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TESTING THE USE OF RADIAL BASIS FUNCTION AUGMENTED WITH POLYNOMIALS AS BASIS FUNCTIONS IN THE BOUNDARY ELEMENT METHOD FOR HEAT TRANSFER PROBLEMS

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Abstract. *The accuracy of the numerical solution obtained by the Boundary Element Method (BEM) is directly affected by the type of interpolation function used. Meanwhile, interpolation by radial basis function augmented with polynomials has been shown to be more accurate than Lagrange interpolation for a range of different functions.*

Therefore, this paper is concerned with the application of such functions as the interpolation functions for all boundary values in the boundary element method for the numerical solution of two-dimensional heat transfer problems. Numerical examples with different geometries and temperature distributions are presented and comparisons with both isogeometric and classical formulation are made to demonstrate the accuracy of the proposed method.

1 INTRODUCTION

The Boundary element method (BEM) is one of most commonly used methods for solving problems in continuum mechanics [1] and can be used in different areas of engineering. The method begins by obtaining the solution only at the boundary. After obtaining the solution at the boundary, the potential inside the domain can be calculated, using the boundary integral equation, without further approximations, with the from the boundary [2]. Therefore, a improvement at of the solution at the boundary also improves the solution at the domain of the problem. For this reason, a good boundary solution is paramount. One of the known improvements in the solution at the boundary is the improvement in the interpolation used during the solution. The most common method of interpolation is by polynomial functions, being more common the use of polynomials of first, second or third order. On the other hand, using NURBS in a isogeometric setting has already shown to have a superior accuracy to that obtained by the polynomials with the same number of collocation points.[3].

Another popular interpolation method is interpolation by radial basis function (RBF), RBF methods appeared in the early 2000s as an alternative for solving differential equations in irregular domains [4, 5, 6]. Since then, different methods, such as the finite element method [7], the finite difference method [8] and BEM[9], have shown good results with its use. In [9] BEM with RBF interpolation was used to solve heat transfer problems. It presented a formulation that used Indirect RBF to interpolate the temperature and its normal gradient. The proposed method obtained improved better solutions not only in terms of the accuracy but also in terms of the rate of convergence. However as the indirect RBFS introduced new unknowns the method could only be solved in the least squares sense by using Singular Value Decomposition technique. This makes the method unsuited for larger problems, as this the computational cost increases rapidly at $O(n^3)$.

Recently it was proposed the addition of a polynomial element to the RBF [10], and thus improving the accuracy of the method.

This work applies RBF augmented with polynomials as the interpolating function of the BEM for solving heat transfer problems in 2d. The accuracy and computational cost of the solution will be compared with both the isogeometric and the classical formulation.

2 RBF INTERPOLATION AUGMENTED WITH POLYNOMIALS

Considering a set of known points $s_1, s_2, ..., s_n$ of a function $y(s)$ unknown, the purpose of the RBF interpolation is to obtain a function $s(x)$ continuous such that:

$$y(x) \approx s(x) = \sum_{j=1}^{n} \lambda_j \phi(||x - x_j||) \tag{1}$$

where $\phi(||x - x_j||)$ is a radial basis function centered on x_j , $||.||$ is the standard euclidean distance, n is the number of radial basis functions and $\lambda^{(i)}$ is the set of weights to be found.

When considering polynomial augmentation, the function $s(x)$ is described by [11]:

$$y(x) \approx s(x) = \sum_{j=1}^{n} \lambda_j \phi(||x - x_j||) + \sum_{k=1}^{s} \beta_k p_k(x) \tag{2}$$

with the conditions

$$\sum_{j=1}^{n} \lambda_j p_k(x_j) = 0, k = 1, ..., s, \tag{3}$$

where p_k corresponds to the polynomial augmentation, β is the polynomial coefficients and s is the order of the polynomial.

Using Equation 2 for the specific points, $s(x_i) = f_i, i = 1, ..., n$ together with Equation 3, we can generate the following system of equations:

$$\begin{bmatrix} A & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \beta \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \tag{4}$$

where A is the square matrix with elements

$$A_{i,j} = \phi(||x_i - x_j||), i, j = 1, ..., n, \tag{5}$$

and P is the $n \times s$ matrix

$$P = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \dots & p_s(x_1) \\ p_1(x_2) & p_2(x_2) & \dots & p_s(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \dots & p_s(x_n) \\ \end{bmatrix}. \quad (6)$$

Using Equation 2 directly on the BEM is not straightforward, as the unknowns would only be calculated

2.1 Expression for radial basis function augmented with polynomial

In [10] an alternative equation for $s(x)$ where the coefficients are not explicitly needed is obtained. The most important results for this work are summarized in this section.

