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CONTACT PROBLEMS IN INDUSTRIAL APPLICATIONS USING FREEFEM

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Abstract. This paper presents an algorithm and a method to solve mechanical contact problems between two bodies or more, for linear elastic and finite deformation problems. The contact problem is considered as an optimization one, more specifically as a minimization problem. The interior point method is used to solve the minimization problem. This algorithm is symmetric and the user does not need to specify anymore a slave and a master body. The algorithm was developed using FreeFEM and IPOPT software.

1 INTRODUCTION

The mechanical contact between two bodies is one of the difficult problems in solid mechanics, indeed the contact area is unknown. There exist several algorithms to solve the contact problems [8, 9], most of them involve the concept of master/slave, which prevents the penetration of the slave body into the master one, and therefore causes the non-symmetry of the algorithm. Otherwise the interior point method was not widely used in contact problems, we can cite for example [10, 11, 12].

In this paper, we restrict ourselves to the contact between two hyperelastic bodies (where the linear elastic body is a particular case). The frictional case is not considered here. One of the goals of this work is to propose a method that uses the FreeFEM [1] software and its tools to solve the contact problem. As it is formulated as an optimization problem, the interior point method is used and the Interior Point OPTimizer algorithm (IPOPT) [5], which is already interfaced with FreeFEM, is used to solve the optimization problem and to reach the solution. The second goal of this work is to obtain a symmetric formulation, in order to allow the user to no longer distinguish between slave and master bodies. An algorithm to solve

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Signorini's problem using FreeFEM and IPOPT software can be found in [2].

This paper is organized as follows: in section 2, we present briefly the contact problem and the symmetric formulation. In section 3, we present the numerical methods, used to solve the constrained minimization problems and we show that it will not generate numerical difficulties, in addition to the convergence of the method. We also introduce briefly alternatives symmetric formulations of the contact problems in section 4, and we present briefly the contact algorithm in section 5. Finally, in section 6, two examples are studied in order to show that the algorithm can handle large deformations, and multi-body contact in 3D.

2 Contact Problem and discretization

In the following we consider the contact between two bodies Ω_1 and Ω_2, where Ω_i denotes the body i in its reference configuration, for the sake of simplicity we suppose that $\Omega_i \subset \mathbb{R}^2$, the same work can be done for three-dimensional space. We note Γ_d^i the part of the boundary $\partial\Omega_i$ where a displacement is imposed, $\Gamma_l^i \subset \partial\Omega_i$ where the load is imposed, $\Gamma_c^i \subset \partial\Omega_i$ the potential contact part of the boundary $\partial\Omega_i$. We also suppose that $\Gamma_d^i, \Gamma_l^i, \Gamma_c^i$ are disjoint and $\partial\Omega_i = \Gamma_d^i \cup \Gamma_l^i \cup \Gamma_c^i$ where Γ_c^i has a non null area. Let $\mathbf{X}$ be a material point in $\Omega = \Omega_1 \cup \Omega_2$, then the displacement field $\mathbf{u}$ is defined by $\mathbf{u} = \mathbf{x} - \mathbf{X}$, where $\mathbf{x} = \phi(\mathbf{X})$ is the actual displacement of $\mathbf{X}$, and ϕ describes the transformation of the initial or reference configuration Ω into the actual configuration $\phi(\Omega)$, due let's say to a load applied on Ω.

Let $\mathbf{f}$ be a surface traction applied at $\Gamma_l = \Gamma_l^1 \cup \Gamma_l^2$, and a null displacement imposed at $\Gamma_d = \Gamma_d^1 \cup \Gamma_d^2$, thus the contact problem can be written into the following constrained minimization problem

$$\mathbf{u} = \underset{\mathbf{v}\in K}{\operatorname{argmin}}(E(\mathbf{v})) \tag{1}$$

where E is the total potential energy defined by:

$$\begin{cases} E(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - f(\mathbf{v}) \text{ for linear elastic problems} \\ E(\mathbf{v}) = \int_\Omega \psi(\mathbf{v})dv - f(\mathbf{v}) \text{ for large deformations problems and for hyperelastic materials} \end{cases} \tag{2}$$

