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Abstract

The classical incremental theory of plasticity is not able to predict plastic strain accumulation during cyclic loading. This because plastic deformation may occur only during loading conditions and when the stress point lies on the yield surface F. On the other hand, F remains fixed in the stress space during unloading conditions, so that successive loading does not produce any plastic deformation until the stress point does not reach again F. This work presents a generalization of the classical theory, which allows to describe plastic strain accumulation during cyclic loading. This is obtained postulating that F follows always the stress point. Moreover, it is assumed the existence of a surface $\overline{F}$, which bounds always F and of an *elastic* surface $\widehat{F}$, which bounds the stress states at which only elastic strain may occur. In the limit case $\widehat{F} \equiv \overline{F}$ the presented generalized theory recovers the classical one.

1 Introduction

Experimental evidence shows that many solid materials present progressive accumulation of plastic (irreversible) deformations under cyclic loading. In some cases this progressive accumulation may cause serious effects; for example, in a saturated soil, it may eventually cause the *liquefaction* of the solid skeleton.

The *standard Incremental Theory of Plasticity* (MELAN [1938]) is at the present the most popular constitutive theory for the mathematical description of irreversible deformations in solid materials. This is a purely mechanical theory, which has been developed with the following main assumptions:

1. time independence of the material response;
2. *small* deformations and displacements.

However, this incremental theory, as originally formulated, cannot describe a progressive accumulation of plastic deformation under cyclic loading.

For example, consider an isotropic material subjected to a triaxial cyclic loading test in which the deviatoric stress invariant value q is confined within the range $[0, q^*]$. Assume for simplicity that the yield surface of this material is represented by a Von Mises surface type, that is

$$F(\sigma, k) = q - N(k)$$

where the initial value of the hardening parameter k is equal to zero. During the first loading phase, the yield surface expands following the stress point and, consequently, plastic deformations are mobilized and the hardening parameter increases its value reaching eventually a limit value k^* such that

$$F(\sigma, k^*) = q^* - N(k^*) = 0$$

According to the standard incremental theory, the space region bounded by this yield surface is the new elastic region. Thus, all next loading phases cannot mobilize any plastic deformation, being the relative stress path always confined within the elastic stress region.

In the last 20 years a number of theoretical works have aimed to overcome the above described limitation of the standard theory. Some of them assume the existence of three, generally distinct, stress surfaces, (DAFALIAS et al. [1975], DAFALIAS et al. [1977], KRIEG et al. [1975], MROZ et al. [1978b], MROZ et al. [1978], PREVOST [1977]):

- a *yield surface* F which is assumed to follow always the stress point, even during unloading conditions;
- a *bounding surface* $\overline{F}$;
- an *elastic surface* $\widehat{F}$;

and

$$\begin{aligned} F &\subseteq \overline{F} \\ \widehat{F} &\subseteq \overline{F} \end{aligned}$$

Loading conditions whose relative stress point increments are directed outside F may mobilize plastic deformations, unless the current stress point lies inside the elastic region bounded by $\widehat{F}$. The size of the plastic deformations results to be a function of a relative distance of the current stress point from $\overline{F}$.

As a rule, the above proposed modification of the standard theory allows to describe the accumulation of plastic deformations during cyclic loading. However, we still have to identify the law of variation of the size of the plastic deformation and the interrelationship between the evolution of these three stress surfaces.

In this paper we first deeply investigate the standard incremental theory; then, we present a possible theoretical framework which allows to answer the above two questions. The practical use of this theory is explained by means of a small example.

2 Notation Convention

The (symmetric) Cauchy stress tensor σ_{ij} and the relative deviatoric stress tensor

$$s_{ij} = \sigma_{ij} - \delta_{ij}\frac{\sigma_{kk}}{3}$$

where δ_{ij} is the usual *Kronecker symbol*, are respectively indicated as

$$\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{21}, \sigma_{31}, \sigma_{32}\}^T \tag{1}$$

$$\mathbf{s} = \{s_{11}, s_{22}, s_{33}, s_{12}, s_{13}, s_{23}, s_{21}, s_{31}, s_{32}\}^T \tag{2}$$

As stress invariant quantities we elect

$$\begin{cases} p &= \frac{I_1^{(\sigma)}}{3}; & \text{Mean pressure} \\ q &= \left(3J_2^{(\sigma)}\right)^{\frac{1}{2}}; & \text{Equivalent shear stress} \\ \theta &= \frac{1}{3}\arcsin\left[-\frac{3\sqrt{3}}{2}\frac{J_3^{(\sigma)}}{J_2^{(\sigma)3/2}}\right]; & \text{Angular invariant of stress} \end{cases} \tag{3}$$

where

$$\begin{cases} I_1^{(\sigma)} &= \sigma_{ii} &= \widehat{\mathbf{m}}^T\boldsymbol{\sigma} \\ J_2^{(\sigma)} &= \frac{1}{2}s_{ij}s_{ij} &= \frac{1}{2}\mathbf{s}^T\mathbf{s} \\ J_3^{(\sigma)} &= \det\left[s_{ij}\right] \end{cases} \tag{4}$$

and

$$-\pi/6 \leq \theta \leq \pi/6$$

$$\widehat{\mathbf{m}} = \{1,1,1,0,0,0,0,0,0\}^T$$

With the notations

$$\delta p \ , \ \delta q \ , \ \delta\theta$$

we indicate the infinitesimal variations of p, q, θ.

The (symmetric) linear Lagrangian strain tensor ϵ_{ij} and the relative deviatoric stress tensor

$$e_{ij} = \epsilon_{ij} - \delta_{ij}\frac{\epsilon_{kk}}{3}$$

are respectively indicated as

$$\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}, \epsilon_{21}, \epsilon_{31}, \epsilon_{32}\}^T \tag{5}$$

$$\mathbf{e} = \{e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}, e_{21}, e_{31}, e_{32}\}^T \tag{6}$$

As strain invariant quantities we elect

$$\begin{cases} \epsilon_v &= I_1^{(\epsilon)}; & \text{Volumetric strain} \\ \epsilon_s &= 2\left(\frac{J_2^{(\epsilon)}}{3}\right)^{\frac{1}{2}}; & \text{Equivalent shear strain} \\ \epsilon_\theta &= \frac{1}{3}\arcsin\left[-\frac{3\sqrt{3}}{2}\frac{J_3^{(\epsilon)}}{J_2^{(\epsilon)3/2}}\right]; & \text{Angular invariant of strain} \end{cases} \tag{7}$$

where

$$\left\{\begin{array}{lll} I_1^{(\epsilon)} & = \epsilon_{ii} & = \mathbf{m}^T\boldsymbol{\epsilon} \\ J_2^{(\epsilon)} & = \frac{1}{2}e_{ij}e_{ij} & = \frac{1}{2}\mathbf{e}^T\mathbf{e} \\ J_3^{(\epsilon)} & = \det\left[e_{ij}\right] & \end{array}\right. \tag{8}$$

and

$$-\pi/6 \leq \epsilon_\theta \leq \pi/6$$

With the notations

$$\delta\epsilon_v \;\; , \;\; \delta\epsilon_s \;\; , \;\; \delta\epsilon_\theta$$

we indicate the invariants of the incremental strain $\delta\boldsymbol{\epsilon}$. In general, these quantities *do not coincide* with the incremental variations of the strain invariants unless for the case of volumetric strain.

