New assumed strain triangles for non linear shell analysis*

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Abstract A comparison between new and existing triangular finite elements based on the shell theory proposed by Juan Carlos Simo and co-workers is presented. Particular emphasis is put on the description of new triangles which show a promising behaviour for linear and non linear shell analysis.

1 Introduction
Considerable effort has been devoted in recent years to the development of efficient and reliable shell elements for linear and non linear analysis. Mainly two different approaches for the formulation of the elements have been used: the so called “degenerated solid” elements (Ahmad, Irons and Zienkiewicz 1970; Stanley, Park and Hughes 1986) and elements developed within the frame of a shell theory. Although the debate between “degenerate” versus “shell” formulation is still open, it is not the object of this work to discuss this subject. The interested reader is addressed to recent work of Büchter and Ramm (1992) and Büchter, Ramm and Roehl (1994) on this matter.

A general non linear shell theory very adequate for finite element computations has been recently proposed by Simo and Fox (1989), Simo, Fox and Rifai (1989, 1990), Simo, Rifai and Fox (1990) and Simo and Kennedy (1992) and excellent results were shown for the well known four node assumed shear strain quadrilateral. This fact motivated the authors to explore the possibilities of triangular elements based on this shell theory. Thus, the object of this work is to assess the behaviour of some linear assumed strain and quadratic triangular elements in the context of Simo’s shell formulation. Particular emphasis is put on the extension of some successful assumed shear strain plate bending triangles developed by Oñate and Zienkiewicz, Suarez and Taylor (1992), Oñate, Zarate and Flores (1994) for linear and non linear shell analysis.

The layout of the paper is as follows. In next section a brief summary of Simo’s shell theory is presented. Full details can be found in Simo and Fox (1989) and Simo, Fox and Rifai (1989). Then the different triangular shell elements analysed are described. Finally, examples of applications showing the performance of the triangular elements for some linear and non linear shell problems are given.

2 Summary of the shell theory chosen
A brief description of the shell theory developed by Simo and co-workers is presented here.

The configuration of the shell in $\mathbb{R}^3$ is defined by

\[ x = \varphi^t \xi, \xi \in [h^-, h^+] \]

where $[h^-, h^+]$ defines the shell thickness. Using a standard basis $[e_x, e_y, e_z]$ in $\mathbb{R}^3$ we can write (greek indices vary from 1 to 2)

\[ \varphi = \varphi^i e_i, \quad \varphi^i = \varphi^i x^i, \quad t = t^i e_i \]

The following measures over the mid-surface are defined

\[ d\mu^0 = \frac{1}{2} d\xi^1 d\xi^2 \quad d\mu = \frac{1}{2} d\xi^1 d\xi^2 \]

with

\[ j^0 = (\varphi^i \times \varphi^j) \cdot t, \quad j = (\varphi^i \times \varphi^j) \cdot t \]

and

\[ \bar{f} = \bar{f}^0 j^0 \]
The resultant stresses can be written in terms of stress measures over the deformed or current configurations as

\[
\begin{align*}
\mathbf{n}^s &= \frac{1}{J^s} \int_h \mathbf{g}^s \mathbf{g}^s j \, d\xi^s = \frac{1}{J^s} \int_h \mathbf{P} \mathbf{g}^s \mathbf{g}^s j \, d\xi^s \\
\mathbf{m}^s &= \mathbf{t} \times \frac{1}{J^s} \int_h \mathbf{g}^s \mathbf{g}^s j \, d\xi^s = \mathbf{t} \times \frac{1}{J^s} \int_h \mathbf{P} \mathbf{g}^s \mathbf{g}^s j \, d\xi^s = \mathbf{t} \times \mathbf{m}^s
\end{align*}
\]  

(14) (15)

\[
\begin{align*}
\mathbf{l} &= \int_h \mathbf{g}^s \, d\xi^s = \frac{1}{J^s} \int_h \mathbf{P} \mathbf{g}^s \, d\xi^s
\end{align*}
\]  