Considering A and P to be full rank, the radial basis function augmented with polynomials will be

$$s(x) = \mathbf{f}^T[I + WP^T]A^{-1}\phi(x) + \mathbf{f}^T W\mathbf{p}(x) \quad (7)$$

where W is an $n \times s$ array defined by:

$$W = A^{-1}P(P^T A^{-1}P)P^{-1} \quad (8)$$

the vector $\phi(x)$ contains the spatial basis for the RBF:

$$\phi(x) = [\phi(||x - x_1||), \phi(||x - x_2||), ..., \phi(||x - x_n||)]^T \quad (9)$$

and $p(x)$ is the polynomial space Π_l^d

$$\mathbf{p}(x) = [p_1(x), p_2(x), ..., p_n(x)] \quad (10)$$

Equation 7 can be rewritten as:

$$s(x) = E(x)\mathbf{f} \quad (11)$$

Where $E(x)$ is defined by:

$$E(x) = [I - WP^T][A^{-1}\phi(x)]^T + W^T\mathbf{p}(x)^T \quad (12)$$

and is the shape function that will be used for the proposed BEM formulation.

2.2 Interpolating examples

To test the radial basis functions augmented with polynomials, two functions will be interpolated:

$$y_1 = 5s + 10 \quad (13)$$

and

$$y_2 = 0.02(12 + 3s - 3.5s^2 + 7.2s^3)(1 + \cos(4\pi s)(1 + 0.8\sin(3\pi s)). \quad (14)$$

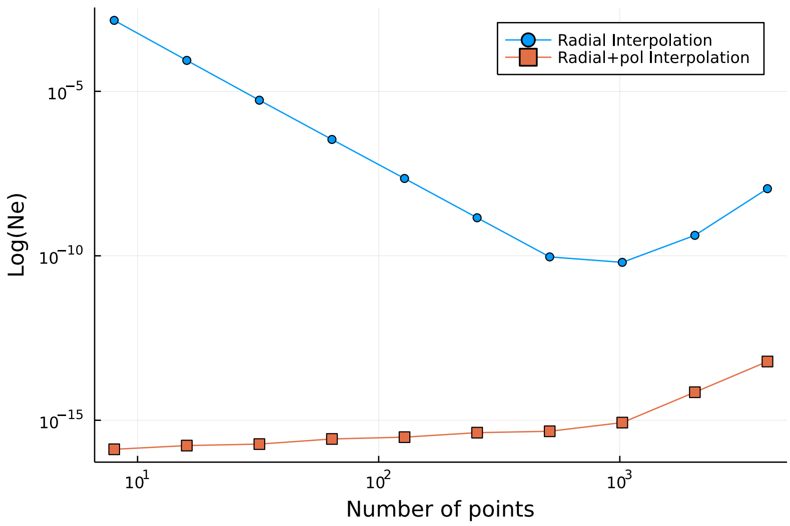
In both examples the independent variable s will be analyzed from 0 to 1. The system accuracy will be measured by the relative error norm defined by:

$$N_e = \left[\frac{(\sum_{i=1}^{q}(y(s^i) - f(s^i))^2}{\sum_{i=1}^{q} y(s^i)^2)} \right]^{1/2}. \quad (15)$$

2.2.1 A linear function

The first example is intended to demonstrate the difficulty that the standard RBF has when trying to interpolate a simple straight line, y_1. In Figure 1 it becomes clear that by augmenting the RBF with a polynomial the interpolant is now able to correctly interpolate y_1 even with a the lowest number of points. An significant increase in the error is observed for both options when the number of points becomes larger than 10^3. For this we will use the Equation 13.

Figure 1: Interpolation error of y_1 for straight lines



It is known that the interpolation by radial basis functions has a weakness, which is the interpolation of straight lines, this can be verified in Figure 1, where the error for the interpolation by RBF has errors greater than RBF+pol, even we can see that the interpolating functions after a certain time have a rapidly growing error. Thus, we can say that the argument with the polynomial is important since cross sections are widely used in engineering.