Incidentally, we remark that, according to the above notation convention, deviatoric stress and strain vectors may be respectively calculated as

$$\mathbf{s} \;=\; \boldsymbol{\sigma} - \widehat{\mathbf{m}}p \tag{9}$$

$$\mathbf{e} \;=\; \boldsymbol{\epsilon} - \widehat{\mathbf{m}}\frac{\epsilon_v}{3} \tag{10}$$

3 Elasto-Plastic Strain Definition

According to the incremental theory of plasticity, an infinitesimal stress increment $\delta\boldsymbol{\sigma}$ causes an infinitesimal strain increment $\delta\boldsymbol{\epsilon}$, which is the sum of an elastic (fully recoverable) component $\delta\boldsymbol{\epsilon}^{(e)}$ and a plastic (irreversible) component $\delta\boldsymbol{\epsilon}^{(p)}$, namely

$$\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\epsilon}^{(e)} + \delta\boldsymbol{\epsilon}^{(p)} \tag{11}$$

The elastic strain may be calculated according to the *Generalized Hooke's Law*:

$$\delta\boldsymbol{\epsilon}^{(e)} = \mathbf{D}^{(e)}\delta\boldsymbol{\sigma} = \left(\mathbf{C}^{(e)}\right)^{-1}\delta\boldsymbol{\sigma} \tag{12}$$

where $\mathbf{D}^{(e)}$ and $\mathbf{C}^{(e)}$, respectively the *Compliance* and the *Stiffness* elastic matrices, may be function of the current $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$. The plastic strain instead is defined as

$$\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda\mathbf{b} \tag{13}$$

where

$$\mathbf{b} = \frac{\partial G}{\partial\boldsymbol{\sigma}} = \left\{\frac{\partial G}{\partial\sigma_{11}}, \frac{\partial G}{\partial\sigma_{22}}, \frac{\partial G}{\partial\sigma_{33}}, \frac{\partial G}{\partial\sigma_{12}}, \frac{\partial G}{\partial\sigma_{13}}, \frac{\partial G}{\partial\sigma_{23}}, \frac{\partial G}{\partial\sigma_{21}}, \frac{\partial G}{\partial\sigma_{31}}, \frac{\partial G}{\partial\sigma_{32}}\right\}^T$$

and

$$G = G(\boldsymbol{\sigma}, \mathbf{k})$$

is a scalar function known as the *Potential Function*. The vector $\mathbf{k}$ collects *n hardening parameters* k_i functions of the stress history, Section 5. Finally, $\delta\lambda$, known as the *plastic multiplier*, is a non negative, even indeterminate, scalar; its value, Section 6, depends on the so called *Yield Function*, Section 4.

Notice that, since the stress $\boldsymbol{\sigma}$ can be expressed as, Eq. 9,

$$\boldsymbol{\sigma} = \mathbf{s} + \widehat{\mathbf{m}}p$$

the potential function may be written in the form

$$G = G(p, \mathbf{s}, \mathbf{k})$$

According to the plastic strain definition in Eq. 13, it is possible to prove that the plastic strain invariants result to be given by (FUSCO [1993])

$$\delta\epsilon_v^{(p)} = \delta\lambda \frac{\partial G}{\partial p} \tag{14}$$

$$\delta\epsilon_s^{(p)} = \delta\lambda \sqrt{\frac{2}{3}} ||\nabla_s G|| \tag{15}$$

where

$$\nabla_s G = \left\{ \frac{\partial G}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \right\}^T = \left\{ \frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \right\}^T$$

$$||\nabla_s G|| = (\nabla_s G^T \nabla_s G)^{1/2} = \left[\frac{\partial G}{\partial s_{ij}} \frac{\partial G}{\partial s_{ij}} - \frac{1}{3} \left(\frac{\partial G}{\partial s_{kk}} \right)^2 \right]^{1/2}$$

If G is function of invariant quantities only, i.e. $G = G(p, q, \theta, \mathbf{k})$, it results

$$\mathbf{b} = \left\{ \frac{\partial G}{\partial \sigma_{ij}} \right\}^T = c_1 \widehat{\mathbf{m}} + c_2 \mathbf{s} + c_3 \mathbf{v} \tag{16}$$

where

$$c_1 = \frac{1}{3} \frac{\partial G}{\partial p}$$

$$c_2 = \frac{3}{2q} \left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q} \frac{\partial G}{\partial \theta} \right)$$

$$c_3 = -\frac{9}{2q^3 \cos 3\theta} \frac{\partial G}{\partial \theta}$$

and the element of $\mathbf{v}$ are given by

$$v_{ij} = w_{ij} - \frac{\delta_{ij}}{3} w_{kk}$$

where

$$[w_{ij}] = \left[\frac{\partial J_3}{\partial s_{ij}}\right] = \begin{bmatrix} (s_{22}s_{33} - s_{23}^2) & (s_{13}s_{23} - s_{33}s_{12}) & (s_{12}s_{23} - s_{22}s_{13}) \\ & (s_{11}s_{33} - s_{13}^2) & (s_{12}s_{13} - s_{11}s_{23}) \\ \text{Symmetric} & & (s_{11}s_{22} - s_{12}^2) \end{bmatrix}$$

$$w_{kk} = \frac{\partial J_3}{\partial s_{kk}} = -J_2 = -\frac{q^2}{3}$$

Moreover,

$$\|\nabla_s G\| = \sqrt{\frac{3}{2}} \left[\left(\frac{\partial G}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial G}{\partial \theta}\right)^2 \right]^{1/2}$$

4 The Yield Function

The incremental theory of plasticity assumes that the type of material response is controlled by a so called *Yield Surface*, of equation

$$F = F(\boldsymbol{\sigma}, \mathbf{k}) \tag{17}$$

where $\mathbf{k}$, Section 5, collects the same hardening parameters of the potential function G. Sometimes F is assumed to coincide with G, in which case the constitutive model is said to obey to an *associative flow rule.*

The yield surface is assumed to bound always the location of each current stress point so that, if $(\boldsymbol{\sigma}, \mathbf{k})$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) = 0; & \boldsymbol{\sigma} \text{ lies on } F \\ F(\boldsymbol{\sigma}, \mathbf{k}) < 0; & \boldsymbol{\sigma} \text{ lies inside } F \end{cases} \tag{18}$$

while

$$F(\boldsymbol{\sigma}, \mathbf{k}) > 0$$

is not admissible. Consider a loading process of an infinitesimal increment $(\delta\boldsymbol{\sigma}, \delta\mathbf{k})$, starting from a current configuration $(\boldsymbol{\sigma}, \mathbf{k})$. By definition:

- Elasto-plastic deformations occur if, and only if, during the loading process the current stress point always lies on the yield surface, that is

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \qquad \text{and} \qquad F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) = 0 \tag{19}$$

Thus, only in these cases, $\delta\lambda \geq 0$,

- In any other case, that is

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0 \qquad \text{and} \qquad F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) \leq 0 \tag{20}$$
$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \qquad \text{and} \qquad F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) < 0 \tag{21}$$

the material response is purely elastic and, consequently, $\delta\lambda = 0$.

It is immediate to verify that the rule in Eq. 19 for establishing elasto-plastic conditions is *equivalent* to requiring:

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) &= 0 \\ \delta F(\boldsymbol{\sigma}, \mathbf{k}) &= \mathbf{a}^T \delta\boldsymbol{\sigma} + \dfrac{\partial F}{\partial k_i} \delta k_i = 0 \end{cases} \tag{22}$$

where $i = 1, 2, \ldots, n$, and

$$\mathbf{a} = \frac{\partial F}{\partial \boldsymbol{\sigma}} = \left\{ \frac{\partial F}{\partial \sigma_{11}}, \frac{\partial F}{\partial \sigma_{22}}, \frac{\partial F}{\partial \sigma_{33}}, \frac{\partial F}{\partial \sigma_{12}}, \frac{\partial F}{\partial \sigma_{13}}, \frac{\partial F}{\partial \sigma_{23}}, \frac{\partial F}{\partial \sigma_{21}}, \frac{\partial F}{\partial \sigma_{31}}, \frac{\partial F}{\partial \sigma_{32}} \right\}^T$$

The second equation, known as the *Consistency Equation*, can be derived expressing $F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k})$ by the *exact* Taylor series

$$F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) = F(\boldsymbol{\sigma}, \mathbf{k}) + \delta F(\boldsymbol{\sigma}, \mathbf{k})$$

5 The Hardening Parameters

The hardening parameters collected in $\mathbf{k}$ have to represent the past plastic history of the material point. In general, therefore, they must depend on the total plastic deformation. However, the incremental theory of plasticity does not necessarily need an explicit functional relationship for $\mathbf{k}$; in fact, we will see that it is sufficient to establish incremental relationships of the type

$$\delta k_i = \frac{\partial k_i}{\partial h_j} \delta h_j \tag{23}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where:

- The partial derivatives $\frac{\partial k_i}{\partial h_j}$ and the initial values of k_i have to be known quantities.
- The infinitesimal increments δh_j of the *internal* variables h_j *must* be expressible as

$$\delta h_j = f_j(\delta \epsilon^{(p)}) = c_j \delta\lambda \tag{24}$$

for $j = 1, 2, \ldots, m$, where c_j are m scalar values. The notation $f_j(\delta\epsilon^{(p)})$ indicates any scalar function of the nine scalar variables $\delta\epsilon_{rs}^{(p)}$ $(r, s = 1, 2, 3)$.

