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INFORMATION

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ABSTRACT

The formulation of constrained families of probability distributions is crucial for analyzing data inherently limited to the unit interval, such as proportions, rates, and probabilities. These models have extensive applications across various domains, such as epidemiology, dependability, finance, and environmental research, where adaptable and resilient statistical instruments are essential for accurately representing intricate real-world phenomena. Motivated by this, our work presents the bounded Uma distribution (BUmD), an innovative probability model obtained by transforming the Uma distribution to the unit interval (0, 1). The suggested distribution maintains the flexibility of the Uma family while broadening its applicability to data represented as proportions, probabilities, and rates. We examine the essential statistical characteristics of the BUmD, encompassing its moments, quantile function, extropy, and reliability metrics, with a focus on its capacity to represent various hazard rate behaviors. Sixteen traditional and contemporary estimation techniques are employed to estimate the model parameters, and their efficacy is assessed through comprehensive Monte Carlo simulations using criteria such as bias, mean squared error, and goodness-of-fit measures. The simulation findings indicate that the maximum likelihood and maximum product of spacings estimators are the most efficient. The practical relevance of the BUmD is evidenced by its application to epidemiological and public health datasets, where it demonstrates enhanced flexibility and fitting performance relative to conventional models. The suggested distribution provides a robust and flexible tool for modeling bounded lifetime, reliability, and proportional data in the applied sciences.

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1 Introduction

In all applied sciences, including engineering, medicine, finance, and insurance, lifetime data analysis and modelling are vital. Numerous lifetime distributions have been established in the statistical literature, each possessing distinct advantages and disadvantages. The initial parameter, lifespan distribution, is characterized by an exponential distribution (ED) with a constant hazard rate. In real-life settings, we frequently encounter conditions where the hazard rate is either decreasing, increasing, or occasionally irregular. The Lindley distribution (LinD) is a one-parameter continuous probability distribution introduced by Lindley [1], primarily used for modeling lifetime and survival data. It integrates characteristics of the exponential and gamma distributions, providing enhanced adaptability in modelling data with growing, decreasing, or stable hazard rates. The distribution has various applications in reliability engineering, biological sciences, and survival analysis, owing to its capacity to mimic diverse failure rate forms and its mathematically manageable qualities, including moments, moment generating functions, and hazard functions. Variants and generalizations of the LinD, including multi-parameter extensions, have been developed to improve its fit and applicability to various datasets, thereby offering enhanced modeling flexibility for complex lifespan data situations. Ghitany et al. [2] conducted an extensive study on the LinD and its uses within the reliability study setting. Nevertheless, the LinD isn't always the best fit, whether considered theoretically or in practice. So, A new two-parameter power LinD is introduced, and its properties are discussed by Ghitany et al. [3]. Shanker et al. [4] introduced the expanded LinD, which adjusts the scale and shape parameters to more effectively represent varied lifetime behaviours. Additional generalizations encompass the power LinD (see Nadarajah et al. [5]), which accommodates increasing and bathtub-shaped hazard rates, and the exponentiated LinD (cf. [6]), which incorporates a shape parameter to represent heavy tails. Additional significant extensions include the generalized LinD (see [7,8]), the alpha power LinD (cf. [9]), and various Lindley-generated families (see [10–13]), which incorporate the Lindley law as a foundational distribution within transformation frameworks. In recent years, the classical LinD has been widely generalized to enhance its flexibility in modeling diverse lifetime and reliability data. Several authors have introduced new shape-augmented forms, such as the exponentiated, power Lindley EX-Lindley, ZLindley, and DUS-LinDs, which allow for monotone, bathtub, or unimodal hazard rate shapes.

Numerous writers have devoted much time to researching and writing about the unit LinD and its refinements. After studying the basic features and potential uses of the classical LinD, Mazucheli et al. [14] applied an appropriate transformation to it to create the unit LinD, which they then used to describe proportional data. Improving fits in different datasets, Karakuş et al. [15] investigated the unit power LinD, focusing on its statistical features and estimation techniques. Several authors have made generalizations based on the unit LinD in recent times, as seen in [16–19]. To improve upon previous one-parameter models, such as the exponential and gamma distributions, Shanker [20] created the Uma distribution (UmD), a one-parameter continuous lifetime distribution. Because it is built as a convex mixture of the gamma and EDs, it can capture a greater range of data behaviour. Let X be a continuous random variable (RV) from UmD, then the probability density function (PDF) and cumulative distribution function (CDF) are respectively given by

$$F(x;\zeta) = 1 - \left[1 + \frac{\zeta x\left(\zeta^2 x^2 + 3\zeta x + \zeta^2 + 6\right)}{\zeta^3 + \zeta^2 + 6}\right]\exp\left(-\zeta x\right), \quad x,\zeta > 0, \tag{1}$$

$$f(x;\zeta) = \frac{\zeta^4}{\zeta^3 + \zeta^2 + 6}\left(1 + x + x^3\right)\exp\left(-\zeta x\right). \tag{2}$$

Expanding upon the previously emphasised significance of the LinD and its extensions, this study aims to convert the UmD into its unit form. This transformation seeks to preserve the beneficial characteristics of the UmD while modifying it for data confined to the interval (0, 1), thus broadening its utility in modelling proportions, probabilities, and rates. The UmD has the most special properties of one-parameter lifetime distributions, thus it is suitable for transformation to the unit interval. First, it is cast as a convex combination of gamma and exponential distributions, giving it more flexibility than the exponential or pure gamma while maintaining a single parameter. Normally, we would transform our RV to obtain a classical PDF and CDF on (0, 1) with no new parameters. Gamma or Weibull distributions similarly yield transformations that replace equivalent expressions with integrals of incomplete gamma functions, where $\mathbf{Y} = \frac{X}{1+X}$. Third, the UmD parent is an up-slope and down-slope hazard rate, and even after transformation, the results in BUmD provide bathtub-shaped hazards, which is quite unique to be observed with the one-parameter bounded model used here. This advantage made us choose UmD as the parent.

Motivation of our paper is presented as follows:

- Numerous real-world datasets, including proportions, rates, and probabilities, are inherently confined inside the unit interval (0, 1), necessitating specialized probability models.
- The UmD offers versatility in simulating various hazard rate patterns; however, its unbounded nature restricts its use to proportional data.
- Developing bounded families, such as the proposed BUmD, preserves beneficial properties of the original distribution while extending its applicability to bounded domains.
- Flexible bounded models are particularly pertinent for applications in epidemiology, public health, reliability engineering, and other fields where data is represented as fractions, survival rates, or constrained lifetimes.

The organization of the remaining sections of this study is illustrated in Fig. 1.

Distinctive flexibility of the BUmD: The proposed BUmD stands out from other unit-interval distributions like the unit Lindley, unit Weibull, or beta distribution because it has a unique set of properties that work together in a single-parameter framework. Depending on ζ, the hazard rate function might be flat, growing, or decreasing. Some models, such as unit Burr-XII or exponentiated unit distributions, require two or more parameters for this flexibility. The probability density function of the BUmD can be left-skewed, near-symmetric, or right-skewed, depending on the value of ζ. This makes it valuable for epidemiology, reliability, and environmental investigations using proportional data. Third, the move from the UmD permits bounded support and easy moments with confluent hypergeometric functions. These traits set the BUmD apart from contemporary limited models and make it a competitive tool for unit-interval data processing.

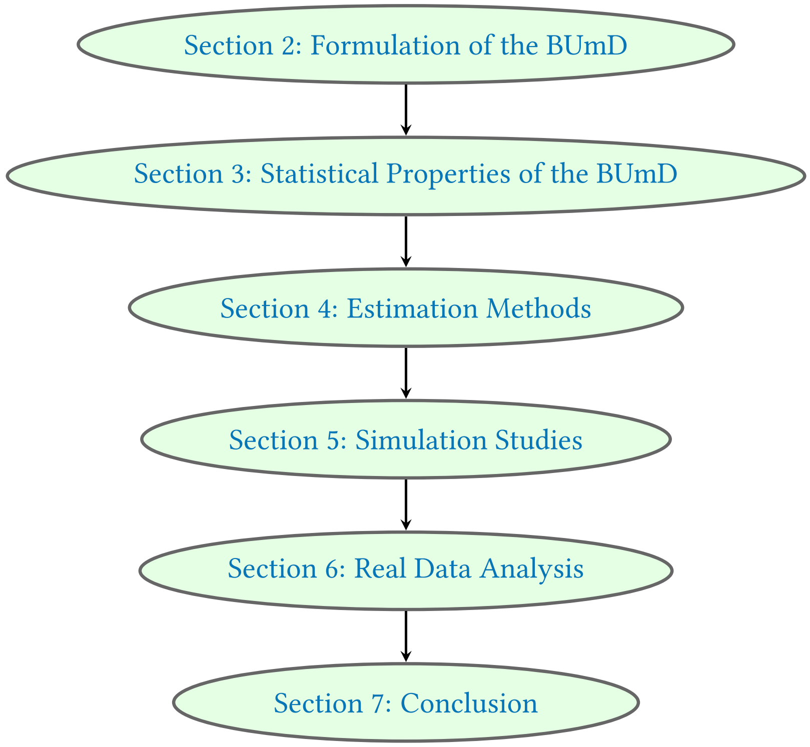

Figure 1: Flowchart of the remaining sections of the study

2 Model Formulation

To extend the application of the UmD, first defined on the positive real line $(0, \infty)$, we transform it into a bounded form on the unit interval $(0, 1)$. This is achieved using the scaled transformation $Y = \frac{X}{1+X}$, where X conforms to the conventional UmD. Utilizing the change of variables technique, the corresponding PDF (2) and CDF (1) of Y can be explicitly obtained, yielding what is referred to as the bounded UmD (BUmD). This restricted variant is particularly beneficial for modeling proportions, rates, and other phenomena inherently confined to the unit interval, while maintaining the flexibility of the original Uma family. The PDF of the BUmD is provided by

$$f(x;\zeta) = \frac{\zeta^4 \left(x^3 + x^2 - 2x + 1\right) \exp\left(-\frac{\zeta x}{1-x}\right)}{\left(\zeta^3 + \zeta^2 + 6\right)(1-x)^5}, \quad 0 < x < 1, \zeta > 0. \tag{3}$$

Also, The CDF is

$$F(x;\zeta) = 1 - \left(\frac{\left(\zeta^2 + \zeta^2 \left(\frac{x}{1-x}\right)^2 + \frac{3\zeta x}{1-x} + 6\right)\zeta x}{\left(\zeta^3 + \zeta^2 + 6\right)(1-x)} + 1\right) \exp\left(-\frac{\zeta x}{1-x}\right). \tag{4}$$

The survival function is given by

$$S(x) = 1 - F(x) = \left(1 + \frac{\frac{\zeta x}{1-x}\left[\zeta^2\left(\frac{x}{1-x}\right)^2 + 3\zeta\left(\frac{x}{1-x}\right) + \zeta^2 + 6\right]}{\zeta^3 + \zeta^2 + 6}\right)\exp\left(-\frac{\zeta x}{1-x}\right). \tag{5}$$

The hazard rate is defined by $h(x) = f(x)/S(x)$ as

$$h(x) = \frac{\zeta^4\left(1 - 2x + x^2 + x^3\right)}{(1-x)^5\left[\zeta^3 + \zeta^2 + 6 + \frac{\zeta x}{1-x}\left(\zeta^2\left(\frac{x}{1-x}\right)^2 + 3\zeta\left(\frac{x}{1-x}\right) + \zeta^2 + 6\right)\right]}. \tag{6}$$

Identifiability: The BUmD is identifiable because $F(x;\zeta)$ in (4) is strictly monotonic in ζ for any fixed $x \in (0,1)$:

$$\frac{\partial F}{\partial \zeta} = \frac{x}{1-x} \cdot \frac{e^{-\frac{\zeta x}{1-x}}}{(\zeta^3 + \zeta^2 + 6)^2} \cdot P(\zeta, x) > 0,$$

where $P(\zeta, x)$ is positive. Hence $\zeta_1 \neq \zeta_2$ implies different distributions.

Limiting behavior: As $\zeta \to 0^+$, $F(x;\zeta) \to 0$ for any $x < 1$ (degeneracy at $x = 1$). As $\zeta \to \infty$, $F(x;\zeta) \to 1$ for any $x > 0$ (degeneracy at $x = 0$). Thus ζ controls concentration near the boundaries.

Fig. 2 displays the PDF graphs for various ζ values, illustrating the flexibility of the BUmD. Larger values of ζ result in more symmetric forms with modest dispersion, in contrast to tiny values of ζ, which exhibit a severely left-skewed distribution centered around the upper bound. This behavior exemplifies how well the proposed distribution can handle non-standard data patterns on the unit interval. The CDF curves show, in Fig. 3, how the probability grows over the unit interval for different ζ values. The CDF descends and gets steeper as ζ rises, suggesting a larger concentration of probability towards the domain's lower half. This proves that the BUmD is good for modeling limited proportions and rates and that it is sensitive to changes in the parameters. Depending on the value of ζ, the hazard rate graphs in Fig. 4 can take on a number of shapes. The hazard rate is initially high and decreases when ζ is small, but it becomes more stable or increasing as ζ grows larger. These patterns demonstrate that the BUmD can accurately predict various survival and reliability scenarios, including decreasing, increasing, and bathtub-shaped hazard rates.

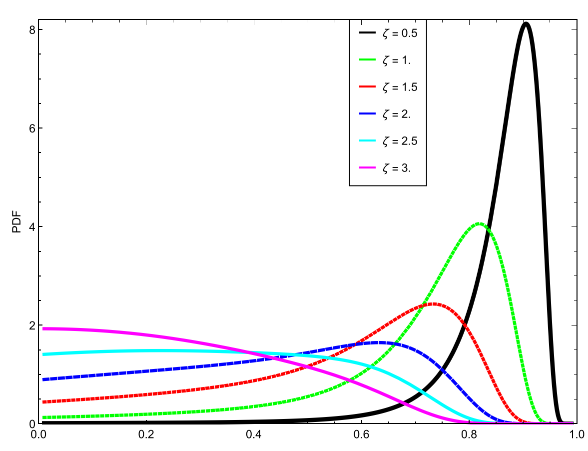

Figure 2: The PDF curves of the BUmD based on parameter selections

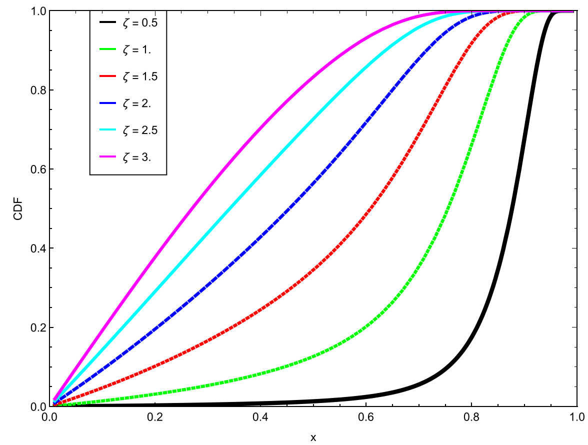

Figure 3: The CDF curves of the BUmD based on parameter selections

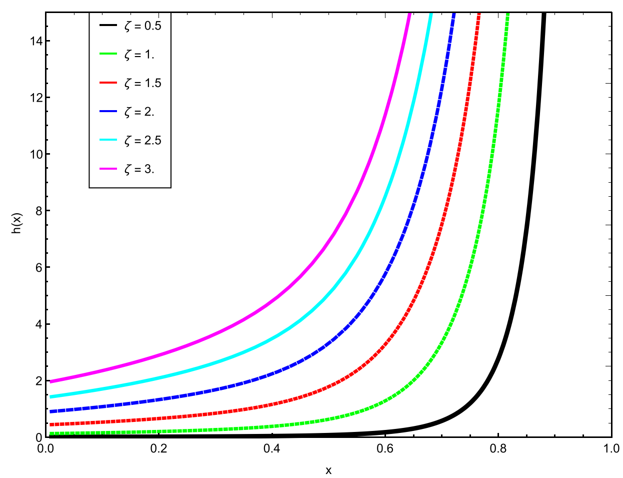

Figure 4: The hazard curves of the BUmD based on parameter selections

3 Statistical Properties

The statistical features of a probability distribution are crucial for understanding its behavior and practical utility. The suggested BUmD must derive and analyze such attributes to demonstrate its flexibility and utility in modeling bounded data. Moments, variance, skewness, and kurtosis reveal the distribution's central tendency, dispersion, asymmetry, and tail behavior. Quantile functions, reliability measures, and information-theoretic indices, such as entropy, further characterize the distribution's structure and expand its applications in reliability analysis, epidemiology, and risk assessment. This section details these features, laying the groundwork for BUmD estimation, modeling, and real data applications.

3.1 Moments of the Bounded Uma Distribution

Moments play an essential role in probability theory and statistical inference by providing critical insights into the characteristics of a distribution. Moments are primarily employed to characterize measurements of central tendency, dispersion, skewness, and kurtosis. The k-th moment about the origin of the BUmD can be articulated in closed form using the integration of its PDF. These moments not only embody the distribution's attributes but also serve as fundamental components for numerous statistical applications, including parameter estimation, reliability analysis, and model fitting. We shall systematically derive the formula for the k-th instant. The k-th moment for BUmD is defined as follows:

$$\mathbb{E}[X^k] = \int_0^1 x^k f(x;\zeta)\,dx.$$

Let $t = \dfrac{x}{1-x}$. Then

$$\begin{aligned}
\mathbb{E}[X^k] &= \frac{\zeta^4}{\zeta^3+\zeta^2+6}\int_0^1 x^k \frac{1-2x+x^2+x^3}{(1-x)^5}\exp\Big(-\frac{\zeta x}{1-x}\Big)\,dx \\
&= \frac{\zeta^4}{\zeta^3+\zeta^2+6}\int_0^\infty \Big(\frac{t}{1+t}\Big)^k \big(t^3+t+1\big)\,(1+t)^2 e^{-\zeta t}\,\frac{dt}{(1+t)^2} \\
&= \frac{\zeta^4}{\zeta^3+\zeta^2+6}\int_0^\infty t^k\,(t^3+t+1)\,(1+t)^{-k}\,e^{-\zeta t}\,dt.
\end{aligned}$$

Thus

$$\mathbb{E}[X^k] = \frac{\zeta^4}{\zeta^3+\zeta^2+6}\int_0^\infty t^k(t^3+t+1)\,(1+t)^{-k}e^{-\zeta t}\,dt.$$

Now use the integral representation of the Tricomi (confluent hypergeometric) function $U(a,c,z)$:

$$U(a,c,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}\,t^{a-1}\,(1+t)^{c-a-1}\,dt, \qquad \Re(a) > 0,\ \Re(z) > 0.$$

Therefore

$$\mathbb{E}[X^k] = \frac{\zeta^4}{\zeta^3+\zeta^2+6}\Big[\Gamma(k+4)\,U(k+4,5,\zeta) + \Gamma(k+2)\,U(k+2,3,\zeta) + \Gamma(k+1)\,U(k+1,2,\zeta)\Big].$$

To calculate the mean and variance, the initial two moments are:

$$\mathbb{E}[X] = \frac{\zeta^4}{\zeta^3+\zeta^2+6}\big[\Gamma(5)U(5,5,\zeta) + \Gamma(3)U(3,3,\zeta) + \Gamma(2)U(2,2,\zeta)\big],$$

$$\mathbb{E}[X^2] = \frac{\zeta^4}{\zeta^3+\zeta^2+6}\big[\Gamma(6)U(6,5,\zeta) + \Gamma(4)U(4,3,\zeta) + \Gamma(3)U(3,2,\zeta)\big].$$

Then $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. The functions $U(a,c,\zeta)$ are available in Mathematica and Maple and can be readily computed for numeric values of ζ and integer k. Higher-order moments offer profound insights into the structure and variability of a probability distribution. The mean and variance encapsulate central tendency and dispersion, respectively, while additional moment-based measures, such as skewness and kurtosis, assess the asymmetry and peakedness of the distribution. The coefficient of variation (CV) is defined as the ratio of the standard deviation to the mean:

$$CV = \frac{\sigma}{\mu},$$

where $\mu = E[X]$ and $\sigma^2 = Var(X)$. The skewness coefficient measures the asymmetry of the distribution and is given by:

$$\gamma_1 = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{\mu_3 - 3\mu\mu_2 + 2\mu^3}{\sigma^3},$$

where $\mu_r = E[X^r]$ denotes the r-th raw moment. The kurtosis coefficient measures the degree of peakedness or flatness relative to the normal distribution, and is expressed as:

$$\gamma_2 = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{\mu_4 - 4\mu\mu_3 + 6\mu^2\mu_2 - 3\mu^4}{\sigma^4}.$$

Table 1 displays the numerical values of the initial four raw moments, variance, skewness (γ_1), kurtosis (γ_2), and CV of the BUmD for various values of the parameter ζ. Several important observations can be made:

- The mean μ diminishes as ζ escalates, signifying that the distribution transitions towards lesser values with an increase in the parameter.
- The variance σ^2 first rises with ζ before stabilizing, whereas the coefficient of variation has a distinct upward trend. This indicates that the relative dispersion of the distribution increases as ζ rises.
- The skewness γ_1 is significantly negative at low values of ζ, indicating pronounced left skewness. As ζ rises, the skewness progressively escalates and ultimately becomes positive, signifying a shift from left-skewed to right-skewed distributions.
- The kurtosis γ_2 is significantly elevated at minimal values of ζ, indicating a sharply peaked distribution. As ζ increases, the kurtosis diminishes and approaches values nearer to those of a normal distribution, however remains over 3, indicating leptokurtic behavior.
- The parameter ζ significantly affects the configuration of the BUmD. Low values of ζ generate distributions characterized by concentration, heavy tails, and pronounced negative skewness, whereas higher values of ζ result in more symmetric and moderately scattered distributions.