2.2.2 A smooth function

This example will be used to compare the interpolation of the smooth function y_2 obtained by the augmented RBF with traditional interpolation methods. The errors for different discretization levels can be viewed in Figure 2.

In this example both RBF and RBF+pol interpolation have a similar error profile, while outperforming interpolating splines of order 3 and 4 and a piece-wise fourth order polynomial interpolation.

Also, for this interpolation method it is important to correctly define the point distribution, as this factor directly affects the quality of the solution. The concentration of points at the end of the segments has the power to improve the quality of the interpolation and of its derivative [8]. So for this work we will use the Gauss-Legendre quadrature to distribute the points on the boundary of the problems. Figure 3 contains the normalized error for the two interpolation methods, note that the interpolation with the Gauss-Legendre distribution proves to be superior for a large range of points, and for this reason this will be the distribution to be used in this article.

Figure 2: Interpolation of function y_2

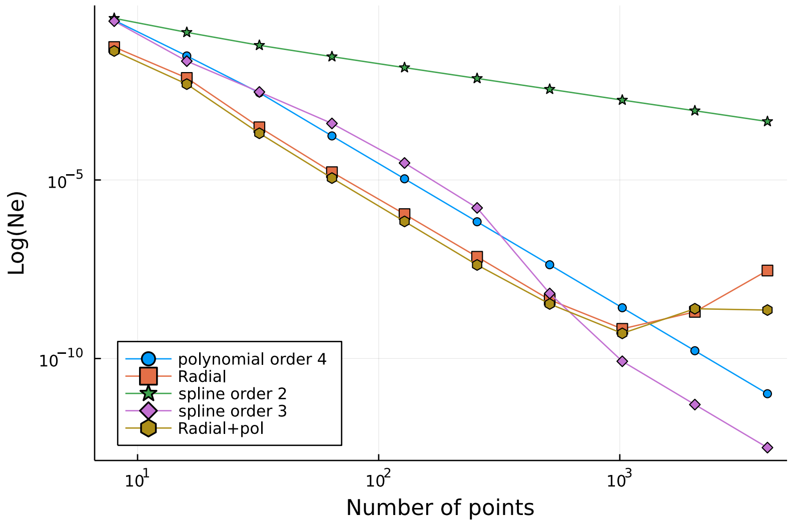
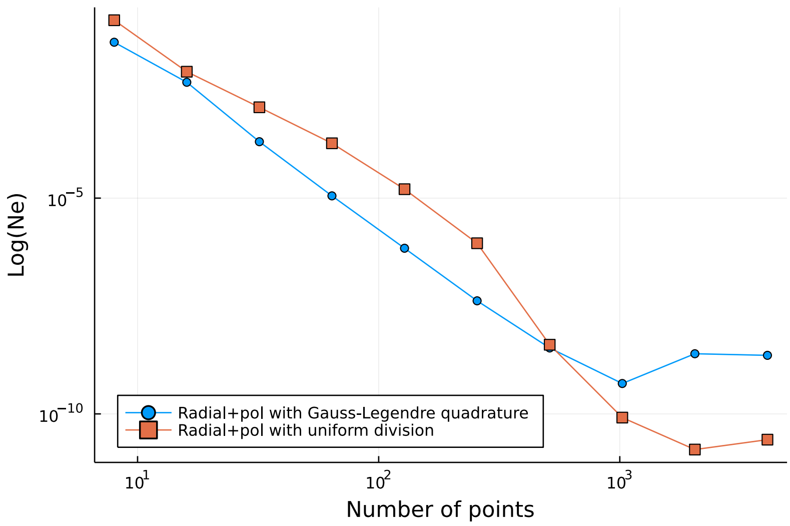


Figure 3: Interpolation for different point distribution



3 THE BOUNDARY ELEMENT METHOD

A direct collocation form of the BEM consists in constructing a system of equations by taking a collocation point for every unknown and integrating along the boundary. For potential problems, the integral equation that relates the potential and its derivative is given as [12]:

$$cu(x') = \int_\Gamma \frac{\partial u}{\partial n}(x) u^*(x,x') d\Gamma - \int_\Gamma u(x) \frac{\partial u^*}{\partial n}(x,x') d\Gamma \tag{16}$$

where c is a jump term that arises from the limiting process of the integral equation and is dependent on the geometry at the source point x', $u(x)$ and $\frac{\partial u}{\partial n}(x)$ are the potential and its derivative in the normal to the boundary direction, respectively, while $u^*(x,x')$ and $\frac{\partial u^*}{\partial n}(x,x')$ refers to fundamental solutions, and finally Γ is the boundary to be studied.