From Eqs. 23 and 24 we obtain

$$\delta k_i = \frac{\partial k_i}{\partial h_j} \delta h_j = \frac{\partial k_i}{\partial h_j} c_j \delta\lambda \tag{25}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Notice that this incremental relationship assures that during elastic processes, where by definition $\delta\lambda = 0$, the hardening parameters do not vary; consequently, F and G remain fixed in the stress space.

Many formulations of elasto-plastic constitutive models assign directly the hardening parameters $\mathbf{k}$, without introducing the internal variables h_j. In our notation, this is equivalent to assume:

$$h_j \equiv k_i$$

for $j = i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, with $m = n$. This implies that

$$\frac{\partial k_i}{\partial h_j} = \delta_{ij}$$

The most common choices of k_i are

- the hardening parameter k_i equal to a constant (*perfectly plastic* model),

$$k_i \equiv h_j = \text{constant}$$

Being $\delta h_j = 0$, it follows

$$c_j = 0$$

- the hardening parameter k_i equal to the total plastic work,

$$k_i \equiv h_j = \int \boldsymbol{\sigma}^T \delta\epsilon^{(p)}$$

Being $\delta h_j = \boldsymbol{\sigma}^T \delta\epsilon^{(p)}$, from Eq. 13 it results:

$$c_j = \boldsymbol{\sigma}^T \mathbf{b}$$

- the hardening parameter k_i equal to the pq-th component of $\epsilon^{(p)}$,

$$k_i \equiv h_j = \epsilon_{pq}^{(p)}$$

Being $\delta h_j = \delta\epsilon_{pq}^{(p)}$, from Eq. 13 it results:

$$c_j = \frac{\partial G}{\partial \sigma_{pq}}$$

- the hardening parameter k_i equal to the total plastic volumetric strain,

$$k_i \equiv h_j = \int \delta\epsilon_v^{(p)}$$

Being $\delta h_j = \delta\epsilon_v^{(p)}$, from Eq. 14 it results:

$$c_j = \frac{\partial G}{\partial p}$$

- the hardening parameter k_i equal to the accumulation of the incremental plastic shear strain

$$k_i \equiv h_j = \epsilon_s^{(p)} = \int \delta\epsilon_s^{(p)}$$

Being $\delta h_j = \delta\epsilon_s^{(p)}$, from Eq. 15 it results:

$$c_j = \sqrt{\frac{2}{3}}\|\nabla_s G\|$$

6 The Plastic Multiplier

By definition, Section 3, the value of the plastic multiplier is never negative and in a purely elastic process

$$\delta\lambda = 0$$

In an elasto-plastic process, instead, its value may be calculated as follows

- if $\delta\sigma$ is assigned

$$\delta\lambda = \begin{cases} \frac{\mathbf{a}^T \delta\boldsymbol{\sigma}}{A} & \text{if } A \neq 0 \\ \text{indeterminate} & \text{if } A = 0 \end{cases} \tag{26}$$

- if $\delta\epsilon$ is assigned

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)}}{A+\mathbf{a}^T\mathbf{c}^{(G)}} & \text{if } A \neq -\mathbf{a}^T\mathbf{c}^{(G)} \\ \text{indeterminate} & \text{if } A = -\mathbf{a}^T\mathbf{c}^{(G)} \end{cases} \tag{27}$$

where

$$\delta\boldsymbol{\sigma}^{(e)} = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} \tag{28}$$

$$\mathbf{c}^{(G)} = \mathbf{C}^{(e)}\mathbf{b} \tag{29}$$

and the scalar quantity A, known as the *Plastic Modulus*, is defined as

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k_i}\delta k_i = -\frac{\partial F}{\partial k_i}\frac{\partial k_i}{\partial h_j}c_j \tag{30}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

In fact, according with the positions in Eqs. 25 and 30, the consistency equation in Eq. 22 takes on the form

$$\mathbf{a}^T\delta\boldsymbol{\sigma} - A\delta\lambda = 0 \tag{31}$$

from which we can verify the statement in Eq. 26.

On the other hand, according to Eqs. 12, 11 and 13, we can express

$$\delta\boldsymbol{\sigma} = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon}^{(e)} = \mathbf{C}^{(e)}\left(\delta\boldsymbol{\epsilon} - \delta\boldsymbol{\epsilon}^{(p)}\right) = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} - \delta\lambda\mathbf{C}^{(e)}\mathbf{b}$$

that is, Eqs. 28 and 29,

$$\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^{(e)} - \delta\lambda\mathbf{c}^{(G)}$$

Substituting this expression of $\delta\boldsymbol{\sigma}$ in Eq. 31 we have the following alternative form for the consistency equation:

$$\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} - \left(\mathbf{a}^T\mathbf{c}^{(G)} + A\right)\delta\lambda = 0 \tag{32}$$

from which we can verify the statement in Eq. 27.

In general, the plastic modulus A value, calculated as in Eq. 30, may be negative, null or positive. In a perfectly plastic model it trivially results

$$A \equiv 0$$

so that, when $\delta\boldsymbol{\sigma}$ is assigned and elasto-plastic response occurs, $\delta\lambda$ is always an indeterminate, non negative, scalar.

7 Stress Based Elasto-Plastic Criterion

By definition, Section 4, for

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0 \tag{33}$$

the material response is always elastic, regardless of the applied stress increment $\delta\boldsymbol{\sigma}$. Elasto-plastic response can occur if, and only if, Eqs. 22 and 31,

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \\ \\ \delta F = \mathbf{a}^T \delta\boldsymbol{\sigma} - A\delta\lambda = 0 \end{cases} \tag{34}$$

Accordingly, we can establish the following *stress based criterion*, which predicts the material response for any given stress increment $\delta\boldsymbol{\sigma}$ applied on any material state $(\boldsymbol{\sigma}, \mathbf{k})$:

1. Elasto-plastic response occurs if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$\begin{array}{lcl} A > 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} \geq 0 \\ A = 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \\ A < 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \end{array}$$

   The case in which

   $$A > 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} > 0$$

   takes the name of *hardening*, since, being the stress increment $\delta\boldsymbol{\sigma}$ directed outside F, it denotes a subsequent expansion of F.

2. Elastic response occurs if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) < 0$$

   or

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$A \geq 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} < 0$$

3. Either elastic or elasto-plastic response may occur if

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0$$

This is the only ambiguous situation which the classical theory of plasticity does not solve by itself. If plasticity occurs, this case takes the name of *softening*, since, being the stress increment $\delta\boldsymbol{\sigma}$ directed inside F, it denotes a subsequent contraction of F.

4. Stress increments by which

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$A \leq 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of stress increment. In any other case the theory provides always a material response, in case with indeterminate plastic strains, when A is zero.

To prove the above statements, consider only the non trivial case $F(\boldsymbol{\sigma}, \mathbf{k}) = 0$. Elasto-plastic response occurs if, and only if, also the second equation in Eq. 34 is satisfied. Since by definition $\delta\lambda \geq 0$, it follows that elasto-plastic response occurs if, and only if,

$$A > 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} \geq 0 \tag{35}$$

$$A = 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} = 0 \tag{36}$$

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} \leq 0 \tag{37}$$

Consequently, elasto-plastic deformations can not occur for

$$A > 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0 \tag{38}$$

$$A = 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} \gtrless 0 \tag{39}$$

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0 \tag{40}$$

These situations may indicate purely elastic or inadmissible material responses. In particular, by definition, in the case of a purely elastic response $\delta\lambda = 0$, consequently, from Eq. 25,

$$\delta k_i = 0 \tag{41}$$

for $i = 1, 2, \ldots, n$. Moreover, according to Eq. 21,

$$F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) < 0$$

which, expressing F into a Taylor series, yields to the conclusion that, in a purely elastic response,

$$F(\boldsymbol{\sigma}, \mathbf{k}) + \mathbf{a}^T\delta\boldsymbol{\sigma} + \frac{\partial F}{\partial k_i}\delta k_i < 0 \tag{42}$$

for $i = 1, 2, \ldots, n$. Being $F(\boldsymbol{\sigma}, \mathbf{k}) = 0$ and $\delta k_i = 0$, we obtain

$$\mathbf{a}^T\delta\boldsymbol{\sigma} < 0 \tag{43}$$

Hence, the situations

$$A = 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} > 0$$
$$A < 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} > 0$$

in Eqs. 39 and 40 can not be elastic; therefore they are not admissible, as stated in 4. The uniqueness of these inadmissible conditions is trivially verified.