Table 1: Numerical values of moments, variance, skewness (γ_1), kurtosis (γ_2), and coefficient of variation (CV) for the BUmD with different parameter values of ζ

ζ	μ	μ_2	μ_3	μ_4	σ^2	γ_1	γ_2	CV
0.1	0.968092	0.937636	0.90852	0.880652	0.000433557	−4.13299	56.074	0.0215083
0.2	0.938226	0.881814	0.83000	0.78223	0.00154692	−4.00327	42.8641	0.0419205
0.3	0.909417	0.830356	0.760442	0.69816	0.00331682	−3.82433	33.2194	0.0633283
0.4	0.880917	0.781825	0.697393	0.624587	0.00581066	−3.5371	25.319	0.0865323
0.5	0.852185	0.735284	0.639294	0.55908	0.00906357	−3.18934	19.147	0.111716
0.6	0.822886	0.690174	0.585173	0.50008	0.0130317	−2.82896	14.5208	0.138727
0.7	0.792875	0.646233	0.53447	0.446587	0.0175816	−2.4847	11.126	0.167234
0.8	0.762183	0.603427	0.486903	0.397963	0.0225041	−2.16946	8.65158	0.196821
0.9	0.730982	0.561879	0.442368	0.353796	0.0275451	−1.88679	6.84509	0.227047
1.	0.699543	0.521804	0.400857	0.313797	0.0324429	−1.6356	5.51898	0.257481
1.1	0.668196	0.48345	0.362396	0.277736	0.0369636	−1.41291	4.53905	0.287729
1.2	0.637279	0.447051	0.326997	0.245399	0.0409256	−1.2152	3.8107	0.317444

(Continued)

Table 1 (continued)

ζ	μ	μ_2	μ_3	μ_4	σ^2	γ_1	γ_2	CV
1.3	0.607111	0.412797	0.294637	0.216558	0.0442124	−1.03909	3.26726	0.346341
1.4	0.577963	0.380814	0.265244	0.190971	0.0467723	−0.881571	2.86143	0.374192
1.5	0.550047	0.351161	0.238698	0.168375	0.0486089	−0.740042	2.55932	0.400828
1.6	0.523513	0.323834	0.214842	0.148501	0.0497681	−0.612325	2.33632	0.426136
1.7	0.498452	0.298777	0.193487	0.131076	0.0503227	−0.496596	2.17429	0.450048
1.8	0.474902	0.275891	0.17443	0.115833	0.0503595	−0.391334	2.05968	0.472538
1.9	0.45286	0.255052	0.157463	0.102521	0.049969	−0.295264	1.98223	0.493612
2.	0.432292	0.236115	0.142376	0.0909038	0.0492381	−0.20731	1.93407	0.513303
2.2	0.395328	0.203345	0.117064	0.0719238	0.0470607	−0.0522434	1.90241	0.548746
2.4	0.363401	0.176394	0.0970672	0.0574423	0.0443335	0.0797091	1.92971	0.579402
2.6	0.335825	0.154179	0.0812093	0.0463342	0.0414009	0.19294	1.99372	0.605887
2.8	0.311941	0.135783	0.0685557	0.0377506	0.0384758	0.290841	2.08000	0.628813
3.	0.291161	0.120453	0.0583824	0.0310603	0.0356785	0.376088	2.17904	0.648738
3.5	0.24969	0.0918952	0.0404802	0.0198517	0.0295499	0.54691	2.44650	0.688456
4.	0.218889	0.0726147	0.0293287	0.0133437	0.0247021	0.674838	2.70882	0.71803
4.5	0.195184	0.0590161	0.0220283	0.00935362	0.0209194	0.774401	2.95041	0.74102
5.	0.176371	0.0490542	0.0170374	0.00678778	0.0179475	0.854621	3.16893	0.759584

3.2 Quantile Function

The quantile function is crucial in statistical modeling, simulation, and descriptive analysis, since it enables the determination of threshold values associated with particular cumulative probabilities. To get the p-th quantile $Q(p;\zeta)$ for $p \in (0,1)$ in the BUmD, we execute the conventional rational change of variable

$$t = \frac{x}{1-x}.$$

Define

$$D(\zeta) = \zeta^3 + \zeta^2 + 6, \qquad B(t;\zeta) = \frac{\zeta\, t\left(\zeta^2 t^2 + 3\zeta t + \zeta^2 + 6\right)}{D(\zeta)}.$$

With this notation the CDF equation $F(x) = p$ is equivalent to the scalar nonlinear equation in t:

$$\left(1 + B(t;\zeta)\right) e^{-\zeta t} = 1 - p. \tag{7}$$

In practice, assuming a constant ζ and probability level p, this equation can be resolved using iterative root-finding techniques, such as the Newton-Raphson method or the bisection method. We utilized Mathematica's built-in solver FindRoot to derive numerical solutions for various values of p and ζ. The resultant quantile estimates are encapsulated in Table 2. The Table 2 illustrates how the quantiles decrease as ζ increases, reflecting the effect of ζ on the distribution's concentration.

Table 2: Quantile values of the bounded Uma distribution

	p				
ζ	0.1	0.25	0.5	0.75	0.9
0.5	0.7550	0.8267	0.8766	0.9092	0.9295
1	0.4455	0.6407	0.7553	0.8221	0.8620
2	0.1068	0.2500	0.4544	0.6196	0.7145
3	0.0519	0.1308	0.2700	0.4336	0.5641

3.3 Extropy of the Bounded Uma distribution

Lad et al. [21] showed that extropy serves as a counterpart to entropy, providing a measure of "order" or informational richness that complements the conventional focus on "disorder" measured by entropy. Unlike entropy, which quantifies uncertainty, extropy denotes a related yet different characteristic of structure and randomness within probability distributions. For a continuous RV with density $f(x)$, the extropy is

$$J(X) = -\frac{1}{2}\int_0^1 f(x)^2 dx.$$

For the BUmD PDF given in (3), let $D(\zeta) = \zeta^3 + \zeta^2 + 6$ and $C = \frac{\zeta^4}{D(\zeta)}$. Using the substitution $x = \frac{t}{1+t}$ (so that $dx = \frac{dt}{(1+t)^2}$), we obtain:

$$J(X) = -\frac{1}{2}C^2\int_0^\infty (t^3 + t + 1)^2(1+t)^2 e^{-2\zeta t} dt.$$

Expanding the polynomial $(t^3 + t + 1)^2(1+t)^2$:

$$(t^3 + t + 1)^2 = t^6 + 2t^4 + 2t^3 + t^2 + 2t + 1,$$

$$(1+t)^2 = 1 + 2t + t^2.$$

Multiplying:

$$(t^3 + t + 1)^2(1+t)^2 = t^8 + 2t^7 + 3t^6 + 4t^5 + 5t^4 + 4t^3 + 3t^2 + 2t + 1.$$

Thus we can write:

$$(t^3 + t + 1)^2(1+t)^2 = \sum_{m=0}^{8} a_m t^m,$$

with explicit coefficients:

$$a_0 = 1,\ a_1 = 2,\ a_2 = 3,\ a_3 = 4,\ a_4 = 5,\ a_5 = 4,\ a_6 = 3,\ a_7 = 2,\ a_8 = 1.$$

Since

$$\int_0^\infty t^m e^{-2\zeta t} dt = \frac{m!}{(2\zeta)^{m+1}},$$

We obtain:

$$J(X) = -\frac{1}{2}\left(\frac{\zeta^4}{D(\zeta)}\right)^2 \sum_{m=0}^{8} a_m \frac{m!}{(2\zeta)^{m+1}}.$$

Substituting the coefficients a_m and simplifying yields:

$$J(X) = -\frac{4\zeta^8 + 8\zeta^7 + 12\zeta^6 + 18\zeta^5 + 42\zeta^4 + 90\zeta^3 + 135\zeta^2 + 315\zeta + 630}{16\zeta\,(\zeta^3 + \zeta^2 + 6)^2},$$

which is always negative, as expected from the definition of extropy. Extropy and entropy work together to give a fuller understanding of how ζ affects both its structure and its uncertainty. Proportional statistics often group together near 0 or 1. Extropy is more responsive to this type of border mass than entropy, facilitating the identification of extreme-value patterns. The BUmD model is different from many other bounded models because it delivers a straightforward closed-form extropy that doesn't need numerical integration to be derived.

4 Estimation Methods

In this section, sixteen different methods are considered for estimating the parameter of the BUmD.

4.1 Method of Maximum Likelihood

Let $x_1, \ldots, x_n$ be a random sample from the BUmD with density given in (3). The log-likelihood function is

$$\begin{aligned}
\ell(\zeta) &= \sum_{i=1}^{n} \log f(x_i;\zeta) \\
&= \sum_{i=1}^{n}\left[4\log\zeta + \log(x_i^3 + x_i^2 - 2x_i + 1) - \frac{\zeta x_i}{1-x_i} - 5\log(1-x_i) - \log(\zeta^3 + \zeta^2 + 6)\right] \\
&= 4n\log\zeta - n\log(\zeta^3 + \zeta^2 + 6) - \zeta\sum_{i=1}^{n}\frac{x_i}{1-x_i} + \sum_{i=1}^{n}\log(x_i^3 + x_i^2 - 2x_i + 1) - 5\sum_{i=1}^{n}\log(1-x_i).
\end{aligned}$$

Differentiating with respect to ζ, the score function is

$$\frac{\partial\ell}{\partial\zeta} = \frac{4n}{\zeta} - \frac{n(3\zeta^2 + 2\zeta)}{\zeta^3 + \zeta^2 + 6} - \sum_{i=1}^{n}\frac{x_i}{1-x_i}.$$

Setting this equal to zero yields the likelihood equation:

$$\frac{4n}{\zeta} - \frac{n(3\zeta^2 + 2\zeta)}{\zeta^3 + \zeta^2 + 6} - \sum_{i=1}^{n}\frac{x_i}{1-x_i} = 0.$$

Thus, the maximum likelihood estimator $\widehat{\zeta}$ is the positive solution of the equation above. Since no closed-form solution is available, numerical methods such as Newton-Raphson are required. Several authors have employed the Newton-Raphson method to obtain parameter estimates for lifetime distributions.

4.2 Method of Anderson-Darling (AD)

The parameter of the BUmD is estimated using the AD method, obtained by minimizing the following function:

$$
\begin{aligned}
A(x_i) &= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S(x_{n-i-1:n})\right] \\
&= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\Bigg[\log\left(1 - \left(\frac{\left((\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right) \\
&\quad + \log\left(\left(\frac{\left((\zeta^2+\zeta^2\left(\frac{x_{n-i-1:n}}{1-x_{n-i-1:n}}\right)^2 + \frac{3\zeta x_{n-i-1:n}}{1-x_{n-i-1:n}} + 6\right)\zeta x_{n-i-1:n}}{(\zeta^3+\zeta^2+6)(1-x_{n-i-1:n})} + 1\right)\exp\left(-\frac{\zeta x_{n-i-1:n}}{1-x_{n-i-1:n}}\right)\right)\Bigg].
\end{aligned}
$$

4.3 Method of Cramer_von_Mises (CVM)

The parameter of the BUmD is estimated using the CVM method, obtained by minimizing the following function:

$$
\begin{aligned}
C(x_i) &= \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 \\
&= \frac{1}{12n} + \sum_{i=1}^{n}\left[\left(1 - \left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{\left(\zeta^3+\zeta^2+6\right)(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right) - \frac{2i-1}{2n}\right]^2.
\end{aligned}
$$

4.4 Method of Maximum Product of Spacings (MPS)

The parameter of the BUmD is estimated using the MPS method, obtained by optimizing the following function:

$$
\begin{aligned}
\delta(x_i) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log I_i(x_i), \\
I_i(x_i) &= F(x_{i:n}) - F(x_{i-1:n}) \\
&= \left[1 - \left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right] \\
&\quad - \left[1 - \left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i-1:n}}{1-x_{i-1:n}}\right)^2 + \frac{3\zeta x_{i-1:n}}{1-x_{i-1:n}} + 6\right)\zeta x_{i-1:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i-1:n})} + 1\right)\exp\left(-\frac{\zeta x_{i-1:n}}{1-x_{i-1:n}}\right)\right] \qquad (8)
\end{aligned}
$$

$F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

4.5 Method of Least Squares (LS)

The parameter of the BUmD is estimated using the LS method, obtained by minimizing the following function:

$$V(x_i) = \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n}\left[\left(1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right)-\frac{i}{n+1}\right]^2.$$

4.6 Method of Percentile (P)

The parameter of the BUmD is estimated using the percentile method, obtained by minimizing the following function:

$$PCE = \sum_{i=1}^{n}[x_{i:n} - Q(p_i)]^2, \sim p_i = \frac{i}{n+1}.$$

4.7 Method of Right_Tail Anderson_Darling (RTAD)

The parameter of the BUmD is estimated using the RTAD method, obtained by minimizing the following function:

$$R(x_i) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{i:n})$$

$$= \frac{n}{2} - 2\sum_{i=1}^{n}\left[1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right]$$

$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right].$$

4.8 Method of Weighted Least Squares (WLS)

The parameter of the BUmD is estimated using the WLS method, obtained by minimizing the following function:

$$W(x_i) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left[\left(1 - \left(\frac{\left(\zeta^2 + \zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right) - \frac{i}{n+1}\right]^2.$$

4.9 Method of Left Tail Anderson Darling (LTAD)

The parameter of the BUmD is estimated using the LTAD method, obtained by minimizing the following function:

$$L(x_i) = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n})$$

$$= -\frac{3}{2}n + 2\sum_{i=1}^{n}\left[1 - \left(\frac{\left(\zeta^2 + \zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right]$$

$$- \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left[1 - \left(\frac{\left(\zeta^2 + \zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2 + \frac{3\zeta x_{i:n}}{1-x_{i:n}} + 6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})} + 1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right].$$

4.10 Method of Minimum Spacing Absolute Distance (MSAD)

The parameter of the BUmD is estimated using the MSAD method, obtained by minimizing the following function:

$$\zeta\,(x_i) = \sum_{i=1}^{n+1} |I_i(x_i) - \frac{1}{n+1}|.$$

4.11 Method of Minimum Spacing Absolute-Log Distance (MSALD)

The parameter of the BUmD is estimated using the MSALD method, obtained by minimizing the following function:

$$\Upsilon\,(x_i) = \sum_{i=1}^{n+1} |\log I_i(x_i) - \log\frac{1}{n+1}|.$$

4.12 Method of Anderson Darling Left Tail Second Order (ADLTSO)

The parameter of the BUmD is estimated using the ADLTSO method, obtained by minimizing the following function:

$$LTS = 2\sum_{i=1}^{n}\log F(x_{i:n}) + \frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{F(x_{i:n})}$$

$$= 2\sum_{i=1}^{n}\log\left[1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right]$$

$$+\frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{\left[1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)\,(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right]}.$$

4.13 *Kolmogorov (K) Method*

The parameter of the BUmD is estimated using the K method, obtained by optimizing the following function:

$$KM = \max_{1\le i\le n}\left[\frac{i}{n}-F(x_{i:n}),\ F(x_{i:n})-\frac{i-1}{n}\right]$$

$$= \max_{1\le i\le n}\left[\frac{i}{n}-\left(1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right),\right.$$

$$\left.\left(1-\left(\frac{\left(\zeta^2+\zeta^2\left(\frac{x_{i:n}}{1-x_{i:n}}\right)^2+\frac{3\zeta x_{i:n}}{1-x_{i:n}}+6\right)\zeta x_{i:n}}{(\zeta^3+\zeta^2+6)(1-x_{i:n})}+1\right)\exp\left(-\frac{\zeta x_{i:n}}{1-x_{i:n}}\right)\right)-\frac{i-1}{n}\right].$$

4.14 *Method of Minimum Spacing Square Distance (MSSD)*

The parameter of the BUmD is estimated using the MSSD method, obtained by minimizing the following function:

$$\phi_{(x_i)} = \sum_{i=1}^{n+1}(I_i(x_i)-\frac{1}{n+1})^2.$$

4.15 *Method of Minimum Spacing Square-Log Distance (MSSLD)*

The parameter of the BUmD is estimated using the MSSLD method, obtained by minimizing the following function:

$$\Psi_{(x_i)} = \sum_{i=1}^{n+1}\left(\log I_i(x_i)-\log\frac{1}{n+1}\right)^2.$$

4.16 Method of Minimum Spacing Linex Distance (MSLD)

The parameter of the BUmD is estimated using the MSLD method, obtained by minimizing the following function:

$$\Delta_{(x_i)} = \sum_{i=1}^{n+1}\left[e^{I_i(x_i)-\frac{1}{n+1}} - \left(I_i(x_i) - \frac{1}{n+1}\right) - 1\right].$$

5 Numerical Simulation

Simulation is a powerful technique in statistics used to generate data that studies the behavior of a specific probability distribution. Simulation studies are especially useful for new or complex distributions as they enable statisticians to understand the practical implications and performance of the distribution. This part evaluates the parameter estimate methodologies outlined in the estimates section and examines their efficacy through meticulously organized experiments. We produced datasets aligned with the theoretical framework outlined in the preceding section to assess these methodologies. This enabled us to evaluate the estimators under the previously defined parameter conditions. We generated numerous datasets that conform to the structural and distributional assumptions of our model through Monte Carlo simulations. To assess the various estimation approaches, we employed a comprehensive evaluation framework based on six essential performance criteria. The selected metrics are intended to capture the fundamental attributes utilized to assess the quality of an estimator. These attributes encompass

- Average bias (BIAS), $|Bias(\widehat{\boldsymbol{T}})| = \dfrac{1}{L}\sum_{i=1}^{L}|\widehat{\boldsymbol{T}} - \boldsymbol{T}|$.
- Mean squared errors (MSE), $MSE = \dfrac{1}{L}\sum_{i=1}^{L}(\widehat{\boldsymbol{T}} - \boldsymbol{T})^2$.
- Mean relative errors (MRE), $MRE = \dfrac{1}{L}\sum_{i=1}^{L}|\widehat{\boldsymbol{T}} - \boldsymbol{T}|/\boldsymbol{T}$.
- Average absolute difference (D_{abs}), $D_{abs} = \dfrac{1}{nH}\sum_{i=1}^{H}\sum_{j=1}^{n}|F(x_{ij};\boldsymbol{T}) - F(x_{ij};\widehat{\boldsymbol{T}})|$.
- Maximum absolute difference (D_{max}), $D_{max} = \dfrac{1}{H}\sum_{i=1}^{H}\max\limits_{j=1,\dots,n}|F(x_{ij};\boldsymbol{T}) - F(x_{ij};\widehat{\boldsymbol{T}})|$.
- Average squared absolute error, $ASAE = \frac{1}{n}\sum_{i=1}^{n}\frac{|x_i-\hat{x}_i|}{x_n-x_1}$.

This work aimed to identify the most efficient parameter estimation approach for our theoretical model through a complete performance evaluation utilizing a Monte Carlo simulation. This simulation involved generating multiple randomized samples, collecting extensive data, and utilizing analytical methods. To establish statistical reliability and draw robust conclusions about the estimator's performance, it was imperative to conduct a substantial number of simulated replications. The results of our comprehensive simulation experiments have been thoroughly recorded in a series of tables, identified as Tables 3–7. To improve interpretability and facilitate pattern recognition, we created detailed visual representations of the numerical data presented in Table 3, illustrated as heat map visualizations in Figs. 5–10. A systematic ranking approach was employed to convert diverse performance outcomes into actionable information. Table 8 displays comprehensive rankings that consider all evaluative elements. Furthermore, it provides partial rankings derived from certain performance factors. From Table 8, we conclude that the most successful strategy for determining the estimators of our proposed model is the MLE, followed by the MPSE.