In order to make Equation 16 suitable for numerical implementation, the continuous fields $u(x)$ and $\frac{\partial u}{\partial n}(x)$ have to be represented in a discrete manner. In the conventional BEM, the idea is to split the boundary into elements using a Lagrange interpolation while in this proposed formulation Equation 11 is used to approximate each variable and the geometry. This leads to:

$$cu(x') = \int_\Gamma u^*(x,x') \left(\frac{\partial u^c}{\partial n} E(x) \right) d\Gamma - \int_\Gamma (u^c E(x))) \frac{\partial u^*}{\partial n}(x,x') u(x) d\Gamma. \tag{17}$$

The boundary integrals in the conventional BEM must be divided into multiple elements whereas in this formulation there is only a need to separate the boundary at the corners or at the change of the type of boundary condition. Each integral can then be easily integrated numerically using Gauss-Legendre quadrature.

4 HEAT TRANSFER EXAMPLES

In this section, the proposed formulation is compared with classical BEM with discontinuous quadratic an cubic elements and with an isogeometric formulation that uses cubic splines. All errors are evaluated according to Equation 15. Finally, first-order polynomial is used to augment all RBFs.

4.1 Heat transfer in a square plate

In this example the capability of the proposed method to represent geometries with straight edges will be tested. A closed square domain, with dimensions of 6 x 6, is used. The prescribed temperatures on the right and on the left are respectively 300 and 0, while at the top and bottom the heat flux is equal to zero. With these boundary conditions, the temperature will vary linearly along the plate. The analytical solution can be written as:

$$u(x_1, x_2) = 300 - 50x_1. \tag{18}$$

The error will be evaluated at 25 evenly spaced internal points, as shown in Figure 4, and thus ensuring that the solution is evaluated at the same coordinates for all methods. The results for the different methods can be seen at Figure 5.

BEM with radial basis function augmented with polynomials presented a smaller error when compared to the other methods. The minimum error of the proposed formulation had a minimum error of $1.09e-10$, while the best of the other methods achieved the smallest error of $1.73e-8$.

4.2 Heat transfer in a hollow cylinder

The second example is a hollow cylinder as shown in Figure 6. The temperature is known at the inner boundary S_i and the flux is known at the outer surface S_e. The analytical solution for the temperature will be given by the following equation:

For this problem, the following will be adopted: $r_i = 1$, $r_2 = 2$, $T_i = 100$, $q_e = 200$ and $k = 1$, the outer and inner contours will have the same number of points. The results for different interpolation methods in BEM can be seen in Figure 7. The analytical solution for the temperature is given by:

$$T(r) = T_i + q_e r_e \log\left(\frac{r}{r_i}\right) \tag{19}$$

Figure 4: Geometry example 1

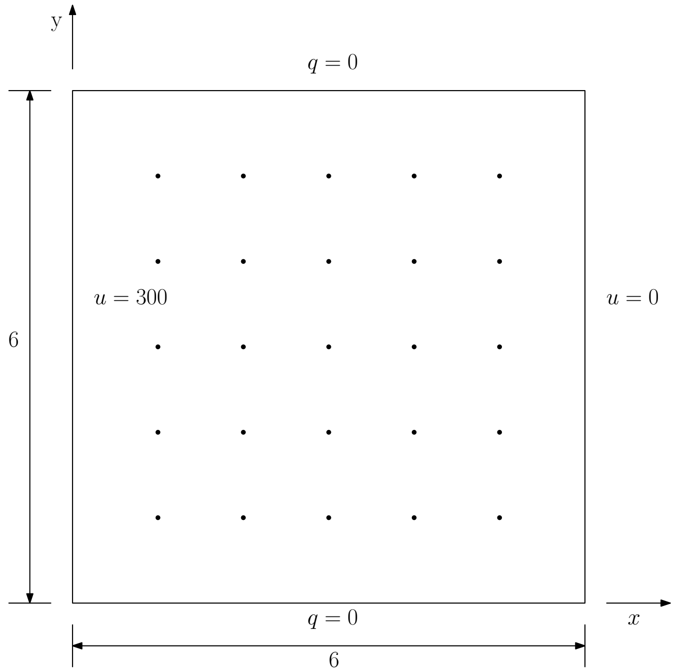
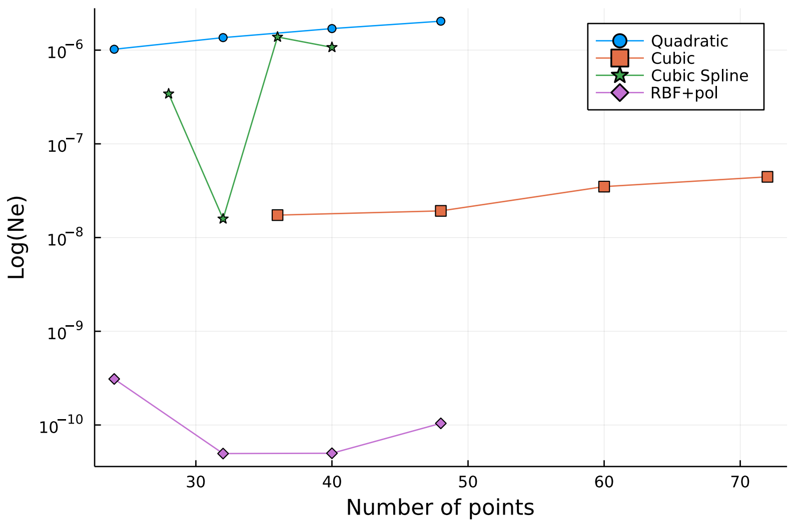


Figure 5: Example numerical 1



and for the flux:

$$q(r) = q_e \frac{r_e}{r} \tag{20}$$

where T_i e q_e are the temperature and the flux at the internal and external boundaries, respectively.

For this example, unlike the first, the method with RBF+pol interpolation did not achieve the lowest errors for the same level of discretization, where BEM with third-order polynomials presented the lowest

Figure 6: Annular region of Numerical example 2

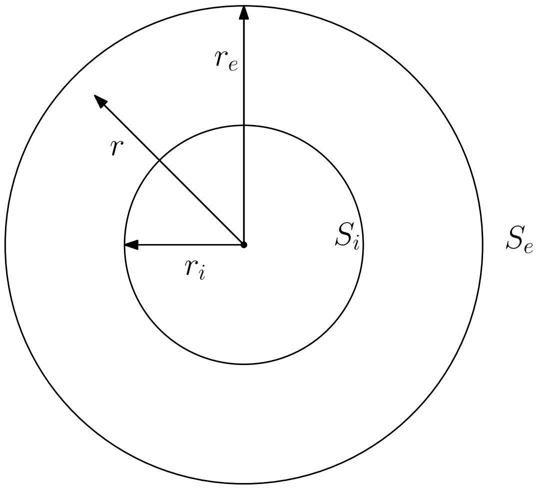
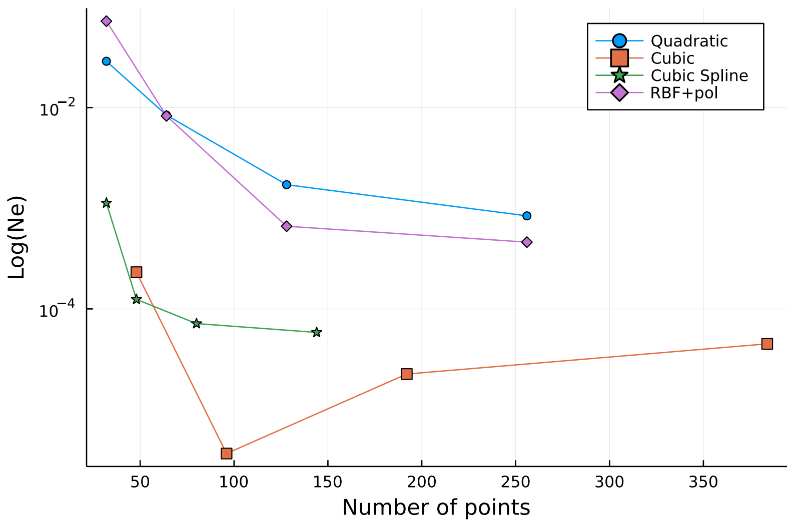


Figure 7: Example numerical 2



errors, the method with RBF+pol interpolation managed to reach a minimum error in the order of $10e-3$ with 128 points.