Then, consider the possible elasto-plastic material response predicted in Eq. 37. According to Eq. 43, the case

$$A < 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} < 0$$

may also represent a purely elastic material response. This justifies therefore the ambiguity stated in item 3.

With the above observations it becomes easy to verify the statements in items 1 and 2.

8 Strain Based Elasto-Plastic Criterion

By definition, Section 4, if

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0 \tag{44}$$

the material response is always elastic, regardless of the applied strain increment $\delta\epsilon$. Elasto-plastic response can occur if, and only if, Eqs. 22 and 32,

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \\ \delta F = \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} - \left(\mathbf{a}^T \mathbf{c}^{(G)} + A\right) \delta\lambda = 0 \end{cases} \tag{45}$$

Accordingly, we can establish the following *strain based criterion*, which predicts the material response for any given strain increment $\delta\epsilon$ applied on any material state $(\boldsymbol{\sigma}, \mathbf{k})$:

1. Elasto-plastic response occurs if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$\begin{array}{lcl} A > -\mathbf{a}^T\mathbf{c}^{(G)} & ; & \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} \geq 0 \\ A = -\mathbf{a}^T\mathbf{c}^{(G)} & ; & \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} = 0 \\ A < -\mathbf{a}^T\mathbf{c}^{(G)} & ; & \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} = 0 \end{array}$$

2. Elastic response occurs if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) < 0$$

   or

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$A \geq -\mathbf{a}^T\mathbf{c}^{(G)} \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} < 0$$

3. Either elastic or elasto-plastic response may occur if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$A < -\mathbf{a}^T\mathbf{c}^{(G)} \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} < 0$$

   This is the only ambiguous situation which the classical theory of plasticity does not solve by itself.

4. Strain increments by which

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$A \leq -\mathbf{a}^T \mathbf{c}^{(G)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of strain increment. In any other case the theory provides always a material response, in case with indeterminate plastic strains and stress increment, when $A = -\mathbf{a}^T \mathbf{c}^{(G)}$.

The proof of the above statements is analogous to that reported in Section 7. It is only important to take into account that, in the case of purely elastic response,

$$\delta\boldsymbol{\sigma}^{(e)} = \mathbf{C}^{(e)} \delta\boldsymbol{\epsilon} = \delta\boldsymbol{\sigma}$$

It is interesting to observe that, in the case of associative flow rule and $\mathbf{C}^{(e)}$ positive definite matrix, it results

$$\mathbf{a}^T \mathbf{c}^{(G)} = \mathbf{a}^T \mathbf{C}^{(e)} \mathbf{a} > 0$$

This implies that the ambiguous and the not admissible situations in items 3 and 4 may occur only when $A < 0$. Moreover, the condition

$$A \leq -\mathbf{a}^T \mathbf{c}^{(G)} = -\mathbf{a}^T \mathbf{C}^{(e)} \mathbf{a}$$

is obviously more restrictive than the condition

$$A \leq 0$$

which, when $\delta\boldsymbol{\sigma}$ is assigned, may cause ambiguous and not admissible situations, Section 7.

9 The Elasto-Plastic Constitutive Equation

The mathematical developments reported in the previous Sections lead to the conclusion that the strain increment $\delta\boldsymbol{\epsilon}$ resulting from a stress increment $\delta\boldsymbol{\sigma}$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated as

$$\delta\boldsymbol{\epsilon} = \mathbf{D}^{(e)} \delta\boldsymbol{\sigma} + \delta\lambda \mathbf{b} \tag{46}$$

where, if a purely elastic response occurs,

$$\delta\lambda = 0$$

while, if plasticity develops,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}}{A} & \text{if } A \neq 0 \\ \text{indeterminate} & \text{if } A = 0 \end{cases}$$

The type of mechanical response is established according to the stress based criterion, Section 7. We remind that, when $A \leq 0$, some $\delta\boldsymbol{\sigma}$ may be not admissible or cause ambiguous situations.

The inversion of Eq. 46 gives

$$\delta\boldsymbol{\sigma} = \mathbf{C}^{(e)}\,(\delta\boldsymbol{\epsilon} - \delta\lambda\mathbf{b})$$

where $\delta\lambda$ may be expressed as in Eq. 27. Hence, the stress increment $\delta\boldsymbol{\sigma}$ resulting from a strain increment $\delta\boldsymbol{\epsilon}$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated as

$$\delta\boldsymbol{\sigma} = \mathbf{C}\delta\boldsymbol{\epsilon} \tag{47}$$

where, if a purely elastic response occurs,

$$\mathbf{C} = \mathbf{C}^{(e)}$$

while, if plasticity develops,

$$\begin{aligned} \mathbf{C} &= \mathbf{C}^{(e)} - \mathbf{C}^{(p)} \\ \mathbf{C}^{(p)} &= \begin{cases} \dfrac{\mathbf{c}^{(G)}\mathbf{c}^{(F)T}}{A + \mathbf{a}^T\mathbf{c}^{(G)}}; & \text{for } A \neq -\mathbf{a}^T\mathbf{c}^{(G)}; \\ \text{indeterminate} & \text{for } A = -\mathbf{a}^T\mathbf{c}^{(G)}; \end{cases} \\ \mathbf{c}^{(F)} &= \mathbf{C}^{(e)T}\mathbf{a} \end{aligned}$$

The type of mechanical response is established according to the strain based criterion, Section 8. We remind that, when $A \leq -\mathbf{a}^T\mathbf{c}^{(G)}$, some $\delta\boldsymbol{\epsilon}$ may be not admissible, or cause ambiguous situations.

It is interesting to remark that

- If $A = 0$ and elasto-plastic response occurs, $\mathbf{C}$ is singular. In fact, in this case, being $\delta\lambda$ indeterminate, Eq. 46, the strain increment $\delta\boldsymbol{\epsilon}$ mobilized by any admissible $\delta\boldsymbol{\sigma}$ is indeterminate; this implies that the matrix $\mathbf{C}$ in Eq. 47 is singular.

- If $\mathbf{C}^{(e)}$ is symmetric and the flow rule is associative, $\mathbf{C}$ is symmetric.

10 The New Proposal: the Main Hypotheses

In this Section we present the general theoretical framework by which it is possible to extend any classical elasto-plastic model in order to account for possible plastic deformations at any stress level. The basic assumptions are:

1. There exists a *Bounding Surface*

$$\overline{F} = \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{48}$$

which, as the name indicates, bounds always the location of the current stress point. Accordingly, if $(\boldsymbol{\sigma}, \overline{\mathbf{k}})$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) = 0; & \text{i.e. } \boldsymbol{\sigma} \text{ lies on } \overline{F}. \\ \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) < 0; & \text{i.e } \boldsymbol{\sigma} \text{ lies inside } \overline{F} \end{cases}$$

while

$$\overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) > 0$$

is not admissible.

2. The vector $\overline{\mathbf{k}}$ collects n hardening parameters $\overline{k}_i$ whose incremental variation is controlled by $\overline{m}$ *internal variables* $\overline{h}_{\overline{j}}$, namely

$$\delta \overline{k}_i = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}} \tag{49}$$

for $i = 1, 2, \ldots, n$ and $\overline{j} = 1, 2, \ldots, \overline{m}$. It is required that the infinitesimal increments $\delta \overline{h}_{\overline{j}}$ *must* be expressible as in Eq. 24, that is

$$\delta \overline{h}_{\overline{j}} = \overline{f}_{\overline{j}}(\delta \boldsymbol{\epsilon}^{(p)}) = \overline{c}_{\overline{j}} \delta \lambda \tag{50}$$

where $\delta \boldsymbol{\epsilon}^{(p)}$ and $\delta \lambda$ are defined as reported in item 9.