(16)

where \( \mathbf{g}^s = \partial \mathbf{x}/\partial \xi^s \), \( \sigma \) and \( \mathbf{P} \) are the Cauchy and 1st Piola-Kirchhoff stress tensors, respectively. On the other hand \( \mathbf{n}^s \) and \( \mathbf{m}^s \) are the resultant stresses and bending moments along a line \( \xi^s \) = constant and \( \mathbf{l} \) is the resultant stress across the thickness. Vector \( \mathbf{m}^s \) is termed “director bending moment” and it allows to define the following “effective resultant stresses” as

\[
\begin{align*}
\mathbf{n}^e &= \mathbf{n}^s - \mathbf{m}^s \\
\mathbf{m}^e &= \mathbf{m}^s
\end{align*}
\]  

(17) (18)

The deformation gradient over the mid-surface (\( \xi^s = 0 \)) in \( \mathbf{F} = \mathbf{a} \mathbf{a} + \mathbf{t} \mathbf{a} = \mathbf{a} \mathbf{a} \mathbf{a} \)

(7)

It can be shown that the resultant stresses \( \mathbf{n}^s \) and \( \mathbf{m}^s \) are conjugate of the generalized strains defined in (5, 6, 7). The constitutive equation between both sets of magnitudes can be written as

\[
\begin{align*}
\mathbf{n}^e &= \rho \mathbf{c} \mathbf{p} \\
\mathbf{m}^e &= \rho \mathbf{c} \mathbf{p} \mathbf{L} \mathbf{c} \mathbf{p} + \mathbf{m}^e \\
\mathbf{m}^e &= \rho \mathbf{c} \mathbf{p} \mathbf{L} \mathbf{c} \mathbf{p} + \mathbf{m}^e
\end{align*}
\]  

where \( \rho \) is the material density. Details on the constitutive relationship for both elasticity and elasto-plasticity using an hyperelastic framework can be found in Simo and Fox (1989) and Simo and Kennedy (1992).

The expression of the internal stress power due to deformation can be written as

\[
\mathcal{W} := \int_{\mathcal{V}} \mathbf{P} : \dot{\mathbf{E}} \, d\mathcal{V}
\]

\[
= \int_{\partial \mathcal{V}} \left[ \mathbf{n}^e \cdot \mathbf{\tilde{v}}^e + \mathbf{m}^e \cdot \mathbf{\tilde{t}}_{\mathcal{V}} + \mathbf{l} \cdot \mathbf{\tilde{t}} \right] \, d\mu
\]

\[
= \int_{\partial \mathcal{V}} \left[ \mathbf{\tilde{n}} \cdot \mathbf{L}_{\alpha} \mathbf{c} + \mathbf{\tilde{q}} : \mathbf{L}_{\alpha} \mathbf{\tilde{\varepsilon}} + \mathbf{\tilde{m}} : \mathbf{L}_{\alpha} \mathbf{\tilde{\gamma}} \right] \, d\mu
\]

(22)

Equation (22) is the basis for the derivation of the finite element formulations following standard procedures. This requires the definition of adequate admissible variations.

Using identical definitions for the reference configuration \( \mathcal{F}^0 \), the following generalized Lagrangian strains can be defined

\[ \varepsilon(\mathbf{\varphi}) = \frac{1}{2} \begin{bmatrix} a_{11} - a_{11}^0 & \ldots & a_{1n} - a_{1n}^0 \\ \vdots & \ddots & \vdots \\ a_{n1} - a_{n1}^0 & \ldots & a_{nn} - a_{nn}^0 \end{bmatrix} \]

(11)

\[ \delta(\mathbf{\varphi}, \mathbf{t}) = \begin{bmatrix} \gamma_{11} - \gamma_{11}^0 \\ \vdots \\ \gamma_{nn} - \gamma_{nn}^0 \end{bmatrix} \]