The application a is continuous, coercive, symmetric, bilinear and is defined by $a(\mathbf{u},\mathbf{v}) = \int_\Omega C\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\, dv$, the application f is linear and continuous and is defined by $f(\mathbf{v}) = \int_{\Gamma_l} \mathbf{f}.\mathbf{v}\, ds$, the material tensor C and the strain energy function ψ describe the behavior of the material. The set K is defined by:

$$K = \left\{ \mathbf{v} \in \left(H^1(\Omega_1)\right)^2 \times \left(H^1(\Omega_2)\right)^2 \;;\; \mathbf{v} = 0 \text{ on } \Gamma_d \;;\; (\mathbf{x} - \bar{\mathbf{x}}_2)\mathbf{n} \geq 0 \text{ on } \Gamma_c^1 \right\} \tag{3}$$

where $\bar{\mathbf{x}}_2$ is the projection of $\mathbf{x} = \mathbf{X} + \mathbf{v} \in \phi(\Gamma_c^1)$ on $\phi(\Gamma_c^2)$ and $\mathbf{n}$ is the outward unit normal vector at $\bar{\mathbf{x}}_2$. The set K describes the non-penetration between the two bodies.

2.1 Discretization

The mesh Ω_h is composed from the triangles family $\{T_i \,|\, i = 1,\ldots,n_T\}$, we then consider the space $V_h = \left\{\mathbf{v} = (v_1,v_2) \in C^1(\Omega_h) \mid \mathbf{v}_{|T_i} \in P^r \times P^r, \forall i = 1,\ldots,n_T \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0\right\}$ where $r = 1$ or 2. The non-penetration constraints are expressed in a weak form as in [3, 4]. The minimization problem in this case becomes

Find $\mathbf{u} \in V_h$ such that

$$\begin{cases} \mathbf{u} = \underset{\mathbf{v}\in V_h}{argmin}(E(\mathbf{v})) \\ \int_{\Gamma_{C1}} ((\mathbf{x}-\bar{\mathbf{x}}_2)\mathbf{n}).\phi_i^{(1)}\, ds \geq 0 \quad \forall i = 1,\ldots,n_{C1} \end{cases} \tag{4}$$

where n_{C1} is the contact nodes number of Γ_c^1 and $\phi_i^{(1)}$ is the shape function at each of these latter. In this formulation, we implicitly choose the first body as slave and the second one as master (otherwise speaking the first discretized body Ω_{h1} is not allowed to penetrate Ω_{h2}). Conversely if we choose the second body as slave and the first one as master (in this case we have n_{C2} constraints) we will not obtain exactly the same results, thus the user must choose the best slave/master. Otherwise there is some criterions to choose a body as slave, for example if it has the finest mesh or if is less stiffer than the second body... But what if this body has the finest mesh and is more stiffer than the second one. Therefore we conclude that there is a difficulty to choose the best slave/master, this issue also occurs in the case of self-contact or multi-body contact problems. Hence the need to create a symmetric formulation in order to not distinguish anymore between slave and master.

One simple way to create a symmetric formulation, is to take into account an additional non-penetration constraints by considering the second body as slave and the first one as master, in other word the problem becomes

$$\begin{cases} \mathbf{u} = \underset{\mathbf{v}\in V_h}{argmin}(E(\mathbf{v})) \\ \int_{\Gamma_{C1}} ((\mathbf{x}-\bar{\mathbf{x}}_2)\mathbf{n}).\phi_i^{(1)}\, ds \geq 0 \quad \forall i = 1,\ldots,n_{C1} \\ \int_{\Gamma_{C2}} ((\mathbf{x}-\bar{\mathbf{x}}_1)\mathbf{n}).\phi_i^{(2)}\, ds \geq 0 \quad \forall i = 1,\ldots,n_{C2} \end{cases} \tag{5}$$

For linear elastic problems, the constraints in the problem (5) are linear, indeed the deformations are supposed to be small. Otherwise for the finite deformation problems and more specifically for hyperelastic materials, it is not the case, but we can transform the problem (5) into a sequence of minimization problems with linear constraints, using a fixed point algorithm. The gradient of the constraints in the problem (5) may be linear dependent, and thus generating a numerical difficulties, we will see next how to avoid these difficulties.