3. There exists a *new* type of *Yield Surface*

$$F = F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \tag{51}$$

which always follows the location of the current stress point. Accordingly, the only admissible material condition is

$$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) = 0$$

while

$$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \neq 0$$

is not admissible.

4. The vectors $\overline{\mathbf{k}}$ and $\mathbf{k}$ are of equal size n and

   (a) There exist n scalar functions of the type

   $$k_i = k_i(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{52}$$

   for $i = 1, 2, \ldots, n$, so that for any given material state $(\boldsymbol{\sigma}, \overline{\mathbf{k}})$, the current location of the yield surface $F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$ can be uniquely determined.

   (b) If plasticity occurs, then the incremental variation of the hardening parameters k_i is controlled by the same $\overline{m}$ internal variables $\overline{h}_{\overline{j}}$ of $\overline{k}_i$ plus other m internal variables h_j, namely

   $$\delta k_i = \frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}} + \frac{\partial k_i}{\partial h_j} \delta h_j \tag{53}$$

   for $i = 1, 2, \ldots, n$, $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$. Analogously to the hypothesis on $\overline{h}_{\overline{j}}$ in item 2, it is required that the infinitesimal increments δh_j *must* be expressible as

   $$\delta h_j = f_j(\delta \epsilon^{(p)}) = c_j \delta \lambda \tag{54}$$

5. The space region bounded by F is a subspace of $\overline{F}$, that is

   $$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \subseteq \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{55}$$

   If the current stress point $\boldsymbol{\sigma}$ lies on $\overline{F}$, then F and $\overline{F}$ must coincide, that is

   $$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \equiv \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{56}$$

   and

   $$k_i = \overline{k}_i \tag{57}$$

   $$\frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}} \tag{58}$$

   $$\frac{\partial k_i}{\partial h_j} = 0 \tag{59}$$

   for all $i = 1, 2, \ldots, n$, $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$.

6. There exists an *Elastic Surface*

$$\widehat{F} = \widehat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{60}$$

defined as

$$\widehat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \equiv F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k} = \widehat{\mathbf{k}})$$

where $\widehat{\mathbf{k}} = \widehat{\mathbf{k}}(\overline{\mathbf{k}})$ collects the n known scalar functions

$$\widehat{k}_i = \widehat{k}_i(\overline{\mathbf{k}}) \tag{61}$$

where $i = 1, 2, \ldots, n$. Notice that the above definition implies that if the current stress point lies on $\widehat{F}$, then F coincides with $\widehat{F}$.

7. In general, for $k_i \to \widehat{k}_i$,

$$\begin{aligned} \frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} &\to \infty \\ \frac{\partial k_i}{\partial h_j} &\to \infty \\ A &\to +\infty \end{aligned}$$

for all $i = 1, 2, \ldots, n$; $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$. The definition of the *Plastic Modulus* A is postponed to item 9. If $\widehat{F}$ and $\overline{F}$ always coincide, then the above assumptions do not apply.

8. There exists a *Potential Function* for plastic deformations, of the form

$$G = G(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \tag{62}$$

9. Analogously to the standard incremental theory of plasticity, it is assumed that an infinitesimal strain increment $\delta\boldsymbol{\epsilon}$ can be expressed as, Eq. 11,

$$\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\epsilon}^{(e)} + \delta\boldsymbol{\epsilon}^{(p)} \tag{63}$$

where:

- $\delta\boldsymbol{\epsilon}^{(e)}$ represents the elastic (fully recoverable) component which may be calculated according to the generalized Hooke's law, Eq. 12,

$$\delta\boldsymbol{\epsilon}^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta\boldsymbol{\sigma} \tag{64}$$

  where $\mathbf{C}^{(e)}$, the tangential elastic stiffness matrix, may be function of the current $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$.

- $\delta\epsilon^{(p)}$ is the plastic (irreversible) component, defined as

$$\delta\epsilon^{(p)} = \delta\lambda \mathbf{b} \tag{65}$$

where

$$\delta\lambda \quad \begin{cases} \geq 0, & \text{if elasto-plastic response occurs.} \\ = 0, & \text{if elastic response occurs.} \end{cases}$$

$$\begin{aligned} \mathbf{b} &= \frac{\partial G}{\partial \boldsymbol{\sigma}} = \\ &= \left\{ \frac{\partial G}{\partial \sigma_{11}}, \frac{\partial G}{\partial \sigma_{22}}, \frac{\partial G}{\partial \sigma_{33}}, \frac{\partial G}{\partial \sigma_{12}}, \frac{\partial G}{\partial \sigma_{13}}, \frac{\partial G}{\partial \sigma_{21}}, \frac{\partial G}{\partial \sigma_{23}}, \frac{\partial G}{\partial \sigma_{31}}, \frac{\partial G}{\partial \sigma_{32}} \right\}^T \end{aligned}$$

The respect of the consistency equation on the yield surface $F(\boldsymbol{\sigma}, \mathbf{k}, \overline{\mathbf{k}})$ yields to the conclusion that, item 4 Section 11, the value plastic multiplier $\delta\lambda$ can be calculated as:

- If $\delta\boldsymbol{\sigma}$ is assigned, Eq. 26,

$$\delta\lambda = \begin{cases} \frac{\mathbf{a}^T \delta\boldsymbol{\sigma}}{A}, & \text{if } A \neq 0. \\ \text{indeterminate}, & \text{if } A = 0. \end{cases} \tag{66}$$

- if $\delta\epsilon$ is assigned, Eq. 27,

$$\delta\lambda = \begin{cases} \frac{\mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)}}{A + \mathbf{a}^T \mathbf{c}^{(G)}}, & \text{if } A \neq -\mathbf{a}^T \mathbf{c}^{(G)}. \\ \text{indeterminate}, & \text{if } A = -\mathbf{a}^T \mathbf{c}^{(G)}. \end{cases} \tag{67}$$

where

$$\begin{aligned} \delta\boldsymbol{\sigma}^{(e)} &= \mathbf{C}^{(e)} \delta\epsilon \\ \mathbf{c}^{(G)} &= \mathbf{C}^{(e)} \mathbf{b} \end{aligned}$$

and the *Plastic Modulus* A is defined as

$$\begin{aligned} A &= -\frac{1}{\delta\lambda} \left[\frac{\partial F}{\partial \overline{k}_i} \delta\overline{k}_i + \frac{\partial F}{\partial k_i} \delta k_i \right] = \\ &= -\left[\frac{\partial F}{\partial \overline{k}_i} \frac{\partial \overline{k}_i}{\partial h_{\overline{j}}} c_{\overline{j}} + \frac{\partial F}{\partial k_i} \frac{\partial k_i}{\partial h_{\overline{j}}} c_{\overline{j}} + \frac{\partial F}{\partial k_i} \frac{\partial k_i}{\partial h_j} c_j \right] \end{aligned} \tag{68}$$

where $i = 1, 2, \ldots, n$; $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$.

According to the above definition of elasto-plastic deformation, the stress increment $\delta\boldsymbol{\sigma}$ resulting from strain increment $\delta\boldsymbol{\epsilon}$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated by a relationship formally identical to that in Eq. 47. The type of mechanical response is established according to the following criterion.

10. By definition, if

$$\widehat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) < 0 \tag{69}$$

the material response is always elastic, regardless of the applied stress increment $\delta\boldsymbol{\sigma}$ or strain increment $\delta\boldsymbol{\epsilon}$. If, instead,

$$\widehat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \geq 0 \tag{70}$$

the type of material response is established according to a stress based criterion or to a strain based criterion, identical with those derived in Sections 7 and 8 for the case of the standard incremental theory of plasticity. However, in the generalized incremental theory, these criteria are given as *definitions*. In fact, according to item 3, the current stress point lies always on the yield surface F; consequently, the mathematical procedure followed in Sections 7 and 8 for deriving the stress and strain criteria cannot be applied.

