \overline{R^{\leq}_{AT}}(D_5) = \{x_2, x_4, x_5\}.$$

**Table 4:** Contains the converted results of the interval-valued defined in Table 3

| $U$ | $a_1$ | $a_2$ | $a_3$ | $d$ |
|---|---|---|---|---|
| $x_1$ | 5.5 | 5.5 | 5.5 | 1 |
| $x_2$ | 3.5 | 5 | 4 | 2 |
| $x_3$ | 5.5 | 6 | 5.5 | 1 |
| $x_4$ | 4.5 | 5 | 5 | 2 |
| $x_5$ | 3.5 | 4 | 4.5 | 2 |

## 3 Approximate Reduction for Inconsistent Interval-Valued Target Information Systems

Since the dominance relation forms a covering on the object set, the approximation functions defined via partitions cannot be directly used. Therefore, we redefine the upper and lower approximation functions as follows.

**Definition 7:** *Let $(U, AT \cup \{d\}, F, g)$ be an interval-valued target information system, $R^{\leq}_{B}$ be the dominance relation generated by the attribute set $B \subseteq AT$, and $R^{\leq}_{d}$ be the dominance relation generated by the target attribute set $d$. Denote $U/R^{\leq}_{d} = \{D_1, D_2, \ldots, D_r\}$. For each $x \in U$, define the upper approximation function as the tuple*

$$\gamma_B(x) = \left(\overline{R^{\leq}_{B}}(D_1), \overline{R^{\leq}_{B}}(D_2), \ldots, \overline{R^{\leq}_{B}}(D_r)\right),$$

*and the lower approximation function as the tuple*

$$\mu_B(x) = \left(\underline{R^{\leq}_{B}}(D_1), \underline{R^{\leq}_{B}}(D_2), \ldots, \underline{R^{\leq}_{B}}(D_r)\right).$$

Thus, for two attribute subsets $B_1, B_2 \subseteq AT$, we have $\gamma_{B_1}(x) = \gamma_{B_2}(x)$ for all $x \in U$ if and only if $\overline{R^{\leq}_{B_1}}(D_i) = \overline{R^{\leq}_{B_2}}(D_i)$ for every decision class $D_i$. The same equivalence holds for lower approximations.

**Definition 8:** *Let $(U, AT \cup \{d\}, F, g)$ be an interval-valued target information system, $\forall x \in U$, $B \subseteq AT$.*

*(1) If $\gamma_B(x) = \gamma_{AT}(x)$, then $B$ is called an upper approximation consistent set. If, in addition, no proper subset of $B$ is an upper approximation consistent set, then $B$ is called an upper approximation reduct.*

*(2) If $\mu_B(x) = \mu_{AT}(x)$, then $B$ is called a lower approximation consistent set. If, in addition, no proper subset of $B$ is a lower approximation consistent set, then $B$ is called a lower approximation reduct.*

**Theorem 2:** *Let $(U, AT \cup \{d\}, F, g)$ be an interval-valued target information system. For all $x \in U$ and $B \subseteq AT$, the following hold:*

*(1) B is an upper approximation consistent set if and only if for all* $D_i \in U/R_d^{\leq}$, *we have* $\overline{R_B^{\leq}}(D_i) = \overline{R_{AT}^{\leq}}(D_i)$.

*(2) B is an upper approximation consistent set if and only if for all* $D_i \in U/R_d^{\leq}$, *whenever* $x_i \in \overline{R_{AT}^{\leq}}(D_i)$ *and* $x_j \notin \overline{R_{AT}^{\leq}}(D_i)$, *we have* $S(x_i) < S(x_j)$.

**Proof:** (1) "⇒" Assume that there exists $D_k \in U/R_d^{\leq}$ such that $\overline{R_B^{\leq}}(D_k) \neq \overline{R_{AT}^{\leq}}(D_k)$.

By $B \subseteq AT$, we have $\overline{R_B^{\leq}}(D_k) \supseteq \overline{R_{AT}^{\leq}}(D_k)$. Therefore, there exists at least one $\overline{R_B^{\leq}}(D_k)$ such that $\overline{R_B^{\leq}}(D_k) \notin \gamma_{AT}(x)$ and $\overline{R_B^{\leq}}(D_k) \in \gamma_B(x)$. Hence $\gamma_B(x) \neq \gamma_{AT}(x)$. Since $B$ is an upper approximation consistent set, by Definition 7 we have $\gamma_B(x) = \gamma_{AT}(x)$, which contradicts the derived conclusion. Thus, when $B$ is an upper approximation consistent set, for all $D_i \in U/R_d^{\leq}$, we have $\overline{R_B^{\leq}}(D_i) = \overline{R_{AT}^{\leq}}(D_i)$.

"⇐" We prove by contradiction. Suppose $B$ is not an upper approximation consistent set, i.e., $\gamma_B(x) \neq \gamma_{AT}(x)$. Since for all $D_i \in U/R_d^{\leq}$, it holds that $\overline{R_B^{\leq}}(D_i) = \overline{R_{AT}^{\leq}}(D_i)$, then by the definition of the upper approximation function, we easily see that $\gamma_B(x) = \gamma_{AT}(x)$, which contradicts the assumption. Hence $B$ is an upper approximation consistent set.

(2) "⇒" Assume there exists $D_i \in U/R_d^{\leq}$ such that when $x_i \in \overline{R_{AT}^{\leq}}(D_i)$ and $x_j \notin \overline{R_{AT}^{\leq}}(D_i)$, the condition $S(x_i) < S(x_j)$ does not hold, i.e., $S(x_j) < S(x_i)$. According to the definition of the dominance relation, we have $x_i \in [x_j]_B^{\leq}$. Since $B$ is an upper approximation consistent set, we have $\overline{R_B^{\leq}}(D_i) = \overline{R_{AT}^{\leq}}(D_i)$. Therefore $x_i \in [x_j]_{AT}^{\leq}$, but $[x_j]_{AT}^{\leq} \cap D_i = \emptyset$, which contradicts $x_i \in \overline{R_{AT}^{\leq}}(D_i)$. Thus, when $x_i \in \overline{R_{AT}^{\leq}}(D_i)$ and $x_j \notin \overline{R_{AT}^{\leq}}(D_i)$, we have $S(x_i) < S(x_j)$.

"⇐" Assume $B$ is not an upper approximation consistent set. Then by part (1), there exists $D_i \in U/R_d^{\leq}$ such that $\overline{R_B^{\leq}}(D_i) \neq \overline{R_{AT}^{\leq}}(D_i)$, i.e., there exists $x \in \overline{R_B^{\leq}}(D_i)$ but $x \notin \overline{R_{AT}^{\leq}}(D_i)$. Hence we have $[x]_B^{\leq} \cap D_i \neq \emptyset$ but $[x]_{AT}^{\leq} \cap D_i = \emptyset$. By the property of the dominance relation, we have $[x]_{AT}^{\leq} \subseteq [x]_B^{\leq}$, so there exists an $x_0 \in ([x]_B^{\leq} \cap D_i)$ but $x_0 \notin [x]_{AT}^{\leq}$, which implies $x_0 \in \overline{R_{AT}^{\leq}}(D_i)$. Consequently, we have $x_0 \in \overline{R_{AT}^{\leq}}(D_i)$ and $x \notin \overline{R_{AT}^{\leq}}(D_i)$. Therefore, there exists some $\alpha_k \in B$ such that $S(x_0) < S(x)$, which contradicts $x_0 \in [x]_B^{\leq}$. Hence, $B$ is an upper approximation consistent set.

**Theorem 3:** *Let* $(U, AT \cup \{d\}, F, g)$ *be an interval-valued target information system. For all* $x \in U$ *and* $B \subseteq AT$, *the following hold: (1) B is a lower approximation consistent set if and only if for all* $D_i \in U/R_d^{\leq}$, *we have* $\underline{R_B^{\leq}}(D_i) = \underline{R_{AT}^{\leq}}(D_i)$. *(2) B is a lower approximation consistent set if and only if for all* $D_i \in U/R_d^{\leq}$, *whenever* $x_i \in \underline{R_{AT}^{\leq}}(D_i)$ *and* $x_j \notin \underline{R_{AT}^{\leq}}(D_i)$, *we have*$S(x_j) < S(x_i)$.