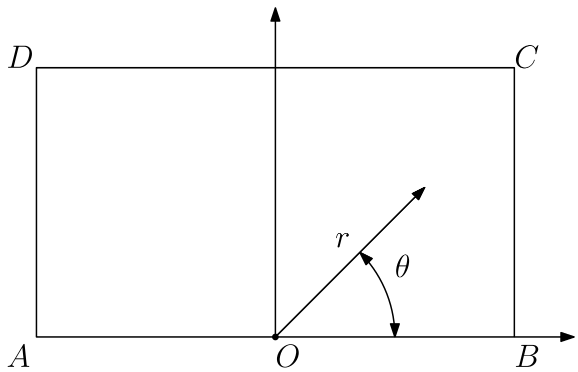
4.3 Heat transfer in a rectangular plate

This example will be used to study a square plate with dimensions of 2×1 that is shown in Figure 8. The following boundary conditions are considered for this structure:

$$q = -\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right) \text{ in } BC, \tag{21}$$

$$q = -\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\cos\theta - \sin\frac{\theta}{2}\sin\theta\right) \text{ in } CD, \tag{22}$$

Figure 8: Geometry example 3



$$q = \frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right) \text{ in } DA, \tag{23}$$

$$T = 0 \text{ in } AO \tag{24}$$

and

$$q = 0 \text{ in } OB. \tag{25}$$

This problem has an analytical solution given by:

$$u = \sqrt{r}\cos\frac{\theta}{2}; \tag{26}$$

$$q_x = \frac{\cos\frac{\theta}{2}}{2\sqrt{r}} \tag{27}$$

and

$$q_y = \frac{\sin\frac{\theta}{2}}{2\sqrt{r}}. \tag{28}$$

The Figure 9 shows the normalized error for the different methods. Where the solution for the proposed method, with a number of points larger than 128, has the lowest error.

4.4 Heat transfer in a square plate under different boundary conditions

In this example a square plate with dimensions 1×1 is analyzed, where in the entire structure $k = 1$, the boundary conditions are defined as in Figure 10.

The analytical solution to this problem is given as:

$$u(x_1, x_2) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x_1}{L}\right)\frac{\sinh\left(\frac{n\pi x_2}{L}\right)}{\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi W}{L}\right)} \tag{29}$$

In Figure 11, it can be seen that the proposed formulation had smaller errors than the other methods.

Figure 9: Example numerical 3

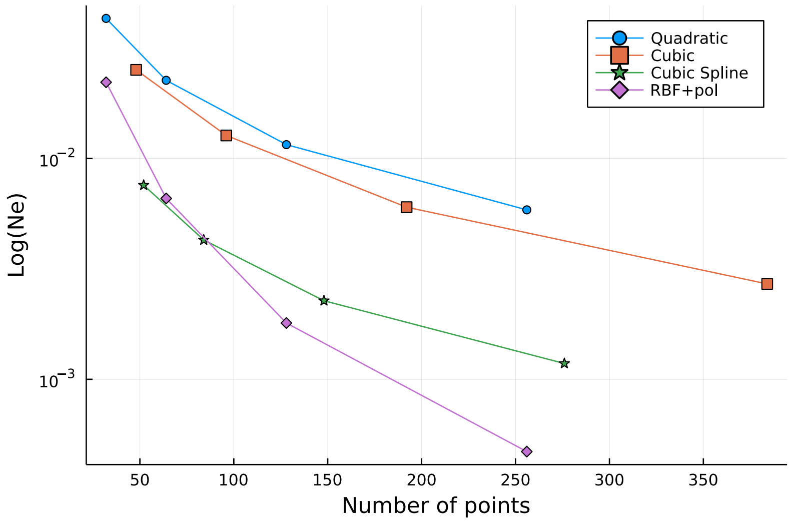
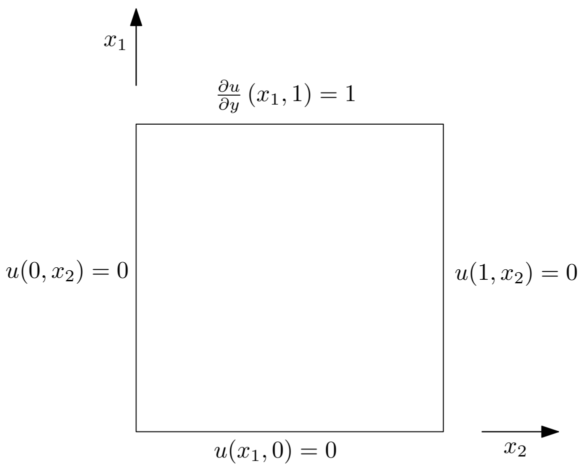