(12)

\[ \chi(\mathbf{\varphi}, \mathbf{t}) = \frac{1}{2} \begin{bmatrix} \kappa_{11} - \kappa_{11}^0 \\ \vdots \\ \kappa_{nn} - \kappa_{nn}^0 \end{bmatrix} \]

(13)

using identical definitions for the reference configuration \( \mathcal{F}^0 \), the following generalized Lagrangian strains can be defined

\[ a_{pp} = \mathbf{\varphi}_p \cdot \mathbf{t}_p \]

(10)
\[ \delta t = \bar{\Lambda} \delta T \]  \hspace{1cm} (23) 

where \( \bar{\Lambda} \) contains the first two rows of the orthogonal matrix which transforms the global vector \( e \) into \( t \) and \( \delta T = [\delta T_x, \delta T_y]^T \). Equation (23) can be written in a more convenient form as

\[ \delta t = \delta \theta \times t \]  \hspace{1cm} (24)

where \( \delta \theta \) is a vector normal to the plane formed by \( t \) and \( \delta t \), i.e. \( \delta \theta \cdot t = 0 \).

3 Description of new triangular shell elements

3.1 Six node quadratic shell triangle with linear assumed transverse shear (TQL1 element)

This element is an extension of the quadratic plate triangle presented in Zienkiewicz, Taylor, Papadopoulos and Oñate (1990) and Oñate, Zienkiewicz, Suarez and Taylor (1992). The geometry of the element is shown in Fig. 2. Both initially flat (subparametric) and curved (isoparametric) versions of the element have been considered. In the subparametric case, the initial geometry is linearly interpolated in terms of the vertex nodal values, whereas a quadratic approximation is used in the isoparametric case. For flat triangles the Jacobian matrix is constant which considerably simplifies some computations. In both flat and curved cases the displacement field is given by

\[ u = \sum_{i=1}^{6} N_i u^i \] 
\[ \tilde{t} = \sum_{i=1}^{6} N^i t^i \text{ with } t = \frac{\tilde{t}}{||\tilde{t}||} \]  \hspace{1cm} (25)

where \( N_i \) are the quadratic shape functions of the standard six node \( C_0 \) triangle (Zienkiewicz and Taylor 1989/1991).

The updated middle surface configuration and the director field are obtained for step \( k+1 \) by

\[ \varphi^{k+1} = \sum_{i=1}^{6} N_i(\xi, \eta) \left( \varphi^i + \mathbf{u}^{k+1}_i \right) \]  \hspace{1cm} (26)

\[ t^{k+1} = \sum_{i=1}^{6} N^i(\xi, \eta) t^i t^{k+1} \]  \hspace{1cm} (27)

with

\[ t^{k+1} = \exp\left[ \frac{\Delta t}{\|\Delta t\|} \right] t^k + \frac{\sin\left( \frac{\Delta t}{\|\Delta t\|} \right)}{\|\Delta t\|} \Delta t^k \]  \hspace{1cm} (28)

Note that \( \Delta t^k \) is the increment between steps \( k \) and \( k+1 \).

The local axes in this element have been defined as follows. Axis \( x_i \) is taken orthogonal to the element plane; \( x_i \) is assumed to lay in the intersection of the element with the global \( x, y \) plane and \( x_i = x_j \times x_k \).

Two versions of the element have been studied in this work. The first one is based on the standard displacement formulation for both bending and membrane fields whereas a linear transverse shear strain field is assumed in the natural coordinate system as

\[ \gamma_i = \gamma_i + \sum_{j=1}^{2} \gamma_{ij} \xi_j + \gamma_{ij} \eta_j \]  \hspace{1cm} (29)

Parameters \( \gamma_i, \gamma_{ij}, \gamma_{ij} \) are obtained by sampling the tangential shears in the six side points shown in Fig. 2b. The derivation of the substitute (bilinear) transverse shear strain matrix for this case follows the lines explained in Oñate, Zienkiewicz, Suarez and Taylor (1992). Further details are given in Flores and Oñate (1993) and in Oñate (1993) where this element is termed TQL1 (for Triangle, Quadratic displacements, Quadratic rotations and Linear assumed transverse shear fields). The flat version of this element is termed TQL0 whereas TQL denotes the curved isoparametric version.