**Proof:** The proof follows similarly to that of Theorem 2.

**Example 3:** *Compute the upper and lower approximation reductions for the information system in Table 2. Let* $B = \{a_1, a_2\}$. *Then* $S_2(x_1) = 4.4, S_2(x_2) = 3.25, S_2(x_3) = 4.55, S_2(x_4) = 3.75, S_2(x_5) = 2.95$. *By comparing the values of the weighted functions and the definition of lower approximation, we have*

$$\underline{R_B^{\leq}}(D_1) = \{x_1, x_2, x_3, x_4, x_5\} = \underline{R_{AT}^{\leq}}(D_1), \underline{R_B^{\leq}}(D_3) = \{x_1, x_2, x_3, x_4, x_5\} = \underline{R_{AT}^{\leq}}(D_3),$$

$$\underline{R_B^{\leq}}(D_2) = \emptyset = \underline{R_{AT}^{\leq}}(D_2), \underline{R_B^{\leq}}(D_4) = \emptyset = \underline{R_{AT}^{\leq}}(D_4), \underline{R_B^{\leq}}(D_5) = \emptyset = \underline{R_{AT}^{\leq}}(D_5).$$

Thus, $\mu_B(x) = \mu_{AT}(x)$, so $B = \{a_1, a_2\}$ is a lower approximation consistent set. Moreover, one can verify that no proper subset of $B$ is a lower approximation consistent set; therefore $B$ is a lower approximation reduct.

Similarly, it can be shown that $B = \{a_i\}$ is an upper approximation reduct.

## 4 Discernibility Matrix Method for Approximate Reduction in Inconsistent Interval-Valued Target Information Systems

In Section 3, equivalent characterizations of upper and lower approximation consistent sets for inconsistent interval-valued target information systems were provided, which give a theoretical basis for judging whether an attribute subset is consistent. However, since the solution process is relatively cumbersome, in this section, we define a discernibility matrix to further develop a discernibility matrix method for upper and lower approximation reductions.

**Definition 9:** *Let $I' = (U, AT \cup \{d\}, F, g)$ be an inconsistent interval-valued target information system. For $\forall D_i \in U/R_d^{\leq}$, denote*

$$D_\mu^* = \{(x_i, x_j) : x_i \in \underline{R_{AT}^{\leq}}(D_i), x_j \notin \underline{R_{AT}^{\leq}}(D_i)\}, D_\gamma^* = \{(x_i, x_j) : x_i \in \overline{R_{AT}^{\leq}}(D_i), x_j \notin \overline{R_{AT}^{\leq}}(D_i)\}.$$

*Sort the pairs according to $\omega$ in descending order. Compare $S_n(x_i) = \sum_{k=1}^{n} \omega_k F_k(x_i)$ and $S_n(x_j) = \sum_{k=1}^{n} \omega_k F_k(x_j)$, and write the first conditional attribute that makes $S_n(x_i) > S_n(x_j)$ hold into $D_\mu(x_i, x_j)$; simultaneously, write all conditional attributes that satisfy $S_n(x_i) > S_n(x_j)$ into $D_\gamma(x_i, x_j)$, where $n = 1, 2, \ldots, m$.*

$$D_\mu(x_i, x_j) = \begin{cases} \{a_k, S_n(x_i) > S_n(x_j)\}, & (x_i, x_j) \in D_\mu^* \\ \emptyset, & (x_i, x_j) \notin D_\mu^* \end{cases}$$

$$D_\gamma(x_i, x_j) = \begin{cases} \{a_k, S_n(x_i) > S_n(x_j)\}, & (x_i, x_j) \in D_\gamma^* \\ \emptyset, & (x_i, x_j) \notin D_\gamma^* \end{cases}$$

We call $D_\mu(x_i, x_j)$ and $D_\gamma(x_i, x_j)$ the lower approximation discernible attribute set and upper approximation discernible attribute set between $x_i$ and $x_j$, respectively. The matrices $M_\mu$ and $M_\gamma$ are called the lower approximation discernibility matrix and upper approximation discernibility matrix of this target information system, respectively.

**Theorem 4:** *Let $I' = (U, AT \cup \{d\}, F, g)$ be an inconsistent interval-valued target information system, and let $A \subseteq AT$. Then*

*(1) $A$ is an upper approximation consistent set if and only if for every $(x, y) \in D_\gamma^*$ with $D_\gamma(x, y) \neq \emptyset$, we have $A \cap D_\gamma(x, y) \neq \emptyset$.*

*(2) $A$ is a lower approximation consistent set if and only if for every $(x, y) \in D_\mu^*$ with $D_\mu(x, y) \neq \emptyset$, we have $A \cap D_\mu(x, y) \neq \emptyset$.*

**Proof:** (1) "$\Rightarrow$" For every $(x, y) \in D_\gamma^*$, there exists $D_i \in U/R_d^{\leq}$ such that $x_i \in \overline{R_{AT}^{\leq}}(D_i)$ and $x_j \notin \overline{R_{AT}^{\leq}}(D_i)$. Since $A$ is an upper approximation consistent set, there exists at least one $a_k \in A$ such that $S(x_i) > S(x_j)$, hence$a_k \in D_\gamma(x, y)$.

Therefore, for every $(x, y) \in D_\gamma^*$ with $D_\gamma(x, y) \neq \emptyset$, we have $A \cap D_\gamma(x, y) \neq \emptyset$.

"$\Leftarrow$" Suppose for every $(x, y) \in D_\gamma^*$ with $D_\gamma(x, y) \neq \emptyset$, we have $A \cap D_\gamma(x, y) \neq \emptyset$. Then there exists at least one $a_k \in D_\gamma(x, y)$, and consequently $S(x_i) > S(x_j)$. Since $x_i \in \overline{R_{AT}^{\leq}}(D_i)$ and $x_j \notin \overline{R_{AT}^{\leq}}(D_i)$, it follows that $A$ is an upper approximation consistent set. $\square$

(2) The proof is similar to that of (1).

**Definition 10:** *Let $I' = (U, AT \cup \{d\}, F, g)$ be an inconsistent interval-valued target information system, and let $M_\mu$ and $M_\gamma$ be its lower approximation and upper approximation discernibility matrices, respectively. Define*

$$F_\mu = \bigwedge \left\{ \bigvee D_\mu(x, y) : x, y \in U \text{ and } (x, y) \in D_\mu^* \right\}, F_\gamma = \bigwedge \left\{ \bigvee D_\gamma(x, y) : x, y \in U \text{ and } (x, y) \in D_\gamma^* \right\}.$$

Then $F_\mu$ and $F_\gamma$ are called the lower approximation discernibility formula and upper approximation discernibility formula of the information system, respectively.

**Theorem 5:** *Let $I' = (U, AT \cup \{d\}, F, g)$ be an inconsistent interval-valued target information system. Suppose the upper approximation discernibility formula $F_\gamma$ can be expressed in minimal disjunctive normal form as $F_\gamma = \bigvee_{k=1}^{t} \left( \bigwedge_{s=1}^{p_k} b_s \right)$. If we denote $B_k = \{b_s : s = 1, 2, \ldots, p_k\}$, then $\{B_k : k = 1, 2, \ldots, t\}$ is the collection of all upper approximation reducts.*

**Proof:** For any $k \leq t$ and $(x, y) \in D_\gamma^*$, the definition of the minimal disjunctive normal form implies that $B_k \cap D_\gamma(x, y) \neq \emptyset$. By Theorem 4, it follows that $B_k$ is an upper approximation consistent set. Moreover, since $F_\gamma = \bigvee_{k=1}^{t}(B_k)$, if we remove any element from $B_k$ to obtain $B_k'$, then there must exist some $(x, y) \in D_\gamma^*$ such that $B_k' \cap D_\gamma(x, y) = \emptyset$. Hence $B_k'$ is no longer an upper approximation consistent set, and consequently $B_k$ is an upper approximation reduct. Because the upper approximation discernibility formula includes all $D_\gamma(x, y)$, there are no other upper approximation reducts beyond those in the collection. $\square$

**Example 4:** Use the method described above to compute the approximate reducts for the inconsistent interval-valued target information system given in Example 1.

