COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING
Commun. Numer. Meth. Engng 2008; **24**:1611–1615
Published online 28 September 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/cnm.1054

Parabolic recovery of boundary gradients

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SUMMARY

A parabolic recovery procedure suited for shear stress and heat flux recovery on surfaces from linear element data is proposed. The information required consists of the usual unknowns at points, as well as gradients recovered at the points that are one layer away from the wall. The procedure has been in use for some time and has consistently delivered superior results as compared with the usual wall shear stress and heat flux obtained from linear finite element method shape functions. Copyright © 2007 John Wiley & Sons, Ltd.

Received 3 April 2007; Revised 20 August 2007; Accepted 20 August 2007

KEY WORDS: higher order gradients; stress recovery; CFD

1. BOUNDARY GRADIENT RECOVERY

Post-processing results of field solvers, such as those used in computational fluid dynamics (CFD) and computational thermodynamics, often lead to the requirement of evaluating derivatives on boundaries. This is particularly the case if the shear stress or the heat flux on the boundary is sought. Many CFD and CSD codes utilize linear elements (triangles, tetrahedral). If we consider a typical boundary element as shown in Figure 1, the most obvious method to evaluate the required stress or heat flux is to take the derivatives of the shape functions and obtain the (constant) gradient of the velocities or temperature in the element and multiply it by the material parameters (viscosity, conductivity).

Denoting the quantity whose gradient is sought by Φ, and the shape functions as N^j, we have

$$\Phi_{,i} = \frac{\partial \Phi}{\partial x_i} = N^j_{,i} \hat{\Phi}_j \tag{1}$$

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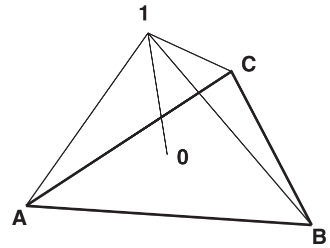

Figure 1. Element with boundary face A–B–C.

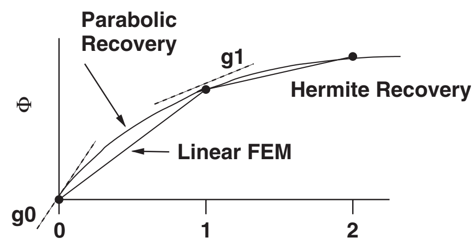

Figure 2. 1-D case.

If we consider the 1-D case shown in Figure 2, this leads to

$$\Phi_{,x}|_0 = \frac{\Phi_1 - \Phi_0}{h} \tag{2}$$

where h is the element size. The recovery of gradients at nodes based on element gradients has been treated before [1]. The basic idea is to represent the gradient of Φ as a linear function itself and then obtain the nodal values *via* a weighted residual method. Suppose that we have gradient information at the points inside the domain. In this case, a better approximation to the gradients on the boundary may be obtained using a parabolic extrapolation.

With the notation of Figure 2, the function Φ in the first element may be represented as

$$\Phi = \Phi_0 + \left[2\frac{\Phi_1 - \Phi_0}{h} - \Phi_{,x}|_1\right]x - \left[\frac{\Phi_1 - \Phi_0}{h} - \Phi_{,x}|_1\right]\frac{x^2}{h} \tag{3}$$

implying

$$\Phi_{,x}|_0 = 2\frac{\Phi_1 - \Phi_0}{h} - \Phi_{,x}|_1 \tag{4}$$

and for 2-/3-D:

$$\Phi_{,n}|_0 = 2\frac{\Phi_1 - \Phi_0}{h} - \Phi_{,n}|_1 \tag{5}$$

DOI: 10.1002/cnm

For the general 3-D case, for each of the boundary faces the following steps are required (see Figure 1):

- obtain the face normal **n**;
- obtain the element adjacent to the face;
- obtain the point $i1$ that is not on the face/surface;
- obtain the normal distance h of point $i1$ to the face;
- obtain the value of Φ_0 at the location the line $i1$, **n** intersects the face;
- obtain the normal derivatives from Equation (5).