Table 3: Numerical values of simulation measures for $\zeta = 0.4$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS ($\hat{\zeta}$)	0.03517 {9}	0.03344 {5}	0.03478 {7}	0.03171 {1}	0.0339 {6}	0.03321 {3}	0.03228 {2}	0.03709 {11}	0.03871 {12}	0.03504 {8}	0.04878 {15}	0.03333 {4}	0.04587 {14}	0.03698 {10}	0.04438 {13}
25	MSE ($\hat{\zeta}$)	0.00205 {9}	0.00177 {5}	0.00189 {7}	0.00159 {1}	0.00181 {6}	0.00174 {4}	0.00172 {3}	0.00226 {11}	0.00257 {12}	0.00198 {8}	0.00383 {15}	0.00171 {2}	0.00338 {14}	0.00218 {10}	0.0032 {13}
25	MRE ($\hat{\zeta}$)	0.08792 {9}	0.08359 {5}	0.08695 {7}	0.07927 {1}	0.08475 {6}	0.08302 {3}	0.08071 {2}	0.09272 {11}	0.09678 {12}	0.08761 {8}	0.12194 {15}	0.08333 {4}	0.11468 {14}	0.09244 {10}	0.11095 {13}
25	D_{abs}	0.04588 {3}	0.04607 {4}	0.04805 {8}	0.04444 {1}	0.04689 {7}	0.04623 {5}	0.0445 {2}	0.05041 {10}	0.05368 {12}	0.04916 {9}	0.06361 {15}	0.04657 {6}	0.06319 {14}	0.05097 {11}	0.06196 {13}
25	D_{max}	0.06485 {3}	0.06563 {5}	0.06819 {8}	0.06257 {1}	0.06614 {7}	0.06581 {6}	0.06296 {2}	0.07155 {10}	0.07629 {12}	0.06915 {9}	0.09134 {15}	0.06555 {4}	0.09012 {14}	0.07229 {11}	0.08819 {13}
25	ASAE	0.04771 {7}	0.04529 {2}	0.04695 {6}	0.0479 {8}	0.0468 {5}	0.04882 {9}	0.04598 {3}	0.04402 {1}	0.05223 {11}	0.05272 {13}	0.04896 {10}	0.04599 {4}	0.05672 {15}	0.05237 {12}	0.05634 {14}
25	$\sum Ranks$	40 {7}	26 {4}	43 {8}	13 {1}	37 {6}	30 {5}	14 {2}	54 {9}	71 {12}	55 {10}	85 {14.5}	24{3}	85{14.5}	64{11}	79{13}
55	BIAS ($\hat{\zeta}$)	0.02295 {7}	0.02085 {1}	0.02186 {4}	0.02089 {2}	0.02204 {5}	0.02155 {3}	0.02256 {6}	0.02329 {8}	0.02886 {13}	0.02623 {11}	0.03598 {15}	0.02366 {9}	0.02766 {12}	0.02395 {10}	0.02988 {14}
55	MSE ($\hat{\zeta}$)	0.00085 {7}	0.00067 {1}	0.00074 {3}	7e-04 {2}	0.00077 {4}	0.00078 {5}	0.00082 {6}	0.00087 {8.5}	0.00132 {13}	0.00112 {11}	0.00227 {15}	0.00087 {8.5}	0.00121 {12}	0.00089 {10}	0.00147 {14}
55	MRE ($\hat{\zeta}$)	0.05738 {7}	0.05212 {1}	0.05466 {4}	0.05222 {2}	0.0551 {5}	0.05388 {3}	0.0564 {6}	0.05822 {8}	0.07215 {13}	0.06559 {11}	0.08996 {15}	0.05916 {9}	0.06916 {12}	0.05987 {10}	0.07471 {14}
55	D_{abs}	0.03061 {6}	0.02898 {1}	0.03028 {4}	0.02925 {2}	0.03052 {5}	0.02987 {3}	0.03124 {7}	0.03211 {8}	0.04008 {13}	0.03659 {11}	0.04723 {15}	0.03297 {9}	0.03827 {12}	0.03339 {10}	0.04112 {14}
55	D_{max}	0.04344 {5}	0.04111 {1}	0.04312 {4}	0.0415 {2}	0.04346 {6}	0.04261 {3}	0.04452 {7}	0.04575 {8}	0.05727 {13}	0.05192 {11}	0.06794 {15}	0.04689 {9}	0.05497 {12}	0.0476 {10}	0.05894 {14}
55	ASAE	0.02744 {7}	0.02681 {5}	0.02637 {1}	0.02754 {8}	0.02664 {4}	0.02794 {10}	0.02656 {3}	0.02639 {2}	0.03138 {13}	0.03079 {12}	0.02781 {9}	0.02697 {6}	0.0323 {14}	0.02985 {11}	0.03355 {15}
55	$\sum Ranks$	38 {7}	10 {1}	19 {2}	31 {5}	28 {4}	26 {3}	34 {6}	41.5 {8}	77 {13}	66 {11}	83 {14}	49.5 {9}	73 {12}	60 {10}	84 {15}
100	BIAS ($\hat{\zeta}$)	0.01732 {7}	0.0157 {2}	0.016 {3}	0.01514 {1}	0.01764 {8}	0.01666 {5}	0.01632 {4}	0.01789 {9}	0.02057 {12}	0.0183 {10}	0.0308 {15}	0.01715 {6}	0.02292 {14}	0.01913 {11}	0.02202 {13}
100	MSE ($\hat{\zeta}$)	0.00044 {6}	0.00038 {2}	4e-04 {3}	0.00036 {1}	0.00048 {8}	0.00044 {6}	0.00041 {4}	0.00049 {9}	0.00067 {12}	0.00054 {10}	0.00182 {15}	0.00044 {6}	0.00084 {14}	0.00058 {11}	0.00076 {13}
100	MRE ($\hat{\zeta}$)	0.0433 {7}	0.03924 {2}	0.04001 {3}	0.03786 {1}	0.0441 {8}	0.04166 {5}	0.04081 {4}	0.04472 {9}	0.05142 {12}	0.04576 {10}	0.07699 {15}	0.04288 {6}	0.0573 {14}	0.04783 {11}	0.05506 {13}
100	D_{abs}	0.02334 {6}	0.02184 {2}	0.02209 {3}	0.02114 {1}	0.02439 {8}	0.02314 {5}	0.02238 {4}	0.02469 {9}	0.02851 {12}	0.02521 {10}	0.04048 {15}	0.02381 {7}	0.03179 {14}	0.02656 {11}	0.03066 {13}
100	D_{max}	0.03323 {6}	0.03109 {2}	0.0316 {3}	0.03012 {1}	0.03467 {8}	0.03293 {5}	0.03209 {4}	0.03515 {9}	0.04062 {12}	0.03598 {10}	0.05818 {15}	0.0339 {7}	0.04541 {14}	0.03782 {11}	0.0438 {13}
100	ASAE	0.01799 {9}	0.01667 {2}	0.0171 {3}	0.01792 {8}	0.01749 {6}	0.01778 {7}	0.01745 {5}	0.0166 {1}	0.01941 {13}	0.0193 {11}	0.0192 {10}	0.01713 {4}	0.02061 {15}	0.01937 {12}	0.02041 {14}
100	$\sum Ranks$	40 {7}	12 {1}	30 {4}	13 {2}	45 {8.5}	32 {5}	24 {3}	45 {8.5}	72 {12}	60 {10}	84 {14.5}	35 {6}	84 {14.5}	66 {11}	78 {13}
140	BIAS ($\hat{\zeta}$)	0.0141 {6}	0.01331 {4}	0.0144 {7}	0.01317 {2}	0.0133 {3}	0.01399 {5}	0.01312 {1}	0.01558 {9}	0.01738 {12}	0.01658 {11}	0.02159 {15}	0.01452 {8}	0.0186 {13}	0.0158 {10}	0.02001 {14}
140	MSE ($\hat{\zeta}$)	0.00033 {7}	0.00028 {2.5}	0.00032 {5.5}	0.00028 {2.5}	0.00028 {2.5}	0.00032 {5.5}	0.00028 {2.5}	0.00038 {9}	0.00047 {12}	0.00042 {11}	0.00087 {15}	0.00034 {8}	0.00054 {13}	0.00041 {10}	0.00063 {14}
140	MRE ($\hat{\zeta}$)	0.03526 {6}	0.03328 {4}	0.036 {7}	0.03294 {2}	0.03324 {3}	0.03498 {5}	0.0328 {1}	0.03894 {9}	0.04344 {12}	0.04144 {11}	0.05398 {15}	0.0363 {8}	0.04651 {13}	0.03951 {10}	0.05001 {14}
140	D_{abs}	0.01902 {5}	0.01848 {4}	0.01997 {7}	0.01842 {2}	0.01843 {3}	0.01935 {6}	0.0182 {1}	0.02147 {9}	0.02406 {12}	0.02304 {11}	0.02929 {15}	0.02021 {8}	0.02593 {13}	0.02185 {10}	0.02772 {14}
140	D_{max}	0.02714 {5}	0.02626 {3.5}	0.02848 {7}	0.02616 {2}	0.02626 {3.5}	0.02756 {6}	0.02592 {1}	0.03067 {9}	0.03436 {12}	0.03281 {11}	0.04173 {15}	0.0288 {8}	0.03699 {13}	0.03104 {10}	0.03952 {14}
140	ASAE	0.01365 {5}	0.0132 {1}	0.0138 {7}	0.01392 {9}	0.0136 {4}	0.01388 {8}	0.01338 {2}	0.01369 {6}	0.0155 {12}	0.01467 {11}	0.01441 {10}	0.01342 {3}	0.01607 {15}	0.01574 {13}	0.01599 {14}
140	$\sum Ranks$	34 {5}	19 {2.5}	40.5 {7}	19.5 {4}	19 {2.5}	35.5 {6}	8.5 {1}	51 {9}	72 {12}	66 {11}	85 {15}	43 {8}	80 {13}	63 {10}	84 {14}
180	BIAS ($\hat{\zeta}$)	0.01143 {1}	0.01157 {3}	0.01252 {8}	0.0115 {2}	0.01229 {6}	0.0125 {7}	0.01192 {4}	0.01336 {9}	0.01503 {12}	0.01385 {10}	0.02082 {15}	0.0122 {5}	0.01662 {13}	0.01493 {11}	0.01673 {14}
180	MSE ($\hat{\zeta}$)	0.00021 {1.5}	0.00022 {3.5}	0.00025 {8}	0.00021 {1.5}	0.00023 {5.5}	0.00024 {7}	0.00022 {3.5}	0.00029 {9}	0.00034 {11.5}	0.00031 {10}	8e-04 {15}	0.00023 {5.5}	0.00042 {13}	0.00034 {11.5}	0.00043 {14}
180	MRE ($\hat{\zeta}$)	0.02857 {1}	0.02893 {3}	0.0313 {8}	0.02875 {2}	0.03072 {6}	0.03124 {7}	0.0298 {4}	0.0334 {9}	0.03759 {12}	0.03462 {10}	0.05205 {15}	0.0305 {5}	0.04156 {13}	0.03734 {11}	0.04183 {14}
180	D_{abs}	0.01554 {1}	0.01609 {3}	0.01728 {7}	0.01602 {2}	0.01704 {6}	0.01738 {8}	0.01656 {4}	0.01845 {9}	0.0209 {12}	0.0192 {10}	0.02811 {15}	0.01702 {5}	0.02316 {13}	0.02074 {11}	0.02328 {14}
180	D_{max}	0.02212 {1}	0.02292 {3}	0.02468 {7}	0.02283 {2}	0.02427 {6}	0.02472 {8}	0.02355 {4}	0.02635 {9}	0.02978 {12}	0.0273 {10}	0.04028 {15}	0.02417 {5}	0.03295 {13}	0.0295 {11}	0.03319 {14}
180	ASAE	0.01132 {6}	0.01105 {4}	0.0113 {5}	0.01145 {8}	0.01089 {3}	0.01166 {9}	0.01077 {2}	0.01072 {1}	0.01336 {14}	0.01265 {11}	0.0121 {10}	0.01141 {7}	0.01349 {15}	0.01277 {12}	0.01314 {13}
180	$\sum Ranks$	11.5 {1}	19.5 {3}	43 {7}	17.5 {2}	32.5 {5.5}	46 {8.5}	21.5 {4}	46 {8.5}	73.5 {12}	61 {10}	85 {15}	32.5 {5.5}	80 {13}	67.5 {11}	83 {14}
225	BIAS ($\hat{\zeta}$)	0.01095 {4}	0.0108 {1}	0.01167 {8}	0.01097 {5}	0.01129 {7}	0.01094 {3}	0.01084 {2}	0.01191 {9}	0.01285 {11}	0.01262 {10}	0.01924 {15}	0.0112 {6}	0.01449 {13}	0.01371 {12}	0.01498 {14}
225	MSE ($\hat{\zeta}$)	0.00019 {4}	0.00018 {1.5}	0.00021 {7.5}	0.00018 {1.5}	2e-04 {6}	0.00019 {4}	0.00019 {4}	0.00022 {9}	0.00027 {11}	0.00026 {10}	0.00071 {15}	0.00021 {7.5}	0.00032 {12.5}	0.00032 {12.5}	0.00036{14}
225	MRE ($\hat{\zeta}$)	0.02737 {4}	0.02701 {1}	0.02917 {8}	0.02742 {5}	0.02823 {7}	0.02734 {3}	0.0271 {2}	0.02978 {9}	0.03214 {11}	0.03155 {10}	0.04809 {15}	0.02799 {6}	0.03623 {13}	0.03429 {12}	0.03744 {14}
225	D_{abs}	0.01484 {1}	0.01508 {3}	0.01621 {8}	0.01523 {5}	0.01569 {7}	0.01515 {4}	0.01497 {2}	0.01652 {9}	0.01774 {11}	0.01761 {10}	0.0259 {15}	0.01558 {6}	0.02019 {13}	0.01887 {12}	0.02086 {14}
225	D_{max}	0.02121 {1}	0.02144 {3}	0.0231 {8}	0.02175 {5}	0.02235 {7}	0.02163 {4}	0.0214 {2}	0.02353 {9}	0.02526 {11}	0.02502 {10}	0.03711 {15}	0.02218 {6}	0.02874 {13}	0.02688 {12}	0.02969 {14}
225	ASAE	0.0098 {7}	0.00955 {6}	0.00928 {2}	0.00998 {8}	0.00928 {2}	0.01028 {9}	0.00941 {4}	0.00928 {2}	0.01105 {13}	0.01058 {11}	0.01032 {10}	0.00948 {5}	0.01123 {14}	0.01128 {15}	0.01099 {12}
225	$\sum Ranks$	21 {3}	15.5 {1}	40.5 {7}	29.5 {5}	45 {8}	27 {4}	16 {2}	46 {9}	67 {11}	60 {10}	84 {15}	35.5 {6}	77.5 {13}	74.5 {12}	81 {14}
270	BIAS ($\hat{\zeta}$)	0.00986 {4}	0.00996 {5}	0.01046 {8}	0.00935 {1}	0.0104 {6}	0.00936 {2}	0.00969 {3}	0.01125 {9}	0.01144 {10}	0.01146 {11}	0.01643 {15}	0.01042 {7}	0.01383 {14}	0.01259 {12}	0.01341 {13}
270	MSE ($\hat{\zeta}$)	0.00016 {4.5}	0.00016 {4.5}	0.00017 {7}	0.00014 {1.5}	0.00017 {7}	0.00014 {1.5}	0.00015 {3}	0.00019 {9}	0.00021 {11}	2e-04 {10}	5e-04 {15}	0.00017 {7}	0.00031 {14}	0.00025 {12}	0.00028 {13}
270	MRE ($\hat{\zeta}$)	0.02465 {4}	0.02491 {5}	0.02616 {8}	0.02337 {1}	0.02599 {6}	0.02341 {2}	0.02422 {3}	0.02813 {9}	0.02861 {10}	0.02865 {11}	0.04108 {15}	0.02604 {7}	0.03457 {14}	0.03149 {12}	0.03351 {13}
270	D_{abs}	0.01344 {3}	0.0138 {5}	0.01449 {8}	0.01303 {2}	0.01441 {6}	0.013 {1}	0.01346 {4}	0.01559 {9}	0.01587 {10}	0.01589 {11}	0.02241 {15}	0.01446 {7}	0.01915 {14}	0.0174 {12}	0.01856 {13}
270	D_{max}	0.01914 {3}	0.01967 {5}	0.02065 {8}	0.01858 {2}	0.02056 {6}	0.01852 {1}	0.01916 {4}	0.02222 {9}	0.02262 {10}	0.02264 {11}	0.03197 {15}	0.02062 {7}	0.02733 {14}	0.02479 {12}	0.02644 {13}
270	ASAE	0.00859 {8}	0.00808 {1}	0.00829 {4}	0.00844 {6}	0.00832 {5}	0.0088 {9}	0.00847 {7}	0.00809 {2}	0.00966 {12}	0.00934 {11}	0.00904 {10}	0.00821 {3}	0.00997 {13}	0.01008 {14}	0.01011 {15}

(Continued)

Table 3 (continued)

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
270	$\sum$ *Ranks*	26.5$^{\{5\}}$	25.5$^{\{4\}}$	43$^{\{8\}}$	13.5$^{\{1\}}$	36$^{\{6\}}$	16.5$^{\{2\}}$	24$^{\{3\}}$	47$^{\{9\}}$	62$^{\{10\}}$	69$^{\{11\}}$	85$^{\{15\}}$	38$^{\{7\}}$	82$^{\{14\}}$	73$^{\{12\}}$	79$^{\{13\}}$
350	BIAS $(\hat{\zeta})$	0.0085$^{\{2\}}$	0.00856$^{\{4\}}$	0.00896$^{\{6\}}$	0.00825$^{\{1\}}$	0.00889$^{\{5\}}$	0.00855$^{\{3\}}$	0.00901$^{\{7\}}$	0.00996$^{\{9\}}$	0.01112$^{\{11\}}$	0.01016$^{\{10\}}$	0.01439$^{\{15\}}$	0.0095$^{\{8\}}$	0.01116$^{\{12\}}$	0.01168$^{\{13\}}$	0.01235$^{\{14\}}$
350	MSE $(\hat{\zeta})$	0.00012$^{\{4\}}$	0.00011$^{\{1.5\}}$	0.00013$^{\{6.5\}}$	0.00011$^{\{1.5\}}$	0.00012$^{\{4\}}$	0.00012$^{\{4\}}$	0.00013$^{\{6.5\}}$	0.00016$^{\{9.5\}}$	2e-04$^{\{11.5\}}$	0.00016$^{\{9.5\}}$	0.00037$^{\{15\}}$	0.00014$^{\{8\}}$	2e-04$^{\{11.5\}}$	0.00022$^{\{13\}}$	0.00024$^{\{14\}}$
350	MRE $(\hat{\zeta})$	0.02124$^{\{2\}}$	0.02141$^{\{4\}}$	0.0224$^{\{6\}}$	0.02062$^{\{1\}}$	0.02222$^{\{5\}}$	0.02136$^{\{3\}}$	0.02253$^{\{7\}}$	0.0249$^{\{9\}}$	0.02781$^{\{11\}}$	0.02539$^{\{10\}}$	0.03598$^{\{15\}}$	0.02374$^{\{8\}}$	0.0279$^{\{12\}}$	0.02919$^{\{13\}}$	0.03087$^{\{14\}}$
350	D_{abs}	0.0116$^{\{2\}}$	0.01187$^{\{4\}}$	0.01247$^{\{7\}}$	0.01146$^{\{1\}}$	0.01234$^{\{5\}}$	0.01186$^{\{3\}}$	0.01246$^{\{6\}}$	0.01385$^{\{9\}}$	0.01535$^{\{11\}}$	0.01408$^{\{10\}}$	0.01955$^{\{15\}}$	0.01318$^{\{8\}}$	0.01548$^{\{12\}}$	0.01604$^{\{13\}}$	0.0171$^{\{14\}}$
350	D_{max}	0.01653$^{\{2\}}$	0.01694$^{\{4\}}$	0.01775$^{\{6\}}$	0.01635$^{\{1\}}$	0.01758$^{\{5\}}$	0.01691$^{\{3\}}$	0.01781$^{\{7\}}$	0.01971$^{\{9\}}$	0.02189$^{\{11\}}$	0.02004$^{\{10\}}$	0.02797$^{\{15\}}$	0.01875$^{\{8\}}$	0.02212$^{\{12\}}$	0.02283$^{\{13\}}$	0.0244$^{\{14\}}$
350	ASAE	0.00698$^{\{5\}}$	0.00702$^{\{6\}}$	0.00682$^{\{3\}}$	0.00708$^{\{7\}}$	0.00686$^{\{4\}}$	0.00721$^{\{9\}}$	0.00663$^{\{1\}}$	0.00668$^{\{2\}}$	0.00791$^{\{12\}}$	0.00776$^{\{11\}}$	0.00735$^{\{10\}}$	0.00711$^{\{8\}}$	0.00816$^{\{13\}}$	0.0083$^{\{15\}}$	0.00829$^{\{14\}}$
350	$\sum$ *Ranks*	17$^{\{2\}}$	23.5$^{\{3\}}$	34.5$^{\{6.5\}}$	12.5$^{\{1\}}$	28$^{\{5\}}$	25$^{\{4\}}$	34.5$^{\{6.5\}}$	47.5$^{\{8\}}$	70.5$^{\{11\}}$	60.5$^{\{10\}}$	83$^{\{15\}}$	48$^{\{9\}}$	75.5$^{\{12\}}$	78$^{\{13\}}$	82$^{\{14\}}$