For the isoparametric (curved) case, a second version of this element uses the following assumed membrane field (aimed to avoid membrane locking)

\[ \mathbf{e}'(\xi, \eta) = \begin{bmatrix} e_{\xi \xi}^c & e_{\xi \eta}^c & 1 \xi \eta \\ e_{\eta \xi}^c & e_{\eta \eta}^c & 1 \eta \xi \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_6 \end{bmatrix} = \mathbf{A}(\xi, \eta) \mathbf{\beta} \]  \hspace{1cm} (30)

where the parameters \( \beta_1, \ldots, \beta_6 \) are obtained by sampling the membrane strains at selected points. Two possibilities have been considered: (i) evaluating the assumed membrane strains at the vertex nodes (element TQLL1) and (ii) evaluating the tangential membrane strains at the two Gauss points along each side and additionally sampling the three membrane strains at the center of the element (element TQLL2). Details of the finite element matrices necessary for non linear computation can be found in Flores and Oñate (1993).
3.2 Linear/quadratic shell triangle with a linear assumed shear strain field (TLQL element)

This element is an extension of the TLQL plate triangle presented in Zienkiewicz, Taylor, Papadopoulos and Öhate (1990) and also in Öhate, Zienkiewicz, Suarez and Taylor (1992). The mid-surface displacements are now linearly interpolated in the terms of the corner values as

$$u = \sum_{i=1}^{3} \xi_i u_i$$  \hspace{1cm} (31)

whereas the following incomplete quadratic approximation is used for the director field

$$\mathbf{t} = \sum_{i=1}^{3} \xi_i \mathbf{t}_i + \sum_{i=4}^{6} N_i^{(6)} \theta_i \mathbf{e}_i$$ \hspace{1cm} (32)

In (31) and (32) \(\xi_i\) are the standard linear shape functions of the 3 node triangle, \(N_i^{(6)}\) are the standard quadratic shape functions for the 6 node triangle and \(\theta_i\) are hierarchical director values. In the plate bending case \(\mathbf{e}_i\) are side vectors, whereas in this case we have taken

$$\mathbf{e}_1 = (\varphi_1 + t^2) - (\varphi_1 + t^3)$$

$$\mathbf{e}_2 = (\varphi_1 + t^2) - (\varphi_2 + t^3)$$

$$\mathbf{e}_3 = (\varphi_1 + t^2) - (\varphi_3 + t^3)$$ \hspace{1cm} (33)

Note that vectors \(\mathbf{e}_i\) must be updated at every solution step. In this way we ensure a smooth director field along each side.

The bending and membrane contributions are obtained in a straightforward manner using a standard displacement formulation. Finally, the transverse shear strain field is assumed in a similar way to the linear transverse shear strain at the side-points shown in Fig. 3. The derivation of the substitute shear strain matrix follows the lines explained in Öhate, Zienkiewicz, Suarez and Taylor (1992) for the analogous plate element and it will not be repeated here.

3.3 Linear shell triangle with linear assumed shear strains (TLLL element)

This element is an extension of the TLLL plate element recently proposed by Öhate, Zarate and Flores (1994). Now the displacement and rotation fields and the assumed transverse shear strains are linearly interpolated in terms of the vertex nodal displacements, the mid-side rotations (defining an incompatible rotation field) and the mid-side tangential shear strain variables, respectively as shown in Fig. 4. Note that the assumed shear field for this element is identical to that of the TLQL of previous section. The director field is interpolated in terms of the director vectors at the mid-side nodes as

$$\mathbf{t} = \sum_{i=4}^{6} N_i^{(6)} \theta_i \mathbf{e}_i$$ \hspace{1cm} (34)

where

$$N_i^{(6)} = 1 - 2 \xi_i$$ \hspace{1cm} (35)

with \(\xi_i\) being the standard linear shape functions of the 3 node triangle.