3 Numerical optimization by the interior point method

After the finite element discretization, the contact problem is transformed into a constrained optimization one, for this purpose the interior point method is used in order to solve it. Let $x \in \mathbb{R}^n$ denotes the degree of freedom of the displacement field $\mathbf{u}$ (using the finite element method), and $c : \mathbb{R}^n \to \mathbb{R}^m$ the linear constraints vector where $m = n_{C1} + n_{C2}$, the contact problem becomes

$$\begin{cases} \underset{x\in\mathbb{R}^n}{Min}\, E(x) \quad \text{such that} \\ c(x) \geq 0 \end{cases} \tag{6}$$

Note that the inequality between two vectors is the inequality between each component of these two vectors. Introducing the slack variables s, the problem (6) is equivalent to

$$\begin{cases} \underset{(x,s)\in\mathbb{R}^n\times\mathbb{R}^m}{Min} E(x) \quad \text{such that} \\ c(x)-s=0 \\ s\geq 0 \end{cases} \tag{7}$$

The constraints are linear, therefore the constraints are qualified for the KKT (Karush–Kuhn–Tucker) system, The first-order optimality or the KKT conditions for the problem (7) are the following

$$\begin{cases} \nabla_x E(x)+\nabla_x c(x)\lambda=0 \\ -\lambda-z=0 \\ c(x)-s=0 \\ SZe=0 \\ s,z\geq 0 \end{cases} \tag{8}$$

where λ and z are respectively the Lagrange multipliers of the equality constraints and the bound constraints, S and Z are two diagonal matrices where $S_{ii}=s_i$ and $Z_{ii}=z_i$, finally $e^T=(1,\dots,1)$. Applying a homotopy method to the problem (8) (see [5]) with the homotopy (also called barrier) parameter μ.We obtain the following equations

$$\begin{cases} \nabla_x E(x)+\nabla_x c(x)\lambda=0 \\ -\lambda-z=0 \\ c(x)-s=0 \\ SZe-\mu e=0 \end{cases} \tag{9}$$

with $s,z\geq 0$ and μ converging to zero. Note if $\mu=0$ then the problem (9) becomes the KKT system of the original problem (8). For each barrier parameter μ, a descent method is applied in order to compute a solution of the primal-dual system (9), indeed at the iteration $k+1$

$$\begin{cases} x_{k+1}=x_k+\alpha_k d_k^x \\ s_{k+1}=s_k+\alpha_k d_k^s \\ \lambda_{k+1}=\lambda_k+\alpha_k d_k^\lambda \\ z_{k+1}=z_k+\alpha_k^z d_k^z \end{cases} \tag{10}$$

where α_k, α_k^z are the step sizes, and d_k^x, d_k^s, d_k^λ, d_k^z are respectively the descent directions of x, s, λ, z. Besides $s_{k+1}\geq 0$, $z_{k+1}\geq 0$ by using the fraction-to-the-boundary rule [5] to choose, α_k, α_k^z. In the following, a Newton's method is applied in order to compute the descent directions. By considering that our constraints are linear, we obtain the following linear system

$$\begin{bmatrix} \nabla_{xx}E(x_k) & 0 & \nabla_x c(x_k) & 0 \\ 0 & 0 & -I & -I \\ \nabla_x c(x_k)^T & -I & 0 & 0 \\ 0 & Z_k & 0 & S_k \end{bmatrix} \begin{pmatrix} d_k^x \\ d_k^s \\ d_k^\lambda \\ d_k^z \end{pmatrix} = -\begin{pmatrix} \nabla_x E(x_k)+\nabla_x c(x_k)\lambda_k \\ -\lambda_k-z_k \\ c(x_k)-s_k \\ S_k Z_k e-\mu e \end{pmatrix} \tag{11}$$

This linear system (11) can be reduced into

$$\begin{bmatrix} \nabla_{xx}E(x_k) & 0 & \nabla_x c(x_k) \\ 0 & S_k^{-1}Z_k & -I \\ \nabla_x c(x_k)^T & -I & 0 \end{bmatrix} \begin{pmatrix} d_k^x \\ d_k^s \\ d_k^\lambda \end{pmatrix} = - \begin{pmatrix} \nabla_x E(x_k) + \nabla_x c(x_k)\lambda_k \\ -\mu S_k^{-1}e - \lambda_k \\ c(x_k) - s_k \end{pmatrix} \tag{12}$$

with the additional equation $d_k^z = \mu S_k^{-1}e - z_k - S_k^{-1}Z_k d_k^s$. The matrix multiplied by the descent directions vector in the equation (12) has the form of

$$K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \tag{13}$$

where $G = \begin{bmatrix} \nabla_{xx}E(x_k) & 0 \\ 0 & S_k^{-1}Z_k \end{bmatrix}$, $A = \begin{bmatrix} \nabla_x c(x_k)^T & -I \end{bmatrix}$.