From the above results in Table 5, we obtain that $\{a_1\}$ is an upper approximation reduct, which is consistent with the result of Example 3. Similarly, the upper approximation reduct can also be derived as $\{a_1, a_2\}$.

**Table 5:** Upper approximation discernibility matrix of Table 3

| $M_r$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $x_1$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |
| $x_2$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |
| $x_3$ | $a_1a_2a_3$ | $a_1a_2a_3$ | $a_1a_2a_3$ | $a_1a_2a_3$ | $a_1a_2a_3$ |
| $x_4$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |
| $x_5$ | $\varnothing$ | $a_1$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |

# 5 Algorithm Design and Experimental Results Analysis

## *5.1 Algorithm Design*

Based on the theoretical foundations of Theorem 2, this section designs algorithms for the discernibility matrix method for approximate reduction in interval-order target information systems with attribute weighting. The algorithms for upper approximation reduction and lower approximation reduction are given below, respectively.

### *5.1.1 Attribute Weight Calculation Based on Gain Ratio*

To quantify the importance of attributes in the reduction process, we adopt the gain ratio measure derived from information theory. For a conditional attribute $a \in AT$ and decision attribute $D$, the information gain is defined as:

$$Gain(a) = H(D) - H(D|a),$$

where $H(D)$ is the entropy of the decision attribute and $H(D|a)$ is the conditional entropy. The intrinsic information of attribute $a$ is defined as:

$$IV(a) = -\sum_{i=1}^{k} p_i \log_2 p_i,$$

where $p_i$ represents the probability distribution of partitions induced by attribute $a$. The gain ratio is then given by:

$$GR(a) = \frac{Gain(a)}{IV(a) + \epsilon},$$

where $\epsilon$ is a small constant to avoid division by zero. Finally, attribute weights are normalized as:

$$w_a = \frac{GR(a)}{\sum_{b \in AT} GR(b)}.$$

These weights are used in the weighted dominance relation and score function in the subsequent reduction algorithms.

Time Complexity Analysis

- Step 2: Computing the score function requires traversing all objects and attributes, with complexity $O(|AT| \cdot |U|)$.
- Steps 3–4: Constructing dominance classes involves pairwise object comparisons, with complexity $O(|U|^2 \cdot |AT|)$.
- Step 6: Building the discernibility matrix has complexity $O(|U|^2 \cdot |AT|)$.
- Step 7: Solving the minimal disjunctive normal form using a greedy Algorithm has worst-case complexity $O(|AT|^2)$.
- Overall complexity: $O(|U|^2 \cdot |AT| + |AT|^2)$, where $|U|$ is the number of objects and $|AT|$ the number of attributes.

**Remarks**

- The score function directly weights the interval midpoints, explicitly integrating weights to enhance the influence of important attributes.

- Dynamic discernibility condition: an attribute is recorded only when the weighted interval difference is significant ($w_k^*|a_k^+(x_i) - a_k^+(x_j)| > 0$), reducing noise interference.
- A greedy strategy is employed to accelerate the solution of the minimal disjunctive normal form, providing heuristic optimization and avoiding exhaustive search.

Time Complexity Analysis

Same as Algorithm 1; overall complexity is $O(|U|^2 \cdot |AT| + |AT|^2)$.

**Remarks**

- Stricter lower approximation condition: An attribute is recorded only when the conditional attribute dominance class is fully contained in the target class ($[x_j]_{AT} \subseteq [x_i]_D$), emphasizing complete inclusion.
- Weight focus on lower bound difference: The discernibility matrix emphasizes interval lower-bound differences ($w_k^*|a_k^-(x_i) - a_k^-(x_j)| > 0$), enhancing sensitivity to conservative estimates.

### *5.1.2 Exact Reduction Algorithm for Small Scale Systems*

To evaluate the optimality of the proposed heuristic reduction Algorithm, we introduce an exact algorithm for computing minimal reducts in small scale systems (i.e., $|U| < 100$). The exact method is based on exhaustive search over the attribute space. Given the conditional attribute set $AT$, all possible subsets are enumerated, and each subset is evaluated based on whether it satisfies the weighted approximate reduction conditions.

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**Algorithm 1:** Upper approximation reduction algorithm for interval order target information systems with attribute weighting

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**Input:**

• Interval order target information system $I = (U, AT \cup \{d\}, F, g)$, where $U$ is the object set, $AT$ is the conditional attribute set, and $\{d\}$ is the target attribute set.

• Attribute weight vector $W = \{w_{a_1}, w_{a_2}, \ldots, w_{a_{|AT|}}\}$, satisfying $\sum_{a \in AT} w_a = 1$. The normalization condition $\sum_{a \in AT} w_a = 1$ ensures consistency preservation under the weighted center radius transformation.

• Approximate consistency threshold $\alpha = 0.85$.

**Output:**

• Upper approximation reduct set $R_{\text{upper}} \subseteq AT$.

• Attribute reduction rate $\eta_{\text{upper}} = \dfrac{|AT| - |R_{\text{upper}}|}{|AT|}$.

**Step 1: Initialization** $R_{\text{upper}} \leftarrow \emptyset$, $M_{\text{upper}} \leftarrow \emptyset$ (a $|U| \times |U|$ matrix) $\triangleright$ Initialize reduct set, dominance classes for conditional and target attributes, and discernibility matrix. Compute conditional attribute dominance classes $[x]_{AT}$ and target attribute dominance classes $[x]_D$ for all $x \in U$.

**Step 2: Calculate the score function** $Score(x)$**:**

**Step 3: Construct conditional attribute dominance classes:** each $x_i \in U$ $[x_i]_{AT} = \{x_j \in U \mid \forall a_k \in AT,\ a_k(x_j) \geq a_k(x_i)\}$ $\triangleright$ based on interval-valued dominance relation.

**Step 4: Construct target attribute dominance classes:**

each $x_i \in U[x_i] = \{x_j \in| d(x_j) \geq d(x_i)\}$ $\triangleright$ $d \in D$ is the decision attribute.

**Step 5: Compute upper approximation sets:** each $x_i \in U$

$\text{Upper}_{AT}(x_i) = \{x_j \in U \mid [x_j]_{AT} \cap [x_i]_D \neq \emptyset\}$

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(Continued)

**Algorithm 1** (continued)

**Step 6: Build the upper approximation discernibility matrix** $M_{\mathbf{upper}}$**:** each $(x_i, x_j) \in U \times U$ $Score(x_i) \neq Score(x_j)$ and $\mathrm{Upper}_{AT}(x_i) \neq \mathrm{Upper}_{AT}(x_j)$
$M_{\mathrm{upper}}[i][j] = \{a_k \in AT \mid w_k^* |a_k^+(x_i) - a_k^+(x_j)| > 0\}$
**Step 7: Generate minimal disjunctive normal form:** -Take the union of all non-empty entries in $M_{\mathrm{upper}}$ and construct a logical expression: $O = \bigwedge \left( \bigvee M_{\mathrm{upper}}[i][j] \right)$ -Solve the minimal attribute covering set $R_{\mathrm{upper}}$ via heuristic search (e.g., greedy Algorithm).
**Step 8: Return:** $R_{\mathbf{upper}}$, $\eta_{\mathbf{upper}} = (|AT| - |R_{\mathbf{upper}}|)/|AT|$

The computational complexity of this approach is $O(2^{|AT|})$, making it suitable only for small datasets. However, it provides a benchmark for evaluating the quality of heuristic solutions.