2. EXAMPLES

The proposed shear stress recovery is shown for two examples. The first is a simple channel with dimensions: length $L=5.0$, height $H=1.0$, and width $W=0.1$. The exact solution is given by the parabola: $u=[1-(2y/H)^2]u_0$, where y is the vertical coordinate and u_0 the maximum velocity. The values were chosen such that the wall shear stress is $\tau=4$. Unstructured and structured grids of different size were generated to observe the convergence to the exact value. Figure 3(a) and (b) shows typical surface grids. Figure 3(b) and (c) shows the shear stress values obtained. Note that, as expected, the linear finite element method (FEM) gradients under-predict the shear. The parabolic

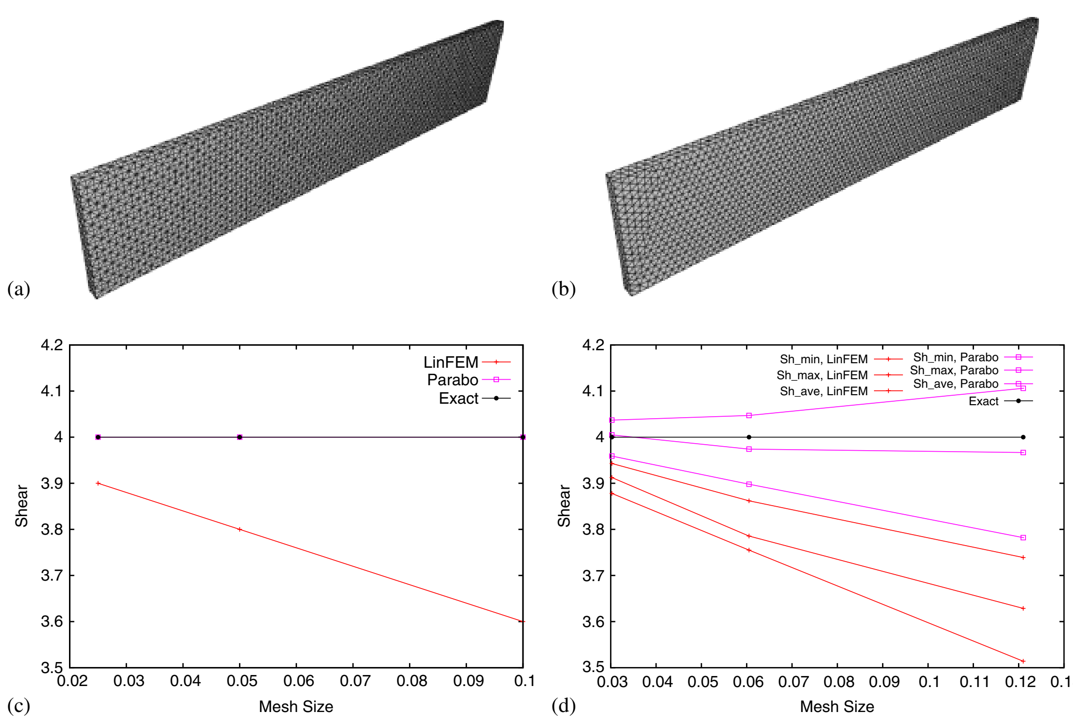

Figure 3. (a), (b) Surface grids of channel (Cartesian, unstructured) and (c), (d) convergence of shear stress (Cartesian, unstructured).