Theorem 1. *Suppose that the matrix $\nabla_{xx}E(x_k)$ is positive definite (for example a linear elastic material), suppose that $s_k > 0$ then for a fixed μ the matrix K is non singular.*

Proof. One has to prove that A is a full row rank matrix (rank= m), and G is positive definite in the null space of A, otherwise speaking if $Z_{(n+m)\times n}$ is the full rank matrix where its columns are a basis for the null space of A, $Z^T GZ$ is positive definite, see [6].

First of all, the matrix A is full row rank (rank= m), and the matrix $S_k^{-1}Z_k$ is positive, thanks to the fraction-to-the boundary rule. Let $X \in \mathbb{R}^n$, we want to prove that $Z^T GZ$ is positive definite. Indeed

$$X^T.(Z^T GZ).X = (ZX)^T.G.ZX = Y^T.G.Y \tag{14}$$

Where $Y = ZX = \begin{bmatrix} U \\ V \end{bmatrix}$, with $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$. Since $\nabla_{xx}E(x_k)$ is positive definite and $S_k^{-1}Z_k$ is a diagonal matrix with positive coefficients then

$$X^T.(Z^T GZ).X = U^T.\nabla_{xx}E(x_k).U + V^T.(S_k^{-1}Z_k).V \geq 0 \tag{15}$$

Suppose that $X^T.(Z^T GZ).X = 0$, thus form the equation (15) and from the positive definiteness of $\nabla_{xx}E(x_k)$ we obtain $U = 0$, and

$$Y = ZX = \begin{bmatrix} 0 \\ V \end{bmatrix} \tag{16}$$

By definition Y belongs to the null space of A then

$$0 = A.Y = \begin{bmatrix} \nabla_x c(x_k)^T & -I_{m\times m} \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix} = -V \tag{17}$$

Therefore $V = 0$ and $Y = ZX = 0$. Because Z has full column rank then $X = 0$. We deduce that the matrix $Z^T GZ$ is positive definite. □

3.1 Some technical discussions

Considering the KKT system (9), we proved that the Lagrange multipliers solutions of this system are bounded for all barrier parameters μ, in addition we showed that the slack variables s_μ also solutions of the KKT system (9), will not converge dramatically to zero, and are greater than a constant multiplied by μ, otherwise speaking $\|s_\mu\|_\infty \geq m.\mu \quad \forall \mu \geq 0$. The proof is very similar to the one in the next section. In the following the vector norm $\|.\|$ denotes the maximum vector norm $\|.\|_\infty$.

3.1.1 Global convergence

In order to solve the minimization problem with linear constraints (7), we will solve a sequence of barrier problems (9) with a decreasing barrier parameter μ, converging to zero. Let $\varepsilon_{tol} > 0$ be a given error tolerance, and $K > 0$ a given positive constant. For each barrier parameter μ, let the error function E_μ be defined as:

$$E_\mu := max\{\|\nabla E + \nabla c.\lambda\|, \|\lambda + z\|, \|c - s\|, \|SZe - \mu e\|\} \tag{18}$$

The algorithm to solve the minimization problem with linear constraints (7) is described briefly in the following box 1.

Algorithm 1 Interior point algorithm to solve the minimization problem with linear constraints

while $E_0 \geq \varepsilon_{tol}$ **do**
  Update the barrier parameter μ (in order to converge towards zero)
  while $E_\mu \geq K.\mu$ **do**
    Solve the barrier problem (9)
  end while
end while

In the algorithm (1) and for each barrier parameter μ, let $(x_\mu, s_\mu, \lambda_\mu, s_\mu)$ be a solution of each barrier problem, where the error E_μ is supposed to satisfy

$$E_\mu(x_\mu, s_\mu, \lambda_\mu, s_\mu) \leq K.\mu \tag{19}$$

We want to prove the existence of a limit point of the sequence $(x_\mu, s_\mu, \lambda_\mu, s_\mu)$ when μ converges to zero, and that this limit point is the solution of the first-order optimality conditions (8) of the original problem which is the minimization problem (7), equivalent to the minimization problem (6). In order to prove it, we will show that the sequence $(x_\mu, s_\mu, \lambda_\mu, s_\mu)$ is bounded.