Table 4: Numerical values of simulation measures for $\zeta = 0.85$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS ($\hat{\zeta}$)	0.06821 {6}	0.07046 {9}	0.06449 {2}	0.06339 {1}	0.06812 {5}	0.06856 {7}	0.06507 {3}	0.07101 {10}	0.07749 {12}	0.07202 {11}	0.09773 {15}	0.06626 {4}	0.08865 {13}	0.06932 {8}	0.09055 {14}
25	MSE ($\hat{\zeta}$)	0.00761 {7}	0.00768 {8}	0.0067 {2}	0.0064 {1}	0.00716 {5}	0.0075 {6}	0.00681 {3}	0.00806 {11}	0.00964 {12}	0.00789 {10}	0.01587 {15}	0.0069 {4}	0.01303 {14}	0.00775 {9}	0.01293 {13}
25	MRE ($\hat{\zeta}$)	0.08025 {6}	0.0829 {9}	0.07588 {2}	0.07458 {1}	0.08015 {5}	0.08066 {7}	0.07655 {3}	0.08354 {10}	0.09116 {12}	0.08474 {11}	0.11497 {15}	0.07796 {4}	0.1043 {13}	0.08155 {8}	0.10653 {14}
25	D_{abs}	0.04618 {5}	0.04889 {10}	0.04474 {2}	0.04386 {1}	0.04678 {6}	0.04699 {7}	0.04496 {3}	0.04833 {9}	0.05351 {12}	0.05001 {11}	0.06401 {15}	0.04562 {4}	0.06024 {13}	0.04792 {8}	0.0615 {14}
25	D_{max}	0.06489 {5}	0.06856 {10}	0.06292 {2}	0.06214 {1}	0.06601 {6}	0.0668 {7}	0.06302 {3}	0.06811 {9}	0.07574 {12}	0.07072 {11}	0.09167 {15}	0.06421 {4}	0.08645 {13}	0.06747 {8}	0.08767 {14}
25	ASAE	0.04735 {7.5}	0.0469 {6}	0.04652 {4}	0.05 {9}	0.04587 {2}	0.05008 {10}	0.04626 {3}	0.04397 {1}	0.05603 {13}	0.05453 {12}	0.04735 {7.5}	0.04665 {5}	0.05721 {15}	0.05366 {11}	0.05659 {14}
25	$\sum$ *Ranks*	36.5 {6}	52 {9.5}	14 {1.5}	14 {1.5}	29 {5}	44 {7}	18 {3}	50 {8}	73 {12}	66 {11}	82.5 {14}	25 {4}	81 {13}	52 {9.5}	83 {15}
55	BIAS ($\hat{\zeta}$)	0.04366 {3}	0.04467 {6}	0.0465 {9}	0.04164 {1}	0.04526 {7}	0.04282 {2}	0.04458 {5}	0.04811 {10}	0.05532 {12}	0.05257 {11}	0.07026 {15}	0.04439 {4}	0.06161 {13}	0.04576 {8}	0.06305 {14}
55	MSE ($\hat{\zeta}$)	0.00294 {3}	0.00322 {6}	0.00358 {9}	0.00274 {1}	0.0032 {5}	0.00284 {2}	0.00326 {7}	0.00368 {10}	0.00485 {12}	0.00444 {11}	0.00824 {15}	0.00308 {4}	0.00607 {13}	0.0033 {8}	0.00656 {14}
55	MRE ($\hat{\zeta}$)	0.05137 {3}	0.05256 {6}	0.0547 {9}	0.04899 {1}	0.05324 {7}	0.05038 {2}	0.05245 {5}	0.0566 {10}	0.06508 {12}	0.06185 {11}	0.08266 {15}	0.05223 {4}	0.07248 {13}	0.05383 {8}	0.07417 {14}
55	D_{abs}	0.02956 {3}	0.03066 {6}	0.03176 {8}	0.02885 {1}	0.03098 {7}	0.02946 {2}	0.0306 {5}	0.03282 {10}	0.03826 {12}	0.03658 {11}	0.04689 {15}	0.03045 {4}	0.04221 {13}	0.03187 {9}	0.0426 {14}
55	D_{max}	0.0419 {3}	0.0435 {6}	0.04509 {9}	0.04095 {1}	0.04414 {7}	0.04182 {2}	0.04339 {5}	0.04668 {10}	0.05437 {12}	0.05163 {11}	0.06691 {15}	0.04337 {4}	0.06026 {13}	0.04502 {8}	0.06103 {14}
55	ASAE	0.02818 {8}	0.02657 {4}	0.02631 {3}	0.0281 {7}	0.0261 {2}	0.02946 {10}	0.0266 {5}	0.02541 {1}	0.03232 {13}	0.03093 {12}	0.02827 {9}	0.02762 {6}	0.03335 {15}	0.02976 {11}	0.03323 {14}
55	$\sum$ *Ranks*	23 {3}	34 {6}	47 {8}	12 {1}	35 {7}	20 {2}	32 {5}	51 {9}	73 {12}	67 {11}	84 {14.5}	26 {4}	80 {13}	52 {10}	84 {14.5}
100	BIAS ($\hat{\zeta}$)	0.03075 {1}	0.03417 {4}	0.03435 {6}	0.03468 {8}	0.03432 {5}	0.03463 {7}	0.0333 {2}	0.03529 {9}	0.03979 {12}	0.03678 {11}	0.05887 {15}	0.0355 {10}	0.0423 {13}	0.03414 {3}	0.04374 {14}
100	MSE ($\hat{\zeta}$)	0.00155 {1}	0.00185 {6}	0.00181 {5}	0.00178 {3}	0.00188 {7}	0.0018 {4}	0.00173 {2}	0.00202 {10}	0.00252 {12}	0.00205 {11}	0.00617 {15}	0.00193 {9}	0.00289 {13}	0.00189 {8}	0.00301 {14}
100	MRE ($\hat{\zeta}$)	0.03618 {1}	0.0402 {4}	0.04041 {6}	0.0408 {8}	0.04038 {5}	0.04074 {7}	0.03918 {2}	0.04152 {9}	0.04681 {12}	0.04327 {11}	0.06926 {15}	0.04176 {10}	0.04976 {13}	0.04017 {3}	0.05146 {14}
100	D_{abs}	0.0208 {1}	0.02346 {3}	0.02367 {6}	0.0239 {8}	0.02359 {4.5}	0.02381 {7}	0.02291 {2}	0.02427 {9}	0.02746 {12}	0.02541 {11}	0.03889 {15}	0.02447 {10}	0.02917 {13}	0.02359 {4.5}	0.03025 {14}
100	D_{max}	0.0295 {1}	0.03334 {3}	0.03362 {6}	0.03406 {8}	0.03359 {5}	0.03391 {7}	0.03257 {2}	0.03453 {9}	0.03894 {12}	0.03607 {11}	0.05574 {15}	0.0347 {10}	0.04156 {13}	0.03342 {4}	0.04289 {14}
100	ASAE	0.01906 {7}	0.01839 {4}	0.01814 {3}	0.01918 {8}	0.01864 {6}	0.0199 {9}	0.01777 {1}	0.01782 {2}	0.02172 {13.5}	0.02126 {12}	0.0211 {11}	0.0184 {5}	0.02172 {13.5}	0.02076 {10}	0.02263 {15}
100	$\sum$ *Ranks*	12 {2}	24 {3}	32 {4}	43 {8}	32.5 {5.5}	41 {7}	11 {1}	48 {9}	73.5 {12}	67 {11}	86 {15}	54 {10}	78.5 {13}	32.5 {5.5}	85 {14}
140	BIAS ($\hat{\zeta}$)	0.02559 {1}	0.02962 {7}	0.02874 {5}	0.02903 {6}	0.02813 {3}	0.027 {2}	0.02859 {4}	0.0297 {8}	0.03262 {12}	0.03185 {11}	0.04531 {15}	0.03031 {9}	0.03482 {14}	0.03103 {10}	0.03471 {13}
140	MSE ($\hat{\zeta}$)	0.00111 {1}	0.00132 {6}	0.00138 {8}	0.00135 {7}	0.00127 {3}	0.00118 {2}	0.00128 {4}	0.0013 {5}	0.00167 {12}	0.00154 {10.5}	0.00356 {15}	0.00144 {9}	0.00198 {13}	0.00154 {10.5}	0.002 {14}
140	MRE ($\hat{\zeta}$)	0.0301 {1}	0.03485 {7}	0.03381 {5}	0.03415 {6}	0.03309 {3}	0.03177 {2}	0.03364 {4}	0.03495 {8}	0.03838 {12}	0.03747 {11}	0.05331 {15}	0.03566 {9}	0.04096 {14}	0.03651 {10}	0.04084 {13}
140	D_{abs}	0.01735 {1}	0.02043 {8}	0.0197 {5}	0.02007 {6}	0.01927 {3}	0.01867 {2}	0.01964 {4}	0.02034 {7}	0.02245 {12}	0.02197 {11}	0.0304 {15}	0.02083 {9}	0.02409 {14}	0.02129 {10}	0.02393 {13}
140	D_{max}	0.02466 {1}	0.029 {8}	0.02805 {5}	0.02859 {6}	0.02748 {3}	0.0265 {2}	0.02799 {4}	0.02899 {7}	0.03198 {12}	0.03114 {11}	0.04338 {15}	0.02963 {9}	0.0342 {14}	0.03029 {10}	0.03412 {13}
140	ASAE	0.01524 {5}	0.01537 {8}	0.01478 {2}	0.01532 {6}	0.01491 {4}	0.01585 {9}	0.01485 {3}	0.01419 {1}	0.01724 {13}	0.01689 {12}	0.01687 {11}	0.01536 {7}	0.01803 {14}	0.01668 {10}	0.01805 {15}
140	$\sum$ *Ranks*	10 {1}	44 {8}	30 {5}	37 {7}	19 {2.5}	19 {2.5}	23 {4}	36 {6}	73 {12}	66.5 {11}	86 {15}	52 {9}	83 {14}	60.5 {10}	81 {13}
180	BIAS ($\hat{\zeta}$)	0.02265 {1}	0.02371 {3}	0.02634 {7}	0.02387 {4}	0.02688 {9}	0.02347 {2}	0.02445 {5}	0.0271 {10}	0.03069 {12}	0.02807 {11}	0.03985 {15}	0.02613 {6}	0.03454 {14}	0.02654 {8}	0.0323 {13}
180	MSE ($\hat{\zeta}$)	0.00083 {1}	0.00094 {5}	0.00111 {7}	9e-04 {3}	0.00117 {10}	0.00088 {2}	0.00093 {4}	0.00112 {8}	0.00143 {12}	0.00119 {11}	0.00308 {15}	0.00109 {6}	0.00187 {14}	0.00113 {9}	0.0016 {13}
180	MRE ($\hat{\zeta}$)	0.02665 {1}	0.0279 {3}	0.03099 {7}	0.02808 {4}	0.03162 {9}	0.02762 {2}	0.02877 {5}	0.03188 {10}	0.0361 {12}	0.03302 {11}	0.04688 {15}	0.03074 {6}	0.04064 {14}	0.03123 {8}	0.038 {13}
180	D_{abs}	0.01541 {1}	0.01624 {3}	0.01804 {7}	0.01651 {4}	0.01847 {9}	0.01619 {2}	0.0168 {5}	0.01854 {10}	0.02107 {12}	0.0192 {11}	0.02669 {15}	0.01794 {6}	0.02369 {14}	0.01825 {8}	0.02233 {13}
180	D_{max}	0.02188 {1}	0.02318 {3}	0.02575 {7}	0.02347 {4}	0.02631 {9}	0.02299 {2}	0.02394 {5}	0.02643 {10}	0.03007 {12}	0.02742 {11}	0.03818 {15}	0.02556 {6}	0.03379 {14}	0.02596 {8}	0.03183 {13}
180	ASAE	0.01367 {9}	0.01317 {6}	0.013 {5}	0.01322 {7}	0.01292 {3}	0.01336 {8}	0.01259 {2}	0.01203 {1}	0.01569 {13}	0.01462 {10}	0.01478 {12}	0.01296 {4}	0.01621 {15}	0.01469 {11}	0.01611 {14}
180	$\sum$ *Ranks*	14 {1}	22 {3}	39 {7}	38 {6}	48 {8.5}	18 {2}	25 {4}	48 {8.5}	72 {12}	64 {11}	86 {15}	33 {5}	84 {14}	51 {10}	78 {13}
225	BIAS ($\hat{\zeta}$)	0.02177 {4}	0.02082 {1}	0.02187 {5}	0.02173 {3}	0.0229 {7}	0.02223 {6}	0.02166 {2}	0.02382 {9}	0.02662 {12}	0.02503 {11}	0.03697 {15}	0.02348 {8}	0.02705 {13}	0.02404 {10}	0.02732 {14}
225	MSE ($\hat{\zeta}$)	0.00075 {4.5}	7e-04 {1}	0.00073 {3}	0.00075 {4.5}	0.00079 {7}	0.00077 {6}	0.00071 {2}	0.00091 {9}	0.00114 {12.5}	0.00101 {11}	0.00237 {15}	0.00082 {8}	0.00114 {12.5}	0.00093 {10}	0.00122 {14}
225	MRE ($\hat{\zeta}$)	0.02562 {4}	0.0245 {1}	0.02573 {5}	0.02557 {3}	0.02694 {7}	0.02615 {6}	0.02549 {2}	0.02803 {9}	0.03132 {12}	0.02945 {11}	0.04349 {15}	0.02762 {8}	0.03182 {13}	0.02828 {10}	0.03214 {14}
225	D_{abs}	0.01483 {2}	0.0143 {1}	0.0151 {5}	0.015 {4}	0.01576 {7}	0.01536 {6}	0.01484 {3}	0.01637 {9}	0.01829 {12}	0.01729 {11}	0.02495 {15}	0.01611 {8}	0.01861 {13}	0.01654 {10}	0.01878 {14}
225	D_{max}	0.02111 {2}	0.02035 {1}	0.02143 {5}	0.02137 {4}	0.02243 {7}	0.0218 {6}	0.02117 {3}	0.02329 {9}	0.02608 {12}	0.02455 {11}	0.03563 {15}	0.02295 {8}	0.02649 {13}	0.02352 {10}	0.02674 {14}
225	ASAE	0.01154 {7}	0.01129 {3}	0.01132 {4}	0.01192 {8}	0.01107 {2}	0.01211 {9}	0.01142 {5}	0.01082 {1}	0.01359 {13}	0.01295 {11}	0.01292 {10}	0.01145 {6}	0.01389 {14}	0.01346 {12}	0.01426 {15}
225	$\sum$ *Ranks*	22.5 {3}	22 {2}	26 {5}	25.5 {4}	36 {6}	38 {7}	16 {1}	45 {8.5}	72.5 {12}	65 {11}	84 {14.5}	45 {8.5}	77.5 {13}	61 {10}	84 {14.5}
270	BIAS ($\hat{\zeta}$)	0.02034 {4}	0.01985 {2}	0.02035 {5}	0.0187 {1}	0.02037 {6}	0.02001 {3}	0.02055 {7}	0.02137 {8}	0.02469 {12}	0.02325 {11}	0.02971 {15}	0.02152 {9}	0.02642 {13}	0.02268 {10}	0.02644 {14}
270	MSE ($\hat{\zeta}$)	0.00069 {7}	0.00061 {2}	0.00065 {4.5}	0.00056 {1}	0.00064 {3}	0.00065 {4.5}	0.00067 {6}	0.00074 {8}	0.00092 {12}	0.00084 {11}	0.0015 {15}	0.00075 {9}	0.00109 {13}	8e-04 {10}	0.0011 {14}
270	MRE ($\hat{\zeta}$)	0.02393 {4}	0.02335 {2}	0.02394 {5}	0.022 {1}	0.02397 {6}	0.02355 {3}	0.02418 {7}	0.02515 {8}	0.02905 {12}	0.02735 {11}	0.03495 {15}	0.02531 {9}	0.03109 {13}	0.02668 {10}	0.0311 {14}
270	D_{abs}	0.01388 {4}	0.01368 {2}	0.01399 {5}	0.01286 {1}	0.01403 {6}	0.01372 {3}	0.01412 {7}	0.01466 {8}	0.01692 {12}	0.01601 {11}	0.02015 {15}	0.01477 {9}	0.0182 {13}	0.01555 {10}	0.01823 {14}
270	D_{max}	0.01977 {4}	0.01944 {2}	0.01991 {5}	0.01832 {1}	0.01993 {6}	0.01953 {3}	0.02009 {7}	0.02091 {8}	0.02412 {12}	0.02277 {11}	0.02875 {15}	0.02104 {9}	0.02592 {13}	0.02215 {10}	0.02593 {14}
270	ASAE	0.0106 {7}	0.01028 {4}	0.01034 {5}	0.01058 {6}	0.00991 {1}	0.01086 {10}	0.01019 {3}	0.00992 {2}	0.01187 {13}	0.01154 {11}	0.0108 {9}	0.01062 {8}	0.01244 {14}	0.01157 {12}	0.01258 {15}

(Continued)

Table 4 (continued)

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
270	$\sum Ranks$	30 $^{\{6\}}$	14 $^{\{2\}}$	29.5 $^{\{5\}}$	11 $^{\{1\}}$	28 $^{\{4\}}$	26.5 $^{\{3\}}$	37 $^{\{7\}}$	42 $^{\{8\}}$	72 $^{\{12\}}$	65 $^{\{10\}}$	83 $^{\{14\}}$	53 $^{\{9\}}$	78 $^{\{13\}}$	67 $^{\{11\}}$	84 $^{\{15\}}$
350	BIAS ($\hat{\zeta}$)	0.01677 $^{\{1\}}$	0.01717 $^{\{4\}}$	0.01813 $^{\{5\}}$	0.0168 $^{\{2\}}$	0.01836$^{\{6.5\}}$	0.01836$^{\{6.5\}}$	0.01845 $^{\{8\}}$	0.02025 $^{\{10\}}$	0.02254 $^{\{13\}}$	0.02013 $^{\{9\}}$	0.02818 $^{\{15\}}$	0.01694 $^{\{3\}}$	0.02179 $^{\{12\}}$	0.02108 $^{\{11\}}$	0.02354 $^{\{14\}}$
350	MSE ($\hat{\zeta}$)	0.00046 $^{\{2.5\}}$	0.00046 $^{\{2.5\}}$	0.00053 $^{\{6.5\}}$	0.00046 $^{\{2.5\}}$	0.00054 $^{\{8\}}$	0.00053$^{\{6.5\}}$	0.00052 $^{\{5\}}$	0.00064 $^{\{9.5\}}$	0.00082 $^{\{13\}}$	0.00064 $^{\{9.5\}}$	0.00139 $^{\{15\}}$	0.00046 $^{\{2.5\}}$	0.00075 $^{\{11.5\}}$	0.00075 $^{\{11.5\}}$	0.00086 $^{\{14\}}$
350	MRE ($\hat{\zeta}$)	0.01973 $^{\{1\}}$	0.0202 $^{\{4\}}$	0.02133 $^{\{5\}}$	0.01977 $^{\{2\}}$	0.0216 $^{\{6\}}$	0.02161 $^{\{7\}}$	0.02171 $^{\{8\}}$	0.02383 $^{\{10\}}$	0.02651 $^{\{13\}}$	0.02368 $^{\{9\}}$	0.03316 $^{\{15\}}$	0.01993 $^{\{3\}}$	0.02563 $^{\{12\}}$	0.02479 $^{\{11\}}$	0.02769 $^{\{14\}}$
350	D_{abs}	0.01146 $^{\{1\}}$	0.0118 $^{\{4\}}$	0.01244 $^{\{5\}}$	0.01158 $^{\{2\}}$	0.01265 $^{\{7\}}$	0.01264 $^{\{6\}}$	0.01272 $^{\{8\}}$	0.01395 $^{\{10\}}$	0.01549 $^{\{13\}}$	0.01383 $^{\{9\}}$	0.01907 $^{\{15\}}$	0.01166 $^{\{3\}}$	0.01502 $^{\{12\}}$	0.01439 $^{\{11\}}$	0.01624 $^{\{14\}}$
350	D_{max}	0.01631 $^{\{1\}}$	0.0168 $^{\{4\}}$	0.01774 $^{\{5\}}$	0.01647 $^{\{2\}}$	0.01799 $^{\{7\}}$	0.01797 $^{\{6\}}$	0.01809 $^{\{8\}}$	0.01981 $^{\{10\}}$	0.02199 $^{\{13\}}$	0.01965 $^{\{9\}}$	0.02726 $^{\{15\}}$	0.01659 $^{\{3\}}$	0.02135 $^{\{12\}}$	0.02049 $^{\{11\}}$	0.02308 $^{\{14\}}$
350	ASAE	0.00925 $^{\{6\}}$	0.00883 $^{\{4\}}$	0.00862 $^{\{3\}}$	0.00936 $^{\{8\}}$	0.00885 $^{\{5\}}$	0.00943 $^{\{9\}}$	0.00856 $^{\{2\}}$	0.00852 $^{\{1\}}$	0.01071 $^{\{13.5\}}$	0.01028 $^{\{10\}}$	0.01033 $^{\{11\}}$	0.00928 $^{\{7\}}$	0.01071$^{\{13.5\}}$	0.01062 $^{\{12\}}$	0.01086 $^{\{15\}}$
350	$\sum Ranks$	12.5 $^{\{1\}}$	22.5 $^{\{4\}}$	29.5 $^{\{5\}}$	18.5 $^{\{2\}}$	39.5 $^{\{7\}}$	41 $^{\{8\}}$	39 $^{\{6\}}$	50.5 $^{\{9\}}$	78.5 $^{\{13\}}$	55.5 $^{\{10\}}$	86 $^{\{15\}}$	21.5 $^{\{3\}}$	73 $^{\{12\}}$	67.5 $^{\{11\}}$	85 $^{\{14\}}$

Table 5: Numerical values of simulation measures for $\zeta = 1.4$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS ($\hat{\zeta}$)	0.1005 {1}	0.1087 {5}	0.1051 {3}	0.10664 {4}	0.11418 {8}	0.10893 {6}	0.10361 {2}	0.12387 {11}	0.13386 {12}	0.112 {7}	0.15905 {15}	0.11749 {10}	0.13604 {13}	0.11718 {9}	0.14296 {14}
25	MSE ($\hat{\zeta}$)	0.01752 {2}	0.01917 {6}	0.01708 {1}	0.01758 {3}	0.02149 {8}	0.01843 {5}	0.0177 {4}	0.0269 {11}	0.0308 {12}	0.01943 {7}	0.04917 {15}	0.02179 {10}	0.03339 {13}	0.02151 {9}	0.03683 {14}
25	MRE ($\hat{\zeta}$)	0.07179 {1}	0.07764 {5}	0.07507 {3}	0.07617 {4}	0.08155 {8}	0.07781 {6}	0.07401 {2}	0.08848 {11}	0.09561 {12}	0.08 {7}	0.11361 {15}	0.08392 {10}	0.09717 {13}	0.0837 {9}	0.10211 {14}
25	D_{abs}	0.04105 {1}	0.04589 {5}	0.04371 {3}	0.04516 {4}	0.04671 {7}	0.04593 {6}	0.0432 {2}	0.05005 {11}	0.05464 {12}	0.04793 {8}	0.06134 {15}	0.04815 {9}	0.05682 {13}	0.04905 {10}	0.05857 {14}
25	D_{max}	0.05925 {1}	0.06575 {5}	0.0633 {3}	0.06544 {4}	0.06831 {7}	0.06594 {6}	0.06242 {2}	0.07291 {11}	0.0791 {12}	0.06897 {8}	0.08993 {15}	0.07032 {9}	0.08258 {13}	0.0711 {10}	0.08579 {14}
25	ASAE	0.04715 {7}	0.04674 {6}	0.04506 {2}	0.04848 {9}	0.04622 {5}	0.0483 {8}	0.0456 {3}	0.04298 {1}	0.05743 {14}	0.05553 {12}	0.05514 {11}	0.04593 {4}	0.05916 {15}	0.05268 {10}	0.05733 {13}
25	$\sum Ranks$	13 {1}	32 {5}	15 {2.5}	28 {4}	43 {7}	37 {6}	15{2.5}	56 {10}	74 {12}	49 {8}	86 {15}	52 {9}	80 {13}	57 {11}	83 {14}
55	BIAS ($\hat{\zeta}$)	0.0646 {1}	0.07266 {5}	0.07161 {3}	0.07223 {4}	0.07447 {6}	0.0697 {2}	0.07582 {8}	0.07737 {9}	0.08721 {12}	0.08154 {11}	0.13267 {15}	0.07475 {7}	0.09724 {14}	0.07792 {10}	0.09386 {13}
55	MSE ($\hat{\zeta}$)	0.00724 {1}	0.00838 {4}	0.00845 {5}	0.00773 {2}	0.00895 {6}	0.00784 {3}	0.00937 {9}	0.00975 {10}	0.01227 {12}	0.01059 {11}	0.0356 {15}	0.00904 {7}	0.01507 {14}	0.00919 {8}	0.01466 {13}
55	MRE ($\hat{\zeta}$)	0.04614 {1}	0.0519 {5}	0.05115 {3}	0.05159 {4}	0.0532 {6}	0.04979 {2}	0.05416 {8}	0.05526 {9}	0.06229 {12}	0.05825 {11}	0.09476 {15}	0.05339 {7}	0.06946 {14}	0.05565 {10}	0.06704 {13}
55	D_{abs}	0.0267 {1}	0.03036 {4}	0.02993 {3}	0.03076 {5}	0.03114 {6}	0.0291 {2}	0.0317 {8}	0.03233 {9}	0.03644 {12}	0.03448 {11}	0.05107 {15}	0.03125 {7}	0.04113 {14}	0.03273 {10}	0.0393 {13}
55	D_{max}	0.03876 {1}	0.04409 {4}	0.04336 {3}	0.04442 {5}	0.04528 {7}	0.04216 {2}	0.04609 {8}	0.047 {9}	0.05299 {12}	0.05001 {11}	0.07533 {15}	0.04524 {6}	0.05993 {14}	0.04761 {10}	0.05726 {13}
55	ASAE	0.0302 {7}	0.02916 {6}	0.02849 {3}	0.03049 {8}	0.02896 {5}	0.03089 {9}	0.02796 {2}	0.02699 {1}	0.03677 {12}	0.03437 {11}	0.04099 {15}	0.02887 {4}	0.03855 {14}	0.03395 {10}	0.03718 {13}
55	$\sum Ranks$	12 {1}	28 {4.5}	20 {2.5}	28 {4.5}	36 {6}	20{2.5}	43 {8}	47 {9}	72 {12}	66 {11}	90 {15}	38 {7}	84 {14}	58 {10}	78 {13}
100	BIAS ($\hat{\zeta}$)	0.05284 {2}	0.05435 {4}	0.05455 {5}	0.05098 {1}	0.05463 {6}	0.05604 {8}	0.05477 {7}	0.06123 {11}	0.06132 {12}	0.05755 {10}	0.10356 {15}	0.05368 {3}	0.06858 {13}	0.0575 {9}	0.07081 {14}
100	MSE ($\hat{\zeta}$)	0.00451 {2}	0.00478 {7}	0.00472 {6}	0.00397 {1}	0.00468 {4}	0.00479 {8}	0.00469 {5}	0.00627 {12}	0.00601 {11}	0.00536 {9}	0.02059 {15}	0.00454 {3}	0.00738 {13}	0.00541 {10}	0.00795 {14}
100	MRE ($\hat{\zeta}$)	0.03775 {2}	0.03882 {4}	0.03897 {5}	0.03641 {1}	0.03902 {6}	0.04003 {8}	0.03912 {7}	0.04373 {11}	0.0438 {12}	0.04111 {10}	0.07397 {15}	0.03834 {3}	0.04899 {13}	0.04107 {9}	0.05058 {14}
100	D_{abs}	0.02208 {2}	0.02267 {4}	0.02312 {7}	0.02147 {1}	0.02289 {6}	0.02369 {8}	0.02281 {5}	0.02547 {11}	0.02595 {12}	0.02456 {10}	0.041 {15}	0.02253 {3}	0.02896 {13}	0.02431 {9}	0.02961 {14}
100	D_{max}	0.03206 {2}	0.03296 {4}	0.03344 {7}	0.03125 {1}	0.03322 {5}	0.0343 {8}	0.03326 {6}	0.03711 {11}	0.03769 {12}	0.03537 {10}	0.06013 {15}	0.03271 {3}	0.04207 {13}	0.03527 {9}	0.04308 {14}
100	ASAE	0.02112 {8}	0.02023 {3}	0.02016 {2}	0.02254 {9}	0.02039 {4}	0.02111 {7}	0.02057 {6}	0.01958 {1}	0.02591 {12}	0.02465 {11}	0.02954 {15}	0.02046 {5}	0.02733 {13}	0.02436 {10}	0.0276 {14}
100	$\sum Ranks$	18 {2}	26 {4}	32 {6}	14 {1}	31 {5}	47 {8}	36 {7}	57 {10}	71 {12}	60 {11}	90 {15}	20 {3}	78 {13}	56 {9}	84 {14}
140	BIAS ($\hat{\zeta}$)	0.04854 {9}	0.04566 {4}	0.04578 {5}	0.04208 {1}	0.04873 {10}	0.0446 {3}	0.04452 {2}	0.05044 {11}	0.05498 {12}	0.04779 {8}	0.07598 {15}	0.04603 {6}	0.05708 {13}	0.04616 {7}	0.06125 {14}
140	MSE ($\hat{\zeta}$)	0.00367 {8}	0.00327 {3.5}	0.00339 {7}	0.00278 {1}	0.00378 {9}	0.00327{3.5}	0.00317 {2}	0.00382 {11}	0.00474 {12}	0.00381 {10}	0.01046 {15}	0.00332 {6}	0.00519 {13}	0.00328 {5}	0.00596 {14}
140	MRE ($\hat{\zeta}$)	0.03467{9}	0.03261{4}	0.0327{5}	0.03006{1}	0.03481{10}	0.03186{3}	0.0318{2}	0.03603{11}	0.03927{12}	0.03414{8}	0.05427{15}	0.03288{6}	0.04077{13}	0.03297{7}	0.04375{14}
140	D_{abs}	0.02026 {9}	0.01915 {4}	0.01923 {5}	0.01777 {1}	0.02045 {10}	0.01869 {2}	0.01878 {3}	0.02108 {11}	0.0232 {12}	0.0201 {8}	0.0308 {15}	0.01926 {6}	0.02413 {13}	0.01958 {7}	0.02569 {14}
140	D_{max}	0.02935 {9}	0.0278 {4}	0.02792 {5}	0.02577 {1}	0.02973 {10}	0.02714 {2}	0.02722 {3}	0.03067 {11}	0.03366 {12}	0.02922 {8}	0.04514 {15}	0.028 {6}	0.03507 {13}	0.02835 {7}	0.03739 {14}
140	ASAE	0.01773 {7}	0.01685 {3}	0.01692 {4}	0.01808 {8}	0.01657 {2}	0.01862 {9}	0.01727 {6}	0.01637 {1}	0.02102 {12}	0.02049 {11}	0.02222 {13}	0.01694 {5}	0.02269 {14}	0.02015 {10}	0.02309 {15}
140	$\sum Ranks$	51 {8.5}	22.5 {3.5}	31 {5}	13 {1}	51{8.5}	22.5{3.5}	18 {2}	56 {11}	72 {12}	53 {10}	88 {15}	35 {6}	79 {13}	43 {7}	85 {14}
180	BIAS ($\hat{\zeta}$)	0.03847 {3}	0.0397 {5}	0.04294 {9}	0.03836 {2}	0.04003 {6}	0.03772 {1}	0.03965 {4}	0.0412 {7}	0.04865 {12}	0.04423 {11}	0.06607 {15}	0.04159 {8}	0.0511 {13}	0.04415 {10}	0.05269 {14}
180	MSE ($\hat{\zeta}$)	0.00224 {1}	0.00249 {5}	0.0029 {9}	0.00235 {3}	0.00236 {4}	0.00232 {2}	0.00252 {6}	0.00268 {7}	0.0036 {12}	0.00313 {11}	0.00727 {15}	0.00282 {8}	0.00431 {13}	0.003 {10}	0.00452 {14}
180	MRE ($\hat{\zeta}$)	0.02748 {3}	0.02836 {5}	0.03067 {9}	0.0274 {2}	0.02859 {6}	0.02694 {1}	0.02832 {4}	0.02943 {7}	0.03475 {12}	0.03159 {11}	0.04719 {15}	0.02971 {8}	0.0365 {13}	0.03153 {10}	0.03763 {14}
180	D_{abs}	0.01608 {2}	0.01662 {4}	0.01804 {9}	0.01613 {3}	0.01687 {6}	0.01581 {1}	0.01671 {5}	0.01732 {7}	0.02052 {12}	0.01873 {11}	0.0271 {15}	0.01736 {8}	0.02152 {13}	0.01856 {10}	0.02218 {14}
180	D_{max}	0.02339 {2}	0.02416 {4}	0.02618 {9}	0.02343 {3}	0.02444 {6}	0.02292 {1}	0.02417 {5}	0.0251 {7}	0.02969 {12}	0.02703 {11}	0.03958 {15}	0.02528 {8}	0.03122 {13}	0.02692 {10}	0.03223 {14}
180	ASAE	0.01565 {7}	0.01456 {2}	0.0147 {5}	0.01623 {9}	0.01464 {4}	0.01592 {8}	0.01461 {3}	0.01434 {1}	0.01894 {12}	0.01838 {11}	0.01902 {13}	0.01544 {6}	0.02047 {15}	0.0181 {10}	0.0201 {14}
180	$\sum Ranks$	18 {2}	25 {4}	50 {9}	22 {3}	32 {6}	14 {1}	27 {5}	36 {7}	72 {12}	66 {11}	88 {15}	46 {8}	80 {13}	60 {10}	84 {14}
225	BIAS ($\hat{\zeta}$)	0.03412 {1}	0.03528 {3}	0.03869 {9}	0.03544 {4}	0.03625 {6}	0.03516 {2}	0.03746 {7}	0.03967 {10}	0.04128 {12}	0.03974 {11}	0.05738 {15}	0.03567 {5}	0.04901 {14}	0.03855 {8}	0.0471 {13}
225	MSE ($\hat{\zeta}$)	0.00184 {1}	0.00198 {4}	0.00225 {8.5}	0.00185 {2}	0.00212 {6}	0.00189 {3}	0.00222 {7}	0.00236 {10}	0.00261 {12}	0.00245 {11}	0.00549 {15}	0.00206 {5}	0.00381 {14}	0.00225 {8.5}	0.00348 {13}
225	MRE ($\hat{\zeta}$)	0.02437 {1}	0.0252 {3}	0.02763 {9}	0.02532 {4}	0.02589 {6}	0.02512 {2}	0.02676 {7}	0.02833 {10}	0.02948 {12}	0.02839 {11}	0.04098 {15}	0.02548 {5}	0.035 {14}	0.02754 {8}	0.03364 {13}
225	D_{abs}	0.01426 {1}	0.01478 {2}	0.01621 {8}	0.01499{4.5}	0.01523 {6}	0.0148 {3}	0.01572 {7}	0.01672 {11}	0.01744 {12}	0.01671 {10}	0.0236 {15}	0.01499 {4.5}	0.02066 {14}	0.01629 {9}	0.0199 {13}
225	D_{max}	0.0207 {1}	0.02148 {2}	0.02355 {8}	0.02174 {5}	0.02208 {6}	0.02154 {3}	0.02285 {7}	0.02423 {10}	0.02524 {12}	0.02425 {11}	0.03448 {15}	0.02171 {4}	0.03 {14}	0.02356 {9}	0.02882 {13}
225	ASAE	0.0143 {8}	0.01321 {4}	0.01303 {2}	0.0144 {9}	0.01329 {5}	0.01409 {7}	0.01308 {3}	0.01278 {1}	0.01701 {12}	0.01598 {10}	0.01705 {13}	0.01339 {6}	0.0178 {14}	0.01608 {11}	0.01792 {15}
225	$\sum Ranks$	13 {1}	18 {2}	44.5 {8}	28.5 {4}	35 {6}	20 {3}	38 {7}	52 {9}	72 {12}	64 {11}	88 {15}	29.5 {5}	84 {14}	53.5 {10}	80 {13}
270	BIAS ($\hat{\zeta}$)	0.03279 {6}	0.03167 {2}	0.03263 {4}	0.03289 {7}	0.03338 {8}	0.03252 {3}	0.03101 {1}	0.03518 {9}	0.03911 {12}	0.03795 {11}	0.05082 {15}	0.03276 {5}	0.04126 {13}	0.03748 {10}	0.04358 {14}
270	MSE ($\hat{\zeta}$)	0.00159 {2}	0.00161 {3.5}	0.00164 {5}	0.00167 {6}	0.00181 {8}	0.00161{3.5}	0.00156 {1}	0.00201 {9}	0.00238 {12}	0.00215 {10}	0.00418 {15}	0.00171 {7}	0.00266 {13}	0.00222 {11}	0.00296 {14}
270	MRE ($\hat{\zeta}$)	0.02342 {6}	0.02262 {2}	0.02331 {4}	0.02349 {7}	0.02385 {8}	0.02323 {3}	0.02215 {1}	0.02513 {9}	0.02794 {12}	0.02711 {11}	0.0363 {15}	0.0234 {5}	0.02947 {13}	0.02677 {10}	0.03113 {14}
270	D_{abs}	0.01372{4.5}	0.01333 {2}	0.01379 {6}	0.01384 {7}	0.01397 {8}	0.01368 {3}	0.01301 {1}	0.01485 {9}	0.0165 {12}	0.016 {11}	0.02112 {15}	0.01372{4.5}	0.01737 {13}	0.01574 {10}	0.01836 {14}
270	D_{max}	0.01993 {4}	0.01934 {2}	0.01994 {5}	0.02011 {7}	0.02035 {8}	0.01985 {3}	0.01889 {1}	0.0215 {9}	0.02391 {12}	0.0232 {11}	0.03071 {15}	0.01995 {6}	0.02522 {13}	0.02282 {10}	0.02666 {14}
270	ASAE	0.0131 {9}	0.01169 {2}	0.01195 {4}	0.01289 {8}	0.0119 {3}	0.01277 {7}	0.01222 {5}	0.01153 {1}	0.01507 {12}	0.01457 {10}	0.0153 {13}	0.01232 {6}	0.01591 {14}	0.0148 {11}	0.0161 {15}