11 Remarks

It is important to make the following remarks on the main hypotheses listed in Section 10:

1. When purely elastic response occurs, the bounding surface $\overline{F}$ remains fixed in the stress space, that is

$$\overline{k}_i = \text{ constant}$$

for all $i = 1, 2, \ldots, n$. In fact, by definition, in a purely elastic response $\delta\lambda = 0$; consequently, according to the hypothesis in item 2 in Section 10,

$$\delta\overline{k}_i = 0$$

for all $i = 1, 2, \ldots, n$.

2. The requirements on the partial derivatives of k_i in Eqs. 58 and 59 are consequences of Eq. 57. In fact, when the current stress point $\boldsymbol{\sigma}$ lies on $\overline{F}$, from Eq. 57 it results that

$$\delta k_i = \delta \overline{k}_i \tag{71}$$

Hence, if plasticity occurs, Eqs. 53 and 49,

$$\frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}} + \frac{\partial k_i}{\partial h_j} \delta h_j = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}}$$

Since this equality must be true for any arbitrary value of $\delta \overline{h}_{\overline{j}}$ and δh_j, the requirements in Eqs. 58 and 59 immediately follow.

3. The requirement in item 7 on the limit value for A guarantees that, if $\widehat{F}$ does not always coincide with $\overline{F}$ and the stress point lies on the elastic surface, then:

   - the infinity value of A makes $\delta\lambda = 0$, Eqs. 66 and 67, and, consequently $\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda\mathbf{b} = \mathbf{o}$, Eq. 65. This assures the continuity of any deformative process in which the stress point crosses the elastic surface.
   - The positiveness of the A value makes admissible any stress increments applied on a material state lying on $\widehat{F}$. In fact, according to the stress based criterion in item 10 in Section 10, the only not admissible situation may occur if $A \leq 0$, case (d).

4. The expressions of $\delta\lambda$ and A in Eqs. 66-68 may be proved noticing that the expansion of the yield surface F defined in Eq. 51 into the *exact* Taylor series about a material state $(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$ yields to

$$F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \overline{\mathbf{k}} + \delta\overline{\mathbf{k}}, \mathbf{k} + \delta\mathbf{k}) = F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) + \delta F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$$

Since by definition the material state always satisfies the yield surface F, then

$$\begin{aligned} F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \overline{\mathbf{k}} + \delta\overline{\mathbf{k}}, \mathbf{k} + \delta\mathbf{k}) &= 0 \\ F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) &= 0 \end{aligned}$$

and, consequently, we can identify the following *consistency equation*

$$\delta F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) = \mathbf{a}^T \delta\boldsymbol{\sigma} + \frac{\partial F}{\partial \overline{k}_i} \delta \overline{k}_i + \frac{\partial F}{\partial k_i} \delta k_i = 0 \tag{72}$$

that is

$$\mathbf{a}^T \delta\sigma - A\delta\lambda = 0 \tag{73}$$

where

$$A = -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{k}_i}\delta\overline{k}_i + \frac{\partial F}{\partial k_i}\delta k_i\right] \tag{74}$$

The consistency equation in Eq. 73 is formally identical with that reported in Eq. 31. Then, proceeding as in Section 6, we eventually obtain the expressions for $\delta\lambda$ in Eqs. 66 and 67.

In the case of plastic deformation, $\delta\overline{k}_i$ and δk_i may be expressed as Eqs. 49 and 53, respectively. Thus, the expression of A in Eq. 74 takes the form reported in Eq. 68.

12 Recovery of the Classical Theory

It is easy to verify that the generalized theory of plasticity proposed in Section 10 recovers the standard incremental theory if:

- The bounding surface $\overline{F}$ coincides with the yield surface of the standard theory.
- The elastic and the bounding surfaces coincide, i.e.

$$\widehat{F}(\sigma, \overline{\mathbf{k}}) \equiv \overline{\mathbf{F}}(\sigma, \overline{\mathbf{k}})$$

  According to items 5 and 6 in Section 10, this is assured with the position:

$$\widehat{\mathbf{k}} = \overline{\mathbf{k}}$$

- When the current stress point lies on $\overline{F}$, the potential surface G is made to coincide with that of the standard theory.

In fact, we have that:

- Being $\widehat{F} \equiv \overline{F}$, plasticity may occur only when the stress point σ lies on $\overline{F}$.
- In general, if σ lies on $\overline{F}$, it can be proved that the plastic modulus expression in Eq. 68 reduces to the standard form

$$A = -\frac{1}{\delta\lambda}\frac{\partial \overline{F}}{\partial \overline{k}_i}\delta\overline{k}_i = -\frac{\partial \overline{F}}{\partial \overline{k}_i}\frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}}\overline{c}_{\overline{j}} \tag{75}$$

Consequently, the elasto-plastic strain definition and the loading criteria in items 9 and 10 of Section 10 result to be identical with those of the standard theory.

The reduced form of A in Eq. 75 can be proved noticing that, when $\boldsymbol{\sigma}$ lies on $\overline{F}$, Eq. 71,

$$\delta k_i = \delta \overline{k}_i$$

Hence, we can simplify the expression of Eq. 68 into

$$A = -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i}\right]\delta\overline{k}_i = -\left[\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i}\right]\frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}}\overline{c}_{\overline{j}} \tag{76}$$

During a loading process in which the current stress point does not leave the bounding surface $\overline{F} \equiv F$, we have that

$$\delta F = \delta \overline{F}$$

that is

$$\mathbf{a}^T\delta\sigma + \frac{\partial F}{\partial \overline{k}_i}\delta\overline{k}_i + \frac{\partial F}{\partial k_i}\delta k_i = \overline{\mathbf{a}}^T\delta\sigma + \frac{\partial \overline{F}}{\partial \overline{k}_i}\delta\overline{k}_i$$

from which, being $\mathbf{a} \equiv \overline{\mathbf{a}}$ and $\delta k_i = \delta\overline{k}_i$, we can establish that

$$\left[\left(\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i}\right) - \frac{\partial \overline{F}}{\partial \overline{k}_i}\right]\delta\overline{k}_i = 0$$

Since this equality must be true for any arbitrary value of the n increments $\delta\overline{k}_i$, it follows that

$$\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i} = \frac{\partial \overline{F}}{\partial \overline{k}_i} \tag{77}$$

for each $i = 1, 2, \ldots, n$. Substituting Eq. 77 into Eq. 76 we obtain the expression of A in Eq. 75.

13 A Generalized Von-Mises Strain Hardening Model

Consider a material whose uniaxial response is of the type shown in Fig. 1. Assume that the behavior of this material under monotonically increasing loading conditions can be modeled with an isotropic elasto-plastic constitutive equation of the type:

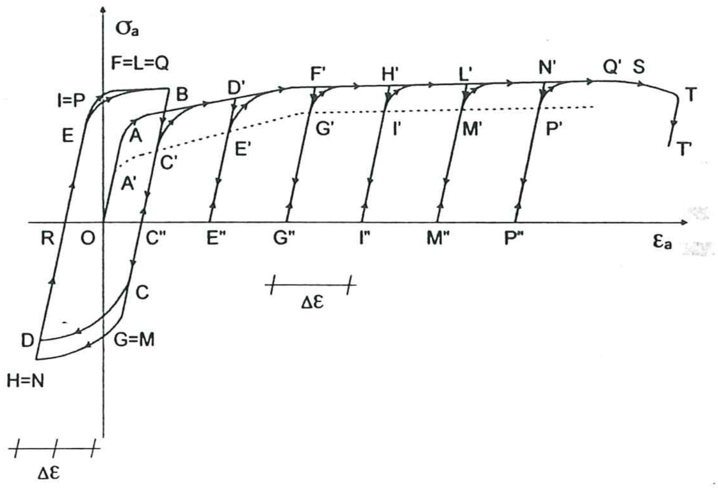

Figure 1: Material response under uniaxial cyclic loading

- Elastic strain obeying the Hooke law for isotropic linear elastic material.
- Failure condition represented by a Von-Mises surface of equation

$$Q = q - N = 0 \tag{78}$$

- Plastic strain obeying an associative flow rule with yield-potential function represented by a Von-Mises surface of equation

$$\overline{F} = q - \overline{q}_y = 0 \tag{79}$$

where $\overline{q}_y$ is a hardening parameter whose incremental variation is controlled by the total plastic shear strain $\epsilon_s^{(p)}$ only, that is

$$\delta\overline{q}_y = \frac{d\overline{q}_y}{d\epsilon_s^{(p)}}\delta\epsilon_s^{(p)} \tag{80}$$

where

$$\begin{aligned} \epsilon_s^{(p)} &= \int \delta\epsilon_s^{(p)} \\ \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} &= \overline{a}_y(\epsilon_s^{(p)}) \end{aligned}$$

and $\overline{a}_y(\epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally.

