Approximate Calculation of the Generalized Erdélyi-Kober Operator Using a Cubic Spline

Shakhobiddin Karimov[1,#], Erkinjon Islamov[1,#] and Elina Shishkina[2,3,4,*,#]

[1] Department of Applied Mathematics and Computer Science, Fergana State University, Fergana, 150100, Uzbekistan

[2] Voronezh State University, Universitetskaya Pl., 1, Voronezh, 394018, Russia

[3] Belgorod State National Research University (“BelGU”), Pobedy Street, 85, Belgorod, 308015, Russia

[4] Kadyrov Chechen State University, Sheripova Str., 32, Grozny, 364024, Russia

[#] These authors contributed equally to this work

INFORMATION

Keywords:
Generalized Erdélyi-Kober operator
fractional calculus
cubic spline
interpolation
numerical approximation
convergence rate
stability
error analysis

DOI: 10.23967/j.rimni.2025.10.68821

SCIPEDIA
Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería
RIMNI

Approximate Calculation of the Generalized Erdélyi-Kober Operator Using a Cubic Spline

Shakhobiddin Karimov[1,#], **Erkinjon Islamov**[1,#] **and Elina Shishkina**[2,3,4,*,#]

[1]Department of Applied Mathematics and Computer Science, Fergana State University, Fergana, 150100, Uzbekistan

[2]Voronezh State University, Universitetskaya Pl., 1, Voronezh, 394018, Russia

[3]Belgorod State National Research University ("BelGU"), Pobedy Street, 85, Belgorod, 308015, Russia

[4]Kadyrov Chechen State University, Sheripova Str., 32, Grozny, 364024, Russia

[#]These authors contributed equally to this work

ABSTRACT

This article investigates the problem of approximating the generalized Erdélyi-Kober fractional operator (often referred to as the Lowndes operator) using cubic splines. A method based on cubic spline interpolation is proposed for approximating the operator on a non-uniform grid. The convergence rate of the proposed method is proven, and its stability is analyzed. Error bounds are established for functions in the class $C^4[0; b]$, providing a mathematical justification for the accuracy of the approximation. The efficiency of the method is validated through practical examples using test functions such as $f(x) = x^{4.7}$ and $f(x) = \cos x$, with results presented in graphical and numerical forms. This approach ensures high accuracy and flexibility in computing fractional integrals, which is of significant importance for solving fractional models used in physics, engineering, and other sciences. The article also provides an overview of the role of the generalized Erdélyi-Kober operator in modern fractional calculus and its applications.

OPEN ACCESS

Received: 06/06/2025

Accepted: 08/09/2025

Published: 27/10/2025

DOI
10.23967/j.rimni.2025.10.68821

Keywords:
Generalized Erdélyi-Kober operator
fractional calculus
cubic spline
interpolation
numerical approximation
convergence rate
stability
error analysis

1 Introduction

In many cases, differential and integral equations are used to describe physical, chemical, biological, and other real-world processes in mathematical terms. These equations represent the mathematical models of the studied phenomen on or process. By analyzing the solutions and properties of these model equations, it is possible to reveal characteristics of the phenomena and processes that are difficult to determine through direct experimentation. In most cases, a model equation developed for a

*Correspondence: Elina Shishkina (ilina_dico@mail.ru).

particular phenomenon or process is later found to represent many other processes as well. Therefore, every new development in the field of differential and integral calculus holds significant practical importance. Consequently, the rapid advancement of this field of mathematics is not coincidental but rather a necessity driven by real-world applications.

By extending the order of derivatives and integrals to fractional values, the concept of fractional-order calculus emerges. Fractional calculus is considered one of the important fields in modern mathematics and is effectively used in solving various applied problems. Numerous examples of the applications of fractional calculus are provided in the literature [1–5].

Fractional calculus and its applications have gained significant attention in mathematics, physics, and engineering over recent decades. The generalized Erdélyi-Kober operator, as an extended form of fractional integrals and derivatives, serves as a key tool, particularly in solving singular differential equations [6,7]. Studies conducted in [6,8,9] demonstrate that the multidimensional generalized Erdélyi-Kober operator is effective in solving higher-order differential equations, particularly those involving the Bessel differential operator, such as polycaloric and polywave equations. The properties of these operators, for instance, their application in finding exact solutions to Cauchy problems, highlight the practical significance of fractional calculus [10,11].

The one-dimensional form of the Erdélyi-Kober operator was initially studied by Erdélyi and Sneddon, with its applications to integral and differential equations extensively covered [12,13]. Subsequently, Lowndes generalized the operator by incorporating the Bessel function into its kernel (termed the generalized Erdélyi-Kober operator), yielding significant results for its one-dimensional case [14,15]. Research on multidimensional Erdélyi-Kober operators was advanced in [6], contributing substantially to the general theory of fractional calculus. These operators are widely applied in modeling diffusion processes, heat conduction, and mechanics of materials.

The problem of approximating the operator holds a significant place in modern computational techniques. Traditional methods, such as those using Riemann-Liouville operators, often lead to high computational costs and complexities [2]. The method of approximation using cubic splines, however, provides high accuracy and flexibility on non-uniform grids [16]. This approach not only enhances computational efficiency but also effectively handles the singular properties of fractional operators [17]. For instance, Diethelm's studies demonstrate the advantages of cubic splines in approximately solving fractional differential equations [17]. Furthermore, the properties of multidimensional Erdélyi-Kober operators, as presented in [18], when adapted for approximation using cubic splines, open new possibilities for solving higher-order singular equations.

In many real-world problems, constructing closed-form analytical solutions is either impossible or, even if an analytical solution is found, it may be too complex for direct use. Therefore, studying numerical methods for solving such problems is of great importance.

For fractional-order differential equations, almost all numerical methods are, in some sense, based on approximating fractional-order differential or integral operators using appropriate formulas. The numerical approximation of fractional-order integrals is often based on polynomial interpolation. Among the works in this field, the book by Oldham and Spanier [19] stands out. Reviews of various known numerical methods for fractional-order integrals and derivatives can be found in the works of Cai and Li [20], Li and Zeng [21], and other literature sources.

Due to the widespread applications of fractional calculus, the development of highly accurate, computationally efficient, and memory-efficient numerical algorithms is becoming increasingly important.

In the process of solving boundary value problems for partial differential equations with singular coefficients using transmutation operators, it becomes necessary to approximate the generalized Erdélyi-Kober operators of fractional order. Therefore, this study investigates the problem of approximating the generalized Erdélyi-Kober operator.

The following sources are noted as being closely related to the present work. In [22], the approximation of left- and right-sided Riemann-Liouville and Riesz fractional integrals using splines was studied. In [23], an optimal quadrature formula was developed for computing fractional-order integrals in a Hilbert space. In [24,25], optimal quadrature formulas were constructed for the approximate computation of fractional-order integrals in the sense of Riemann-Liouville. In [26], the approximation of the fractional-order Erdélyi-Kober integral operator using a cubic spline was investigated, while in [27], a detailed study was conducted on approximating this operator using the rectangle, trapezoidal, and midpoint methods. The article [28] examines the generalized Caputo-Katugampola fractional derivative and related examples, while also addressing the problem of its approximate computation using the ψ-predictor-corrector method.

In conclusion, the study of approximating the generalized Erdélyi-Kober operator using cubic splines significantly contributes to the theoretical and practical development of fractional calculus. This approach is not only rigorously grounded mathematically but also widely applicable in fields such as physics, engineering, and beyond, including modeling diffusion processes, heat conduction, and material mechanics [2,7,17]. Further development of this method is expected to unlock new possibilities in fractional calculus.

In this work, we examine the problem of approximating the generalized Erdélyi-Kober fractional operator $\Im^{\alpha}_{\lambda,\eta}$, as presented in [14], for a function $f(x)$ defined on an arbitrary (uniform or non-uniform) grid:

$$\Im^{\alpha}_{\lambda,\eta} f(x) = 2^{\alpha}\lambda^{1-\alpha}x^{-2\alpha-2\eta}\int_0^x t^{2\eta+1}\left(x^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right) f(t)\,dt, \tag{1}$$

where $\eta,\alpha,\lambda \in R$ and $\alpha > 0$, $\eta \geq -1/2$, $J_{\nu}(z)$-is the first-kind Bessel function [29], which is defined as follows:

$$J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(-z^2/4\right)^k}{k!\,\Gamma(\nu+k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \tag{2}$$

where $\Gamma(\alpha)$ is Euler's gamma function.

In many cases, the following form of the generalized Erdélyi-Kober operator is used:

$$\Im^{\alpha}_{\lambda,\eta} f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x t^{2\eta+1}\left(x^2-t^2\right)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right) f(t)\,dt,$$

where $\bar{J}_{\nu}(z)$-is the Bessel-Clifford function [29], which is defined as follows:

$$\bar{J}_{\nu}(z) = \Gamma(\nu+1)\,(z/2)^{-\nu}J_{\nu}(z) = \sum\nolimits_{k=0}^{\infty}\frac{\left(-z^2/4\right)^k}{(\nu+1)_k\,k!}. \tag{3}$$

Suppose that the approximate calculation of the Lowndes operator (1) at the given point $x = b$ is required.

$$\Im^{\alpha}_{\lambda,\eta} f(x)\Big|_{x=b} = 2^{\alpha}\lambda^{1-\alpha}b^{-2\alpha-2\eta}\int_0^b t^{2\eta+1}\left(b^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{b^2-t^2}\right) f(t)\,dt. \tag{4}$$

2 Cubic Spline Interpolation

Suppose that the integrand function $f(x)$ in (4) is defined on the interval $[0, b]$ and is sufficiently smooth. The considered segment $[0, b]$ is divided into n arbitrary small subintervals $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n$, where the function values $f(x_i)$ can be computed at the node points x_i.