(Continued)

Table 5 (continued)

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
270	$\sum Ranks$	31.5 $^{\{5\}}$	13.5 $^{\{2\}}$	28 $^{\{4\}}$	42 $^{\{7\}}$	43 $^{\{8\}}$	22.5 $^{\{3\}}$	10 $^{\{1\}}$	46 $^{\{9\}}$	72 $^{\{12\}}$	64 $^{\{11\}}$	88 $^{\{15\}}$	33.5 $^{\{6\}}$	79 $^{\{13\}}$	62 $^{\{10\}}$	85 $^{\{14\}}$
350	BIAS $(\hat{\zeta})$	0.02896 $^{\{6\}}$	0.02784 $^{\{2\}}$	0.0294 $^{\{7\}}$	0.02701 $^{\{1\}}$	0.02865 $^{\{4\}}$	0.02892 $^{\{5\}}$	0.02812 $^{\{3\}}$	0.03054 $^{\{9\}}$	0.03439 $^{\{12\}}$	0.03348 $^{\{11\}}$	0.04731 $^{\{15\}}$	0.03007 $^{\{8\}}$	0.03702 $^{\{13\}}$	0.03342 $^{\{10\}}$	0.03742 $^{\{14\}}$
350	MSE $(\hat{\zeta})$	0.00124 $^{\{2.5\}}$	0.00128 $^{\{4\}}$	0.00138 $^{\{6.5\}}$	0.00115 $^{\{1\}}$	0.00136 $^{\{5\}}$	0.00138 $^{\{6.5\}}$	0.00124 $^{\{2.5\}}$	0.00141 $^{\{8\}}$	0.00186 $^{\{12\}}$	0.00177 $^{\{10\}}$	0.00384 $^{\{15\}}$	0.00145 $^{\{9\}}$	0.00229 $^{\{14\}}$	0.00179 $^{\{11\}}$	0.00215 $^{\{13\}}$
350	MRE $(\hat{\zeta})$	0.02069 $^{\{6\}}$	0.01988 $^{\{2\}}$	0.021 $^{\{7\}}$	0.01929 $^{\{1\}}$	0.02046 $^{\{4\}}$	0.02066 $^{\{5\}}$	0.02008 $^{\{3\}}$	0.02181 $^{\{9\}}$	0.02457 $^{\{12\}}$	0.02391 $^{\{11\}}$	0.0338 $^{\{15\}}$	0.02148 $^{\{8\}}$	0.02645 $^{\{13\}}$	0.02387 $^{\{10\}}$	0.02673 $^{\{14\}}$
350	D_{abs}	0.01209 $^{\{5\}}$	0.01166 $^{\{2\}}$	0.01234 $^{\{7\}}$	0.0114 $^{\{1\}}$	0.01203 $^{\{4\}}$	0.01217 $^{\{6\}}$	0.01182 $^{\{3\}}$	0.01282 $^{\{9\}}$	0.0145 $^{\{12\}}$	0.01407 $^{\{11\}}$	0.01957 $^{\{15\}}$	0.01268 $^{\{8\}}$	0.01556 $^{\{13\}}$	0.01406 $^{\{10\}}$	0.01578 $^{\{14\}}$
350	D_{max}	0.01755 $^{\{5\}}$	0.01692 $^{\{2\}}$	0.0179 $^{\{7\}}$	0.01652 $^{\{1\}}$	0.01746 $^{\{4\}}$	0.01765 $^{\{6\}}$	0.01714 $^{\{3\}}$	0.01862 $^{\{9\}}$	0.02102 $^{\{12\}}$	0.0204 $^{\{11\}}$	0.02849 $^{\{15\}}$	0.01837 $^{\{8\}}$	0.02258 $^{\{13\}}$	0.02039 $^{\{10\}}$	0.02289 $^{\{14\}}$
350	ASAE	0.0115 $^{\{9\}}$	0.01043 $^{\{3\}}$	0.0103 $^{\{2\}}$	0.01132 $^{\{8\}}$	0.01045 $^{\{4\}}$	0.01111 $^{\{7\}}$	0.01054 $^{\{5\}}$	0.00978 $^{\{1\}}$	0.01333 $^{\{13\}}$	0.01298 $^{\{11\}}$	0.01329 $^{\{12\}}$	0.01067 $^{\{6\}}$	0.01417 $^{\{14\}}$	0.0128 $^{\{10\}}$	0.01437 $^{\{15\}}$
350	$\sum Ranks$	33.5 $^{\{5\}}$	15 $^{\{2\}}$	36.5 $^{\{7\}}$	13 $^{\{1\}}$	25 $^{\{4\}}$	35.5 $^{\{6\}}$	19.5 $^{\{3\}}$	45 $^{\{8\}}$	73 $^{\{12\}}$	65 $^{\{11\}}$	87 $^{\{15\}}$	47 $^{\{9\}}$	80 $^{\{13\}}$	61 $^{\{10\}}$	84 $^{\{14\}}$