DOI: 10.1002/cnm

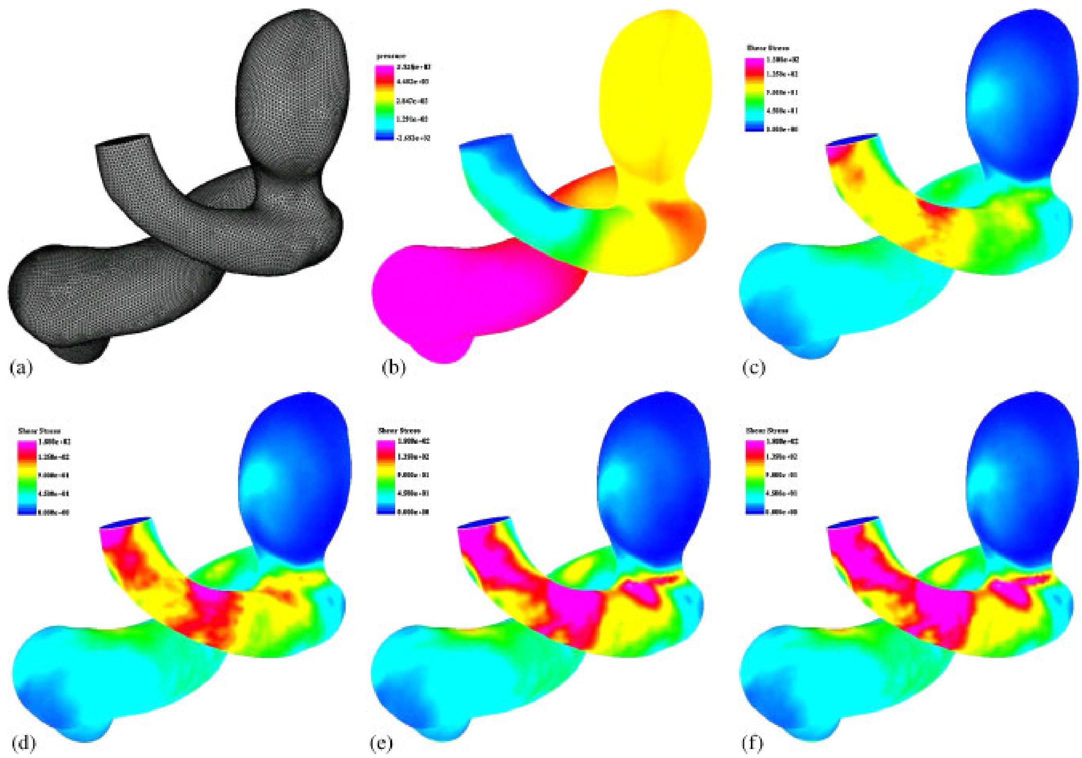

Figure 4. (a), (b) Aneurysm: surface mesh, surface pressure; (c), (d) aneurysm: shear stress (linear FEM, parabolic recovery); and (e), (f) aneurysm: shear stress (linear FEM, parabolic recovery, RANS mesh).

recovery is able to obtain the exact result for the structured grids. For the unstructured grids, some points exhibit shear values that are larger or smaller than the exact solution. This is because the gradient recovery at the first point off the wall is performed *via* linear elements, leading to inexact gradients. Still, the overall accuracy of the parabolic recovery is noticeably higher than that of the linear FEM.

The second example shows the shear stresses obtained from an aneurysm simulation. The mesh, whose surface is displayed in Figure 4(a), had approximately 1.14 Mels. The surface pressures are displayed in Figure 4(b). The shear stresses obtained using the linear FEM and parabolic recovery are shown (with the same scale) in Figure 4(c) and (d). Note the marked increase in shear stress recorded by the parabolic recovery in the neck region.

In order to assess grid convergence, a grid was generated that had the same isotropic element size distribution as the previous one, but with an additional 10 layers of highly stretched elements in the wall region to resolve possible boundary layers. The results obtained on this mesh (again, with the same scale) are displayed in Figure 4(e) and (f). Note that in this case the linear and parabolic stresses are similar, indicating convergence. The run was repeated once more on a globally refined mesh (with boundary layer gridding) of approximately 10 Mels. No discernable differences with the results shown in Figure 4(e) and (f) were observed, again indicating convergence. Overall, it is remarkable how much better (in absolute terms) the shear stresses obtained from the parabolic recovery on the isotropic mesh are in comparison with the linear FEM stresses.

3. CONCLUSIONS

A parabolic recovery procedure suited for shear stress and heat flux recovery on surfaces from linear element data has been presented. The information required consists of the usual unknowns at points, as well as gradients recovered at the points that are one layer away from the wall. The procedure has been in use for some time and has consistently delivered superior results as compared with the usual shear stress obtained from linear FEM shape functions. It may also be used as an error indicator to assess if grid resolution in wall regions is adequate.

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DOI: 10.1002/cnm