Figure 10: Geometry example 4



5 COMPUTATIONAL COST

Another important factor is how computationally expensive each method is. To measure this cost, the necessary time that each method takes to solve numerical example 3 for different discretizations will be compared. The simulations were made in a laptop with a Intel Core I7-10750H processor with $8GB$ of RAM.

In Figure 12 it can be seen that the proposed method was slower than traditional BEM and faster than the isogeometric formulation.

Figure 11: Example numerical 4

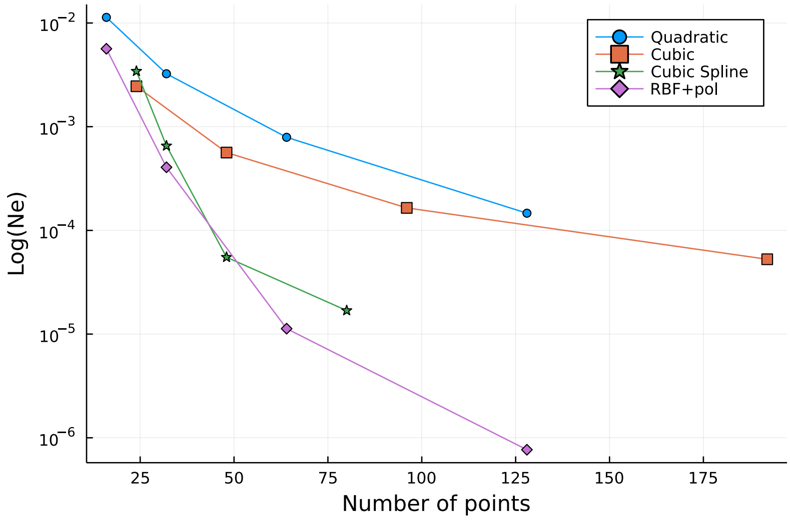
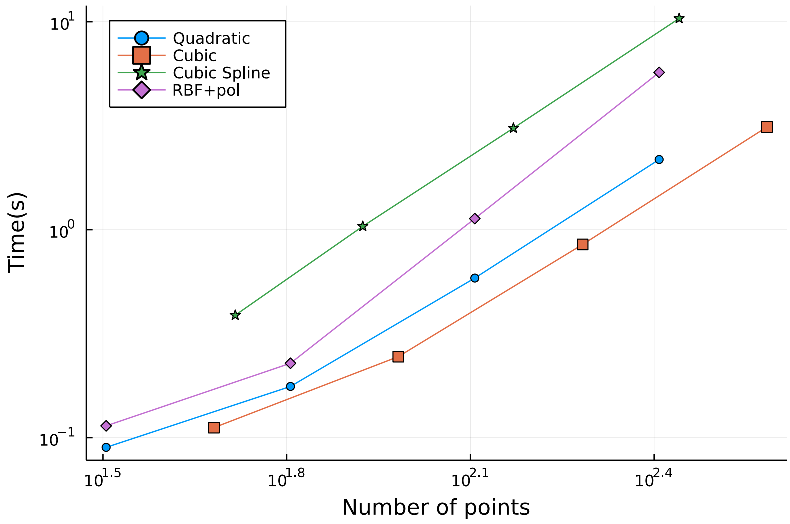


Figure 12: Solution time for example 4



6 CONCLUSION

In this work, BEM with radial basis functions augmented with polynomials as interpolation function was used to solve heat transfer problems. The proposed method presented the lowest error for all examples with straight geometries. It did not perform as well on the problem with circular section, this behavior might be explained by the difficulty on representing that geometry exactly. Finally, the proposed method was also faster then the isogeometric formulation.

The method can be further improved by testing different radial basis functions, including functions with compact support. The resulting interpolating matrix presented very high condition number as the number of points was increased. Testing different nodes distributions is also desired, as the condition number is directly related to the smallest distance between two nodes.

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