Table 6: Numerical values of simulation measures for $\zeta = 2.5$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS ($\hat{\zeta}$)	$0.21354^{\{2\}}$	$0.24946^{\{7\}}$	$0.24251^{\{5\}}$	$0.21164^{\{1\}}$	$0.25897^{\{10\}}$	$0.23111^{\{3\}}$	$0.23904^{\{4\}}$	$0.27668^{\{12\}}$	$0.26516^{\{11\}}$	$0.25753^{\{9\}}$	$0.42535^{\{15\}}$	$0.25209^{\{8\}}$	$0.28956^{\{13\}}$	$0.24847^{\{6\}}$	$0.29122^{\{14\}}$
25	MSE ($\hat{\zeta}$)	$0.08139^{\{2\}}$	$0.10678^{\{7\}}$	$0.09979^{\{5\}}$	$0.07332^{\{1\}}$	$0.11514^{\{10\}}$	$0.08857^{\{3\}}$	$0.09721^{\{4\}}$	$0.14456^{\{14\}}$	$0.11979^{\{11\}}$	$0.11234^{\{9\}}$	$0.35458^{\{15\}}$	$0.10887^{\{8\}}$	$0.13698^{\{12\}}$	$0.10355^{\{6\}}$	$0.14113^{\{13\}}$
25	MRE ($\hat{\zeta}$)	$0.08542^{\{2\}}$	$0.09978^{\{7\}}$	$0.097^{\{5\}}$	$0.08466^{\{1\}}$	$0.10359^{\{10\}}$	$0.09244^{\{3\}}$	$0.09562^{\{4\}}$	$0.11067^{\{12\}}$	$0.10607^{\{11\}}$	$0.10301^{\{9\}}$	$0.17014^{\{15\}}$	$0.10084^{\{8\}}$	$0.11582^{\{13\}}$	$0.09939^{\{6\}}$	$0.11649^{\{14\}}$
25	D_{abs}	$0.03823^{\{1\}}$	$0.04553^{\{6\}}$	$0.04374^{\{5\}}$	$0.04043^{\{2\}}$	$0.04574^{\{7\}}$	$0.04229^{\{3\}}$	$0.0433^{\{4\}}$	$0.04821^{\{11\}}$	$0.04983^{\{12\}}$	$0.04806^{\{10\}}$	$0.06945^{\{15\}}$	$0.04576^{\{8\}}$	$0.0549^{\{13\}}$	$0.04668^{\{9\}}$	$0.05496^{\{14\}}$
25	D_{max}	$0.05625^{\{1\}}$	$0.0669^{\{6\}}$	$0.06466^{\{5\}}$	$0.05961^{\{2\}}$	$0.06763^{\{8\}}$	$0.06226^{\{3\}}$	$0.0639^{\{4\}}$	$0.07165^{\{11\}}$	$0.07329^{\{12\}}$	$0.07044^{\{10\}}$	$0.10308^{\{15\}}$	$0.06717^{\{7\}}$	$0.08142^{\{14\}}$	$0.06818^{\{9\}}$	$0.08119^{\{13\}}$
25	ASAE	$0.04799^{\{8\}}$	$0.04382^{\{2\}}$	$0.0439^{\{3\}}$	$0.04925^{\{9\}}$	$0.04359^{\{1\}}$	$0.04495^{\{4\}}$	$0.04499^{\{5\}}$	$0.04773^{\{7\}}$	$0.05913^{\{12\}}$	$0.0552^{\{11\}}$	$0.06709^{\{15\}}$	$0.04748^{\{6\}}$	$0.06021^{\{14\}}$	$0.05409^{\{10\}}$	$0.05985^{\{13\}}$
25	$\sum Ranks$	$16^{\{1.5\}}$	$35^{\{6\}}$	$28^{\{5\}}$	$16^{\{1.5\}}$	$46^{\{8.5\}}$	$19^{\{3\}}$	$25^{\{4\}}$	$67^{\{11\}}$	$69^{\{12\}}$	$58^{\{10\}}$	$90^{\{15\}}$	$45^{\{7\}}$	$79^{\{13\}}$	$46^{\{8.5\}}$	$81^{\{14\}}$
55	BIAS ($\hat{\zeta}$)	$0.14844^{\{1\}}$	$0.165^{\{7\}}$	$0.17477^{\{9\}}$	$0.15509^{\{2\}}$	$0.17679^{\{10\}}$	$0.16206^{\{4\}}$	$0.16143^{\{3\}}$	$0.18419^{\{11\}}$	$0.18765^{\{12\}}$	$0.169^{\{8\}}$	$0.27615^{\{15\}}$	$0.16453^{\{6\}}$	$0.20237^{\{13\}}$	$0.16286^{\{5\}}$	$0.20452^{\{14\}}$
55	MSE ($\hat{\zeta}$)	$0.03666^{\{1\}}$	$0.04268^{\{5\}}$	$0.04973^{\{9\}}$	$0.03692^{\{2\}}$	$0.05156^{\{10\}}$	$0.04477^{\{7\}}$	$0.04168^{\{3\}}$	$0.0547^{\{11\}}$	$0.06167^{\{12\}}$	$0.04512^{\{8\}}$	$0.15844^{\{15\}}$	$0.04472^{\{6\}}$	$0.063^{\{13\}}$	$0.04209^{\{4\}}$	$0.07199^{\{14\}}$
55	MRE ($\hat{\zeta}$)	$0.05938^{\{1\}}$	$0.066^{\{7\}}$	$0.06991^{\{9\}}$	$0.06203^{\{2\}}$	$0.07072^{\{10\}}$	$0.06482^{\{4\}}$	$0.06457^{\{3\}}$	$0.07367^{\{11\}}$	$0.07506^{\{12\}}$	$0.0676^{\{8\}}$	$0.11046^{\{15\}}$	$0.06581^{\{6\}}$	$0.08095^{\{13\}}$	$0.06514^{\{5\}}$	$0.08181^{\{14\}}$
55	D_{abs}	$0.02682^{\{1\}}$	$0.03038^{\{6\}}$	$0.03218^{\{9.5\}}$	$0.02903^{\{2\}}$	$0.03218^{\{9.5\}}$	$0.02999^{\{4\}}$	$0.02998^{\{3\}}$	$0.03382^{\{11\}}$	$0.03495^{\{12\}}$	$0.03206^{\{8\}}$	$0.04592^{\{15\}}$	$0.03032^{\{5\}}$	$0.03861^{\{14\}}$	$0.03062^{\{7\}}$	$0.03852^{\{13\}}$
55	D_{max}	$0.03985^{\{1\}}$	$0.04481^{\{5\}}$	$0.04733^{\{9\}}$	$0.04316^{\{2\}}$	$0.04788^{\{10\}}$	$0.04435^{\{3.5\}}$	$0.04435^{\{3.5\}}$	$0.05003^{\{11\}}$	$0.0516^{\{12\}}$	$0.04726^{\{8\}}$	$0.06882^{\{15\}}$	$0.04491^{\{6\}}$	$0.05707^{\{13\}}$	$0.04518^{\{7\}}$	$0.05726^{\{14\}}$
55	ASAE	$0.03016^{\{8\}}$	$0.02758^{\{1\}}$	$0.02805^{\{2\}}$	$0.03202^{\{9\}}$	$0.02913^{\{4\}}$	$0.02931^{\{5\}}$	$0.02896^{\{3\}}$	$0.02954^{\{6\}}$	$0.03745^{\{12\}}$	$0.036^{\{11\}}$	$0.04437^{\{15\}}$	$0.02957^{\{7\}}$	$0.03843^{\{13\}}$	$0.03357^{\{10\}}$	$0.03997^{\{14\}}$
55	$\sum Ranks$	$13^{\{1\}}$	$31^{\{5\}}$	$47.5^{\{8\}}$	$19^{\{3\}}$	$53.5^{\{10\}}$	$27.5^{\{4\}}$	$18.5^{\{2\}}$	$61^{\{11\}}$	$72^{\{12\}}$	$51^{\{9\}}$	$90^{\{15\}}$	$36^{\{6\}}$	$79^{\{13\}}$	$38^{\{7\}}$	$83^{\{14\}}$
100	BIAS ($\hat{\zeta}$)	$0.1113^{\{1\}}$	$0.11754^{\{3\}}$	$0.12597^{\{8\}}$	$0.11419^{\{2\}}$	$0.12193^{\{6\}}$	$0.11802^{\{4\}}$	$0.12416^{\{7\}}$	$0.13909^{\{11\}}$	$0.14764^{\{12\}}$	$0.13219^{\{10\}}$	$0.22934^{\{15\}}$	$0.12809^{\{9\}}$	$0.15289^{\{13\}}$	$0.12105^{\{5\}}$	$0.15909^{\{14\}}$
100	MSE ($\hat{\zeta}$)	$0.01994^{\{1\}}$	$0.02128^{\{2\}}$	$0.02515^{\{8\}}$	$0.02166^{\{3\}}$	$0.0235^{\{6\}}$	$0.02201^{\{4\}}$	$0.02416^{\{7\}}$	$0.03196^{\{11\}}$	$0.03499^{\{12\}}$	$0.02731^{\{9\}}$	$0.11637^{\{15\}}$	$0.02796^{\{10\}}$	$0.04127^{\{14\}}$	$0.02312^{\{5\}}$	$0.0398^{\{13\}}$
100	MRE ($\hat{\zeta}$)	$0.04452^{\{1\}}$	$0.04702^{\{3\}}$	$0.05039^{\{8\}}$	$0.04568^{\{2\}}$	$0.04877^{\{6\}}$	$0.04721^{\{4\}}$	$0.04966^{\{7\}}$	$0.05563^{\{11\}}$	$0.05905^{\{12\}}$	$0.05288^{\{10\}}$	$0.09174^{\{15\}}$	$0.05124^{\{9\}}$	$0.06116^{\{13\}}$	$0.04842^{\{5\}}$	$0.06364^{\{14\}}$
100	D_{abs}	$0.02039^{\{1\}}$	$0.02186^{\{3\}}$	$0.02324^{\{8\}}$	$0.02131^{\{2\}}$	$0.02247^{\{5\}}$	$0.02198^{\{4\}}$	$0.02301^{\{7\}}$	$0.02558^{\{11\}}$	$0.0277^{\{12\}}$	$0.02476^{\{10\}}$	$0.03802^{\{15\}}$	$0.02356^{\{9\}}$	$0.02838^{\{13\}}$	$0.02285^{\{6\}}$	$0.02975^{\{14\}}$
100	D_{max}	$0.03033^{\{1\}}$	$0.03234^{\{3\}}$	$0.03434^{\{8\}}$	$0.0316^{\{2\}}$	$0.03341^{\{5\}}$	$0.03261^{\{4\}}$	$0.03405^{\{7\}}$	$0.03787^{\{11\}}$	$0.041^{\{12\}}$	$0.03667^{\{10\}}$	$0.05735^{\{15\}}$	$0.03501^{\{9\}}$	$0.04236^{\{13\}}$	$0.03389^{\{6\}}$	$0.04429^{\{14\}}$
100	ASAE	$0.02208^{\{9\}}$	$0.02003^{\{1\}}$	$0.02063^{\{4\}}$	$0.02201^{\{8\}}$	$0.02045^{\{3\}}$	$0.0209^{\{5\}}$	$0.02029^{\{2\}}$	$0.02197^{\{7\}}$	$0.02758^{\{12\}}$	$0.02663^{\{11\}}$	$0.03362^{\{15\}}$	$0.02169^{\{6\}}$	$0.02852^{\{13\}}$	$0.02507^{\{10\}}$	$0.02961^{\{14\}}$
100	$\sum Ranks$	$14^{\{1\}}$	$15^{\{2\}}$	$44^{\{8\}}$	$19^{\{3\}}$	$31^{\{5\}}$	$25^{\{4\}}$	$37^{\{6.5\}}$	$62^{\{11\}}$	$72^{\{12\}}$	$60^{\{10\}}$	$90^{\{15\}}$	$52^{\{9\}}$	$79^{\{13\}}$	$37^{\{6.5\}}$	$83^{\{14\}}$
140	BIAS ($\hat{\zeta}$)	$0.09709^{\{3\}}$	$0.10086^{\{6\}}$	$0.10327^{\{8\}}$	$0.09693^{\{2\}}$	$0.09791^{\{4\}}$	$0.0984^{\{5\}}$	$0.09306^{\{1\}}$	$0.11747^{\{11\}}$	$0.11817^{\{12\}}$	$0.11036^{\{9\}}$	$0.16749^{\{15\}}$	$0.11285^{\{10\}}$	$0.12556^{\{14\}}$	$0.10213^{\{7\}}$	$0.12382^{\{13\}}$
140	MSE ($\hat{\zeta}$)	$0.01421^{\{1\}}$	$0.01674^{\{7\}}$	$0.0178^{\{8\}}$	$0.01427^{\{3\}}$	$0.01567^{\{5\}}$	$0.0149^{\{4\}}$	$0.01424^{\{2\}}$	$0.02258^{\{12\}}$	$0.02207^{\{11\}}$	$0.01938^{\{9\}}$	$0.04811^{\{15\}}$	$0.01952^{\{10\}}$	$0.02568^{\{14\}}$	$0.01643^{\{6\}}$	$0.02413^{\{13\}}$
140	MRE ($\hat{\zeta}$)	$0.03884^{\{3\}}$	$0.04035^{\{6\}}$	$0.04131^{\{8\}}$	$0.03877^{\{2\}}$	$0.03916^{\{4\}}$	$0.03936^{\{5\}}$	$0.03723^{\{1\}}$	$0.04699^{\{11\}}$	$0.04727^{\{12\}}$	$0.04415^{\{9\}}$	$0.067^{\{15\}}$	$0.04514^{\{10\}}$	$0.05022^{\{14\}}$	$0.04085^{\{7\}}$	$0.04953^{\{13\}}$
140	D_{abs}	$0.01793^{\{2\}}$	$0.01858^{\{6\}}$	$0.01902^{\{7\}}$	$0.01815^{\{4\}}$	$0.01803^{\{3\}}$	$0.01822^{\{5\}}$	$0.01727^{\{1\}}$	$0.02178^{\{11\}}$	$0.0221^{\{12\}}$	$0.02062^{\{9\}}$	$0.03066^{\{15\}}$	$0.02099^{\{10\}}$	$0.02346^{\{13\}}$	$0.01934^{\{8\}}$	$0.02355^{\{14\}}$
140	D_{max}	$0.0265^{\{2\}}$	$0.02758^{\{6\}}$	$0.0283^{\{7\}}$	$0.02702^{\{5\}}$	$0.02672^{\{3\}}$	$0.02694^{\{4\}}$	$0.02562^{\{1\}}$	$0.03228^{\{11\}}$	$0.03278^{\{12\}}$	$0.03044^{\{9\}}$	$0.04517^{\{15\}}$	$0.03106^{\{10\}}$	$0.03493^{\{14\}}$	$0.02855^{\{8\}}$	$0.03487^{\{13\}}$
140	ASAE	$0.01833^{\{7\}}$	$0.01647^{\{1\}}$	$0.01725^{\{5\}}$	$0.01877^{\{9\}}$	$0.01674^{\{2\}}$	$0.01722^{\{4\}}$	$0.01712^{\{3\}}$	$0.01798^{\{6\}}$	$0.02336^{\{12\}}$	$0.02086^{\{11\}}$	$0.02619^{\{15\}}$	$0.01835^{\{8\}}$	$0.02372^{\{13\}}$	$0.02053^{\{10\}}$	$0.02472^{\{14\}}$
140	$\sum Ranks$	$18^{\{2\}}$	$32^{\{6\}}$	$43^{\{7\}}$	$25^{\{4\}}$	$21^{\{3\}}$	$27^{\{5\}}$	$9^{\{1\}}$	$62^{\{11\}}$	$71^{\{12\}}$	$56^{\{9\}}$	$90^{\{15\}}$	$58^{\{10\}}$	$82^{\{14\}}$	$46^{\{8\}}$	$80^{\{13\}}$
180	BIAS ($\hat{\zeta}$)	$0.08296^{\{3\}}$	$0.08772^{\{4\}}$	$0.09408^{\{9\}}$	$0.07748^{\{1\}}$	$0.09238^{\{7\}}$	$0.08119^{\{2\}}$	$0.08779^{\{5\}}$	$0.09941^{\{10\}}$	$0.10721^{\{12\}}$	$0.09313^{\{8\}}$	$0.15805^{\{15\}}$	$0.09986^{\{11\}}$	$0.10957^{\{13\}}$	$0.08956^{\{6\}}$	$0.11166^{\{14\}}$
180	MSE ($\hat{\zeta}$)	$0.01076^{\{3\}}$	$0.01204^{\{4\}}$	$0.01426^{\{9\}}$	$0.00945^{\{1\}}$	$0.01325^{\{7\}}$	$0.01069^{\{2\}}$	$0.01234^{\{5\}}$	$0.01531^{\{10\}}$	$0.0183^{\{12\}}$	$0.01373^{\{8\}}$	$0.04458^{\{15\}}$	$0.01626^{\{11\}}$	$0.01894^{\{13\}}$	$0.01298^{\{6\}}$	$0.01975^{\{14\}}$
180	MRE ($\hat{\zeta}$)	$0.03318^{\{3\}}$	$0.03509^{\{4\}}$	$0.03763^{\{9\}}$	$0.03099^{\{1\}}$	$0.03695^{\{7\}}$	$0.03248^{\{2\}}$	$0.03512^{\{5\}}$	$0.03976^{\{10\}}$	$0.04288^{\{12\}}$	$0.03725^{\{8\}}$	$0.06322^{\{15\}}$	$0.03994^{\{11\}}$	$0.04383^{\{13\}}$	$0.03583^{\{6\}}$	$0.04466^{\{14\}}$
180	D_{abs}	$0.01529^{\{3\}}$	$0.0163^{\{4\}}$	$0.01742^{\{8\}}$	$0.0146^{\{1\}}$	$0.01718^{\{7\}}$	$0.01512^{\{2\}}$	$0.01642^{\{5\}}$	$0.01853^{\{11\}}$	$0.01994^{\{12\}}$	$0.0175^{\{9\}}$	$0.02817^{\{15\}}$	$0.0184^{\{10\}}$	$0.02057^{\{13\}}$	$0.01688^{\{6\}}$	$0.02097^{\{14\}}$
180	D_{max}	$0.02268^{\{3\}}$	$0.02415^{\{4\}}$	$0.02582^{\{8\}}$	$0.02162^{\{1\}}$	$0.02543^{\{7\}}$	$0.02243^{\{2\}}$	$0.02432^{\{5\}}$	$0.02742^{\{11\}}$	$0.0296^{\{12\}}$	$0.02591^{\{9\}}$	$0.04201^{\{15\}}$	$0.02732^{\{10\}}$	$0.03052^{\{13\}}$	$0.02499^{\{6\}}$	$0.03115^{\{14\}}$
180	ASAE	$0.01611^{\{8\}}$	$0.01497^{\{2.5\}}$	$0.01488^{\{1\}}$	$0.01607^{\{7\}}$	$0.01537^{\{5\}}$	$0.01509^{\{4\}}$	$0.01497^{\{2.5\}}$	$0.01581^{\{6\}}$	$0.01989^{\{12\}}$	$0.01808^{\{10\}}$	$0.0239^{\{15\}}$	$0.01615^{\{9\}}$	$0.02084^{\{13\}}$	$0.01859^{\{11\}}$	$0.02095^{\{14\}}$
180	$\sum Ranks$	$23^{\{4\}}$	$22.5^{\{3\}}$	$44^{\{8\}}$	$12^{\{1\}}$	$40^{\{6\}}$	$14^{\{2\}}$	$27.5^{\{5\}}$	$58^{\{10\}}$	$72^{\{12\}}$	$52^{\{9\}}$	$90^{\{15\}}$	$62^{\{11\}}$	$78^{\{13\}}$	$41^{\{7\}}$	$84^{\{14\}}$
225	BIAS ($\hat{\zeta}$)	$0.06991^{\{1\}}$	$0.07528^{\{2\}}$	$0.07954^{\{5\}}$	$0.07573^{\{3\}}$	$0.08889^{\{11\}}$	$0.07812^{\{4\}}$	$0.08327^{\{8\}}$	$0.08876^{\{10\}}$	$0.09222^{\{12\}}$	$0.08488^{\{9\}}$	$0.13055^{\{15\}}$	$0.08151^{\{6\}}$	$0.10136^{\{13\}}$	$0.08226^{\{7\}}$	$0.10256^{\{14\}}$
225	MSE ($\hat{\zeta}$)	$0.00771^{\{1\}}$	$0.00887^{\{3\}}$	$0.00998^{\{5\}}$	$0.00876^{\{2\}}$	$0.01223^{\{11\}}$	$0.0099^{\{4\}}$	$0.01043^{\{7\}}$	$0.01199^{\{10\}}$	$0.01388^{\{12\}}$	$0.0116^{\{9\}}$	$0.02863^{\{15\}}$	$0.01114^{\{8\}}$	$0.01605^{\{13\}}$	$0.01036^{\{6\}}$	$0.0167^{\{14\}}$
225	MRE ($\hat{\zeta}$)	$0.02796^{\{1\}}$	$0.03011^{\{2\}}$	$0.03182^{\{5\}}$	$0.03029^{\{3\}}$	$0.03556^{\{11\}}$	$0.03125^{\{4\}}$	$0.03331^{\{8\}}$	$0.03551^{\{10\}}$	$0.03689^{\{12\}}$	$0.03395^{\{9\}}$	$0.05222^{\{15\}}$	$0.0326^{\{6\}}$	$0.04055^{\{13\}}$	$0.0329^{\{7\}}$	$0.04102^{\{14\}}$
225	D_{abs}	$0.01298^{\{1\}}$	$0.01397^{\{2\}}$	$0.0148^{\{5\}}$	$0.01412^{\{3\}}$	$0.01636^{\{10\}}$	$0.01446^{\{4\}}$	$0.01549^{\{8\}}$	$0.01651^{\{11\}}$	$0.01731^{\{12\}}$	$0.0159^{\{9\}}$	$0.02394^{\{15\}}$	$0.01504^{\{6\}}$	$0.01909^{\{13\}}$	$0.01544^{\{7\}}$	$0.01928^{\{14\}}$
225	D_{max}	$0.01919^{\{1\}}$	$0.0207^{\{2\}}$	$0.02192^{\{5\}}$	$0.02097^{\{3\}}$	$0.02431^{\{10\}}$	$0.02144^{\{4\}}$	$0.02295^{\{8\}}$	$0.02446^{\{11\}}$	$0.02571^{\{12\}}$	$0.02361^{\{9\}}$	$0.03549^{\{15\}}$	$0.02233^{\{6\}}$	$0.02833^{\{13\}}$	$0.02286^{\{7\}}$	$0.02859^{\{14\}}$
225	ASAE	$0.01403^{\{6\}}$	$0.01365^{\{5\}}$	$0.01307^{\{1\}}$	$0.01475^{\{9\}}$	$0.01341^{\{3\}}$	$0.01356^{\{4\}}$	$0.01313^{\{2\}}$	$0.01414^{\{7\}}$	$0.01811^{\{12\}}$	$0.01735^{\{11\}}$	$0.01995^{\{15\}}$	$0.01434^{\{8\}}$	$0.01883^{\{13\}}$	$0.01618^{\{10\}}$	$0.01941^{\{14\}}$
225	$\sum Ranks$	$11^{\{1\}}$	$16^{\{2\}}$	$26^{\{5\}}$	$23^{\{3\}}$	$56^{\{9.5\}}$	$24^{\{4\}}$	$41^{\{7\}}$	$59^{\{11\}}$	$72^{\{12\}}$	$56^{\{9.5\}}$	$90^{\{15\}}$	$40^{\{6\}}$	$78^{\{13\}}$	$44^{\{8\}}$	$84^{\{14\}}$
270	BIAS ($\hat{\zeta}$)	$0.06252^{\{1\}}$	$0.06878^{\{4\}}$	$0.07581^{\{7\}}$	$0.0671^{\{3\}}$	$0.07737^{\{8\}}$	$0.0666^{\{2\}}$	$0.07002^{\{5\}}$	$0.08494^{\{12\}}$	$0.08377^{\{11\}}$	$0.08076^{\{10\}}$	$0.11486^{\{15\}}$	$0.07765^{\{9\}}$	$0.09655^{\{14\}}$	$0.07554^{\{6\}}$	$0.0931^{\{13\}}$
270	MSE ($\hat{\zeta}$)	$0.00649^{\{1\}}$	$0.00735^{\{4\}}$	$0.00884^{\{6\}}$	$0.00708^{\{3\}}$	$0.0093^{\{8\}}$	$0.007^{\{2\}}$	$0.00766^{\{5\}}$	$0.01176^{\{12\}}$	$0.01127^{\{11\}}$	$0.00999^{\{10\}}$	$0.02201^{\{15\}}$	$0.00974^{\{9\}}$	$0.01538^{\{14\}}$	$0.00895^{\{7\}}$	$0.01389^{\{13\}}$
270	MRE ($\hat{\zeta}$)	$0.02501^{\{1\}}$	$0.02751^{\{4\}}$	$0.03032^{\{7\}}$	$0.02684^{\{3\}}$	$0.03095^{\{8\}}$	$0.02664^{\{2\}}$	$0.02801^{\{5\}}$	$0.03398^{\{12\}}$	$0.03351^{\{11\}}$	$0.0323^{\{10\}}$	$0.04594^{\{15\}}$	$0.03106^{\{9\}}$	$0.03862^{\{14\}}$	$0.03022^{\{6\}}$	$0.03724^{\{13\}}$
270	D_{abs}	$0.01154^{\{1\}}$	$0.01283^{\{4\}}$	$0.01411^{\{6\}}$	$0.01257^{\{3\}}$	$0.01453^{\{9\}}$	$0.01243^{\{2\}}$	$0.01299^{\{5\}}$	$0.01576^{\{12\}}$	$0.01558^{\{11\}}$	$0.01515^{\{10\}}$	$0.02107^{\{15\}}$	$0.01434^{\{8\}}$	$0.01804^{\{14\}}$	$0.01417^{\{7\}}$	$0.01755^{\{13\}}$
270	D_{max}	$0.01712^{\{1\}}$	$0.01903^{\{4\}}$	$0.02094^{\{6\}}$	$0.01864^{\{3\}}$	$0.02151^{\{9\}}$	$0.01846^{\{2\}}$	$0.01928^{\{5\}}$	$0.02339^{\{12\}}$	$0.02311^{\{11\}}$	$0.02247^{\{10\}}$	$0.0313^{\{15\}}$	$0.02132^{\{8\}}$	$0.02678^{\{14\}}$	$0.02102^{\{7\}}$	$0.02599^{\{13\}}$
270	ASAE	$0.01299^{\{9\}}$	$0.01191^{\{2\}}$	$0.01202^{\{3\}}$	$0.0127^{\{7\}}$	$0.01219^{\{4\}}$	$0.01234^{\{5\}}$	$0.01185^{\{1\}}$	$0.01253^{\{6\}}$	$0.01567^{\{12\}}$	$0.01541^{\{11\}}$	$0.01804^{\{15\}}$	$0.01275^{\{8\}}$	$0.01802^{\{14\}}$	$0.01501^{\{10\}}$	$0.01734^{\{13\}}$

(Continued)

Table 6 (continued)

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
270	$\sum$ *Ranks*	$14^{\{1\}}$	$22^{\{3.5\}}$	$35^{\{6\}}$	$22^{\{3.5\}}$	$46^{\{8\}}$	$15^{\{2\}}$	$26^{\{5\}}$	$66^{\{11\}}$	$67^{\{12\}}$	$61^{\{10\}}$	$90^{\{15\}}$	$51^{\{9\}}$	$84^{\{14\}}$	$43^{\{7\}}$	$78^{\{13\}}$
350	BIAS ($\hat{\zeta}$)	$0.05462^{\{1\}}$	$0.06022^{\{3\}}$	$0.06784^{\{8\}}$	$0.05684^{\{2\}}$	$0.06596^{\{6\}}$	$0.06324^{\{5\}}$	$0.0617^{\{4\}}$	$0.07149^{\{10\}}$	$0.07335^{\{12\}}$	$0.07182^{\{11\}}$	$0.10313^{\{15\}}$	$0.06786^{\{9\}}$	$0.08064^{\{14\}}$	$0.06723^{\{7\}}$	$0.0803^{\{13\}}$
350	MSE ($\hat{\zeta}$)	$0.00481^{\{1\}}$	$0.00596^{\{3\}}$	$0.00706^{\{7\}}$	$0.00513^{\{2\}}$	$0.0067^{\{6\}}$	$0.00636^{\{5\}}$	$0.00602^{\{4\}}$	$0.00818^{\{11\}}$	$0.00865^{\{12\}}$	$0.00803^{\{10\}}$	$0.0186^{\{15\}}$	$0.00724^{\{8\}}$	$0.01037^{\{14\}}$	$0.00741^{\{9\}}$	$0.01019^{\{13\}}$
350	MRE ($\hat{\zeta}$)	$0.02185^{\{1\}}$	$0.02409^{\{3\}}$	$0.02714^{\{8\}}$	$0.02274^{\{2\}}$	$0.02638^{\{6\}}$	$0.0253^{\{5\}}$	$0.02468^{\{4\}}$	$0.02859^{\{10\}}$	$0.02934^{\{12\}}$	$0.02873^{\{11\}}$	$0.04125^{\{15\}}$	$0.02715^{\{9\}}$	$0.03226^{\{14\}}$	$0.02689^{\{7\}}$	$0.03212^{\{13\}}$
350	D_{abs}	$0.01012^{\{1\}}$	$0.01118^{\{3\}}$	$0.01261^{\{8\}}$	$0.01064^{\{2\}}$	$0.01224^{\{6\}}$	$0.01174^{\{5\}}$	$0.0115^{\{4\}}$	$0.01324^{\{10\}}$	$0.01361^{\{12\}}$	$0.01349^{\{11\}}$	$0.01882^{\{15\}}$	$0.01258^{\{7\}}$	$0.01517^{\{14\}}$	$0.01266^{\{9\}}$	$0.01508^{\{13\}}$
350	D_{max}	$0.01498^{\{1\}}$	$0.01659^{\{3\}}$	$0.01873^{\{8.5\}}$	$0.01577^{\{2\}}$	$0.01818^{\{6\}}$	$0.01741^{\{5\}}$	$0.01706^{\{4\}}$	$0.01966^{\{10\}}$	$0.0202^{\{12\}}$	$0.02^{\{11\}}$	$0.02801^{\{15\}}$	$0.01867^{\{7\}}$	$0.02248^{\{14\}}$	$0.01873^{\{8.5\}}$	$0.02241^{\{13\}}$
350	ASAE	$0.0112^{\{8.5\}}$	$0.01051^{\{2\}}$	$0.01012^{\{1\}}$	$0.0112^{\{8.5\}}$	$0.01071^{\{5\}}$	$0.01069^{\{4\}}$	$0.01054^{\{3\}}$	$0.01105^{\{7\}}$	$0.01373^{\{12\}}$	$0.01315^{\{11\}}$	$0.01572^{\{15\}}$	$0.01104^{\{6\}}$	$0.01517^{\{14\}}$	$0.01288^{\{10\}}$	$0.01512^{\{13\}}$
350	$\sum$ *Ranks*	$13.5^{\{1\}}$	$17^{\{2\}}$	$40.5^{\{7\}}$	$18.5^{\{3\}}$	$35^{\{6\}}$	$29^{\{5\}}$	$23^{\{4\}}$	$58^{\{10\}}$	$72^{\{12\}}$	$65^{\{11\}}$	$90^{\{15\}}$	$46^{\{8\}}$	$84^{\{14\}}$	$50.5^{\{9\}}$	$78^{\{13\}}$