We replace the function $y = f(x)$ with an interpolating spline [30], composed of piecewise-defined polynomials connected at the points (x_0, y_0), $(x_1, y_1), \ldots, (x_n, y_n)$, and define it as follows:

$$f(x) \cong s(x) = \begin{cases} s_0(x), & \text{if } x \in [x_0, x_1], \\ s_1(x), & \text{if } x \in [x_1, x_2], \\ \ldots \\ s_{n-1}(x), & \text{if } x \in [x_{n-1}, x_n], \end{cases} \tag{5}$$

where $s_i(x) = \sum_{j=0}^{3} C_{i,j}(x - x_i)^{3-j}$, $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ represents cubic polynomials in each small interval (Fig. 1) [30]. Here, $C_{i,j}$ are the unknown coefficients of the polynomial $s_i(x)$ in the small interval $[x_i, x_{i+1}]$ for $j = 0, 1, 2, 3$.

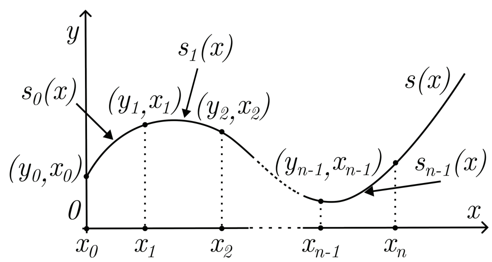


Figure 1: Cubic spline

To determine the values of the $4n$ coefficients $C_{i,j}$, we need to solve the following system of $4n$ equations [30]:

$$\begin{cases} s_i(x_i) = f(x_i), & i = 0, \ldots, n-1, \\ s_i(x_{i+1}) = f(x_{i+1}), & i = 0, \ldots, n-1, \\ s_i'(x_{i+1}) = s_{i+1}'(x_{i+1}), & i = 0, \ldots, n-2, \\ s_i''(x_{i+1}) = s_{i+1}''(x_{i+1}), & i = 0, \ldots, n-2, \\ s_0'(x_0) = f'(x_0), & s_{n-1}'(x_n) = f'(x_n). \end{cases} \tag{6}$$

Or, this is the following system of equations:

$$\begin{cases} C_{i,3} = \varphi(x_i), & i = 0, \ldots, n-1, \\ C_{i,0}(x_i - x_{i+1})^3 + C_{i,1}(x_i - x_{i+1})^2 + C_{i,2}(x_i - x_{i+1}) + C_{i,3} = \varphi(x_{i+1}), & i = 0, \ldots, n-1, \\ 3C_{i,0}(x_i - x_{i+1})^2 + 2C_{i,1}(x_i - x_{i+1}) + C_{i,2} = C_{i+1,2}, & i = 0, \ldots, n-2, \\ 6C_{i,0}(x_i - x_{i+1}) + 2C_{i,1} = C_{i+1,1}, & i = 0, \ldots, n-2, \\ C_{0,2} = \varphi'(x_0), \\ 3C_{n-1,0}(x_{n-1} - x_n)^2 + 2C_{n-1,1}(x_{n-1} - x_n) + C_{n-1,2} = \varphi'(x_n). \end{cases}$$

If the values of the function derivative at the points $x = x_0$ and $x = x_n$ are unknown for applying condition (6), a spline called the "Natural Spline" can be constructed using the following conditions:

$$\begin{cases} s_i(x_i) = \varphi(x_i), & i = 0, \dots, n-1, \\ s_i(x_{i+1}) = \varphi(x_{i+1}), & i = 0, \dots, n-1, \\ s_i'(x_{i+1}) = s_{i+1}'(x_{i+1}), & i = 0, \dots, n-2, \\ s_i''(x_{i+1}) = s_{i+1}''(x_{i+1}), & i = 0, \dots, n-2, \\ s_0''(x_0) = 0, & s_{n-1}''(x_n) = 0. \end{cases}$$

By solving the system of Eqs. (6) using an appropriate method (e.g., the Gaussian method), we determine the values of the unknown coefficients $C_{i,j}$ $(i = 0, \dots, n-1,\ j = 0, \dots, 3)$. To express the polynomials $s_i(x)$ $i = 0, \dots, n-1$ in the standard form $s_i(x) = A_{i,0}x^3 + A_{i,1}x^2 + A_{i,2}x + A_{i,3}$, the coefficients $A_{i,j}$ $(i = 0, \dots, n-1,\ j = 0, \dots, 3)$ are computed using the following formulas.

$$A_{i,0} = C_{i,0}, \quad i = 0, \dots, n-1,$$

$$A_{i,1} = C_{i,1} - 3C_{i,0}x_i, \quad i = 0, \dots, n-1,$$

$$A_{i,2} = 3C_{i,0}x_i^2 - 2C_{i,1}x_i + C_{i,2}, \quad i = 0, \dots, n-1,$$

$$A_{i,3} = -C_{i,0}x_i^3 + C_{i,1}x_i^2 - C_{i,2}x_i + C_{i,3}, \quad i = 0, \dots, n-1.$$

It should be noted that many modern mathematical systems already have built-in methods for computing cubic splines. By providing function values at the mesh node points, these methods allow constructing cubic splines, obtaining the coefficients of their piecewise polynomials, and using them for various calculations.

Once the cubic spline (5), which approximates the function over the given interval, is determined, the integrand function of the generalized Erdélyi-Kober fractional integral operator is replaced with a piecewise spline defined on a mesh with $n + 1$ nodes. Subsequently, in the fractional integral, the function $f(x)$ is replaced by the spline $s(x)$, which is then integrated. The numerical errors in computing the generalized Erdélyi-Kober fractional integral operator mainly arise due to the approximation of the integrand function using splines.

3 Cubic Spline Interpolation Error

Based on the theorems in [16,31], the interpolation errors of the presented spline approximations can be estimated.

Theorem 1: *If a spline $s(x)$ satisfying conditions (6) is constructed for the function $f(x) \in C^4[a, b]$ on the mesh $\{x_i;\ i = 0, \dots, n;\ x_0 = a < x_1 < \dots < x_n = b\}$, then the following inequality holds:*

$$\left\| f^{(m)}(x) - s^{(m)}(x) \right\| \le C_m \left\| f^{(4)}(x) \right\| h^{4-m} \quad (m = 0, 1, 2, 3),$$

where $h = \max\limits_{i=0,\dots,n-1} (x_{i+1} - x_i)$,

$$C_0 = \frac{5}{384}, \quad C_1 = \frac{9 + \sqrt{3}}{216}, \quad C_2 = \frac{3\beta + 1}{12}, \quad C_3 = \frac{\beta^2 + 1}{2}, \tag{7}$$

β is the ratio of the largest step size to the smallest step size in the mesh, $\beta = \frac{h}{\min\limits_{i=0,\dots,n-1}(x_{i+1} - x_i)}$.

The proof of the theorem is given in [31].

If we take $m = 0$, then according to (7), we obtain the following estimate:

$$\|f(x) - s(x)\| \leq \frac{5}{384}\left\|f^{(4)}(x)\right\| h^4, \tag{8}$$

where $h = \max\limits_{i=0,\dots,n-1}(x_{i+1} - x_i)$.

4 Approximate Computation of the Generalized Erdélyi-Kober Operator

In this section, before performing the computations, we address the question of whether the generalized Erdélyi-Kober operator, which is a fractional integral involving the first-kind Bessel function in its kernel, can be approximated using cubic spline interpolation. The kernel of this integral operator is given by $K(x,t) = t^{2\eta+1}\left(x^2 - t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right)$, where it is smoothly integrable over any interval $[0, x]$, $x \geq 0$. The validity of this statement can be substantiated for $f(x) = 1$ using the following formula provided in [29]:

$$\Im^{\alpha}_{\lambda,\eta} 1 = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} \bar{J}_{\alpha+\eta}(\lambda x), \tag{9}$$

where $\eta, \alpha, \lambda \in \mathbb{R}$, $\alpha > 0$, $\eta \geq -1/2$, and $\bar{J}_\nu(z)$ denotes the Bessel-Clifford function. It is known that the Bessel-Clifford function possesses derivatives of arbitrary order, i.e., it is a smooth function. Given that for $z \in \mathbb{R}$ and $\nu \geq -\frac{1}{2}$, $\left\|\bar{J}_\nu(z)\right\| \leq 1$ as established in [29], it follows that the boundedness of $\Im^{\alpha}_{\lambda,\eta} 1$ is given by $\left\|\Im^{\alpha}_{\lambda,\eta} 1\right\| \leq \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}$.

This implies that the function $K(x,t)$ is integrable over any subinterval of $[0, x]$, $x \in \mathbb{R}_+$. Based on these considerations, for a fixed value $x = b$, the function $K(x,t)$ can be regarded as a single-variable function that is integrable over any subinterval of $[0, b]$. This allows us to replace the function $f(x) \in C^2[0, b]$ with a cubic spline $s(x)$.