Consider two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we say that $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i\ \forall i \leq n$, and $\mathbf{x} < \mathbf{y}$ if $x_i < y_i\ \forall i \leq n$. In the proof we will need a theorem of the alternative, due to Gordan [13], which states the following theorem.
Theorem 2. *Consider a matrix $A_{n\times m}$. Exactly one of the following propositions has a solution:*
1-There exists $x \in \mathbb{R}^m$ such that $A.x > 0$
2-There exists $y \in \mathbb{R}^n$ such that $y^T.A = 0$, $y \geq 0$, $y \neq 0$

In addition, we consider the following assumptions:
Assumption 1.

- *The constraints vector $c : \mathbb{R}^n \to \mathbb{R}^m$ is linear*
- *There exists $\bar{x} \in \mathbb{R}^n$ such that $c(\bar{x}) > 0$*

First x_μ is assumed to be bounded and thus belongs to a compact set Ω. From the inequality (19) we have

$$\|c(x_\mu) - s_\mu\| \leq K.\mu \tag{20}$$

The application c is continuous on the compact Ω, thus from the inequality (20) we deduce that s_μ is bounded.

In the algorithm s_μ and z_μ are chosen to be positive, thanks to the fraction-to-boundary rule. From the inequality (19) we obtain

$$\lambda_\mu \leq K\mu - z_\mu \tag{21}$$

Next we will be inspired by a proof which can be found in [7], by proving that the Lagrange multiplier λ_μ is bounded. Suppose that it is not the case and $\|\lambda_\mu\| \to +\infty$. As was mentioned, $z_\mu \geq 0$, we notice from the inequality (21) that if a component of λ_μ is positive then it is of the order of $K\mu$ and converges to zero. Therefore $\exists N > 0$ such that $\forall n \geq N \Rightarrow e^T.\lambda_{\mu_n} \leq 0$, moreover $d_n := \frac{\lambda_{\mu_n}}{e^T.\lambda_{\mu_n}}$ is bounded. We deduce that there exists a subsequence of d_n also denoted d_n such that

$$d_n \underset{n\to\infty}{\longrightarrow} d^* \text{ with } d^* \geq 0 \tag{22}$$

Otherwise $e^T.d^* = 1$, thus $d^* \geq 0$ and $d^* \neq 0$.

From the error definition (18) and from the inequation (19) we have

$$\left\| \frac{\nabla E(x_{\mu_n})}{e^T.\lambda_{\mu_n}} + \nabla c(x_{\mu_n}).\frac{\lambda_{\mu_n}}{e^T.\lambda_{\mu_n}} \right\| = \frac{1}{|e^T.\lambda_{\mu_n}|} \left\| \nabla E(x_{\mu_n}) + \nabla c(x_{\mu_n}).\lambda_{\mu_n} \right\| \leq \frac{K\mu_n}{|e^T.\lambda_{\mu_n}|} \tag{23}$$

Thus if $n \to \infty$ then

$$\|\nabla c(x^*).d^*\| = 0 \Rightarrow \nabla c(x^*).d^* = 0 \tag{24}$$

Doing the same thing

$$\left\| \frac{z_{\mu_n}}{e^T.\lambda_{\mu_n}} + \frac{\lambda_{\mu_n}}{e^T.\lambda_{\mu_n}} \right\| = \frac{1}{|e^T.\lambda_{\mu_n}|} \left\| z_{\mu_n} + \lambda_{\mu_n} \right\| \leq \frac{K\mu_n}{|e^T.\lambda_{\mu_n}|} \tag{25}$$

As $d_n = \frac{\lambda_{\mu_n}}{|e^T.\lambda_{\mu_n}|} \to d^*$ then $\frac{z_{\mu_n}}{|e^T.\lambda_{\mu_n}|} \to -d^*$.