Fig. 4. TLLL triangular shell element. Nodes and sampling points for the tangential transverse shear strain \(\gamma_t\).

It is worth noting that many interesting analogies can be found between this element and that derived by Van Keulen (1993) (see also Van Keulen and Öhate, 1995) as an extension of the plate element derived by Roldán and Zienkiewicz (1971). More details about the derivation of the element matrices for the TLLL element can be found in Öhate, Zarate and Flores (1994) and in Flores and Óhate (1993).

4 Examples

4.1 Cylindrical roof

The well known Scordelis-Lo cylindrical roof shown in Fig. 5 is chosen first to compare the behaviour of the triangular shell elements previously described with that of the popular

![Fig. 5. Scordelis-Lo cylindrical roof. Geometry and material properties](https://www.scipedia.com.com)
four noded quadrilateral with linearly assumed transverse shear strains (termed here QLLL) (Dvorkin and Bathe 1984; Simo, Fox and Rifai 1989; Oñate, Zienkiewicz, Suarez and Taylor 1992).

Table 1 shows the convergence of the normalized vertical displacements of the free edge of the central section for different meshes. Elements TQL and TQL2 show slow convergence due to membrane locking. Element TQQL converges to the correct result but it is rather flexible.

Table 1. Convergence of the vertical displacement at the mid-side point of the edge

<table>
<thead>
<tr>
<th>DOF</th>
<th>QLLL</th>
<th>TQQL</th>
<th>TQQL</th>
<th>TQQL1</th>
<th>TQQL2</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>1.083</td>
<td>1.349</td>
<td>0.544</td>
<td>1.000</td>
<td>0.484</td>
</tr>
<tr>
<td>198</td>
<td>-</td>
<td>1.095</td>
<td>0.747</td>
<td>0.983</td>
<td>0.694</td>
</tr>
<tr>
<td>344</td>
<td>1.015</td>
<td>1.044</td>
<td>0.987</td>
<td>0.983</td>
<td>0.860</td>
</tr>
<tr>
<td>1344</td>
<td>1.000</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2060</td>
<td>-</td>
<td>1.000</td>
<td>-</td>
<td>1.005</td>
<td>0.994</td>
</tr>
</tbody>
</table>

Table 2. Curved cantilever beam. Bending moment at Gauss points. Numbers show the range of variation of bending moment values in the mesh. Exact solution is 1.0

<table>
<thead>
<tr>
<th>DOF</th>
<th>TLQL</th>
<th>TQQL</th>
<th>TQQL1</th>
<th>TQQL2</th>
<th>TLQL</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>1.135</td>
<td>0.745</td>
<td>0.742</td>
<td>0.916</td>
<td>0.977</td>
</tr>
<tr>
<td>90</td>
<td>1.479</td>
<td>0.800</td>
<td>0.762</td>
<td>0.920</td>
<td>0.977</td>
</tr>
<tr>
<td>156</td>
<td>0.798</td>
<td>0.745</td>
<td>0.762</td>
<td>0.920</td>
<td>0.977</td>
</tr>
<tr>
<td>930</td>
<td>0.900</td>
<td>0.990</td>
<td>0.990</td>
<td>0.989</td>
<td>0.989</td>
</tr>
<tr>
<td>3660</td>
<td>0.977</td>
<td>0.977</td>
<td>0.977</td>
<td>0.977</td>
<td>0.977</td>
</tr>
</tbody>
</table>

4.2 Curved cantilever beam

Fig. 6. Cantilever curved beam. Geometry and material properties

$$\rho \text{ given by the relation}$$

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

Using a displacement control Newton-Raphson algorithm (in this case the rotation of the free edge is taken as the control d.o.f.) the shell is deformed into a complete circle (Fig. 7). The value of the normalized applied bending moment for the different triangular elements is presented in Table 3.