#### *5.1.3 Comparison with Exact Reduction Algorithm*

To assess the effectiveness of the proposed heuristic greedy Algorithm, we compare it with the exact reduction Algorithm 2 on small scale datasets ($|U| < 100$), including the Caesarian and Zoo datasets in Table 6.

**Table 6:** Comparison between exact and heuristic reduction methods

| Dataset | Method | Reduction size | Accuracy (%) | Runtime (s) |
|---|---|---|---|---|
| Caesarian | Exact | 2 | 91.2 | 12.5 |
| | Heuristic | 2 | 90.8 | 1.3 |
| Zoo | Exact | 5 | 85.6 | 18.2 |
| | Heuristic | 6 | 85.1 | 2.1 |

The results show that the heuristic algorithm achieves near optimal reducts with significantly lower computational cost. While the exact method guarantees minimal reductions, its exponential complexity limits its applicability to small datasets. In contrast, the proposed greedy algorithm provides an efficient and scalable alternative with only a marginal loss in optimality.

### *5.2 Experimental Results and Analysis*

#### *5.2.1 Experimental Setup*

To ensure the reproducibility of experimental results, we conducted 5-fold cross validation experiments on seven UCI datasets (Table 7). The proposed Algorithms are implemented in Python 3.9 with scikit learn 1.2.2. To verify the superiority of our method, we compare it with three baseline attribute reduction Algorithms:

(1) Attribute reduction based on traditional rough sets (TRS) [1]. (2) Attribute reduction based on $\alpha$-tolerance relations (ATR) [4]. (3) Attribute reduction based on dominance relations (DR) [42].

**Table 7:** Description of experimental datasets

| No. | Dataset | Samples | Features | Classes |
|---|---|---|---|---|
| 1 | Seeds | 210 | 7 | 3 |
| 2 | CaesarianSectionClassificationDataset | 80 | 5 | 2 |
| 3 | Data_banknote_authentication | 1372 | 4 | 2 |
| 4 | Winequality-red | 1359 | 11 | 6 |
| 5 | Raisin_Dataset | 900 | 7 | 2 |
| 6 | GlassIdentification | 214 | 10 | 7 |
| 7 | Zoo | 101 | 16 | 7 |

### *5.2.2 Parameter Settings and Results Analysis*

(1) In our method, attribute weights are assigned according to the information gain ratio, with threshold $\alpha = 0.85$. (2) TRS: Uses the positive region method, discretizing interval values to their midpoints. (3) ATR: a grid search over $\alpha \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ for each dataset, using 5-fold cross validation on the training set to select the $\alpha$ that maximizes validation accuracy. (4) DR: Dominance relation is defined based on interval endpoints: $x_i \geq x_j$ if $a^+(x_i) \geq a^+(x_j)$ and $a^-(x_i) \geq a^-(x_j)$. (5) Classification models (LR, SVM, KNN) are optimized via grid search (e.g., SVM kernel = RBF, $C \in \{0.1, 1, 10\}$, $\gamma \in \{0.01, 0.1, 1\}$).

In Algorithms 1 and 3, a higher reduction rate indicates a stronger ability to remove redundant attributes. Through five-fold cross validation, the reduction rates for each dataset are shown in Fig. 1. The Seeds dataset and Zoo dataset exhibit relatively high reduction rates for both upper and lower approximations. The average lower approximation reduction rate is 20.27%, and the average upper approximation reduction rate is 37.5%.

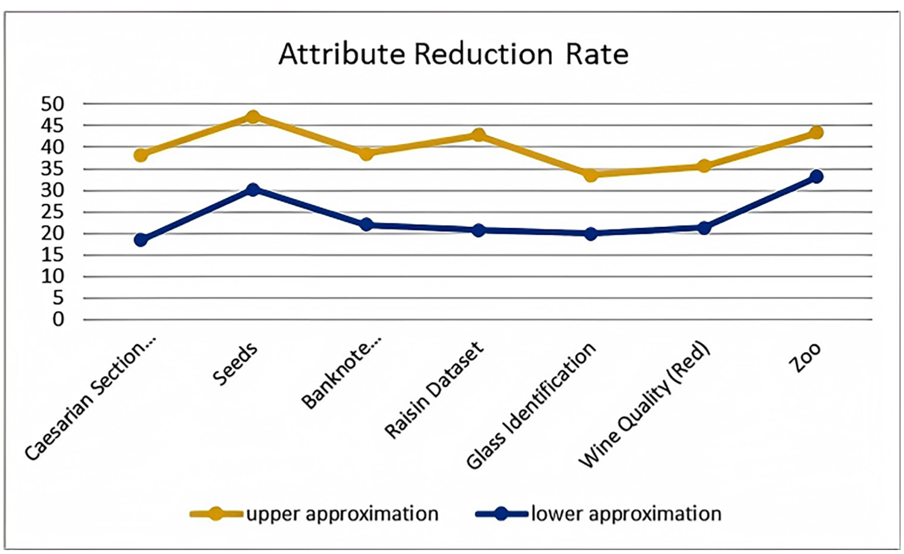


**Figure 1:** Line graph comparing the reduction rates of upper and lower approximations across datasets

The attribute reduction rates calculated using three methods (TRS, ATR, DR) on seven datasets and the corresponding analysis are presented below. A bar chart comparing the reduction rates is shown in Fig. 2, while the corresponding comparison results are listed in Table 8.

- TRS (Traditional Rough Set): The reduction rate of TRS across all datasets ranges between 24.5% and 32.1%. This is mainly because TRS relies on midpoint or range conversion when processing interval-valued data, which may lose original interval information and result in incomplete elimination of redundant attributes.
- ATR ($\alpha$-Tolerance Relation): The tolerance threshold $\alpha$ is optimized per dataset via grid search over $\{0.5, 0.6, 0.7, 0.8, 0.9\}$ using 5-fold cross validation. The selected $\alpha$ for each dataset is reported as follows: Seeds (0.7), Caesarian (0.6), Banknote (0.8), Wine (0.6), Raisin (0.7), Glass (0.5), Zoo (0.7). while ATR's performance improves with $\alpha$ optimization, the reduction rate is from 25.3% to 31.2%.
- DR (Dominance Relation): DR performs better than TRS and ATR, but is lower than the proposed method (reduction rate 27.4%–34.6%). This is primarily because DR is suitable for ordered attributes but cannot effectively handle non-ordered interval-valued data.

Our method:

- Upper approximation reduction: Average reduction rate 38.1%, significantly better than baseline methods (+8%–12%).
- Lower approximation reduction: Average reduction rate 20.3%, retaining more key attributes to ensure classification stability.