We have also from the error definition, that for each component i

$$|s^{(i)}_{\mu_n}.z^{(i)}_{\mu_n} - \mu| \leq K.\mu \tag{26}$$

If we multiply the equation (26) by $\frac{1}{|e^T.\lambda_{\mu_n}|}$ and taking $n \to \infty$ we obtain

$$s_i^*.d_i^* = 0 \quad \forall i = 1,\dots,m \tag{27}$$

Let $I = \{i \,|\, s_i^* = 0\}$, this set is not empty because if it is not the case, then using the fact that $d^* \neq 0$ we will have a contradiction with the equation (27). If we take a vector $v \in \mathbb{R}^m$, v_I denotes the vector with the components v_i where $i \in I$. We have $c_I(x^*) = s_I^* = 0$ and from equation (27) if $i \notin I$, $d_i^* = 0$. Thus

$$e_I^T.d_I^* = e^T.d^* = 1 \text{ and } d_I^* \geq 0 \tag{28}$$

We conclude that

$$d_I^* \neq 0 \text{ and } d_I^* \geq 0 \tag{29}$$

Otherwise from the equation (24)

$$\nabla c(x^*)d^* = \nabla c_I(x^*)d_I^* = 0 \tag{30}$$

In conclusion we obtain

$$\begin{cases} (d_I^*)^T (\nabla c_I(x^*))^T = 0 \\ d_I^* \geq 0 \\ d_I^* \neq 0 \end{cases} \tag{31}$$

By the theorem of the alternative, we deduce that there is no vector p such that

$$(\nabla c_I(x^*))^T p > 0 \tag{32}$$

We will prove next that we have a contradiction.

Otherwise, let $c_I^{(i)}$ a component of c_I, we know that $c_I^{(i)}$ is linear, thus

$$c_I^{(i)}(\bar{x}) - c_I^{(i)}(x^*) = (\nabla c_I^{(i)}(x^*))^T.(\bar{x} - x^*) \tag{33}$$

We have $c_I^{(i)}(x^*) = 0$, and by the assumption (1) $c_I^{(i)}(\bar{x}) > 0$. Therefore

$$(\nabla c_I^{(i)}(x^*))^T.(\bar{x} - x^*) > 0 \tag{34}$$

besides

$$(\nabla c_I(x^*))^T.(\bar{x} - x^*) = \begin{bmatrix} \vdots \\ (\nabla c_I^{(i)}(x^*))^T \\ \vdots \end{bmatrix} .(\bar{x} - x^*) = \begin{bmatrix} \vdots \\ (\nabla c_I^{(i)}(x^*))^T.(\bar{x} - x^*) \\ \vdots \end{bmatrix} > 0 \tag{35}$$

which contradicts the theorem of the alternative, and thus λ_μ is bounded. This fact implies that the Lagrange multiplier z_μ is also bounded.

In conclusion $y_\mu := (x_\mu, s_\mu, \lambda_\mu, z_\mu)$ is bounded, and thus belongs to a compact set. Then there exists a subsequence of y_μ converging to $y^* = (x^*, s^*, \lambda^*, z^*)$. Finally using the error definition and the inequality (19), $(x^*, s^*, \lambda^*, z^*)$ satisfies the original KKT system (8)

$$\begin{cases} \nabla E(x^*) + \nabla c(x^*)\lambda^* = 0 \\ z^* = -\lambda^* \\ c(x^*) - s^* = 0 \\ S^* Z^* e = 0 \\ s^* \geq 0 \,,\, z^* \geq 0 \end{cases} \tag{36}$$

4 Alternative formulations

Another symmetric formulations were developed in order to solve the contact problems, more precisely the minimization of the energy with linear constraints. These formulations are a mixture of the penalty method and the interior point method.

4.1 First alternative formulation

Instead of using the minimization problem (7), we will penalize the slack variables to be negative, and we obtain the following problem

$$\begin{cases} \underset{(u,s)\in\mathbb{R}^n\times\mathbb{R}^m}{Min} E(u) + \sum_{i=1}^{m} \mu.\eta(s_i) \quad \text{such that} \\ c(u) - s = 0 \\ s \geq -\varepsilon \end{cases} \tag{37}$$

where μ the penalty factor ($\mu \to \infty$ in this case), $\varepsilon > 0$ and, $\eta : \mathbb{R} \to \mathbb{R}_+$ a C^2 and a convex function with the following property: $\eta(x) > 0\ \forall x < 0$ and $\eta(x) = 0\ \forall x \geq 0$. An example of a penalty function η

$$\eta(x) = \begin{cases} 0 \quad \text{if} \quad x \geq 0 \\ -x^3 \quad \text{if} \quad -1 \leq x < 0 \\ 3x^2 + 3x + 1 \quad \text{if} \quad x \leq -1 \end{cases} \tag{38}$$

In this formulation we ensure that the inequality in the problem (37) will not be activated. Note that the components values of the Hessian matrix due to the penalty terms are of the order $\mu.s_i$, thus near the solution these values are not high.