4.4 Hinged cylindrical panel under a point load

A shallow cylindrical panel, pinned at two edges and free at the other two, subjected to a central point load leads to

Table 3. Roll-up a clamped beam. Normalized bending moment for a complete circle configuration

<table>
<thead>
<tr>
<th>TQQL</th>
<th>TQQL1</th>
<th>TQQL2</th>
<th>TLQL</th>
<th>TLLL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.002</td>
<td>0.980</td>
<td>1.016</td>
<td>0.990</td>
<td>0.989</td>
</tr>
</tbody>
</table>

Fig. 7. Roll-up of a clamped strip. Reference and deformed configurations
a snap-through behaviour with a reversal of curvature. Depending on the thickness a snap-back is also possible. Two different thicknesses have been considered for $R/h = 200$ and $R/h = 400$. The geometry of the panel and the material properties are shown in Fig. 7. For $R/h = 200$ two regular meshes (25 and 49 nodes) have been used to test convergence. The displacement of the point under the load vs. the value of the load is plotted in Fig. 9 for the triangular elements presented here. It can be seen that for the fine mesh (Fig. 9b) the results are almost coincident and all elements converge to the correct results. Results for the case $R/h = 400$ using the fine mesh are presented in Fig. 10. The TQQL1 element shows a very flexible behaviour in this case.

4.5 Impact dynamics test
The triangular shell elements proposed seem particularly advantageous for crashworthiness and sheet stamping problems where the triangular discretization of complex shell geometries is typically required.

A simple example of the ability of the shell triangles for analysis of impact dynamics problem presented is given here.

The problem is the study of the impact of a low-velocity ($V_0 = 6.94$ m/s) cylindrical bar against a clamped square plate (Hallet 1987). The plate dimensions are $600 \times 600$ mm and the thickness is 5.11 mm, whilst the projectile has a diameter of 40 mm and a mass of 40.5 kg. Interface conditions are frictionless and the material properties for both target and projectile are $E = 211 \times 10^6$ N/m$^2$, $v = 0.3$, initial yield stress = $280 \times 10^6$ N/m$^2$ and hardening modulus = $690 \times 10^6$ N/m$^2$. The elasto-plastic model used is described in Simo and Kennedy (1993).

Due to symmetry conditioning an eighth of the plate is considered. Six finite strain 8-node solid elements (Garcia Garino 1993) have been used for the projectile and 24 TQQL shell triangles for the plate. Vertical displacement contours are shown in Fig. 11 for the instant of maximum deformation. Also the displacements of the center of the plate and the blunt surface of the projectile are plot with respect to time. The maximum displacement of the plate was computed as 27.5 mm at 6.3 ms in comparison with the experimental values of 27.6 mm at 5.47 ms which represents errors of $-0.4\%$ and $14.5\%$ respectively. Similar good behaviour was found for the TLQL and TLLL elements. Further details on this example and other non linear dynamic studies using the new shell triangles can be found in Flores and Oñate (1993).

5 Concluding remarks
A family of assumed strain triangles for non linear thick/thin shell analysis has been presented. The elements follow the shell theory by Simo and co-workers and can account for large displacements, large rotations and plasticity effects.

All the elements converge to the correct results when the mesh is refined. For coarse meshes it has been observed that...
Fig. 11. Impact of a metallic bar against a square plate. Evolution of central displacement: continuous line (—) for plate and dashed (—) for projectile. Displacement contours for t = 0.006

(a) Both the flat TQQ1 element and the assumed membrane strain TQQL1 element behave rather flexible.
(b) The isoparametric TQQ1 element exhibits membrane locking as expected.
(c) The TQQ2 element with assumed membrane strains behaves rather stiff.
(d) The TLQL element has an excellent bending behaviour but for membrane dominated problems requires fine meshes due to the intrinsic constant membrane strain field.
(e) The TLQL2 exhibits a behaviour similar to that of the TLQL element although its formulation is much simpler. Indeed this element is a promising candidate for large scale non-linear computations of shells.

References:

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