Table 7: Numerical values of simulation measures for $\zeta = 3.5$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS ($\hat{\zeta}$)	0.37192 {2}	0.41929 {7}	0.437 {8}	0.35386 {1}	0.4723 {11}	0.41505 {6}	0.41429 {5}	0.49736 {13}	0.4419 {9}	0.39555 {3}	0.65816 {15}	0.44811 {10}	0.49291 {12}	0.40019 {4}	0.55122 {14}
25	MSE ($\hat{\zeta}$)	0.24792 {2}	0.30431 {7}	0.34104 {9}	0.21242 {1}	0.42219 {12}	0.28797 {6}	0.28016 {5}	0.44838 {13}	0.3691 {10}	0.26237 {3}	0.81528 {15}	0.32647 {8}	0.38469 {11}	0.27176 {4}	0.52091 {14}
25	MRE ($\hat{\zeta}$)	0.10626 {2}	0.1198 {7}	0.12486 {8}	0.1011 {1}	0.13494 {11}	0.11859 {6}	0.11837 {5}	0.1421 {13}	0.12626 {9}	0.11301 {3}	0.18805 {15}	0.12803 {10}	0.14083 {12}	0.11434 {4}	0.15749 {14}
25	D_{abs}	0.03834 {1}	0.04392 {5}	0.04517 {8}	0.03907 {2}	0.0478 {10}	0.04423 {6}	0.04385 {4}	0.05158 {12}	0.04671 {9}	0.0444 {7}	0.06383 {15}	0.04782 {11}	0.05531 {13}	0.04355 {3}	0.05997 {14}
25	D_{max}	0.05607 {1}	0.06444 {4}	0.06646 {8}	0.05697 {2}	0.07091 {11}	0.06521 {7}	0.0645 {5}	0.07546 {12}	0.06847 {9}	0.06508 {6}	0.09411 {15}	0.07013 {10}	0.08116 {13}	0.06394 {3}	0.08911 {14}
25	ASAE	0.04729 {7}	0.0439 {2}	0.0446 {4}	0.04573 {6}	0.04541 {5}	0.0444 {3}	0.043 {1}	0.04838 {9}	0.05446 {12}	0.05389 {11}	0.06544 {15}	0.04799 {8}	0.05977 {13}	0.05048 {10}	0.06295 {14}
25	$\sum$ *Ranks*	15 {2}	32 {5}	45 {8}	13 {1}	60 {11}	34 {7}	25 {3}	72 {12}	58 {10}	33 {6}	90 {15}	57 {9}	74 {13}	28 {4}	84 {14}
55	BIAS ($\hat{\zeta}$)	0.23741 {1}	0.2489 {3}	0.28597 {7}	0.24119 {2}	0.29024 {9}	0.25292 {4}	0.26519 {5}	0.33069 {12}	0.3144 {10}	0.27146 {6}	0.5012 {15}	0.32061 {11}	0.33407 {13}	0.28652 {8}	0.34959 {14}
55	MSE ($\hat{\zeta}$)	0.09776 {2}	0.10181 {3}	0.13246 {8}	0.08761 {1}	0.13805 {9}	0.10246 {4}	0.12202 {6}	0.19244 {13}	0.16609 {10}	0.11925 {5}	0.48774 {15}	0.18219 {12}	0.17834 {11}	0.12802 {7}	0.20916 {14}
55	MRE ($\hat{\zeta}$)	0.06783 {1}	0.07112 {3}	0.0817 {7}	0.06891 {2}	0.08292 {9}	0.07226 {4}	0.07577 {5}	0.09448 {12}	0.08983 {10}	0.07756 {6}	0.1432 {15}	0.0916 {11}	0.09545 {13}	0.08186 {8}	0.09988 {14}
55	D_{abs}	0.02494 {1}	0.02697 {3}	0.03071 {7}	0.02662 {2}	0.03148 {9}	0.02764 {4}	0.02791 {5}	0.03419 {11}	0.03424 {12}	0.02988 {6}	0.04902 {15}	0.03331 {10}	0.03754 {13}	0.03139 {8}	0.03854 {14}
55	D_{max}	0.03683 {1}	0.03958 {3}	0.04525 {7}	0.03922 {2}	0.0463 {9}	0.04061 {4}	0.04144 {5}	0.05086 {12}	0.05045 {11}	0.04414 {6}	0.07278 {15}	0.04935 {10}	0.0552 {13}	0.04603 {8}	0.0568 {14}
55	ASAE	0.02926 {6}	0.02771 {1}	0.02785 {3}	0.02927 {7}	0.02839 {4}	0.02778 {2}	0.02852 {5}	0.03106 {9}	0.03611 {12}	0.03387 {11}	0.04692 {15}	0.0298 {8}	0.0367 {13}	0.03247 {10}	0.03816 {14}
55	$\sum$ *Ranks*	12 {1}	16 {2.5}	39 {6}	16 {2.5}	49 {8.5}	22 {4}	31 {5}	69 {12}	65 {11}	40 {7}	90 {15}	62 {10}	76 {13}	49 {8.5}	84 {14}
100	BIAS ($\hat{\zeta}$)	0.17584 {1}	0.20301 {5}	0.20708 {6}	0.18344 {2}	0.20936 {7}	0.18506 {3}	0.21082 {8}	0.23195 {12}	0.217 {11}	0.2128 {9}	0.36647 {15}	0.21612 {10}	0.2562 {14}	0.1968 {4}	0.25136 {13}
100	MSE ($\hat{\zeta}$)	0.05142 {1}	0.06572 {5}	0.07127 {7}	0.05294 {2}	0.06926 {6}	0.0548 {3}	0.07207 {9}	0.08604 {12}	0.07159 {8}	0.07544 {11}	0.24981 {15}	0.07504 {10}	0.10582 {14}	0.06077 {4}	0.09626 {13}
100	MRE ($\hat{\zeta}$)	0.05024 {1}	0.058 {5}	0.05916 {6}	0.05241 {2}	0.05982 {7}	0.05287 {3}	0.06023 {8}	0.06627 {12}	0.062 {11}	0.0608 {9}	0.10471 {15}	0.06175 {10}	0.0732 {14}	0.05623 {4}	0.07182 {13}
100	D_{abs}	0.01888 {1}	0.02206 {5}	0.02207 {6}	0.02025 {3}	0.02282 {8}	0.02012 {2}	0.02266 {7}	0.02497 {12}	0.02393 {11}	0.0233 {9}	0.03647 {15}	0.02336 {10}	0.02816 {14}	0.02178 {4}	0.02788 {13}
100	D_{max}	0.02783 {1}	0.03235 {5}	0.03278 {6}	0.02984 {3}	0.03364 {8}	0.02965 {2}	0.03339 {7}	0.03679 {12}	0.03527 {11}	0.03433 {9}	0.0545 {15}	0.03438 {10}	0.04158 {14}	0.03214 {4}	0.04113 {13}
100	ASAE	0.02128 {8}	0.01995 {2}	0.0197 {1}	0.02112 {6}	0.02012 {4}	0.02004 {3}	0.02017 {5}	0.02213 {9}	0.02577 {12}	0.02447 {11}	0.03376 {15}	0.02121 {7}	0.0268 {14}	0.02353 {10}	0.02661 {13}
100	$\sum$ *Ranks*	13 {1}	27 {4}	32 {6}	18 {3}	40 {7}	16 {2}	44 {8}	69 {12}	64 {11}	58 {10}	90 {15}	57 {9}	84 {14}	30 {5}	78 {13}
140	BIAS ($\hat{\zeta}$)	0.15171 {1}	0.16646 {5}	0.17081 {6}	0.16048 {2}	0.17402 {7}	0.16607 {4}	0.16223 {3}	0.20806 {12}	0.1867 {11}	0.18052 {10}	0.30265 {15}	0.17843 {9}	0.21995 {14}	0.17699 {8}	0.20826 {13}
140	MSE ($\hat{\zeta}$)	0.03859 {1}	0.0435 {4}	0.04926 {8}	0.04012 {2}	0.04892 {7}	0.04384 {5}	0.04321 {3}	0.06789 {12}	0.05818 {11}	0.0495 {9}	0.14773 {15}	0.05236 {10}	0.0832 {14}	0.04865 {6}	0.07082 {13}
140	MRE ($\hat{\zeta}$)	0.04335 {1}	0.04756 {5}	0.0488 {6}	0.04585 {2}	0.04972 {7}	0.04745 {4}	0.04635 {3}	0.05945 {12}	0.05334 {11}	0.05158 {10}	0.08647 {15}	0.05098 {9}	0.06284 {14}	0.05057 {8}	0.0595 {13}
140	D_{abs}	0.01632 {1}	0.01799 {4}	0.01831 {6}	0.01762 {3}	0.01873 {7}	0.01808 {5}	0.01759 {2}	0.0224 {12}	0.02046 {11}	0.01986 {10}	0.03193 {15}	0.01912 {8}	0.02404 {14}	0.01951 {9}	0.02297 {13}
140	D_{max}	0.02415 {1}	0.02655 {4}	0.02706 {6}	0.02599 {3}	0.02775 {7}	0.02665 {5}	0.02595 {2}	0.03317 {12}	0.03015 {11}	0.02927 {10}	0.04721 {15}	0.02828 {8}	0.0355 {14}	0.02874 {9}	0.03396 {13}
140	ASAE	0.01778 {7}	0.01639 {2}	0.01661 {3}	0.01755 {6}	0.01698 {5}	0.01674 {4}	0.01629 {1}	0.01831 {9}	0.02093 {12}	0.02019 {11}	0.02797 {15}	0.01784 {8}	0.02313 {13}	0.01988 {10}	0.02329 {14}
140	$\sum$ *Ranks*	12 {1}	24 {4}	35 {6}	18 {3}	40 {7}	27 {5}	14 {2}	69 {12}	67 {11}	60 {10}	90 {15}	52 {9}	83 {14}	50 {8}	79 {13}
180	BIAS ($\hat{\zeta}$)	0.1303 {1}	0.13937 {3}	0.1552 {6}	0.13991 {4}	0.16633 {11}	0.13832 {2}	0.1519 {5}	0.17397 {12}	0.16459 {10}	0.15826 {7}	0.25801 {15}	0.16168 {8}	0.18654 {13}	0.16364 {9}	0.20087 {14}
180	MSE ($\hat{\zeta}$)	0.02875 {1}	0.03203 {4}	0.03734 {6}	0.02956 {3}	0.04336 {10}	0.0295 {2}	0.03636 {5}	0.04783 {12}	0.04318 {9}	0.03806 {7}	0.11081 {15}	0.04104 {8}	0.05806 {13}	0.04378 {11}	0.06502 {14}
180	MRE ($\hat{\zeta}$)	0.03723 {1}	0.03982 {3}	0.04434 {6}	0.03997 {4}	0.04752 {11}	0.03952 {2}	0.0434 {5}	0.04971 {12}	0.04703 {10}	0.04522 {7}	0.07372 {15}	0.04619 {8}	0.0533 {13}	0.04675 {9}	0.05739 {14}
180	D_{abs}	0.01406 {1}	0.01521 {3}	0.01697 {6}	0.01539 {4}	0.01811 {10}	0.01517 {2}	0.01639 {5}	0.01867 {12}	0.01802 {9}	0.01743 {7}	0.02728 {15}	0.01748 {8}	0.02044 {13}	0.01818 {11}	0.02187 {14}
180	D_{max}	0.02078 {1}	0.02232 {2}	0.02496 {6}	0.02273 {4}	0.02667 {10}	0.02233 {3}	0.02419 {5}	0.02764 {12}	0.02654 {9}	0.02565 {7}	0.04033 {15}	0.02575 {8}	0.03023 {13}	0.02682 {11}	0.03236 {14}
180	ASAE	0.01522 {6}	0.01437 {1.5}	0.01454 {4}	0.01523 {7}	0.01441 {3}	0.01458 {5}	0.01437 {1.5}	0.01544 {9}	0.01864 {12}	0.01751 {10}	0.02295 {15}	0.01541 {8}	0.01998 {14}	0.01757 {11}	0.01985 {13}
180	$\sum$ *Ranks*	11 {1}	16.5 {3}	34 {6}	26 {4}	55 {9}	16 {2}	26.5 {5}	69 {12}	59 {10}	45 {7}	90 {15}	48 {8}	79 {13}	62 {11}	83 {14}
225	BIAS ($\hat{\zeta}$)	0.12004 {1}	0.13605 {5}	0.14151 {6}	0.12215 {2}	0.14158 {7}	0.12235 {3}	0.13304 {4}	0.15143 {11}	0.15025 {10}	0.1518 {12}	0.23297 {15}	0.14986 {9}	0.17016 {14}	0.14484 {8}	0.16241 {13}
225	MSE ($\hat{\zeta}$)	0.02372 {3}	0.02815 {4}	0.03335 {8}	0.02296 {1}	0.03142 {6}	0.02328 {2}	0.02897 {5}	0.03755 {12}	0.03573 {10}	0.03643 {11}	0.10422 {15}	0.03452 {9}	0.04674 {14}	0.03235 {7}	0.04188 {13}
225	MRE ($\hat{\zeta}$)	0.0343 {1}	0.03887 {5}	0.04043 {6}	0.0349 {2}	0.04045 {7}	0.03496 {3}	0.03801 {4}	0.04327 {11}	0.04293 {10}	0.04337 {12}	0.06656 {15}	0.04282 {9}	0.04862 {14}	0.04138 {8}	0.0464 {13}
225	D_{abs}	0.01294 {1}	0.01476 {5}	0.01512 {6}	0.01342 {3}	0.01531 {7}	0.01328 {2}	0.01441 {4}	0.01646 {11}	0.01644 {10}	0.01677 {12}	0.02464 {15}	0.01633 {9}	0.01874 {14}	0.01598 {8}	0.01788 {13}
225	D_{max}	0.01908 {1}	0.02179 {5}	0.02236 {6}	0.01981 {3}	0.02262 {7}	0.01965 {2}	0.02129 {4}	0.02427 {10}	0.02429 {11}	0.02472 {12}	0.03636 {15}	0.0241 {9}	0.02766 {14}	0.02357 {8}	0.02645 {13}
225	ASAE	0.01331 {7}	0.01267 {2}	0.01282 {4}	0.01297 {6}	0.01276 {3}	0.01291 {5}	0.01249 {1}	0.01415 {9}	0.0168 {12}	0.01536 {10}	0.02072 {15}	0.01392 {8}	0.01777 {14}	0.01562 {11}	0.01763 {13}
225	$\sum$ *Ranks*	14 {1}	26 {5}	36 {6}	17 {2.5}	37 {7}	17 {2.5}	22 {4}	64 {11}	63 {10}	69 {12}	90 {15}	53 {9}	84 {14}	50 {8}	78 {13}
270	BIAS ($\hat{\zeta}$)	0.10564 {1}	0.12007 {5}	0.1253 {8}	0.11289 {2}	0.12265 {6}	0.11795 {4}	0.1158 {3}	0.15094 {12}	0.13798 {10}	0.13924 {11}	0.20191 {15}	0.1329 {9}	0.15679 {13}	0.12396 {7}	0.16009 {14}
270	MSE ($\hat{\zeta}$)	0.01786 {1}	0.02312 {5}	0.02508 {8}	0.01957 {2}	0.02392 {6}	0.02232 {4}	0.02124 {3}	0.03573 {12}	0.02926 {10}	0.03046 {11}	0.06855 {15}	0.02838 {9}	0.03758 {13}	0.0246 {7}	0.03943 {14}
270	MRE ($\hat{\zeta}$)	0.03018 {1}	0.03431 {5}	0.0358 {8}	0.03225 {2}	0.03504 {6}	0.0337 {4}	0.03309 {3}	0.04313 {12}	0.03942 {10}	0.03978 {11}	0.05769 {15}	0.03797 {9}	0.0448 {13}	0.03542 {7}	0.04574 {14}
270	D_{abs}	0.01149 {1}	0.013 {5}	0.01366 {7}	0.01238 {2}	0.01324 {6}	0.01279 {4}	0.01258 {3}	0.01626 {12}	0.01511 {10}	0.01525 {11}	0.02159 {15}	0.01435 {9}	0.01712 {13}	0.01368 {8}	0.01749 {14}
270	D_{max}	0.01691 {1}	0.01918 {5}	0.02015 {7}	0.01824 {2}	0.01959 {6}	0.01889 {4}	0.01856 {3}	0.02407 {12}	0.02233 {10}	0.02255 {11}	0.03189 {15}	0.0212 {9}	0.02534 {13}	0.02019 {8}	0.02589 {14}
270	ASAE	0.01202 {6}	0.0116 {4}	0.01151 {3}	0.0122 {7}	0.01161 {5}	0.01137 {1.5}	0.01137 {1.5}	0.01292 {9}	0.01453 {12}	0.01436 {11}	0.01843 {15}	0.01247 {8}	0.01632 {14}	0.0142 {10}	0.01631 {13}

(Continued)

Table 7 (continued)

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
270	$\sum Ranks$	11 {1}	29 {5}	41 {7}	17 {3}	35 {6}	21.5 {4}	16.5 {2}	69 {12}	62 {10}	66 {11}	90 {15}	53 {9}	79 {13}	47 {8}	83 {14}
350	BIAS ($\hat{\zeta}$)	0.09877 {2}	0.10636 {5}	0.11261 {6.5}	0.09707 {1}	0.11727 {10}	0.09922 {3}	0.10618 {4}	0.12087 {11}	0.12403 {12}	0.11355 {8}	0.17612 {15}	0.11666 {9}	0.13813 {13}	0.11261 {6.5}	0.14428 {14}
350	MSE ($\hat{\zeta}$)	0.01555 {3}	0.01773 {4}	0.02068 {7}	0.01468 {1}	0.02212 {10}	0.01554 {2}	0.01809 {5}	0.02309 {11}	0.02436 {12}	0.02142 {9}	0.05131 {15}	0.02131 {8}	0.02985 {13}	0.02024 {6}	0.03245 {14}
350	MRE ($\hat{\zeta}$)	0.02822 {2}	0.03039 {5}	0.03217 {6.5}	0.02773 {1}	0.0335 {10}	0.02835 {3}	0.03034 {4}	0.03453 {11}	0.03544 {12}	0.03244 {8}	0.05032 {15}	0.03333 {9}	0.03947 {13}	0.03217 {6.5}	0.04122 {14}
350	D_{abs}	0.01073 {2}	0.01151 {4}	0.01213 {6}	0.01065 {1}	0.01272 {10}	0.01078 {3}	0.01156 {5}	0.01313 {11}	0.01354 {12}	0.01243 {7}	0.01894 {15}	0.01266 {9}	0.01516 {13}	0.01244 {8}	0.01573 {14}
350	D_{max}	0.01579 {2}	0.01699 {4}	0.01795 {6}	0.01569 {1}	0.01879 {10}	0.01591 {3}	0.01702 {5}	0.01939 {11}	0.02 {12}	0.01834 {7}	0.02795 {15}	0.01872 {9}	0.0224 {13}	0.01839 {8}	0.02329 {14}
350	ASAE	0.01047 {6}	0.0102 {4}	0.01029 {5}	0.01079 {7}	0.01009 {3}	0.00994 {2}	0.00974 {1}	0.01095 {8}	0.01306 {12}	0.01227 {10}	0.01603 {15}	0.01096 {9}	0.01447 {14}	0.01289 {11}	0.01408 {13}
350	$\sum Ranks$	17 {3}	26 {5}	37 {6}	12 {1}	53 {9.5}	16 {2}	24 {4}	63 {11}	72 {12}	49 {8}	90 {15}	53 {9.5}	79 {13}	46 {7}	83 {14}