In the following Section we present the extension of this simple Von-Mises strain hardening elasto-plastic model to account for the uniaxial response under cyclic load of the type in Fig. 1.

13.1 The constitutive equation

According to the recipe listed in Section 10, a generalization of the Von-Mises strain hardening model to account for plastic accumulation can be obtained as follows:

1. The bounding surface has equation

$$\overline{F} = \overline{F}(q, \overline{q}_y) = q - \overline{q}_y \tag{81}$$

   where $\overline{q}_y$ is the hardening parameter whose value is assumed to be always a positive quantity.

2. The incremental variation of the hardening parameter $\overline{q}_y$ is controlled by the total plastic shear strain $\epsilon_s^{(p)}$ only, that is

$$\delta\overline{q}_y = \frac{d\overline{q}_y}{d\epsilon_s^{(p)}}\delta\epsilon_s^{(p)} \tag{82}$$

   where

$$\frac{d\overline{q}_y}{d\epsilon_s^{(p)}} = \overline{a}_y(\epsilon_s^{(p)}) \tag{83}$$

   and $\overline{a}_y(\epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally.

3. The yielding surface has equation

$$F = F(q, \overline{q}_y, q_y) = q - t\overline{q}_y - q_y = 0 \tag{84}$$

   with the conditions

$$\begin{aligned} t &= 0 \\ 0 \leq\ q_y &\leq \overline{q}_y \end{aligned}$$

   In practice, F does not depend on $\overline{q}_y$.

4. With regard to the hardening parameter q_y, we have that:

   - From the above defined yield function equation we find that q_y can be expressed as

   $$q_y = q_y(q, \overline{q}_y) = q - t\overline{q}_y \tag{85}$$

   where $t = 0$, that is

   $$q_y = q \tag{86}$$

   - If plasticity occurs, the incremental variation of q_y is controlled by the variable $\epsilon_s^{(p)}$ only, that is

   $$\delta q_y = \frac{dq_y}{d\epsilon_s^{(p)}} \delta\epsilon_s^{(p)} \tag{87}$$

   where

   $$\frac{dq_y}{d\epsilon_s^{(p)}} = a_y(q_y, \epsilon_s^{(p)}) \tag{88}$$

   and $a_y(q_y, \epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally. However, the mathematical framework of the model requires that, items 5 and 7 in Section 10,

   $$a_y(q_y, \epsilon_s^{(p)}) \begin{cases} = \overline{a}_y(\epsilon_s^{(p)}); & \text{for } q_y = \overline{q}_y. \\ \to \infty; & \text{for } q_y \to \widehat{q}_y. \end{cases} \tag{89}$$

   The parameter $\widehat{q}_y$ is defined in item 6.

5. It is easy to verify that:

   - the space region bounded by F is a subspace of $\overline{F}$, that is

   $$F(q, \overline{q}_y, q_y) \subseteq \overline{F}(q, \overline{q}_y) \tag{90}$$

   - if the current stress point $\boldsymbol{\sigma}$ lies on $\overline{F}$, then F coincides with $\overline{F}$, that is

   $$F(q, \overline{q}_y, q_y) \equiv \overline{F}(q, \overline{q}_y) \tag{91}$$

   and

   $$\begin{aligned} q_y &= \overline{q}_y \\ \frac{dq_y}{d\epsilon_s^{(p)}} &= \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} \end{aligned}$$

6. The elastic surface $\widehat{F}$ has equation

$$\widehat{F} = \widehat{F}(q, \overline{q}_y) = q - \widehat{q}_y \tag{92}$$

where

$$\widehat{q}_y = \widehat{a}_y(\overline{q}_y) \tag{93}$$

and $\widehat{a}_y(\widehat{q}_y)$, which has the dimension of a stress, is a function to be determined experimentally. Notice that, according to item 6 in Section 10, $\widehat{F}$ is derived from F as

$$\widehat{F}(q, \overline{q}_y) \equiv F(q, \overline{q}_y, q_y = \widehat{q}_y)$$

Moreover, we require

$$\widehat{a}_y(\overline{q}_y) > 0$$

for any $\overline{q}_y$, so that for $\boldsymbol{\sigma} = \mathbf{o}$ the mechanical response is purely elastic.

7. According to the definition of F and assuming associative flow rule as stated in the next item 8, the plastic modulus results to be given by

$$A = \frac{dq_y}{d\epsilon_s^{(p)}} \tag{94}$$

Note that, if the experimental function in Eq. 88 respect the conditions in Eq. 89, then for $q_y \rightarrow \widehat{q}_y$

$$A \rightarrow +\infty$$

as required in item 7 in Section 10.

8. The potential function for plastic deformations coincides with the yield surface, associative flow rule, that is

$$G \equiv F(q, \overline{q}_y, q_y) = q - t\overline{q}_y - q_y = 0 \tag{95}$$

with $t = 0$; in practice G does not depend on $\overline{q}_y$.

9. According to the definition in item 9 in Section 10, the infinitesimal strain increment $\delta\epsilon$ is given by

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)} \tag{96}$$

where:

- the elastic (fully recoverable) strain increment can be calculated according to the Hooke law for isotropic material,

$$\delta \epsilon^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta \sigma \tag{97}$$

where the elastic stiffness matrix $\mathbf{C}^{(e)}$ is a constant symmetric positive definite matrix function of two constants E and ν to be determined experimentally:

$$\mathbf{C}^{(e)} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & . & 0 \\ \nu & (1-\nu) & \nu & 0 & . & 0 \\ \nu & \nu & (1-\nu) & 0 & . & 0 \\ 0 & 0 & 0 & (1-2\nu) & . & 0 \\ . & . & . & . & & . \\ 0 & 0 & 0 & 0 & . & (1-2\nu) \end{bmatrix}_{(9\times 9)} \tag{98}$$

- the plastic (irreversible) strain increment can be calculated as

$$\delta \epsilon^{(p)} = \delta \lambda \mathbf{b} \tag{99}$$

where, according to the above listed hypotheses, it eventually results that

$$\delta \lambda = \frac{\delta q}{A} \tag{100}$$

$$\mathbf{b} \equiv \mathbf{a} = \frac{\partial F}{\partial \sigma} = \frac{3}{2q}\mathbf{s} \tag{101}$$

$$A = a_y(q_y, \epsilon_s^{(p)}) \tag{102}$$

The above expression for $\delta\lambda$ can be obtained taking into account that, for a stress invariant function $F = F(p, q, \theta, \mathbf{k})$, it results

$$\mathbf{a}^T \delta \sigma = \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial \theta}\delta \theta$$

10. If the current stress point $\boldsymbol{\sigma}$ lies inside $\widehat{F}$, that is

$$q < \widehat{q}_y$$

the material response is purely elastic. Otherwise the type of mechanical response is determined according to the criteria in item 10 in Section 10.