Suppose that the values of the function $f(x)$, which is the integrand of the generalized Erdélyi-Kober operator, are given at the node points of an arbitrary grid $\omega = \{x_i;\ i = 0, \dots, n;\ x_0 = 0 < x_1 < \dots < x_n = b\}$ obtained by dividing the interval $[0, b]$ into n subintervals. Using these given values $(x_i, f(x_i))$ $(i = 0, 1 \dots, n)$, we compute the cubic spline $s(x)$that approximates $f(x)$ over the interval $[0, b]$ as described in Section 2. Then, we can approximately compute the numerical values of the generalized Erdélyi-Kober fractional integral operator at the points $x \in \omega$.

We replace the function $f(x)$ in the generalized Erdélyi-Kober operator (1) with the spline $s(x)$:

$$\Im^{\alpha}_{\lambda,\eta} f(x) \cong \Im^{\alpha}_{\lambda,\eta} s(x) = 2^{\alpha}\lambda^{1-\alpha} x^{-2\alpha-2\eta} \int\limits_0^x t^{2\eta+1}\left(x^2 - t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right) s(t)\, dt, \tag{10}$$

then, substituting the general form of the spline (5) into Eq. (10) for $x = x_r$, we integrate the corresponding spline segments for each interval $[0, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{r-1}, x_r]$. As a result, we obtain the following:

$$\Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} = 2^{\alpha}\lambda^{1-\alpha}x_r^{-2\alpha-2\eta}\int\limits_0^{x_r} t^{2\eta+1}\left(x_r^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x_r^2-t^2}\right) s(t)\,dt$$

$$= 2^{\alpha}\lambda^{1-\alpha}x_r^{-2\alpha-2\eta}\sum_{i=0}^{r-1}\int\limits_{x_i}^{x_{i+1}} t^{2\eta+1}\left(x_r^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x_r^2-t^2}\right) s_i(t)\,dt,\ \ r=1,\ldots n.$$

We replace $s_i(t)$ with the corresponding polynomials:

$$\Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} = 2^{\alpha}\lambda^{1-\alpha}x_r^{-2\alpha-2\eta}\sum_{i=0}^{r-1}\int\limits_{x_i}^{x_{i+1}} t^{2\eta+1}\left(x_r^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x_r^2-t^2}\right)\sum_{j=0}^{3}A_{i,j}t^{3-j}dt$$

$$= 2^{\alpha}\lambda^{1-\alpha}x_r^{-2\alpha-2\eta}\sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}\int\limits_{x_i}^{x_{i+1}} t^{2\eta+4-j}\left(x_r^2-t^2\right)^{\frac{\alpha-1}{2}} J_{\alpha-1}\left(\lambda\sqrt{x_r^2-t^2}\right)dt,\quad r=1,\ldots n.$$

By applying Eq. (2) to the final expression and performing some calculations, we obtain the following:

$$\Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} = \frac{2^{\alpha}\lambda^{1-\alpha}}{x_r^{2\alpha+2\eta}}\sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}\int\limits_{x_i}^{x_{i+1}} t^{2\eta+4-j}\left(x_r^2-t^2\right)^{\frac{\alpha-1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\lambda\sqrt{x_r^2-t^2}}{2}\right)^{\alpha-1+2k}}{k!\,\Gamma(\alpha+k)}dt$$

$$= 2x_r^{-2\alpha-2\eta}\sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}\sum_{k=0}^{\infty}\frac{\left(-\lambda^2/4\right)^k}{k!\,\Gamma(\alpha+k)}\int\limits_{x_i}^{x_{i+1}} t^{2\eta+4-j}\left(x_r^2-t^2\right)^{\alpha+k-1}dt,\ \ r=1,\ldots n.$$

We substitute the integral variable as $t^2 = x_r^2\tau,\ dt = \dfrac{x_r d\tau}{2\sqrt{\tau}}$ and perform the calculations.

$$\Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} = 2x_r^{-2\alpha-2\eta}\sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}\sum_{k=0}^{\infty}\frac{\left(-\lambda^2/4\right)^k}{k!\,\Gamma(\alpha+k)}\int\limits_{\left(x_i/x_r\right)^2}^{\left(x_{i+1}/x_r\right)^2}\left(x_r^2\tau\right)^{2\eta+4-j}\left(x_r^2-x_r^2\tau\right)^{\alpha+k-1}\frac{x_r}{2\sqrt{\tau}}d\tau$$

$$= \sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}x_r^{3-j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma(\alpha+k)}\int\limits_0^{\left(x_{i+1}/x_r\right)^2}\tau^{\eta+2-\frac{j+1}{2}}(1-\tau)^{\alpha+k-1}d\tau$$

$$-\sum_{i=0}^{r-1}\sum_{j=0}^{3}A_{i,j}x_r^{3-j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma(\alpha+k)}\int\limits_0^{\left(x_i/x_r\right)^2}\tau^{\eta+2-\frac{j+1}{2}}(1-\tau)^{\alpha+k-1}d\tau,\ \ r=1,\ldots n.$$

Taking into account that $\left(\frac{x_i}{x_r}\right)^2 < 1$ and $\left(\frac{x_{i+1}}{x_r}\right)^2 \le 1$ for $i=0,1,\ldots,r-1$ in the final formula, we can express the integrals in both summations using the incomplete beta function:

$$\Im_{\lambda,\eta}^{\alpha} s\left(x\right)\big|_{x=x_r} = \sum_{i=0}^{r-1}\sum_{j=0}^{3} A_{i,j} x_r^{3-j} \sum_{k=0}^{\infty} \frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)} B_{(x_{i+1}/x_r)^2}\left(\eta+\frac{5-j}{2},\alpha+k\right)$$
$$-\sum_{i=0}^{r-1}\sum_{j=0}^{3} A_{i,j} x_r^{3-j} \sum_{k=0}^{\infty} \frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)} B_{(x_i/x_r)^2}\left(\eta+\frac{5-j}{2},\alpha+k\right). \quad r=1,\ldots n.$$

where $B_z\left(a,b\right)$ is the incomplete beta function, which is defined as follows:

$$B_z\left(a,b\right)=\int_0^z u^{a-1}(1-u)^{b-1}du,\ 0\le z\le 1.$$

By using equation $B_z\left(a,b\right)=\frac{z^a}{a}{}_2F_1\left(a,1-b;a+1;z\right)$ given in [32], we obtain the following:

$$\Im_{\lambda,\eta}^{\alpha} s\left(x\right)\big|_{x=x_r} = 2\sum_{i=0}^{r-1}\left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i+1,j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)}{}_2F_1\left(Q_j,1-\alpha-k;Q_j+1;\left(\frac{x_{i+1}}{x_r}\right)^2\right)$$
$$-2\sum_{i=0}^{r-1}\left(\frac{x_i}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i,j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)}{}_2F_1\left(Q_j,1-\alpha-k;Q_j+1;\left(\frac{x_i}{x_r}\right)^2\right). \quad r=1,\ldots n,$$

where $K_{i,j}=\frac{x_i^{3-j}}{2\eta-j+5}$, $Q_j=\eta+\frac{5-j}{2}$.

Here, ${}_2F_1\left(a,b;c;z\right)$ is a hypergeometric function, defined in the following series form:

$${}_2F_1\left(a,b;c;z\right)=\sum_{i=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n.$$

Expressing the hypergeometric function ${}_2F_1$ in series form, we rewrite the final equation as follows:

$$\Im_{\lambda,\eta}^{\alpha} s\left(x\right)\big|_{x=x_r} = 2\sum_{i=0}^{r-1}\left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i+1,j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)}\sum_{l=0}^{\infty}\frac{\left(Q_j\right)_l(1-\alpha-k)_l}{\left(Q_j+1\right)_l l!}\left(\left(\frac{x_{i+1}}{x_r}\right)^2\right)^l$$
$$-2\sum_{i=0}^{r-1}\left(\frac{x_i}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i,j}\sum_{k=0}^{\infty}\frac{\left(-(\lambda x_r/2)^2\right)^k}{k!\,\Gamma\left(\alpha+k\right)}\sum_{l=0}^{\infty}\frac{\left(Q_j\right)_l(1-\alpha-k)_l}{\left(Q_j+1\right)_l l!}\left(\left(\frac{x_i}{x_r}\right)^2\right)^l. \quad r=1,\ldots n.$$

Using the following equality for

$$(1-\alpha-k)_l=\frac{(1-\alpha-k+l)\,!}{\Gamma\left(1-\alpha-k\right)}=\frac{\Gamma\left(1-\alpha\right)\left(1-\alpha\right)_{l-k}}{\Gamma\left(1-\alpha\right)\left(1-\alpha\right)_{-k}}=\frac{(1-\alpha)_{l-k}}{\left((-1)^k/(\alpha)_k\right)}=(-1)^k(1-\alpha)_{l-k}(\alpha)_k,$$

we obtain the following:

$$\Im_{\lambda,\eta}^{\alpha} s\left(x\right)\big|_{x=x_r} = \frac{2}{\Gamma\left(\alpha\right)}\sum_{i=0}^{r-1}\left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i+1,j}\sum_{k,l=0}^{\infty}\frac{\left(Q_j\right)_l(1-\alpha)_{l-k}}{\left(Q_j+1\right)_l k!\,l!}\left(\left(\frac{\lambda x_r}{2}\right)^2\right)^k\left(\left(\frac{x_{i+1}}{x_r}\right)^2\right)^l$$
$$-\frac{2}{\Gamma\left(\alpha\right)}\sum_{i=0}^{r-1}\left(\frac{x_i}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3} A_{i,j}K_{i+1,j}\sum_{k,l=0}^{\infty}\frac{\left(Q_j\right)_l(1-\alpha)_{l-k}}{\left(Q_j+1\right)_l k!\,l!}\left(\left(\frac{\lambda x_r}{2}\right)^2\right)^k\left(\left(\frac{x_i}{x_r}\right)^2\right)^l. \quad r=1,\ldots n.$$