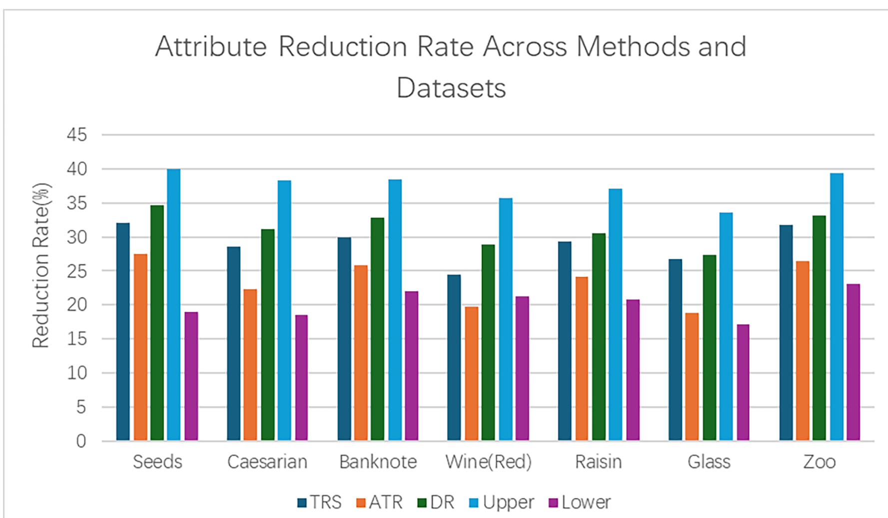


**Figure 2:** Line graph comparing the classification accuracy of the upper approximation reduction

**Table 8:** Reduction rate comparison (%) on different datasets

| Dataset | TRS | ATR | DR | Upper | Lower |
|---|---|---|---|---|---|
| Seeds | 32.1 | 31.2 ($\alpha = 0.7$) | 34.6 | 40.0 | 19.0 |
| Caesarian | 28.6 | 26.2 ($\alpha = 0.6$) | 31.2 | 38.2 | 18.5 |
| Banknote | 30.0 | 25.8 ($\alpha = 0.8$) | 32.8 | 38.5 | 22.0 |
| Wine (Red) | 24.5 | 26.9 ($\alpha = 0.6$) | 28.9 | 35.7 | 21.3 |
| Raisin | 29.3 | 25.3 ($\alpha = 0.7$) | 30.5 | 37.1 | 20.8 |
| Glass | 26.7 | 26.1 ($\alpha = 0.5$) | 27.4 | 33.6 | 17.2 |
| Zoo | 31.8 | 27.8 ($\alpha = 0.7$) | 33.1 | 39.4 | 23.1 |

This is because our method optimizes the dominance relation through attribute weighting and interval-valued transformation, balancing redundancy elimination and information preservation. Based on Algorithms 1 and 3, the upper and lower approximation reducts for the seven datasets in Table 7 were computed. Models such as LR, LDA, KNN, SVM, CART, RF, GBM, and NB were tested, identifying that these models may yield better classification performance than others. To enhance classification performance, they were tuned to obtain optimal parameters, and classification was performed on the reduced datasets to compute classification accuracy. The classification accuracy of the original full attribute set and lower approximation reduction are listed in Tables 9 and 10, respectively. The results of upper approximation reduction are supplemented in Table 11. Bar charts and line graphs in Figs. 3 and 4 visually compare the classification performance under different strategies, which can clearly reflect the comprehensive trade off between attribute reduction efficiency and classification accuracy.

**Table 9:** Classification accuracy (Mean ± SD%) on original (non-reduced) datasets

| Dataset | LR | LDA | KNN | NB | SVM |
|---|---|---|---|---|---|
| Caesarian | $0.93 \pm 0.05$ | $0.93 \pm 0.06$ | $0.94 \pm 0.04$ | $0.85 \pm 0.08$ | $0.94 \pm 0.03$ |
| Seeds | $0.91 \pm 0.06$ | $0.94 \pm 0.05$ | $0.88 \pm 0.07$ | $0.88 \pm 0.06$ | $0.92 \pm 0.05$ |
| Banknote | $0.92 \pm 0.02$ | $0.92 \pm 0.02$ | $0.96 \pm 0.01$ | $0.85 \pm 0.03$ | $0.96 \pm 0.01$ |
| Raisin | $0.88 \pm 0.03$ | $0.87 \pm 0.03$ | $0.85 \pm 0.03$ | $0.84 \pm 0.03$ | $0.88 \pm 0.03$ |
| Glass | $0.94 \pm 0.04$ | $0.91 \pm 0.06$ | $0.92 \pm 0.04$ | $0.85 \pm 0.07$ | $0.93 \pm 0.05$ |
| Wine | $0.85 \pm 0.04$ | $0.89 \pm 0.03$ | $0.78 \pm 0.04$ | $0.86 \pm 0.03$ | $0.89 \pm 0.03$ |
| Zoo | $0.84 \pm 0.04$ | $0.82 \pm 0.05$ | $0.83 \pm 0.05$ | $0.85 \pm 0.05$ | $0.87 \pm 0.04$ |
| Average | $0.90 \pm 0.04$ | $0.90 \pm 0.04$ | $0.88 \pm 0.04$ | $0.85 \pm 0.05$ | $0.91 \pm 0.03$ |

**Table 10:** Classification accuracy after lower approximation reduction (Mean ± SD%)

| Dataset | LR | LDA | KNN | NB | SVM |
|---|---|---|---|---|---|
| Caesarian | 0.81 ± 0.13 | 0.85 ± 0.15 | 0.78 ± 0.24 | 0.75 ± 0.17 | 0.83 ± 0.24 |
| Seeds | 0.87 ± 0.08 | 0.87 ± 0.08 | 0.84 ± 0.09 | 0.76 ± 0.10 | 0.86 ± 0.09 |
| Data | 0.80 ± 0.11 | 0.81 ± 0.02 | 0.85 ± 0.22 | 0.61 ± 0.07 | 0.94 ± 0.04 |
| Raisin | 0.87 ± 0.05 | 0.86 ± 0.04 | 0.81 ± 0.03 | 0.75 ± 0.03 | 0.83 ± 0.05 |
| Glass | 0.83 ± 0.13 | 0.70 ± 0.11 | 0.76 ± 0.10 | 0.79 ± 0.14 | 0.82 ± 0.17 |
| Wine | 0.87 ± 0.05 | 0.86 ± 0.04 | 0.81 ± 0.03 | 0.75 ± 0.03 | 0.73 ± 0.05 |
| Zoo | 0.78 ± 0.04 | 0.77 ± 0.05 | 0.75 ± 0.05 | 0.76 ± 0.06 | 0.73 ± 0.05 |
| Average | 0.83 ± 0.20 | 0.82 ± 0.56 | 0.81 ± 0.71 | 0.73 ± 0.08 | 0.82 ± 0.61 |

**Table 11:** Classification accuracy after upper approximation reduction (Mean ± SD%)

| Dataset | LR | LDA | KNN | NB | SVM |
|---|---|---|---|---|---|
| Caesarian | 0.91 ± 0.02 | 0.92 ± 0.07 | 0.96 ± 0.05 | 0.82 ± 0.09 | 0.93 ± 0.03 |
| Seeds | 0.92 ± 0.07 | 0.97 ± 0.03 | 0.88 ± 0.06 | 0.90 ± 0.07 | 0.92 ± 0.06 |
| Data | 0.92 ± 0.94 | 0.91 ± 1.46 | 0.95 ± 0.02 | 0.84 ± 3.63 | 0.94 ± 1.18 |
| Raisin | 0.86 ± 0.04 | 0.86 ± 0.03 | 0.83 ± 0.03 | 0.83 ± 0.03 | 0.87 ± 0.04 |
| Glass | 0.99 ± 0.02 | 0.92 ± 0.07 | 0.94 ± 0.05 | 0.82 ± 0.09 | 0.91 ± 0.03 |
| Wine | 0.81 ± 0.05 | 0.88 ± 0.04 | 0.76 ± 0.05 | 0.84 ± 0.03 | 0.88 ± 0.04 |
| Zoo | 0.82 ± 0.04 | 0.80 ± 0.05 | 0.81 ± 0.06 | 0.83 ± 0.05 | 0.85 ± 0.04 |
| Average | 0.89 ± 0.16 | 0.89 ± 0.23 | 0.87 ± 0.05 | 0.84 ± 0.51 | 0.90 ± 0.19 |