4.2 Second alternative formulation

This formulation uses the penalty method in order to penalize the penetration between the two bodies, when we consider the first body as slave and the second one as master and vice-versa, in order to have a symmetric formulation. Let $\mathbf{v} \in \mathbf{V}$ be an admissible displacement field, the symmetric formulation is the following

$$\underset{\mathbf{v}\in\mathbf{V}}{min} \left(E(\mathbf{v}) + \mu \int_{\Gamma_C^1} \eta\left((\mathbf{x} - \bar{\mathbf{x}}_2)\mathbf{n}_2\right) ds_1 + \mu \int_{\Gamma_C^2} \eta\left((\mathbf{x} - \bar{\mathbf{x}}_1)\mathbf{n}_1\right) ds_2 \right) \tag{39}$$

where μ the penalty factor, $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ are respectively the projection of the actual position $\mathbf{x}$ on the first body and the second body. $\mathbf{n}_i$ is the outward unit normal vector at $\bar{\mathbf{x}}_i$. Because the penalty terms are equal to zero when there is no contact, a bound on the displacement must be specified in the optimization process, and this bound is not obvious.

5 Contact algorithm

A fixed point method is used in order to transform the contact problem into a sequence of a minimization ones, with linear constraints, especially for the finite deformation problems. Indeed for the linear elastic problems, the constraints are linear by definition. More precisely for each iteration of the fixed point algorithm, the previous displacement is used in order to search for each integration point of the slave contact area, the closest segment (in 2D) or the closest triangle (in 3D) of the master body, and its position on this closest segment or triangle. The integration points have been employed, because the constraints are written in term of integrals. Finally the algorithm to solve the contact problems is briefly described in the following box (2). Note that the resolution of the linear system generated by the interior point method is solved using direct methods, for example the solver MUMPS (MUltifrontal Massively Parallel sparse direct Solver).

Algorithm 2 Symmetric algorithm using the fixed point method

Initialization of the displacement $\mathbf{U}_0 = 0$ and setting the tolerance $TOLER$

while *error* $\geq TOLER$ **do**

1. Using the displacement vector $\mathbf{U}_{n-1}$ of the previous iteration $n-1$:
   Compute the projection points' parameters $\{\eta_i^* \,|\, i = 1,\ldots,nS\}$ of all slave integration points
   Compute the normal at the projection points $\{n_i \,|\, i = 1,\ldots,nS\}$ (Using smoothing techniques)
2. For each integration point, its projection point $\bar{\mathbf{x}}_i$ depends linearly on the actual displacement
3. Reverse the role of the master and the slave bodies
4. Form the Energy and the symmetric linear constraints
5. Use one of the previous methods in order to solve the minimization problem with linear constraints, and to obtain the actual displacement $\mathbf{U}_n$
6. error=$\frac{\|\mathbf{U}_n - \mathbf{U}_{n-1}\|_\infty}{\|\mathbf{U}_{n-1}\|_\infty}$

end while

6 Numerical examples

The algorithm was developed using FreeFEM [1] and IPOPT [5] software, and validated against several test cases.

6.1 disc-in-disc

The following example, which can be found in [8], is considered in order to show that our algorithm can handle large deformations. In this example the contact is between a hollow disc and an inner disc, as shown in the Figure (1), it is a quasi-static study where the inertia is not taken into account. The first formulation was used to consider the contact, and linear finite elements were used. The outer and the inner radius of the hollow disc are respectively $r_{ho} = 2.UL$, $r_{hi} = 0.7UL$, the radius of the inner disc is $r_{in} = 0.6UL$. Neo-Hookean material is considered for the two discs, with the following properties $E_h = 1000.\frac{UF}{UL^2}$, $\nu_h = 0$ for the hollow disc, and $E_i = 2000.\frac{UF}{UL^2}$, $\nu_i = 0.3$ for the inner disc, note that UF, UL denote respectively the force and the length units. We impose a vertical downward displacement at all nodes of the inner disc, the outside of the hollow disc is fixed. The maximal displacement imposed is $1.125UL$ and is done by 100 load increments. The deformation states of the two discs at the steps 0, 50, 100 are depicted in the following figure (Figure (1)).