It is known that Horn's $\mathbf{H}_3(\alpha,\beta,\delta;x,y)$ confluent hypergeometric function [33] is defined in the following series form:

$$\mathbf{H}_3(\alpha,\beta,\delta,x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m}{(\delta)_m m!\,n!}x^m y^n,\quad |x|<1.$$

Taking into account that for $i=0,\dots,r-1$, $\left(\frac{x_i}{x_r}\right)^2<1$ and for $i=0,\dots,r-2$, $\left(\frac{x_{i+1}}{x_r}\right)^2<1$, the above expression can be rewritten as follows according to the last formula:

$$\Im^{\alpha}_{\lambda,\eta}s(x)\Big|_{x=x_r}=\Im^{\alpha}_{\lambda,\eta}s_{r-1}(x)\Big|_{x=x_r}+\frac{2}{\Gamma(\alpha)}\sum_{i=0}^{r-2}\left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3}A_{i,j}K_{i+1,j}H_{\alpha,\lambda}(i+1,j,r)$$

$$-\frac{2}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\left(\frac{x_i}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3}A_{i,j}K_{i,j}H_{\alpha,\lambda}(i,j,r),\quad r=1,\dots n,$$

where $H_{\alpha,\lambda}(i,j,r)=\mathbf{H}_3\left(1-\alpha,Q_j,Q_j+1,\left(\frac{x_i}{x_r}\right)^2,\left(\frac{\lambda x_r}{2}\right)^2\right)$.

We can write the first term $\Im^{\alpha}_{\lambda,\eta}s_{r-1}(x)\Big|_{x=x_r}$ on the right side of the equation as follows using the

$$\Im^{\alpha}_{\lambda,\eta}Cx^{2\beta}=Cx^{2\beta}\frac{\Gamma(\eta+\beta+1)}{\Gamma(\alpha+\eta+\beta+1)}\bar{J}_{\alpha+\eta+\beta}(\lambda x),\tag{11}$$

formula provided in literature [29]:

$$\Im^{\alpha}_{\lambda,\eta}s_{r-1}(x)\Big|_{x=x_r}=\sum_{j=0}^{3}\Im^{\alpha}_{\lambda,\eta}A_{r-1,j}x_r^{3-j}=\sum_{j=0}^{3}A_{r-1,j}x_r^{3-j}\frac{\Gamma(Q_j)}{\Gamma(\alpha+Q_j)}\bar{J}_{\alpha+Q_j-1}(\lambda x_r),$$

here, $\bar{J}_\nu(z)$ is the Bessel-Clifford function (3).

We obtained the following formula for the approximate calculation of the generalized Erdélyi-Kober operator (Lowndes operator) using a cubic spline

$$\begin{aligned}\Im^{\alpha}_{\lambda,\eta}s(x)\Big|_{x=x_r}&=\sum_{j=0}^{3}A_{r-1,j}\,x_r^{3-j}\frac{\Gamma(Q_j)}{\Gamma(\alpha+Q_j)}\bar{J}_{\alpha+\eta+\frac{3-j}{2}}(\lambda x_r)\\&+\begin{cases}0, & \text{if } r=1;\\ \dfrac{2}{\Gamma(\alpha)}\displaystyle\sum_{i=0}^{r-2}\left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3}A_{i,j}K_{i+1,j}H_{\alpha,\lambda}(i+1,j,r), & \text{if } r>1\end{cases}\\&-\frac{2}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\left(\frac{x_i}{x_r}\right)^{2\eta+2}\sum_{j=0}^{3}A_{i,j}K_{i,j}H_{\alpha,\lambda}(i,j,r),\quad r=1,\dots,n.\end{aligned}\tag{12}$$

Using Formula (12), we numerically compute the action of the generalized Erdélyi-Kober operator on a function given on an arbitrary grid (in tabular form).

Instead of Formula (12), one can also use the formula

$$\Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} = \frac{2}{\Gamma(\alpha)} \left[\begin{array}{c} \sum_{i=0}^{r-1} \left(\frac{x_{i+1}}{x_r}\right)^{2\eta+2} \sum_{j=0}^{3} A_{i,j} K_{i+1,j} H_{\alpha,\lambda}(i+1,j,r) - \\ - \sum_{i=0}^{r-1} \left(\frac{x_i}{x_r}\right)^{2\eta+2} \sum_{j=0}^{3} A_{i,j} K_{i,j} H_{\alpha,\lambda}(i,j,r) \end{array} \right], \quad r = 1, \ldots n,$$

taking into account the equality

$$\mathbf{H}_3(a,b,c;1,y) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_1F_2(c-a-b; 1-a, c-a; -y), \mathrm{Re}(c-a-b) > 0,$$

presented in reference [34].

5 Error Analysis of the Numerical Method

It is natural for computational errors to occur in numerical methods. Knowing the approximation errors in spline interpolation (see Section 3), we can also estimate the errors arising from the approximate computation of generalized Erdélyi-Kober operators.

Theorem 2: *If a spline $s(x)$satisfying condition (6) is constructed for the function$f(x) \in C^4[0,l]$ on the grid $\omega_r = \{x_i;\ i = 0,\ldots,r;\ x_0 = 0 < x_1 < \ldots < x_r = l\}$ then the following estimate holds:*

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\Big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} \right\| \le \begin{cases} C_0 h^4 \left\|f^{(4)}(x)\right\| \dfrac{(\alpha+1)(\alpha+\eta+1) + (\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}}, & 0 < \alpha \le \dfrac{1}{2}, \\ \dfrac{C_0 h^4}{(\eta+1)_\alpha} \left\|f^{(4)}(x)\right\|, & \alpha > \dfrac{1}{2}, \end{cases} \tag{13}$$

where $C_0 = \dfrac{5}{384}$, $h = \max\limits_{i=0,1,\ldots,r-1} |x_{i+1} - x_i|$, $\left\|f^{(4)}(x)\right\| = \max\limits_{x\in[0,l]} \left|f^{(4)}(x)\right|$.

Proof of Theorem 2: For the fractional order generalized Erdélyi-Kober integral operator $\Im^{\alpha}_{\lambda,\eta} s(x)\big|_{x=x_r}$ $(r > 0)$, we determine the estimate of the approximation error as follows:

$$Err = \left\| \Im^{\alpha}_{\lambda,\eta} f(x)\Big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta} s(x)\Big|_{x=x_r} \right\| = \left\| \Im^{\alpha}_{\lambda,\eta} (f(x) - s(x))\Big|_{x=x_r} \right\|$$

$$= \left\| \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\|.$$

First, let's consider the case where $0 < \alpha \le \frac{1}{2}$. We apply the formula

$$\bar{J}_{\nu-1}(x) = \bar{J}_\nu(x) - \frac{x^2}{4\nu(\nu+1)} \bar{J}_{\nu+1}(x),$$

given in reference [29].

$$Err = \left\| \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \left(\begin{array}{c} \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) \\ -\frac{\lambda^2\left(x_r^2-t^2\right)}{4\alpha(\alpha+1)} \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) \end{array} \right) (f(t) - s(t))\, dt \right\|$$

$$= \left\| \begin{array}{c} \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int\limits_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \\ -\frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \int\limits_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \end{array} \right\|.$$

Using the property of the norm $\|A - B\| \le \|A\| + \|B\|$, we refine the inequality:

$$Err \le \left\| \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\| +$$

$$+ \left\| \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\|$$

$$= \left\| \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\| +$$

$$+ \left\| \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\|$$

$$\le \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \left\| \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t)) \right\| dt +$$

$$+ \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \left\| \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t)) \right\| dt$$

$$\le \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \left\| \bar{J}_\alpha \left(\lambda\sqrt{x_r^2 - t^2}\right) \right\| \|(f(t) - s(t))\| \, dt +$$

$$+ \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \left\| \bar{J}_{\alpha+1} \left(\lambda\sqrt{x_r^2 - t^2}\right) \right\| \|(f(t) - s(t))\| \, dt.$$