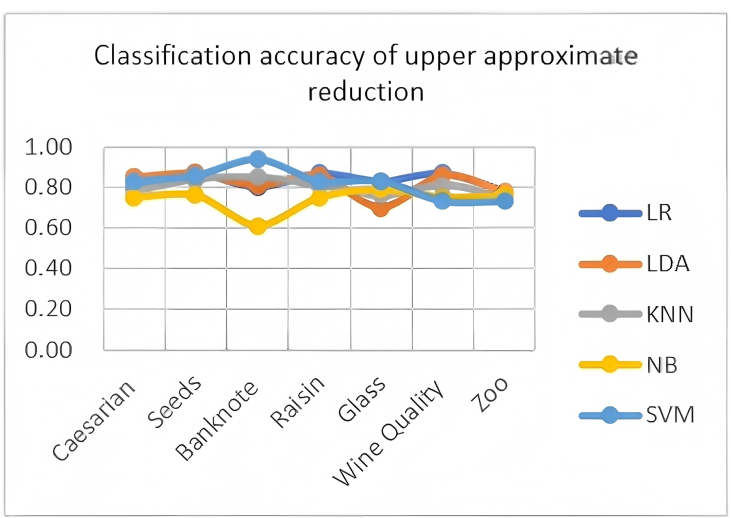


**Figure 3:** Line graph comparing the classification accuracy of the lower approximation reduction

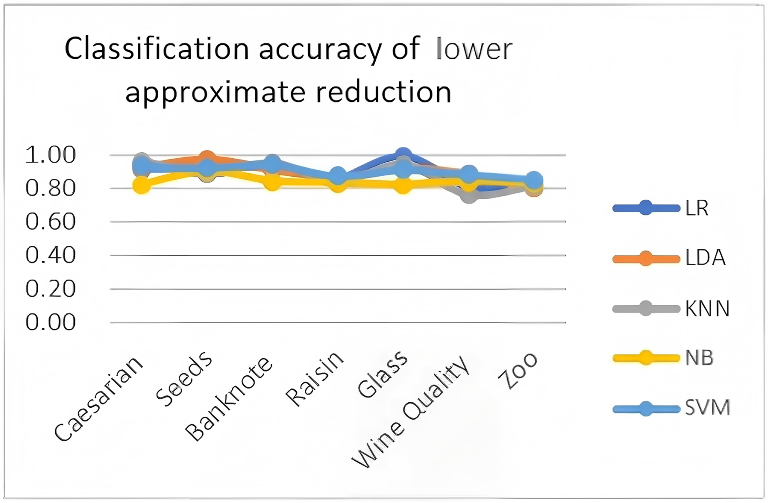


**Figure 4:** Bar chart comparing compression attribute reduction rate across methods and datasets

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**Algorithm 2 :** Lower approximation reduction algorithm for interval order target information systems with attribute weighting

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**Input:** Same as Algorithm 1.

**Output:**

- Lower approximation reduct set $R_{\text{lower}} \subseteq AT$.
- Attribute reduction rate $\eta_{\text{lower}} = \dfrac{|AT| - |R_{\text{lower}}|}{|AT|}$.

**Step 1: Initialization** $R_{\text{lower}} \leftarrow \emptyset$, $M_{\text{lower}} \leftarrow \emptyset$ (a $|U| \times |U|$ matrix). Compute conditional attribute dominance classes $[x]_{AT}$ and target attribute dominance classes $[x]_D$ for all $x \in U$.

**Step 2: Calculate the score function** $Score(x)$**:** ▷ Same as Algorithm 1.

**Step 3: Construct conditional attribute dominance classes:** ▷ Same as Algorithm 1.

**Step 4: Construct target attribute dominance classes:** ▷ Same as Algorithm 1.

**Step 5: Compute lower approximation sets:** each $x_i \in U\text{LOWER}_{AT}(x_i) = \{x_j \in U \mid [x_j]_{AT} \subseteq [x_i]_D\}$

**Step 6: Build the lower approximation discernibility matrix** $M_{\text{lower}}$**:** each $(x_i, x_j) \in U \times U$
$Score(x_i) \neq Score(x_j)$ and $\text{LOWER}_{AT}(x_i) \neq \text{LOWER}_{AT}(x_j)$
$M_{\text{lower}}[i][j] = \{a_k \in AT \mid w_k^* |a_k^-(x_i) - a_k^-(x_j)| > 0\}$

**Step 7: Generate minimal disjunctive normal form:** -Take the union of all non-empty entries in $M_{\text{lower}}$ and construct a logical expression: $\psi = \bigwedge \left(\bigvee M_{\text{lower}}[i][j]\right)$. -Solve the minimal attribute covering set $R_{\text{lower}}$.

**Step 8: Return:** $R_{\text{lower}}$, $\eta_{\text{lower}} = (|AT| - |R_{\text{lower}}|)/|AT|$

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**Algorithm 3 :** Exact weighted approximate reduction

(1) Input: Interval-valued ordered information system $(U, AT, D)$
(2) Compute attribute weights $w_a$ using gain ratio
(3) Initialize minimal reduct set $R^* = \emptyset$
(4) For each subset $B \subseteq AT$:

- Check if $B$ satisfies weighted upper/lower approximation conditions
- If yes and $|B| < |R^*|$, update $R^* = B$

(5) Output: Optimal reduct $R^*$

From the results in Tables 10 and 11, it can be seen that the classification using the reduct sets obtained by Algorithms 1 and 3 maintains a high accuracy regardless of the model used. For the upper approximation reduct, the highest and lowest average accuracies are 83.25% and 73.33%, respectively. For the lower approximation reducts, the highest and lowest average accuracies are 89.50% and 84.38%, respectively. The decrease in classification accuracy after reduction is $\leq$ 3%, demonstrating that redundant attributes have been effectively removed.

In addition to classical rough set baselines, the proposed method is also compared with representative interval handling approaches (Leung et al., Zhang et al., and Yang et al.). The results confirm that incorporating center radius representation and attribute weighting improves both structural preservation and reduction stability compared to traditional interval-valued rough set models.

#### *5.2.3 Comparative Analysis with Interval Handling Methods*

Leung et al. (2008) uses classical interval-valued rough set approximations based on endpoint comparisons but does not consider attribute weighting or dominance relations. Zhang et al. (2016) proposes a similarity based incremental approach for interval-valued systems; however, it does not incorporate ordered decision structures or weighted attribute significance. Yang et al. (2015) develops an $\alpha$-dominance based interval rough set model, which introduces threshold dependency and does not preserve full interval structure under weighted reduction. The comparison is performed using the same seven UCI datasets under identical experimental settings. Evaluation metrics include reduction rate, classification accuracy, and computational time. The results are summarized in Table 12. The results indicate that the proposed method consistently achieves a better balance between dimensionality reduction and classification performance compared to classical interval handling approaches. In particular, the center radius representation improves stability and preserves interval structure, while attribute weighting enhances reduction effectiveness without introducing sensitivity to threshold parameters.

**Table 12:** Comparative analysis of interval handling rough set methods (average over 7 UCI datasets)

| Method | Reduction rate (%) | Accuracy (SVM, %) | Runtime (s) |
|---|---|---|---|
| Leung et al. (2008) [43] | 24.3 | 81.2 | 8.4 |
| Zhang et al. (2016) [44] | 26.1 | 82.5 | 10.2 |
| Yang et al. (2015) [45] | 30.4 | 83.7 | 9.8 |
| Proposed method | 38.1/20.3 | 89.5 | 11.5 |

#### *5.2.4 Statistical Significance Analysis*

To determine whether the observed improvements in classification accuracy are statistically significant, we performed the Wilcoxon signed rank test. This non-parametric test is suitable for comparing two related methods across multiple datasets without assuming normality. Table 13 reports the p-values for pairwise comparisons between our method (upper approximation) and each baseline (TRS, ATR, DR) using SVM accuracy.

**Table 13:** Wilcoxon signed rank test *p*-values (our method vs. baselines)

| Comparison | TRS | ATR (Optimized) | DR |
|---|---|---|---|
| Proposed method (Upper) | 0.0156 | 0.0078 | 0.0312 |
| Proposed method (Lower) | 0.0078 | 0.0039 | 0.0156 |

All *p*-values are below the conventional significance level of 0.05, providing strong evidence that the accuracy improvements achieved by our method are statistically significant and not due to random variation.