13.2 The uniaxial response of the model

In an uniaxial stress test, the material sample is subjected to a vertical uniform stress field σ_a. In this case, therefore, the stress vector field is given by

$$\boldsymbol{\sigma} = \sigma_a \{1,0,0,0,0,0,0,0,0\}^T \tag{103}$$

from which it follows that

$$\begin{aligned} \mathbf{s} &= \boldsymbol{\sigma} - \widehat{\mathbf{m}}\frac{\sigma_{kk}}{3} = \frac{\sigma_a}{3}\{2,-1,-1,0,0,0,0,0,0\}^T & (104) \\ q &= \left(\frac{3}{2}\mathbf{s}^T\mathbf{s}\right)^{1/2} = |\sigma_a| & (105) \end{aligned}$$

Under a uniform axial stress field, the model presented in Section 13.1 predicts a strain vector field

$$\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\epsilon}^{(e)} + \delta\boldsymbol{\epsilon}^{(p)} \tag{106}$$

where, Eqs. 97 and 99,

$$\delta\boldsymbol{\epsilon}^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1}\delta\boldsymbol{\sigma} = \frac{\delta\sigma_a}{E}\{1,-\nu,-\nu,0,0,0,0,0,0\}^T \tag{107}$$

$$\delta\boldsymbol{\epsilon}^{(p)} = \frac{3}{2q}\delta\lambda\mathbf{s} = \begin{cases} \mathbf{o}; & \text{for pure elastic response} \\ \frac{\delta\sigma_a}{A}\left\{1,-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\right\}^T; & \text{if } A \neq 0 \\ \text{indeterminate}; & \text{if A} = 0 \end{cases} \tag{108}$$

and,

$$\begin{aligned} A &= a_y(|\sigma_a|, \epsilon_s^{(p)}) \\ \epsilon_s^{(p)} &= \int \left|\delta\epsilon_a^{(p)}\right| \end{aligned} \tag{109}$$

It is useful to note that, when plasticity occurs, it results, Eq. 108,

$$A = a_y(|\sigma_a|, \epsilon_s^{(p)}) = \frac{\delta\sigma_a}{\delta\epsilon_a^{(p)}} \tag{110}$$

13.3 Experimental determination of the material properties

The constitutive model in Section 13.2 requires the experimental determination of the following material properties:

- the constant elastic parameters E and ν;
- the plastic functional relationships, Eqs. 83, 88 and 93,

$$\begin{aligned} \overline{a}_y(\epsilon_s^{(p)}) &= \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} \\ a_y(q_y, \epsilon_s^{(p)}) &= \frac{dq_y}{d\epsilon_s^{(p)}} \\ \widehat{a}_y(\overline{q}_y) &= \widehat{q}_y \end{aligned}$$

All these material properties can be determined from the simple uniaxial test shown in Fig. 1 as follows:

1. We identify in the linear paths in Fig. 1 below the dashed line A'C'E'G'I'M'P' the purely elastic response of the material.

2. With reference to these linear elastic paths and the elastic relationships in Eq. 107, we can calculate the elastic material constants as

$$\begin{aligned} E &= \frac{\delta\sigma_a}{\delta\epsilon_a} \\ \nu &= \frac{1}{2}\left(1 - E\frac{\delta\epsilon_v}{\delta\sigma_a}\right) \end{aligned}$$

3. We draw the diagram σ_a vs. $\epsilon_a^{(p)}$ in Fig. 2 scaling the abscissa in Fig. 1 by the elastic strain, that is

$$\epsilon_a^{(p)} = \epsilon_a - \epsilon_a^{(e)} = \epsilon_a - \frac{\sigma_a}{E}$$

We note that, according to Eq. 109, for $\sigma_a > 0$, it results, Fig. 2,

$$\epsilon_s^{(p)} = \int \left|\delta\epsilon_a^{(p)}\right| = \epsilon_a^{(p)}$$

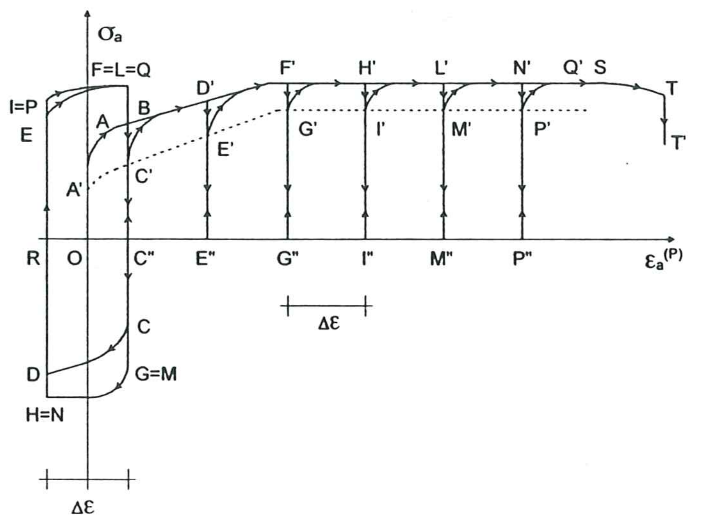

Figure 2: The uniaxial material response scaled by the elastic strain

4. We identify in the stress path ABD'F'H'L'N'Q'ST in Fig. 2 the plastic response of the material when the stress point lies on the bounding surface; then:

   - According to Eq. 81, along this path $\sigma_a = |\sigma_a| = q = \overline{q}_y$.
   - According to Eqs. 89 and 110, the functional relationship $\overline{a}_y(\epsilon_s^{(p)})$ can be identified by interpolating, with an opportune function, various pairs

   $$(\overline{a}_y(\epsilon_s^{(p)}) = \sigma', \epsilon_s^{(p)} = \epsilon_a^{(p)})$$

   where

   $$\sigma' = \frac{d\sigma_a}{d\epsilon_a^{(p)}}$$

   is the slope of the stress path measured at $\epsilon_a^{(p)}$.

5. We identify in the dashed line A'C'E'G'I'M'P' in Fig. 2 the boundary for pure elastic deformation; the functional relationship $\widehat{a}_y(\overline{q}_y)$ can be

identified by interpolating, with an opportune function, the pairs

$$(\widehat{a}_y(\overline{q}_y) = \sigma_a^{C'}, \overline{q}_y = \sigma_a^{B})$$
$$(\widehat{a}_y(\overline{q}_y) = \sigma_a^{E'}, \overline{q}_y = \sigma_a^{D'})$$
$$\vdots$$

6. We identify in the stress paths within the continuous line ABD'F'H'L'N'Q'ST and the dashed line A'C'E'G'I'M'P' in Fig. 2 the plastic response of the material when the stress point lies inside the bounding surface; then:

   - according to Eqs. 86 and 105 along this path $q_y = q = |\sigma_a| = \sigma_a$;
   - the functional relationship $a_y(q_y, \epsilon_s^{(p)})$ can be identified by interpolating, with an opportune function, various pairs

   $$(a_y(q_y, \epsilon_s^{(p)}) = \sigma', q_y = \sigma_a, \epsilon_s^{(p)} = \epsilon_a^{(p)})$$

   where

   $$\sigma' = \frac{d\sigma_a}{d\epsilon_a^{(p)}}$$

   is the slope of the stress path, for example C'D', measured at $\epsilon_a^{(p)}$.

13.4 Note on the hysteretic loop

We recall that the σ_a vs. $\epsilon_a^{(p)}$ graph shown in Fig. 2 is obtained scaling the elastic strain component from the assumed material behavior in Fig. 1, item 3 in Subsection 13.3.

In particular, the hysteretic loop sketched in Fig. 2 presents the following geometrical characteristics:

- The paths C"CD, C"GH and C"MN result symmetric to the paths C"C'D', G"G'H' and M"M'N', with respect to the $\epsilon_a^{(p)}$ axis.
- The paths REF,RIL and RPQ are identical to the paths E"E'F', I"I'L' and P"P'Q', with an opportune shift.

It is possible to prove that the proposed model is able to reproduce exactly such type of response.

In fact, we note that, according to Eq. 110, the slope of the material plastic response depends only on the absolute value of the uniaxial stress σ_a and plastic strain $\epsilon_a^{(p)}$, namely

$$\frac{\delta\sigma_a}{\delta\epsilon_a^{(p)}} = A = a_y(q_y, \epsilon_s^{(p)})$$

where

$$\begin{aligned} q_y &= q = |\sigma_a| \\ \epsilon_s^{(p)} &= \int \delta\epsilon_s^{(p)} = \int |\delta\epsilon_a^{(p)}| \end{aligned}$$

Thus, for example, the q_y and $\epsilon_s^{(p)}$ values at the point C' are equal to those at the point C. This implies that the slope at the point C and C' coincide and consequently the path CD has to result symmetric to the path C'D'.

14 Conclusions

The theoretical framework presented in Section 10 is a complete recipe to set up a constitutive equation.

The remarks in Section 12 provide the mathematical condition for recovering the classical incremental theory of plasticity.

The small example in Section 13 shows that the proposed generalized incremental theory permits, in line of principle, a simple extension of any existing plastic model.

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