Using the fact that for the Bessel-Clifford function, for any $\nu > -\frac{1}{2}$, the inequality $\left\| \bar{J}_\nu(x) \right\| \le 1$ holds [29], we further strengthen the above inequality:

$$Err \le \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \|(f(t) - s(t))\| \, dt +$$

$$+ \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \sum_{i=0}^{r-1} \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} \|(f(t) - s(t))\| \, dt$$

$$\le \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \|f(\tilde{x}_i) - s(\tilde{x}_i)\| \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} dt +$$

$$+ \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \sum_{i=0}^{r-1} \|f(\tilde{x}_i) - s(\tilde{x}_i)\| \int_{x_i}^{x_{i+1}} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} dt,$$

where $\tilde{x}_i \in [x_i, x_{i+1}]$. If we assume that $\|f(\tilde{x}) - s(\tilde{x})\| = \max\limits_{i=0,1,\ldots,r-1} \|f(\tilde{x}_i) - s(\tilde{x}_i)\|$ holds for $\tilde{x} \in [0, x_r]$, then the next estimate of the error Err takes the following form:

$$Err \le \|f(\tilde{x}) - s(\tilde{x})\| \left(\frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} dt + \frac{\lambda^2 x_r^{-2\alpha-2\eta}}{2\Gamma(\alpha+2)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha} dt \right)$$

$$= \|f(\tilde{x}) - s(\tilde{x})\| \left(I_{2,\eta}^{\alpha} 1\big|_{x=x_r} + \frac{\lambda^2 x_r^2}{4(\alpha+1)} I_{2,\eta}^{\alpha+1} 1\big|_{x=x_r} \right).$$

Here, $I_{\sigma,\eta}^{\alpha} f(x)$ is the fractional order Erdélyi-Kober operator, and $I_{\sigma,\eta}^{\alpha} C = \dfrac{C\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}$ is the formula valid for [29]. We have the following:

$$Err \le \|f(\tilde{x}) - s(\tilde{x})\| \left(\frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} + \frac{\lambda^2 x_r^2 \Gamma(\eta+1)}{4\Gamma(\alpha+\eta+2)(\alpha+1)} \right)$$

$$= \|f(\tilde{x}) - s(\tilde{x})\| \left(\frac{(\alpha+1)(\alpha+\eta+1) + (\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}} \right). \tag{14}$$

For the estimate in (14), applying the Formula (8) gives us the following:

$$Err \le \frac{5}{384} \left\|f^{(4)}(\tilde{x})\right\| h^4 \frac{(\alpha+1)(\alpha+\eta+1) + (\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}},$$

where $h = \max\limits_{i=0,1,\ldots,r-1} |x_{i+1} - x_i|$, $\left\|f^{(4)}(\tilde{x})\right\| = \max\limits_{i=0,1,\ldots,r-1} \left|f^{(4)}(\tilde{x}_i)\right|$.

Now, let's consider the case where $\alpha > \frac{1}{2}$.

$$Err = \left\| \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t))\, dt \right\|$$

$$\leq \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \left\| \bar{J}_{\alpha-1}\left(\lambda\sqrt{x_r^2 - t^2}\right) (f(t) - s(t)) \right\| dt$$

$$\leq \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \left\| \bar{J}_{\alpha-1}\left(\lambda\sqrt{x_r^2 - t^2}\right) \right\| \|f(t) - s(t)\|\, dt.$$

Using the property that for any $v > -\frac{1}{2}$, the inequality $\left\|\bar{J}_v(x)\right\| \leq 1$ holds [29], we strengthen the above inequality:

$$Err \leq \frac{2x_r^{-2\alpha-2\eta}}{\Gamma(\alpha)} \int_0^{x_r} t^{2\eta+1} \left(x_r^2 - t^2\right)^{\alpha-1} \|f(t) - s(t)\|\, dt.$$

As in the previous case, performing some calculations, we arrive at the following result:

$$Err \leq \frac{5\left\|f^{(4)}(\tilde{x})\right\| h^4}{384(\eta+1)_\alpha}. \quad \square$$

Theorem is proved.

From the evaluation in (13), we draw the following conclusions:

1) For $0 < \alpha \leq \frac{1}{2}$, the error estimate is directly proportional to $(x_r)^2$. This may lead to an increase in error for sufficiently large $x = x_r$. In this case, it is recommended to choose h proportionally smaller relative to $\left(\frac{1}{x_r}\right)^2$.

2) For sufficiently large values of η, the numerical method provides good convergence.

3) The error is directly proportional to h^4, which allows for easy error control and indicates that the method is convergent, as $\lim\limits_{h\to 0} Err = 0$.

6 Checking the Stability of Numerical Methods

For the function $y = f(x)$, the numerical value of the generalized Erdelyi-Kober operator $\Im_{\lambda,\eta}^{\alpha} f(x)$ at the point $x = l$ is required to be calculated. To use the cubic spline interpolation method, we need to know the values of the function $f(x)$ at the grid nodes ω_r. Suppose that when calculating the values of this function at the grid nodes, corresponding errors $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r$ have occurred, and these errors can be expressed using some function $\varepsilon(x)$. In this case, we will be approximating the operator value $\Im_{\lambda,\eta}^{\alpha} g(x)$ for the function $g(x) = f(x) + \varepsilon(x)$, rather than for the function $f(x)$. The values of the function $g(x)$ at the grid nodes are calculated as follows:

$g(x_i) = f(x_i) + \varepsilon(x_i). \quad i = 0, 1, \ldots, r.$

Let ${}_{\varepsilon}s(x)$ be the cubic spline that interpolates the function $g(x)$ on the mesh ω_r. According to Theorem 2, we have the following:

$$\left\| \Im^{\alpha}_{\lambda,\eta} g(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| \leq \frac{5h^4}{384} \left\| g^{(4)}(x) \right\| W, \tag{15}$$

where $W = \begin{cases} \frac{(\alpha+1)(\alpha+\eta+1)+(\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}}, & 0 < \alpha \leq \frac{1}{2}; \\ \frac{1}{(\eta+1)_{\alpha}}, & \alpha > \frac{1}{2}. \end{cases}$ In estimate (15), we replace the function $g(x)$ with $f(x) + \varepsilon(x)$, and perform the following calculation:

$$\left\| \Im^{\alpha}_{\lambda,\eta}\left(f(x) + \varepsilon(x)\right)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| \leq \frac{5h^4}{384} \left\| f^{(4)}(x) + \varepsilon^{(4)}(x) \right\| W,$$

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} + \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| \leq \frac{5h^4}{384} \left\| f^{(4)}(x) + \varepsilon^{(4)}(x) \right\| W.$$

According to the property of the norm $\|X\| - \|Y\| \leq \|X + Y\|$ the following inequality holds:

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| - \left\| \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} \right\|$$

$$\leq \left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} + \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\|.$$

From this, by further strengthening the inequality, we obtain the following:

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| - \left\| \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} \right\| \leq \frac{5h^4}{384} \left\| f^{(4)}(x) + \varepsilon^{(4)}(x) \right\| W,$$

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| \leq \frac{5h^4}{384} \left\| f^{(4)}(x) + \varepsilon^{(4)}(x) \right\| W + \left\| \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} \right\|.$$

As a result of the calculations similar to those in the previous paragraph, we can conclude that

$$\left\| \Im^{\alpha}_{\lambda,\eta}\varepsilon(x)\big|_{x=x_r} \right\| \leq \|\varepsilon(x)\| W.$$

Using the property of the norm $\|X + Y\| \leq \|X\| + \|Y\|$ and the last inequality, we obtain the following strengthened inequality:

$$\left\| \Im^{\alpha}_{\lambda,\eta} f(x)\big|_{x=x_r} - \Im^{\alpha}_{\lambda,\eta}\left({}_{\varepsilon}s(x)\right)\big|_{x=x_r} \right\| \leq \left\| f^{(4)}(x) \right\| W + \left(\frac{5h^4}{384} \left\| \varepsilon^{(4)}(x) \right\| + \|\varepsilon(x)\| \right) W. \tag{16}$$

From the error estimates (3) and (16), it can be concluded that the value of the function $f(x)$ at the nodes of the given grid, when computed with the error $\varepsilon(x_i)$ $(i = 0, 1, \ldots, r)$, results in an increase in the approximation error of the generalized Erdélyi-Kober operator $\Im^{\alpha}_{\lambda,\eta} f(x)$, calculated using the cubic spline interpolation method, by an amount

$$W_{\varepsilon} = \left(\frac{5h^4}{384} \left\| \varepsilon^{(4)}(x) \right\| + \|\varepsilon(x)\| \right) \begin{cases} \dfrac{(\alpha+1)(\alpha+\eta+1) + (\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}}, & 0 < \alpha \leq \dfrac{1}{2}; \\ \dfrac{1}{(\eta+1)_{\alpha}}, & \alpha > \dfrac{1}{2}. \end{cases}$$

We can define the function $\varepsilon(x)$ as a collection of linear functions $\varepsilon_i(x)$, $x \in [x_i, x_{i+1}]$, connecting the points $(0, \varepsilon_0)$ and (x_1, ε_1), (x_1, ε_1) and (x_2, ε_2), $\ldots$, $(x_{n-1}, \varepsilon_{n-1})$ and (x_n, ε_n) on the respective subintervals $[0, x_1], [x_1, x_2], \ldots, [x_{r-1}, x_r]$. This provides a simpler form for W_ε:

$$W_\varepsilon = \|\varepsilon(x)\| \begin{cases} \dfrac{(\alpha+1)(\alpha+\eta+1)+(\lambda x_r/2)^2}{(\alpha+1)(\eta+1)_{\alpha+1}}, & 0 < \alpha \le \dfrac{1}{2}; \\ \dfrac{1}{(\eta+1)_\alpha}, & \alpha > \dfrac{1}{2}. \end{cases}$$

From the obtained results, we draw the following conclusions:

1) The computation of the function $f(x)$ at the grid points ω_r with an error $\varepsilon(x)$, $x \in \omega_r$, alters the approximation of the generalized Erdélyi-Kober operator using cubic splines by an amount independent of h.