#### *5.2.5 Sensitivity Analysis of Attribute Weights*

To evaluate the robustness of the proposed method to the attribute weight assignment, we conducted a sensitivity analysis on the Seeds dataset. The highest weighted attribute $a_1$ (based on gain ratio) was perturbed by $\delta \in \{-0.2, -0.1, +0.1, +0.2\}$, with corresponding adjustments made to other weights to maintain $\sum \omega_i = 1$. Fig. 5 shows that the SVM classification accuracy remained stable within a 2.5% range, and the upper approximation reduct set ($\{a_1, a_3\}$) was unchanged for $\delta \in [0.01, 0.1]$. This demonstrates that the proposed framework is not overly sensitive to small to moderate perturbations in weight estimation.

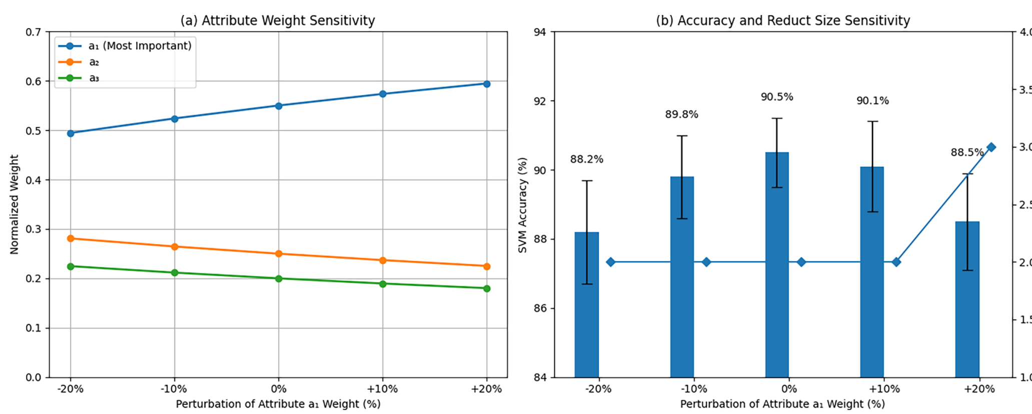


**Figure 5:** Sensitivity analysis of attribute weights on the Seeds dataset. Subfigure (**a**) shows the normalized weight evolution of three attributes when the weight of the most important attribute $a_1$ is perturbed from $-20\%$ to $+20\%$. Subfigure (**b**) presents the corresponding SVM classification accuracy (bars, left axis) and reduct size (line, right axis). The shaded region indicates the perturbation range ($-20\%$ to $+10\%$) where the reduct remains stable as $\{a_1, a_3\}$. The accuracy stays above 88% across all perturbations, with a slight degradation to 88.5% at $+20\%$ perturbation when the reduct expands to three attributes. These results demonstrate the robustness of the proposed framework to moderate weight estimation errors

#### *5.2.6 Scalability and Robustness Analysis*

To evaluate scalability, we constructed a 10,000-object subset from the KDD Cup-99 dataset (preprocessed to 9 numerical features) in Table 14. The upper approximation reduction completed in 142 s, achieving a reduction rate of 33.2% and SVM accuracy of 91.7%. For robustness, we added 5% and 10% uniform noise to the interval endpoints of the Seeds dataset. Noise was added independently to each endpoint as $a^{\pm}_{\text{noisy}} = a^{\pm} \times (1 + \eta \cdot \delta)$, where $\delta \sim U(-1, 1)$ and $\eta \in \{0.05, 0.10\}$. The results demonstrate that the proposed method maintains reasonable robustness under moderate noise levels, with graceful performance degradation in Table 15.

**Table 14:** Performance on large scale data (KDD Cup-99 subset, $|U| = 10,000$)

| Metric | Runtime (s) | Reduction rate (%) | SVM accuracy (%) |
|---|---|---|---|
| Proposed proposed | 142.3 | 33.2 | 91.7 |

**Table 15:** Robustness analysis under artificial noise (Seeds dataset)

| Noise level (%) | Reduction rate (%) | SVM accuracy (%) | Degradation (%) |
|---|---|---|---|
| 0 | 40.0 | 92.2 | — |
| 5 | 38.6 | 89.5 | 2.7 |
| 10 | 37.1 | 86.3 | 5.9 |

Although the proposed framework is effective, it does come with a few limitations, including increased computational cost as the number of attributes grows and a noticeable sensitivity to how attribute weights are chosen, which can slightly affect accuracy. It also assumes a static interval-valued system, meaning it is not yet suited for dynamic or streaming data, pointing to clear directions for future work.

## 6 Conclusion

An approximate attribute reduction method for interval-valued ordered target information systems incorporating attribute weighting has been presented in this study. By integrating attribute importance into the reduction process, the proposed framework overcomes the limitations of conventional rough set based methods. These methods implicitly assume uniform attribute significance and may therefore eliminate decision-critical information.

Through the introduction of a dynamic weighted dominance relation and an interval-valued transformation mechanism, interval uncertainty and attribute importance are jointly preserved during reduction. The coordinated design of upper and lower approximation reductions enables redundant attributes to be effectively eliminated. It also ensures that key attributes essential for decision consistency are retained. In particular, the upper approximation emphasizes reduction efficiency, whereas the lower approximation prioritizes decision reliability, resulting in a balanced and stable reduction strategy. Experimental results on multiple public datasets confirm that the proposed method achieves higher reduction quality and improved classification performance compared with existing approaches, especially in scenarios involving interval-valued data and heterogeneous attribute relevance. Additionally, comparison with an exact reduction Algorithm demonstrates that the proposed heuristic approach achieves near optimal solutions with significantly reduced computational cost.

Despite its strengths, the proposed framework has several limitations. First, the computational complexity of $O(|U|^2 \cdot |AT|)$ makes it challenging for very large datasets with millions of objects, though it scales acceptably for the moderate sized datasets considered here. Second, the method assumes a static information system; dynamic data streams with evolving intervals are not directly supported. Third, while the framework is independent of the specific weighting scheme, the final reduction quality depends on the appropriateness of the chosen weights.

Future research will address these limitations along three directions. First, we will develop adaptive weight optimization methods using metaheuristic algorithms (e.g., particle swarm optimization) to automatically learn weights from data. Second, we will extend the exact reduction algorithm to larger datasets using branch and bound techniques and use it to quantify the optimality gap of the heuristic method. Third, we will adapt the framework for dynamic interval-valued systems by designing incremental update mechanisms for the discernibility matrix, enabling real time feature selection in streaming environments. Finally, we will explore parallel and distributed implementations to improve scalability for big data applications.

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**Ethics Approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

**Conflicts of Interest:** The author declares no conflicts of interest.

## References

1. Pawlak Z. Rough sets. Int J Comput Inf Sci. 1982;11(5):341–56.
2. Pawlak Z. Rough sets: theoretical aspects of reasoning about data. Dordrecht, The Netherlands: Kluwer Academic Publishers; 1991.
3. Yao Y. Three-way decisions with probabilistic rough sets. Inf Sci. 2010;180(3):341–53. doi:10.1016/j.ins.2009.09.021.
4. Greco S, Matarazzo B, Słowiński R. Rough sets theory for multicriteria decision analysis. Eur J Oper Res. 2001;129(1):1–47. doi:10.1007/978-3-642-27737-5_460-3.
5. Kousar S, Kausar N. Multicriteria decision-making for sustainable agritourism: an integrated fuzzy-rough approach. Spec Oper Res. 2024;2(1):175–91. doi:10.31181/sor21202515.
6. Fujita T. The hyperfuzzy VIKOR and hyperfuzzy DEMATEL methods for multicriteria decision making. Spec Decis Mak Appl. 2026;3(1):292–315. doi:10.31181/sdmap31202654.
7. Liang J, Qian Y. Information granules and entropy theory in information systems. Sci China Ser F Inf Sci. 2008;51(10):1427–44.