6.2 Multi-body contact between three beams

In this example we consider the bending of three cantilever beams as shown in the Figure (2), the goal of this example is to show that the contact algorithm can work in three-dimensional space and can handle multi-body contact problems. The first formulation was used to consider the contact, and linear finite elements were used. The dimension of the three beams are the same, indeed the length of each beam is $200.UL$ and each cross section is a square of dimension $25.UL$. The three beams have the same elastic

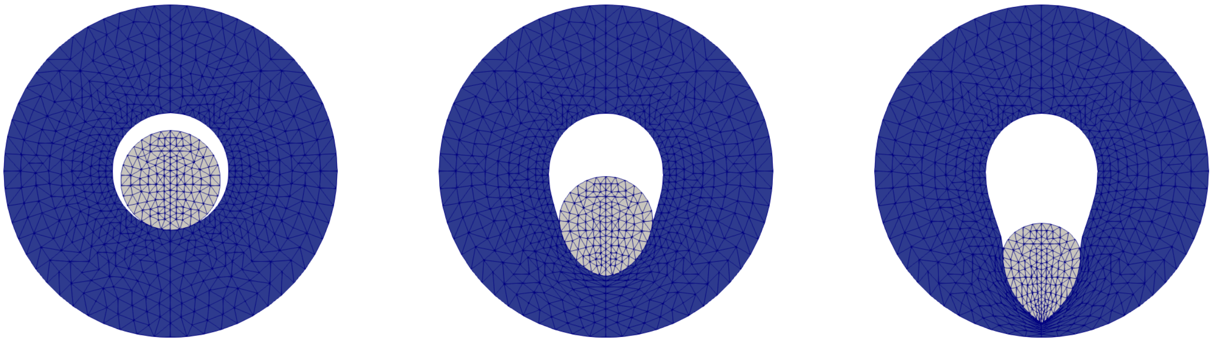

Figure 1: The deformation states at the steps 0, 50, 100

material properties $E = 2.10^5 \frac{UF}{UL^2}$, $\nu = 0$. We apply a downward vertical surface load of $f = 100 \frac{UF}{UL^2}$ at the upper surface of the top beam. We can see in the Figure (2) the deformations and the sliding between the beams.

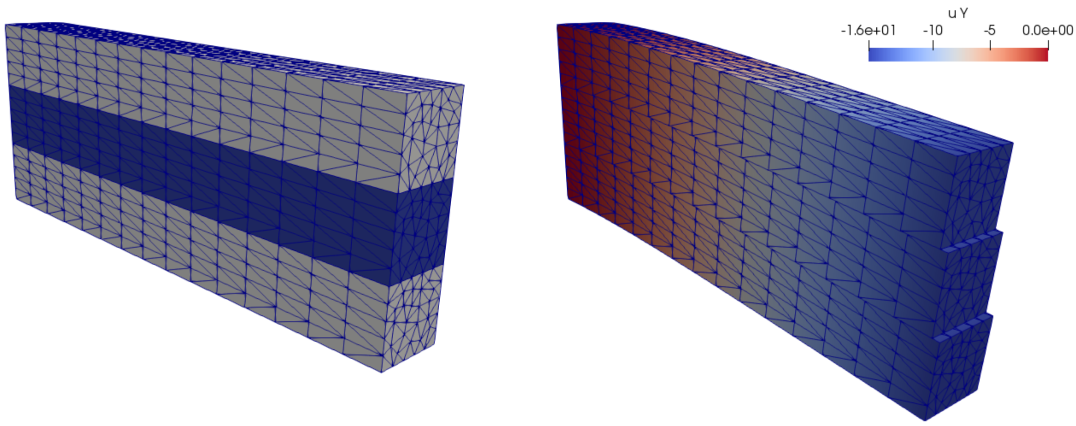

Figure 2: The sliding between beams. The initial configuration (on the left), the deformations and the sliding (on the right)

7 CONCLUSIONS

We developed an algorithm to solve frictionless contact problems, using FreeFEM and IPOPT software. The contact problem is written in the form of a sequence of minimization problems with linear constraints, and each minimization problem is solved using the interior point method. The advantages of this algorithm is its simplicity, in addition the algorithm is symmetric, therefore the user does not distinguish anymore between the slave and the master body, which is very useful for the self-contact and multi-body contact problems.

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