2) For $0 < \alpha \le \frac{1}{2}$, the stability of the numerical method depends on the value of x_r, with the condition $\|\varepsilon(x)\| < \left(\frac{2}{\lambda x_r}\right)^2$ ensuring stability.

3) For $\alpha > \frac{1}{2}$, given that $\varepsilon(x) \sim W_\varepsilon$, the value of the generalized Erdélyi-Kober operator $\Im^\alpha_{\lambda,\eta} f(x)$ at the point $x = x_r$, approximated via cubic spline interpolation as $\Im^\alpha_{\lambda,\eta}(s_\varepsilon(x))\big|_{x=x_r}$, is stable with respect to the error $\varepsilon(x)$ in determining the integrand $f(x)$.

7 Sample Calculations

To analyze the efficiency of the numerical method, we utilize the values of the parameters $\alpha = \{0.2, 1.7, 2.5\}$, $\eta = -0.5$, $\lambda = \{0.2, 1.7, 2.5\}$, and $h = \{0.5, 0.1, 0.02, 0.01\}$. The exact solution $\Im^\alpha_{\lambda,\eta} f(x)$, the approximate solution $\Im^\alpha_{\lambda,\eta} s(x)$, and the absolute error function

$$R(x) = \left|\Im^\alpha_{\lambda,\eta} f(x) - \Im^\alpha_{\lambda,\eta} s(x)\right|, \quad x \in \omega_r,$$

are used for this analysis.

All computations were carried out in Python using the scipy, mpmath, and math libraries with the float64 64-bit floating-point type. For plotting the graphs, the matplotlib library was used.

To compare the results of numerical computations, analytical formulas that allow for the calculation of the exact value of the generalized Erdélyi-Kober operator are required. Below, we select test functions for the computations (Tables 1 and 2).

Example 1. If we select the test function $f(x) = x^{4.7}$, we can use the following formula, as given in (11), to compute the exact value of the generalized Erdélyi-Kober operator:

$$\Im^\alpha_{\lambda,\eta} x^{4.7} = x^{4.7} \frac{\Gamma\left(\eta + 2\frac{7}{20} + 1\right)}{\Gamma\left(\alpha + \eta + 2\frac{7}{20} + 1\right)} \bar{J}_{\alpha+\eta+2\frac{7}{20}}(\lambda x). \tag{17}$$

We now justify this formula. In [14], the following formula is provided:

$$\Im^\alpha_{\kappa,\eta} x^{2\beta} f(x) = x^{2\beta} \Im^\alpha_{\kappa,\eta+\beta} f(x).$$

By setting $\kappa = \lambda$ and $f(x) = 1$ in the above formula and applying (9), we obtain:

$$\Im^\alpha_{\lambda,\eta} x^{2\beta} = x^{2\beta} \frac{\Gamma(\eta+\beta+1)}{\Gamma(\alpha+\eta+\beta+1)} \bar{J}_{\alpha+\eta+\beta}(\lambda x).$$

Table 1: We calculate the action of the generalized Erdélyi-Kober operator on the function $f(x) = x^{4.7}$. To compute the exact value of the operator, we use Formula (17)

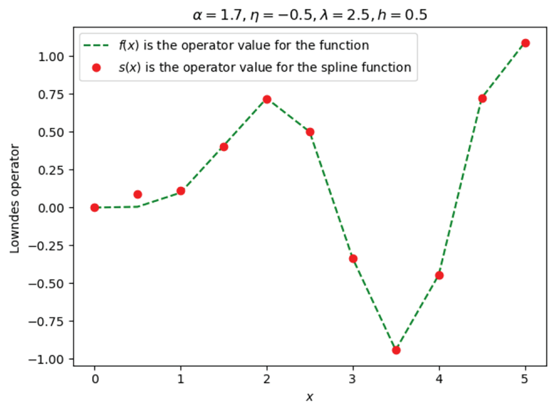
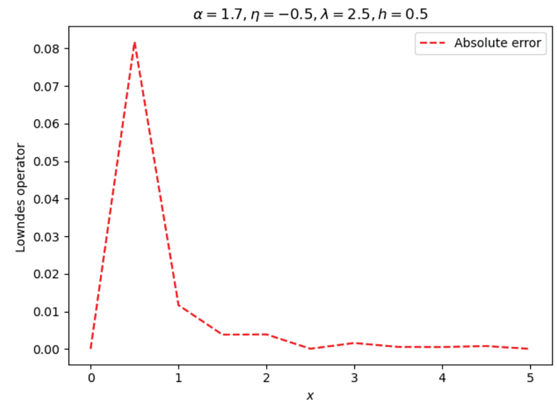
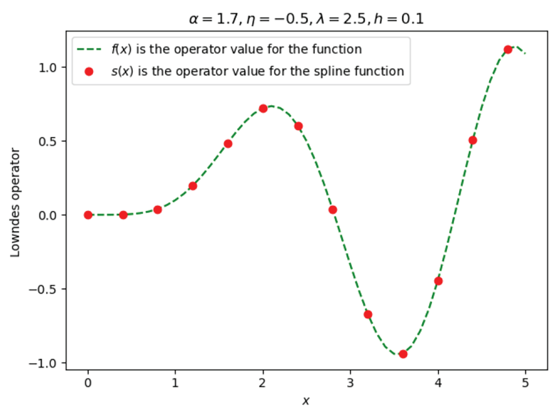
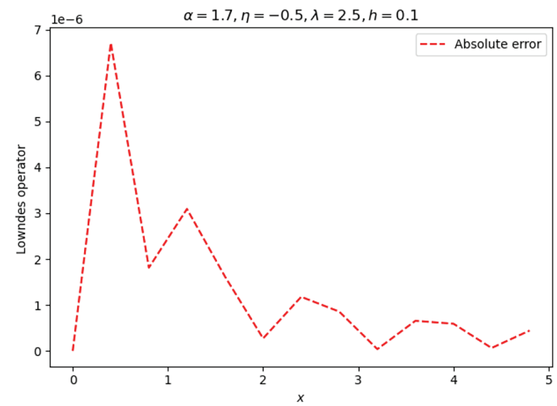
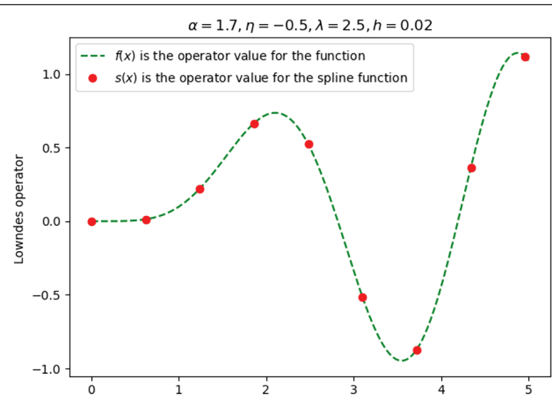
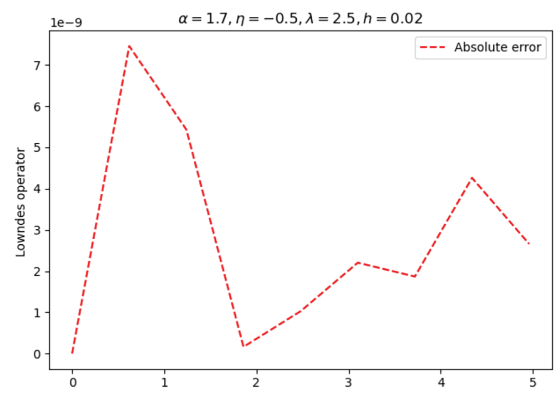
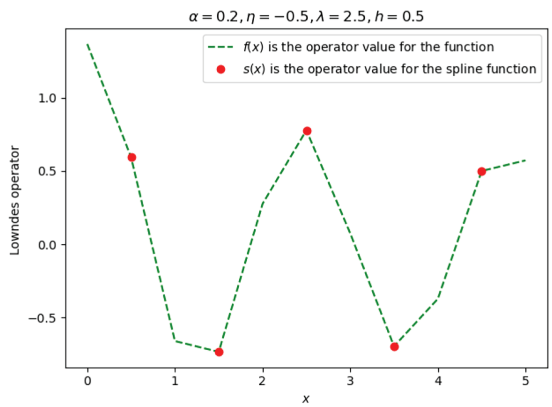
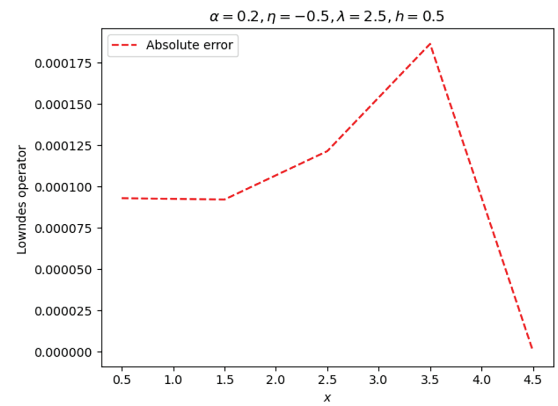
Exact and approximate computation graph	Absolute error
	