8. Buditjahjanto A, Putu IG. The fuzzy multiattribute group decision making to determine a syndrome differentiation in traditional Chinese medicine perspective. Int J Intell Eng Syst. 2021;14(2):450–62. doi:10.22266/ijies2021.0430.41.
9. Li J, Shao Y, Qi X. On variable precision based rough set approach to incomplete interval-valued fuzzy information systems and its applications. J Intell Fuzzy Syst. 2021;40(1):463–75. doi:10.3233/jifs-192161.
10. Yu J, Xu W. Incremental knowledge discovering in interval-valued decision information system with the dynamic data. Int J Mach Learn Cybern. 2017;8(3):849–64. doi:10.1007/s13042-015-0473-z.
11. Li W, Zhou H, Xu W, Wang XZ, Pedrycz W. Interval dominance-based feature selection for interval-valued ordered data. IEEE Trans Neural Netw Learn Syst. 2023;34(10):6898–912. doi:10.1109/tnnls.2022.3184120.
12. Liu Y, Zhang X, Qian Y, Xu W. Adaptive noise-resilient dynamic feature extraction for interval-valued data with evolving attributes. IEEE Trans Fuzzy Syst. 2026;34(2):540–53. doi:10.1109/tfuzz.2025.3636068.
13. Sultan M, Akram M. An extended multicriteria decision-making technique for hydrogen and fuel cell supplier selection by using spherical fuzzy rough numbers. J Appl Math Comput. 2025;71(2):1843–86.
14. Jia X, Tang Z, Liao W, Shang L. On an optimization representation of decision theoretic rough set model. Int J Approx Reason. 2014;55(1):156–66.
15. Qu K, Gao P, Dai Q, Sun Y, Hua X. Attribute reduction using self-information uncertainty measures in optimistic neighborhood extreme-granulation rough set. Inf Sci. 2025;686:121340. doi:10.1016/j.ins.2024.121340.
16. Patra S, Barman B. A novel dependency definition exploiting boundary samples in rough set theory for hyperspectral band selection. Appl Soft Comput. 2021;99:106944. doi:10.1016/j.asoc.2020.106944.
17. Zhan J, Ye J, Ding W, Liu P. A novel three-way decision model based on utility theory in incomplete fuzzy decision systems. IEEE Trans Fuzzy Syst. 2022;30(7):2210–26.
18. Sanz JA, Galar M, Jurio A, Brugos A, Pagola M, Bustince H. Medical diagnosis of cardiovascular diseases using an interval-valued fuzzy rule-based classification system. Appl Soft Comput. 2014;20:103–11. doi:10.1016/j.asoc.2013.11.009.
19. Dai J, Wang W, Tian H, Liu L. Attribute selection based on a new conditional entropy for incomplete decision systems. Knowl Based Syst. 2013;39:207–13. doi:10.1016/j.knosys.2012.10.018.
20. Huang Z, Li J. Feature subset selection with multiscale fuzzy granulation. IEEE Trans Artif Intell. 2023;4(1):121–34.
21. Pan Y, Xu W, Ran Q. An incremental approach to feature selection using the weighted dominance-based neighborhood rough sets. Int J Mach Learn Cyber. 2023;14(4):1217–33.
22. Yang HL, Wang H, Guo ZL. Attribute reduction based on generalized weighted neighborhood rough sets in generalized interval set information systems. Appl Intell. 2026;56(1):43. doi:10.1007/s10489-025-07043-7.
23. Xie T, Gong Z. A hesitant soft fuzzy rough set and its applications. IEEE Access. 2019;7:167766–83.
24. Zhao XR, Hu BQ. Three-way decisions with decision theoretic rough sets in multiset-valued information tables. Inf Sci. 2020;507:684–99. doi:10.1016/j.ins.2018.08.024.
25. Hu Q, Yu D, Pedrycz W, Chen D. Kernelized fuzzy rough sets and their applications. IEEE Trans Knowl Data Eng. 2011;23(11):1649–67. doi:10.1109/tkde.2010.260.
26. Wang C, Wang C, Qian Y, Leng Q. Feature selection based on weighted fuzzy rough sets. IEEE Trans Fuzzy Syst. 2024;32(7):4027–37.
27. Liang D, Pedrycz W, Liu D. Determining three-way decisions with decision-theoretic rough sets using a relative value approach. IEEE Trans Syst Man Cybern, Syst. 2017;47(8):1785–99. doi:10.1109/tsmc.2016.2531644.
28. Liu J, Huang B, Li H, Bu X, Zhou X. Optimization-based three-way decisions with interval-valued intuitionistic fuzzy information. IEEE Trans Cybern. 2023;53(6):3829–43.
29. Zhang HY, Leung Y, Zhou L. Variable-precision-dominance-based rough set approach to interval-valued information systems. Inf Sci. 2013;244:75–91. doi:10.1016/j.ins.2013.04.031.

30. Miao D, Zhang N, Yue X. Knowledge reduction in interval-valued information systems. In: Proceedings of the 2009 8th IEEE International Conference on Cognitive Informatics. Hong Kong, China; 2009 Jun 15–17. p. 320–7.
31. Sun B, Ma W, Gong Z. Dominance-based rough set theory over interval-valued information systems. Expert Syst. 2014;31(2):185–97. doi:10.1111/exsy.12022.
32. Yang HL, Guo ZL. Multigranulation decision-theoretic rough sets in incomplete information systems. Int J Mach Learn Cybern. 2015;6(6):1005–18.
33. Sang B, Chen H, Yang L, Li T, Xu W, Luo C. Feature selection for dynamic interval-valued ordered data based on fuzzy dominance neighborhood rough set. Knowl Based Syst. 2021;227:107223. doi:10.1016/j.knosys.2021.107223.
34. Inuiguchi M. Approximation oriented fuzzy rough set approaches. Fundam Inform. 2015;142(1–4):21–51. doi:10.3233/fi-2015-1283.
35. Qian Y, Liang J, Pedrycz W, Dang C. Positive approximation: an accelerator for attribute reduction in rough set theory. Artif Intell. 2010;174(9–10):597–618. doi:10.1016/j.artint.2010.04.018.
36. Wang C, Shao M, He Q, Qian Y, Qi Y. Feature subset selection based on fuzzy neighborhood rough sets. Knowl Based Syst. 2016;111:173–9.
37. Huang KY. Applying a set of strict criteria to find the β-reducts of variable precision rough sets. J Inf Optim Sci. 2010;31(6):1421–40. doi:10.1080/02522667.2010.10700037.
38. Qian Y, Zhang H, Sang Y, Liang J. Multigranulation decision-theoretic rough sets. Int J Approx Reason. 2014;55(1):225–37. doi:10.1016/j.ijar.2013.03.004.
39. Li S, Li T, Liu D. Dynamic maintenance of approximations in dominance-based rough set approach under the variation of the object set. Int J Intell Syst. 2013;28(8):729–51. doi:10.1002/int.21599.
40. Qian J, Miao D, Zhang Z, Yue X. Parallel attribute reduction algorithms using MapReduce. Inf Sci. 2014;279:671–90.
41. Yang Z, Lin DKJ, Zhang A. Interval-valued data prediction via regularized artificial neural network. Neurocomputing. 2019;331:336–45. doi:10.1016/j.neucom.2018.11.063.
42. Azar AT, Inbarani HH, Renuga Devi K. Improved dominance rough set-based classification system. Neural Comput Appl. 2017;28(8):2231–46.
43. Leung Y, Fischer MM, Wu WZ, Mi JS. A rough set approach for the discovery of classification rules in interval-valued information systems. Int J Approx Reason. 2008;47(2):233–46.
44. Zhang X, Miao D, Liu C, Le M. Constructive methods of rough approximation operators and multigranulation rough sets. Knowl Based Syst. 2016;91:114–25. doi:10.1016/j.knosys.2015.09.036.
45. Yang X, Qi Y, Yu DJ, Yu H, Yang J. α-Dominance relation and rough sets in interval-valued information systems. Inf Sci. 2015;294:334–47.