	
	

Table 2: We calculate the action of the generalized Erdélyi-Kober operator on the function $f(x) = \cos x$. To compute the exact value of the operator, we use Formula (18)

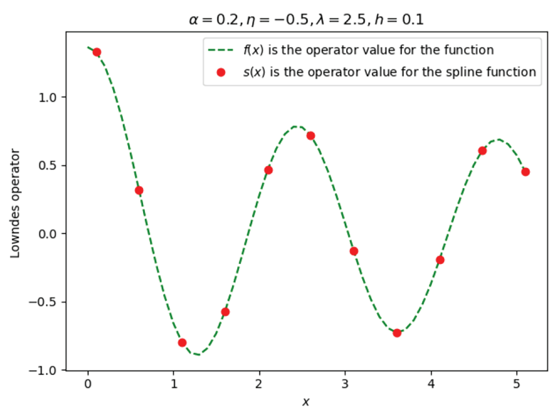
<table>
<tr><th>Exact and approximate computation graph</th><th>Absolute error</th></tr>
<tr><td>



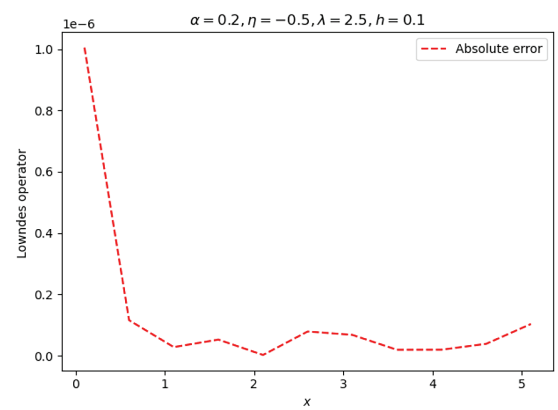
</td><td>



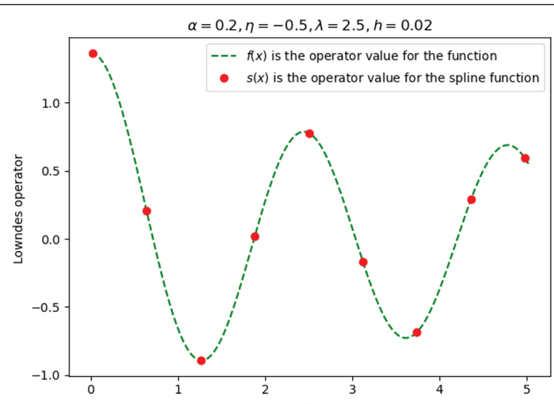
</td></tr>
<tr><td>



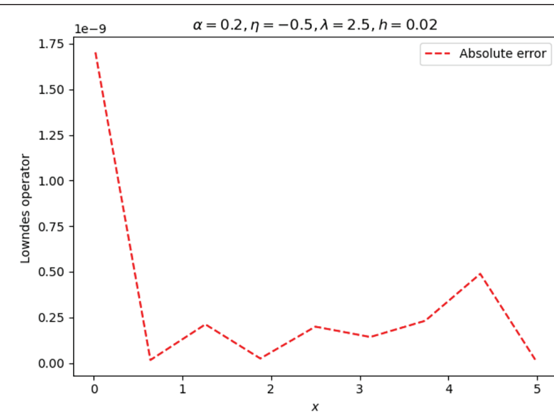
</td><td>



</td></tr>
<tr><td>



</td><td>



</td></tr>
</table>

By substituting $\beta = 2\frac{7}{20}$ into the last formula, we obtain Formula (17).

Example 2. If we select the test function $f(x) = \cos x$, we can utilize the following formula provided in [29] to compute the exact value of the generalized Erdélyi-Kober operator:

$$\Im^{\alpha}_{\lambda,-\frac{1}{2}} \cos x = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \bar{J}_{\alpha-\frac{1}{2}}\left(x\sqrt{\lambda^2+1}\right). \tag{18}$$

We now justify this formula. In [35], the following formula is given:

$$\int_0^a t^{\nu+1}\left(a^2-t^2\right)^{\mu/2} J_\mu\left(b\sqrt{a^2-t^2}\right) I_\nu(ct)\,dt = a^{\mu+\nu+1} b^\mu c^\nu \left(b^2-c^2\right)^{-(\mu-\nu+1)/2} J_{\mu+\nu+1}\left(a\sqrt{b^2-c^2}\right),$$

$$b > 0, \quad \operatorname{Re}\mu, \operatorname{Re}\nu > -1, \tag{19}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, defined as $I_\nu(z) = e^{-\nu\pi i/2} J_\nu\left(e^{\pi i/2} z\right)$ [35]. The following formula is also valid [36]:

$$\bar{I}_\nu(x) = \Gamma(\nu+1)\left(\frac{x}{2}\right)^{-\nu} I_\nu(x), \quad -\infty < x < \infty. \tag{20}$$

Applying Formulas (3) and (20) to (19), we obtain:

$$\int_0^a t^{2\nu+1}\left(a^2-t^2\right)^{\mu} \bar{J}_\mu\left(b\sqrt{a^2-t^2}\right) \bar{I}_\nu(ct)\,dt = \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{2\Gamma(\mu+\nu+2)} a^{2(\mu+\nu+1)} \bar{J}_{\mu+\nu+1}\left(a\sqrt{b^2-c^2}\right),$$

where $b > 0, \quad \operatorname{Re}\mu > -1, \quad \operatorname{Re}\nu > -1.$

By making the substitutions $a = x > 0$, $\nu = \eta \geq -\frac{1}{2}$, $\mu = \alpha - 1$, $\alpha > 0$, $b = \lambda \in \mathbb{R}$, and $c = \rho$, we derive:

$$\int_0^x t^{2\eta+1}\left(x^2-t^2\right)^{\alpha-1} \bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right) \bar{I}_\eta(\rho t)\,dt = \frac{\Gamma(\alpha)\Gamma(\eta+1)}{2\Gamma(\alpha+\eta+1)} x^{2(\alpha+\eta)} \bar{J}_{\alpha+\eta}\left(x\sqrt{\lambda^2-\rho^2}\right).$$

From this last equality, we obtain the result:

$$\Im^{\alpha}_{\lambda,\eta} \bar{I}_\eta(\rho x) = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} \bar{J}_{\alpha+\eta}\left(x\sqrt{\lambda^2-\rho^2}\right).$$

By replacing ρ with $i\rho$, and noting that $\bar{I}_\eta(i\rho x) = \bar{J}_\eta(\rho x)$, we obtain:

$$\Im^{\alpha}_{\lambda,\eta} \bar{J}_\eta(\rho x) = \frac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)} \bar{J}_{\alpha+\eta}\left(x\sqrt{\lambda^2+\rho^2}\right).$$

For $\eta = -\frac{1}{2}$, since $\bar{J}_{-\frac{1}{2}}(\rho x) = \cos \rho x$, we have:

$$\Im^{\alpha}_{\lambda,-\frac{1}{2}} \bar{J}_{-\frac{1}{2}}(\rho x) = \Im^{\alpha}_{\lambda,-\frac{1}{2}} \cos x = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \bar{J}_{\alpha-\frac{1}{2}}\left(x\sqrt{\lambda^2+\rho^2}\right).$$

This final equality yields Formula (18) for $\rho = 1$.

The obtained numerical results and errors are presented in the following table (Tables 3 and 4).

Table 3: Numerical values of the operator $\Im^{\alpha}_{\lambda,\eta} f\,(x)$ $(\eta = -0.5)$ when $f(x) = x^{4.7}$

x	α	λ	$\Im^{\alpha}_{\lambda,\eta} f(x)$	$h = 0.1$		$h = 0.01$	
				$\Im^{\alpha}_{\lambda,\eta} s(x)$	Absolute error	$\Im^{\alpha}_{\lambda,\eta} s(x)$	Absolute error
1.5	1.7	2.5	0.4083551	0.4083530	2.1036×10^{-6}	0.4083551	1.8275×10^{-10}
1.5	2.5	1.7	0.2090830	0.2090807	2.3330×10^{-6}	0.2090830	3.4352×10^{-10}
3.0	1.7	2.5	-0.3356440	-0.3356436	4.2291×10^{-7}	-0.3356440	2.9291×10^{-11}
3.0	2.5	1.7	1.8894576	1.8894571	5.3634×10^{-7}	1.8894576	4.6059×10^{-11}
4.5	1.7	2.5	0.7244580	0.7244580	7.5849×10^{-8}	0.7244580	1.4057×10^{-11}
4.5	2.5	1.7	0.5579314	0.5579318	4.1797×10^{-7}	0.5579314	4.0500×10^{-11}

Table 4: Numerical values of the $\Im^{\alpha}_{\lambda,\eta} f\,(x)$ $(\eta = -0.5)$ operator when $f\,(x) = \cos x$

x	α	λ	$\Im^{\alpha}_{\lambda,\eta} f\,(x)$	$h = 0.1$		$h = 0.01$	
				$\Im^{\alpha}_{\lambda,\eta} s\,(x)$	Absolute error	$\Im^{\alpha}_{\lambda,\eta} s\,(x)$	Absolute error
1.5	0.2	2.5	-0.7327097	-0.7317097	4.7498×10^{-8}	-0.7327097	9.8567×10^{-12}
1.5	2.5	0.2	0.7255687	0.7255687	4.7024×10^{-8}	0.7255687	8.5889×10^{-12}
3.0	0.2	2.5	0.0756358	0.0756357	7.6558×10^{-8}	0.0756358	4.4855×10^{-12}
3.0	2.5	0.2	0.3684999	0.3685000	2.0715×10^{-8}	0.3684999	4.3990×10^{-12}
4.5	0.2	2.5	0.5003536	0.5003535	3.7655×10^{-8}	0.5003536	7.1206×10^{-12}
4.5	2.5	0.2	0.0633754	0.0633754	3.7042×10^{-8}	0.0633754	4.4367×10^{-13}

8 Conclusion

The generalized Erdélyi-Kober fractional-order integral operator is a transmutation operator. Using this operator, many complex boundary value problems involving singular coefficients for differential equations with partial derivatives can be simplified and solved analytically with ease. This method also allows for the numerical solution of such problems. For this, it will be necessary to develop methods for approximating the value of the operator.

In this work, without knowing the analytical expression of the function and its properties, the action of the generalized Erdélyi-Kober fractional-order integral operator on this function was studied based on its given values on an arbitrary uniform grid. For this purpose, the function given on the grid was approximated using a cubic spline. The function, given in tabular form, was replaced by the spline determined analytically, and the value of the operator was computed. The convergence error of the developed method was determined, and the stability estimate was obtained. The numerical method was tested with specific examples, and the obtained results were presented in graphs and tables.

The ability to approximate functions given on a uniform grid using cubic splines in modern mathematical systems, particularly in the Python programming language, as well as the capability to compute the values of confluent hypergeometric functions, makes it possible to easily apply this method in practice.

Acknowledgement: Not applicable.

Funding Statement: The work of the third author was supported by the State Assignment of the Ministry of Education and Science of the Russian Federation under Grant No. FEGS-2023-0003.

Author Contributions: The authors confirm contribution to the paper as follows: Conceptualization, Shakhobiddin Karimov and Erkinjon Islamov; methodology, Erkinjon Islamov; software, Erkinjon Islamov; validation, Shakhobiddin Karimov, Erkinjon Islamov and Elina Shishkina; formal analysis, Elina Shishkina; investigation, Erkinjon Islamov; resources, Shakhobiddin Karimov; data curation, Erkinjon Islamov; writing—original draft preparation, Erkinjon Islamov; writing—review and editing, Elina Shishkina; visualization, Elina Shishkina; supervision, Shakhobiddin Karimov; project administration, Elina Shishkina; funding acquisition, Elina Shishkina. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: The authors confirm that the data supporting the findings of this study are available within the article.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Nakhushev AM. Fractional calculus and its application. Moscow, Russia: FIZMATLIT; 2003. 272 p.
2. Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: theory and applications. Yverdon, Switzerland: Gordon and Breach Science Publishers; 1993. 523 p.
3. Uchaikin VV. Method of fractional derivatives. Ulyanovsk, Russia: Artishok Publ; 2008. 512 p.
4. Marichev O, Shishkina E. Overview of fractional calculus and its computer implementation in Wolfram Mathematica. Fractional Calculus Appl Anal. 2024;27(5):1995–2062. doi:10.1007/s13540-024-00332-x.
5. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. In: North-holland mathematics studies. Vol. 204. Amsterdam, The Netherland: Elsevier; 2006. 540 p. doi:10.1016/s0304-0208(06)x8001-5.
6. Karimov ST, Sitnik SM. On some generalizations of the properties of the multidimensional generalized Erdélyi-Kober operators and their applications, transmutation operators and applications, trends in mathematics. Cham, Switzerland: Springer; 2020. p. 85–112.
7. Kiryakova VS. Generalized fractional calculus and applications. In: Pitman research notes in mathematics. Vol. 301. San Francisco, CA, USA: Longman Scientific & Technical; 1994.
8. Karimov ST. On one method for the solution of an analog of the cauchy problem for a polycaloric equation with singular bessel operator. Ukr Math J. 2018;69(10):1593–606. doi:10.1007/s11253-018-1457-4.
9. Karimov ST. An analog of the cauchy problem for the inhomogeneous multidimensional polycaloric equation containing the bessel operator. J Math Sci. 2021;254(6):703–17. doi:10.1007/s10958-021-05334-1.
10. Urinov AK, Karimov ST. On the cauchy problem for the iterated generalized two-axially symmetric equation of hyperbolic type. Lobachevskii J Math. 2020;41(1):102–10. doi:10.1134/s199508022001014x.
11. Karimov ST, Oripov SA. Solution of the Cauchy problem for a hyperbolic equation of the fourth order with the Bessel operator by the method of transmutation operators. Boletin de la Sociedad Matematica Mexicana. 2023;29(2):28.
12. Bateman H, Erdélyi A. Higher transcendental functions. Vol. 2. New York, NY, USA: McGraw-Hill; 1953. 396 p.

13. Sneddon IN. The use in mathematical analysis of Erdélyi-Kober operators and of some of their applications, In: Fractional calculus and its applications. Lecture notes in mathematics. New York, NY, USA: Springer; 1975. Vol. 457, p. 37–79. doi:10.1007/bfb0067097.
14. Lowndes JS. A generalisation of the Erdélyi-Kober operators. Proc Edinb Math Soc. 1970 1970;17(2): 139–48. doi:10.1017/s0013091500009408.
15. Lowndes JS. An application of some fractional integrals. Proc Edinb Math Soc. 1979;20(1):35–41. doi:10.1017/s0017089500003694.
16. De Boor CA. Practical guide to splines. Revised Edition. Berlin/Heidelberg, Germany: Springer; 2001. 372 p.
17. Diethelm K. The analysis of fractional differential equations: an application-oriented exposition using differential operators of caputo type. Berlin/Heidelberg, Germany: Springer; 2010.
18. Sitnik SM, Karimov ST. Solution of the goursat problem for a fourth-order hyperbolic equation with singular coefficients by the method of transmutation operators. Mathematics. 2023;11(4):951. doi:10.3390/math11040951.
19. Oldham KB, Spanier J. The fractional calculus: theory and applications of differentiation and integration to arbitrary order. San Diego, CA, USA: Academic Press; 1974. 234 p.
20. Cai M, Li C. Numerical approaches to fractional integrals and derivatives: a review. Mathematics. 2020;8(1):43. doi:10.3390/math8010043.
21. Li C, Zeng F. Numerical methods for fractional calculus. New York, NY, USA: Chapman and Hall/CRC; 2015. 300 p.
22. Ciesielski M, Grodzki G. Numerical approximations of the riemann-liouville and riesz fractional integrals. Informatica. 2023;35(1):21–46. doi:10.15388/23-infor540.
23. Hayotov A, Boboev S. An optimal quadrature formula with derivative for weakly singular integrals. Math Methods Appl Sci. 2025;5(36):165. doi:10.1002/mma.70034.
24. Shadimetov K, Nuraliev F, Toshboev O. Optimal quadrature formulas for the approximate calculation of the Riemann-Liouville fractional order integral. AIP Conf Proc. 2024;3004(1):060041. doi:10.1063/5.0199830.
25. Shadimetov K, Nuraliev F, Toshboev O. Optimal methods for approximate calculation of the fractional Riemann-Liouville integral in Sobolev space. AIP Conf Proc. 2024;3147(1):030031. doi:10.1063/5.0210290.
26. Karimov ST, Islamov ER. Approximation of the erdélyi-kober operator using cubic spline interpolation. Lobachevskii J Math. 2024;45(9):4534–47. doi:10.1134/s199508022460523x.
27. Płociniczak Ł., Sobieszek S. Numerical schemes for integro-differential equations with Erdélyi-Kober fractional operator. Numer Algor. 2017;76(1):125–50. doi:10.1007/s11075-016-0247-z.
28. Sadek L, Idris SA, Jarad F. The general Caputo-Katugampola fractional derivative and numerical approach for solving the fractional differential equations. Alex Eng J. 2025;2025(121):539–57. doi:10.1016/j.aej.2025.02.065.
29. Urinov AK, Sitnik SM, Shishkina EL, Karimov SHT. Fractional integrals and derivatives (Generalizations and Applications). Fergana, Uzbekistan: Fergana Publishing; 2022. 192 p.
30. Engeln-Müllges G, Uhlig F. Numerical algorithms with C. Berlin, Germany: Springer; 1996.
31. Hall CA. On error bounds for spline interpolation. J Approx Theory. 1968;1(2):209–18. doi:10.1016/0021-9045(68)90025-7.
32. Gupta AK, Nadarajah S, editors. Handbook of beta distribution and its applications. 1st ed. Boca Raton, FL, USA: CRC Press; 2004. 600 p.
33. Bateman H, Erdélyi A. Higher transcendental functions. Vol. I. New York, NY, USA: McGraw-Hill; 1953. 326 p.
34. Kapilevich MB. Confluent hypergeometric Horn functions. Differ Uravn. 1966;2(9):1239–54.

35. Prudnikov AP, Brichkov YA, Marichev UI. Integrals and series. Special functions. London, UK: Gordon and Breach Science Publishers; 1983. 752 p.
36. Urinov AQ. Special functions and special operators. Fergana, Uzbekistan: Fergana Publishing; 2012. 112 p.