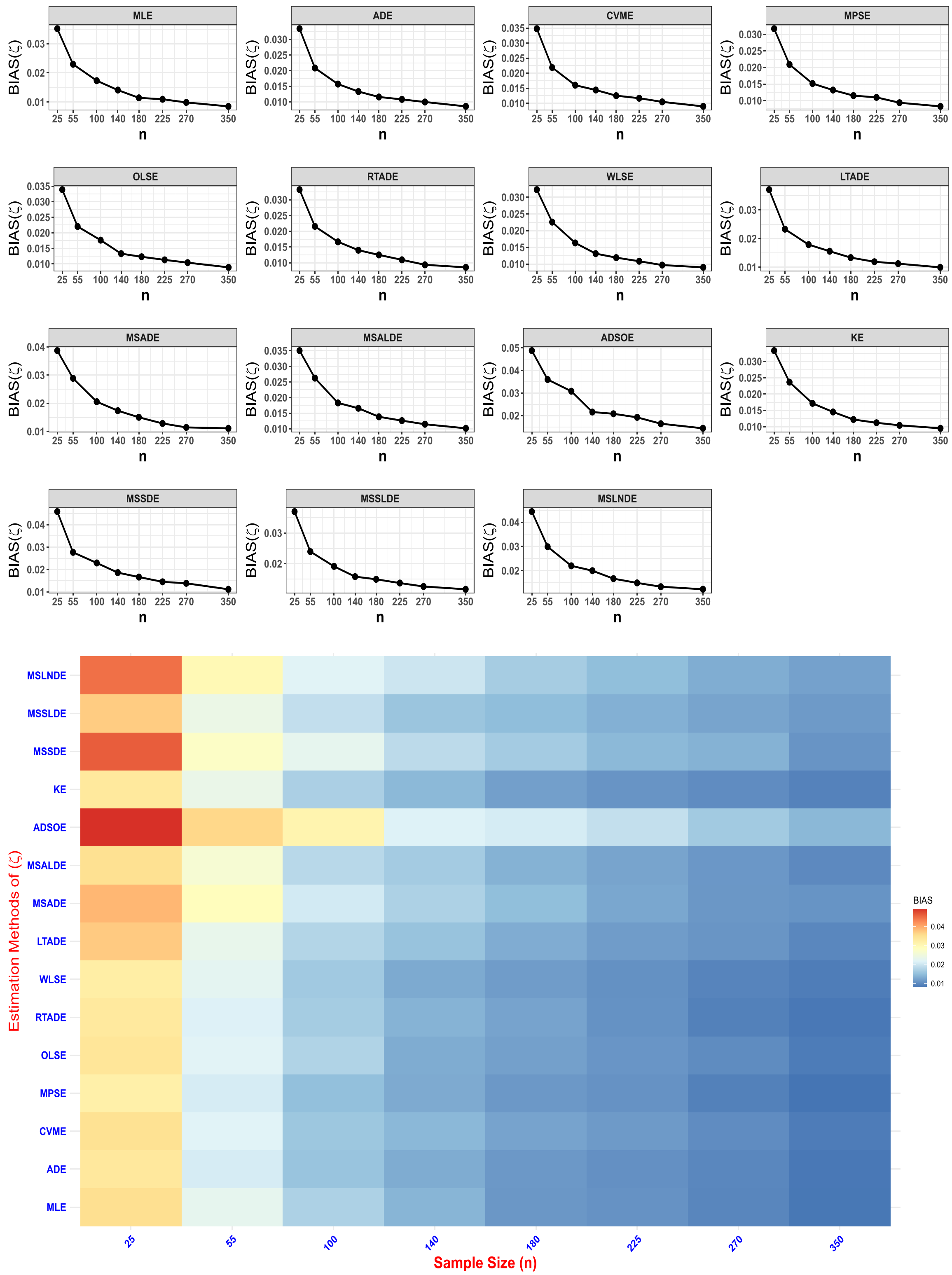

Figure 5: A graphical representation for BIAS values presented in Table 3

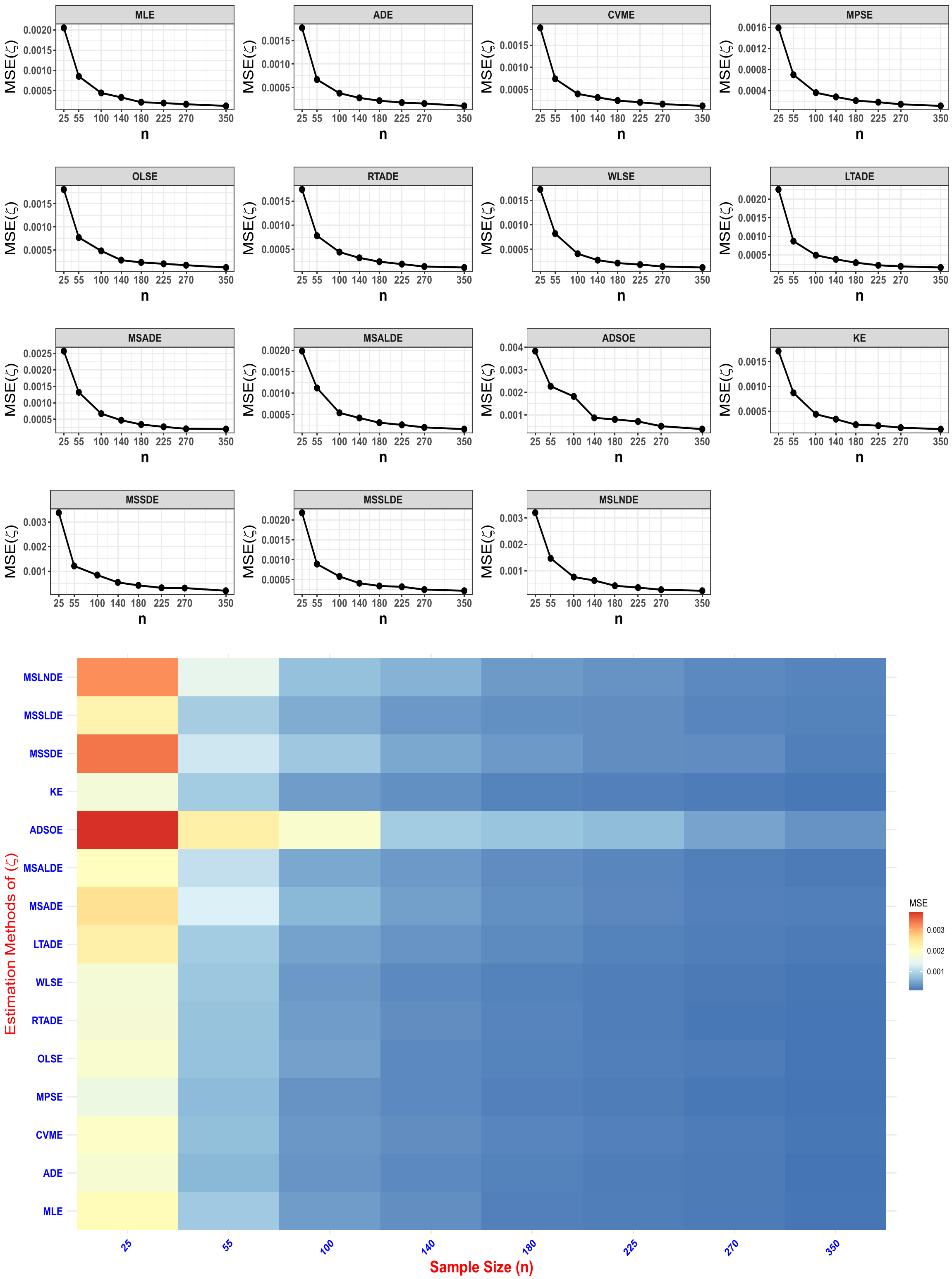

Figure 6: A graphical representation for MSE values presented in Table 3

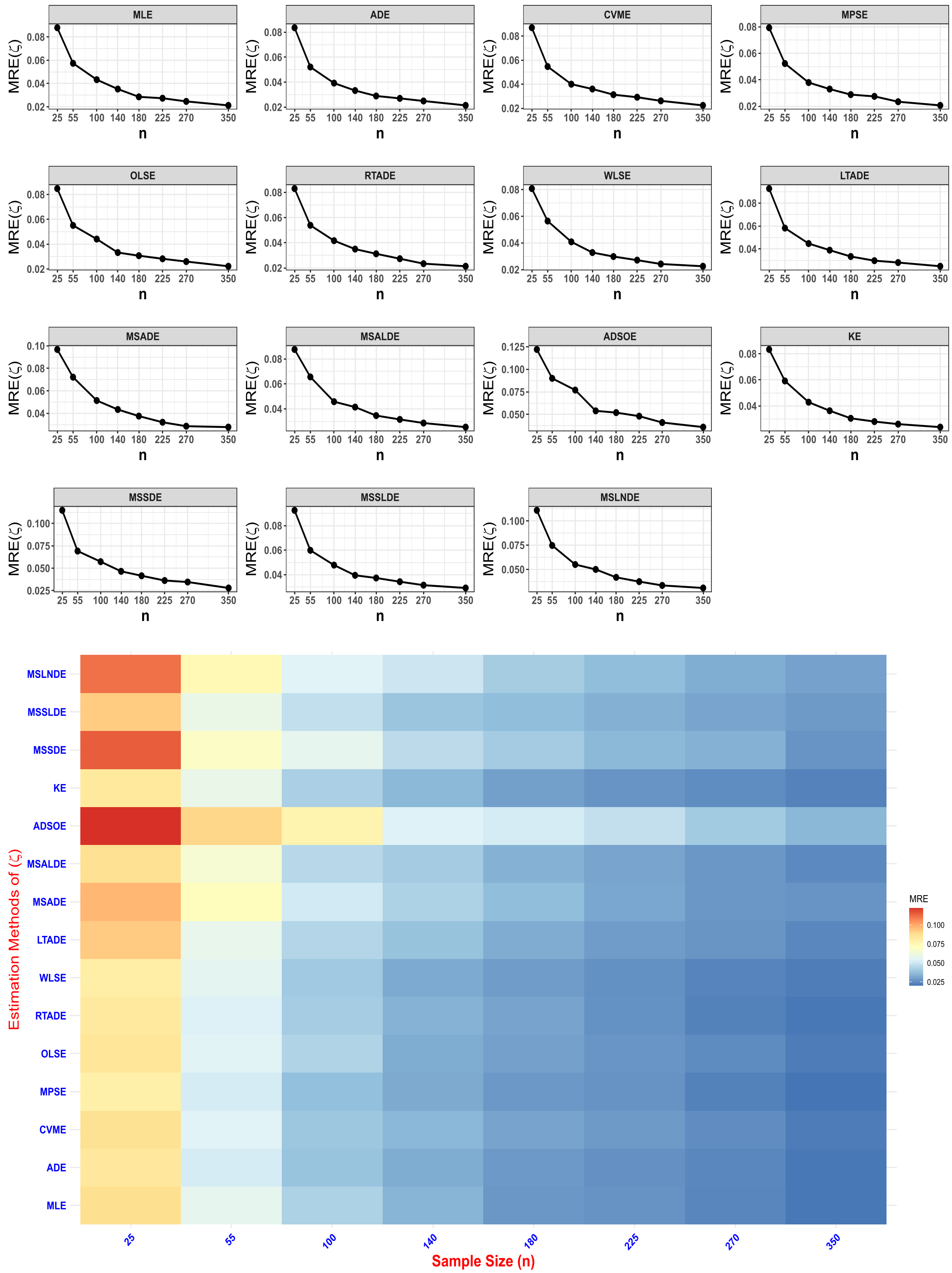

Figure 7: A graphical representation for MRE values presented in Table 3

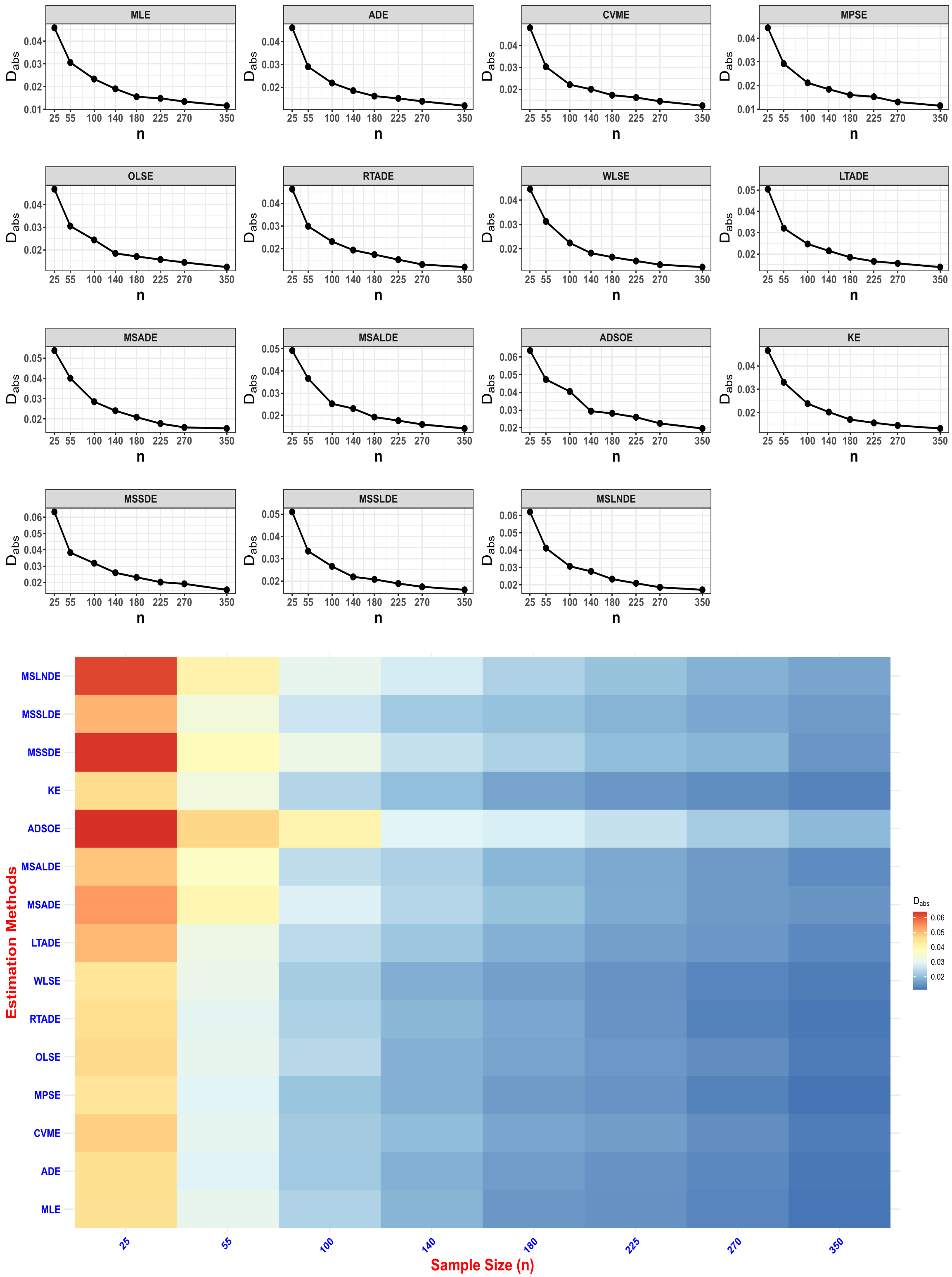

Figure 8: A graphical representation for D_{abs} values presented in Table 3

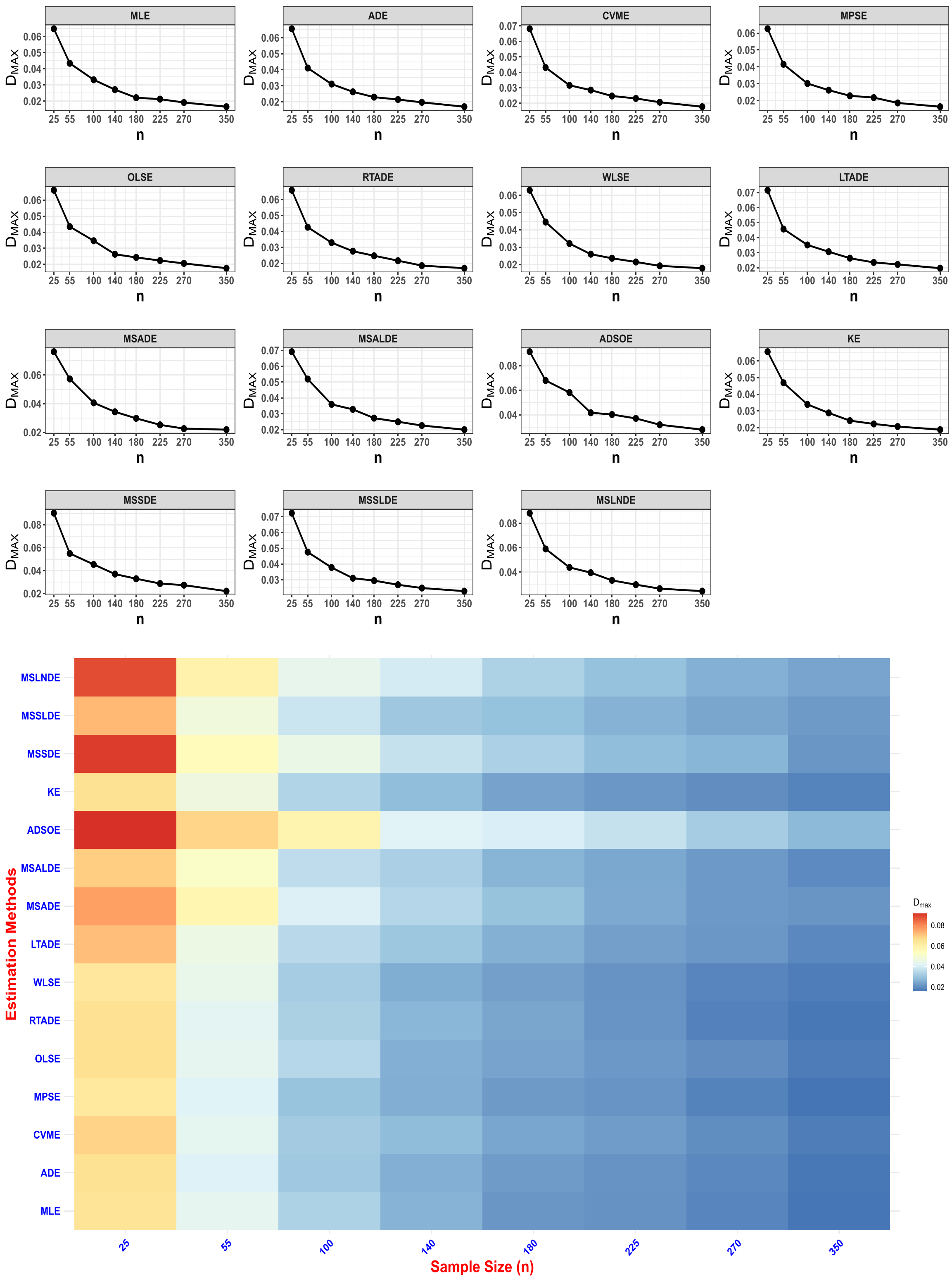

Figure 9: A graphical representation for D_{max} values presented in Table 3

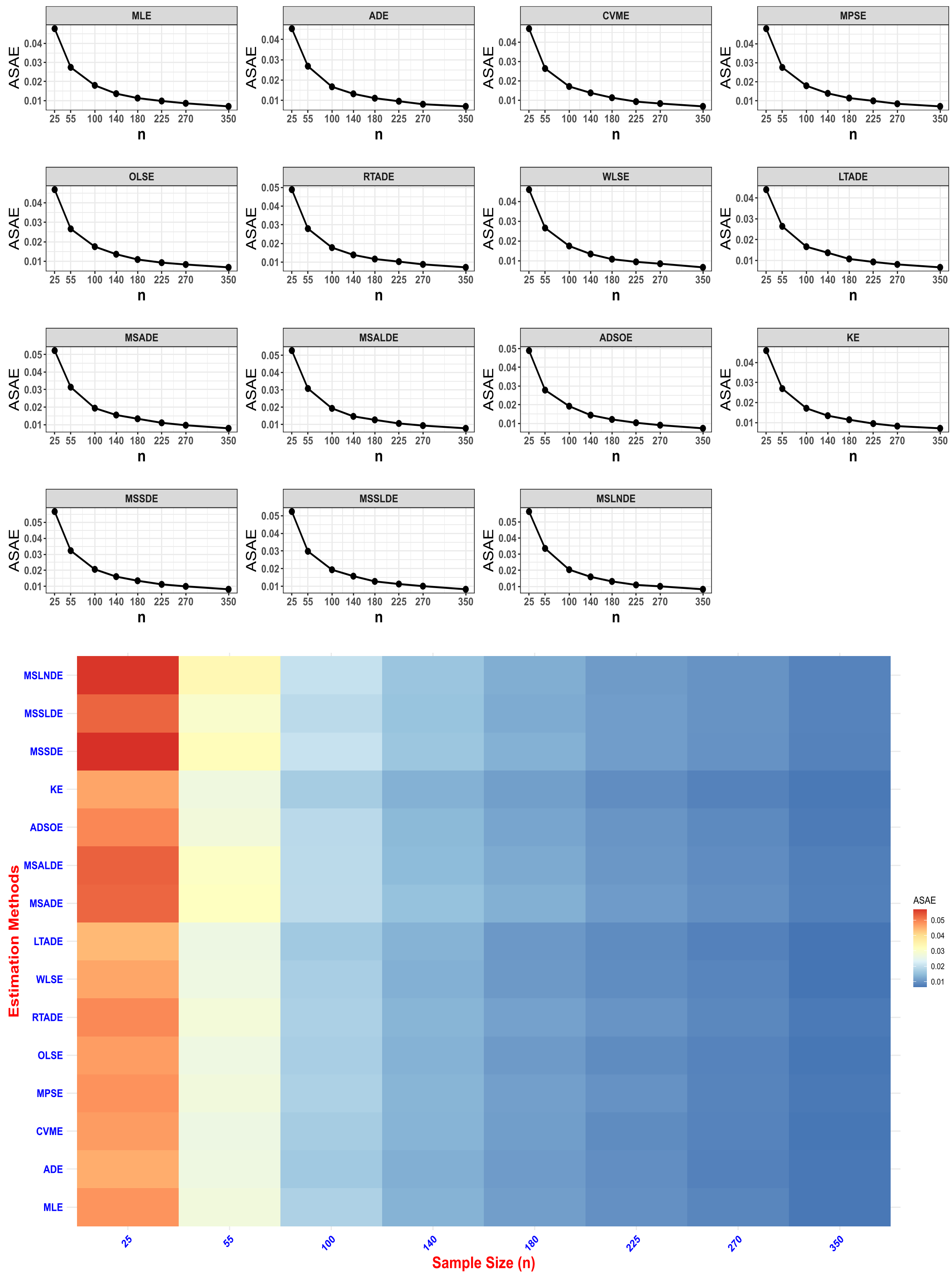

Figure 10: A graphical representation for ASAE values presented in Table 3

Table 8: Partial and overall ranks for all estimation methods of our proposed model

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$\zeta = 0.4$	25	7.0	4.0	8.0	1.0	6.0	5.0	2.0	9.0	12.0	10.0	14.5	3.0	14.5	11.0	13.0
	55	7.0	1.0	2.0	5.0	4.0	3.0	6.0	8.0	13.0	11.0	14.0	9.0	12.0	10.0	15.0
	100	7.0	1.0	4.0	2.0	8.5	5.0	3.0	8.5	12.0	10.0	14.5	6.0	14.5	11.0	13.0
	140	5.0	2.5	7.0	4.0	2.5	6.0	1.0	9.0	12.0	11.0	15.0	8.0	13.0	10.0	14.0
	180	1.0	3.0	7.0	2.0	5.5	8.5	4.0	8.5	12.0	10.0	15.0	5.5	13.0	11.0	14.0
	225	3.0	1.0	7.0	5.0	8.0	4.0	2.0	9.0	11.0	10.0	15.0	6.0	13.0	12.0	14.0
	270	5.0	4.0	8.0	1.0	6.0	2.0	3.0	9.0	10.0	11.0	15.0	7.0	14.0	12.0	13.0
	350	2.0	3.0	6.5	1.0	5.0	4.0	6.5	8.0	11.0	10.0	15.0	9.0	12.0	13.0	14.0
$\zeta = 0.85$	25	6.0	9.5	1.5	1.5	5.0	7.0	3.0	8.0	12.0	11.0	14.0	4.0	13.0	9.5	15.0
	55	3.0	6.0	8.0	1.0	7.0	2.0	5.0	9.0	12.0	11.0	14.5	4.0	13.0	10.0	14.5
	100	2.0	3.0	4.0	8.0	5.5	7.0	1.0	9.0	12.0	11.0	15.0	10.0	13.0	5.5	14.0
	140	1.0	8.0	5.0	7.0	2.5	2.5	4.0	6.0	12.0	11.0	15.0	9.0	14.0	10.0	13.0
	180	1.0	3.0	7.0	6.0	8.5	2.0	4.0	8.5	12.0	11.0	15.0	5.0	14.0	10.0	13.0
	225	3.0	2.0	5.0	4.0	6.0	7.0	1.0	8.5	12.0	11.0	14.5	8.5	13.0	10.0	14.5
	270	6.0	2.0	5.0	1.0	4.0	3.0	7.0	8.0	12.0	10.0	14.0	9.0	13.0	11.0	15.0
	350	1.0	4.0	5.0	2.0	7.0	8.0	6.0	9.0	13.0	10.0	15.0	3.0	12.0	11.0	14.0
$\zeta = 1.4$	25	1.0	5.0	2.5	4.0	7.0	6.0	2.5	10.0	12.0	8.0	15.0	9.0	13.0	11.0	14.0
	55	1.0	4.5	2.5	4.5	6.0	2.5	8.0	9.0	12.0	11.0	15.0	7.0	14.0	10.0	13.0
	100	2.0	4.0	6.0	1.0	5.0	8.0	7.0	10.0	12.0	11.0	15.0	3.0	13.0	9.0	14.0
	140	8.5	3.5	5.0	1.0	8.5	3.5	2.0	11.0	12.0	10.0	15.0	6.0	13.0	7.0	14.0
	180	2.0	4.0	9.0	3.0	6.0	1.0	5.0	7.0	12.0	11.0	15.0	8.0	13.0	10.0	14.0
	225	1.0	2.0	8.0	4.0	6.0	3.0	7.0	9.0	12.0	11.0	15.0	5.0	14.0	10.0	13.0
	270	5.0	2.0	4.0	7.0	8.0	3.0	1.0	9.0	12.0	11.0	15.0	6.0	13.0	10.0	14.0
	350	5.0	2.0	7.0	1.0	4.0	6.0	3.0	8.0	12.0	11.0	15.0	9.0	13.0	10.0	14.0
$\zeta = 2.5$	25	1.5	6.0	5.0	1.5	8.5	3.0	4.0	11.0	12.0	10.0	15.0	7.0	13.0	8.5	14.0
	55	1.0	5.0	8.0	3.0	10.0	4.0	2.0	11.0	12.0	9.0	15.0	6.0	13.0	7.0	14.0
	100	1.0	2.0	8.0	3.0	5.0	4.0	6.5	11.0	12.0	10.0	15.0	9.0	13.0	6.5	14.0
	140	2.0	6.0	7.0	4.0	3.0	5.0	1.0	11.0	12.0	9.0	15.0	10.0	14.0	8.0	13.0
	180	4.0	3.0	8.0	1.0	6.0	2.0	5.0	10.0	12.0	9.0	15.0	11.0	13.0	7.0	14.0
	225	1.0	2.0	5.0	3.0	9.5	4.0	7.0	11.0	12.0	9.5	15.0	6.0	13.0	8.0	14.0
	270	1.0	3.5	6.0	3.5	8.0	2.0	5.0	11.0	12.0	10.0	15.0	9.0	14.0	7.0	13.0
	350	1.0	2.0	7.0	3.0	6.0	5.0	4.0	10.0	12.0	11.0	15.0	8.0	14.0	9.0	13.0
$\zeta = 3.5$	25	2.0	5.0	8.0	1.0	11.0	7.0	3.0	12.0	10.0	6.0	15.0	9.0	13.0	4.0	14.0
	55	1.0	2.5	6.0	2.5	8.5	4.0	5.0	12.0	11.0	7.0	15.0	10.0	13.0	8.5	14.0
	100	1.0	4.0	6.0	3.0	7.0	2.0	8.0	12.0	11.0	10.0	15.0	9.0	14.0	5.0	13.0
	140	1.0	4.0	6.0	3.0	7.0	5.0	2.0	12.0	11.0	10.0	15.0	9.0	14.0	8.0	13.0
	180	1.0	3.0	6.0	4.0	9.0	2.0	5.0	12.0	10.0	7.0	15.0	8.0	13.0	11.0	14.0
	225	1.0	5.0	6.0	2.5	7.0	2.5	4.0	11.0	10.0	12.0	15.0	9.0	14.0	8.0	13.0
	270	1.0	5.0	7.0	3.0	6.0	4.0	2.0	12.0	10.0	11.0	15.0	9.0	13.0	8.0	14.0
	350	3.0	5.0	6.0	1.0	9.5	2.0	4.0	11.0	12.0	8.0	15.0	9.5	13.0	7.0	14.0
$\sum$ Ranks		109.0	147.0	239.0	119.0	262.5	166.5	161.5	388.0	467.0	401.5	595.0	297.5	531.0	364.5	551.0
Overall rank		1	3	6	2	7	5	4	10	12	11	15	8	13	9	14

6 Real Data Analysis

This section presents the performance of the proposed BUm model against several competing models listed below. Two real datasets are used to demonstrate the flexibility and suitability of the distributions. The results are summarized through descriptive statistics, parameter estimates, goodness-of-fit measures, and graphical comparisons such as the fitted PDF, CDF, survival plots, and probability-probability (P-P) plots.

The evaluation of model performance is based on several statistical criteria. The log-likelihood (LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan–Quinn Information Criterion (HQIC) are used to assess goodness-of-fit. Larger LL values indicate a better fit, while smaller AIC, BIC, CAIC, and HQIC values are preferred. The coefficient of determination (R^2) is used to measure explanatory power, with higher values suggesting a stronger fit. Additionally, the root mean square error (RMSE) serves as an error-based measure, where lower values indicate better predictive accuracy. For the Kolmogorov–Smirnov (KS), Anderson–Darling (AD), and Cramér–von Mises (CVM) tests, smaller statistic values and larger p-values suggest a better fit.

The PDFS of the competing models are given below:

- Unit Gompertz distribution (UGoD) by [22]:

$$f(x) = \alpha\beta x^{-(\beta+1)} \exp\left[-\alpha\left(x^{-\beta}-1\right)\right]; x \in (0,1), \alpha, \beta > 0.$$

- Epanechnikov power unit inverse Lindley distribution (EPuD) by [18]:

$$f(x) = \frac{\delta\theta^2 x^{-3\delta-1} e^{\theta-2\theta x^{-\delta}}\left[3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}\left(\theta+x^{\delta}\right)\right]}{2(\theta+1)^2}; x \in (0,1), \delta, \theta > 0.$$

- Power unit inverse Lindley distribution (PUiD) by [16]:

$$f(x) = \frac{\delta\beta^2 e^{\beta}}{1+\beta} x^{-2\delta-1} e^{-\frac{\beta}{x^{\delta}}}; x \in (0,1), \delta, \beta > 0.$$

- Bounded Weighted Exponential Distribution (BWeD) by [23]:

$$f(x) = \frac{\lambda(\alpha+1)}{\alpha} x^{\lambda-1}\left(1-x^{\alpha\lambda}\right); x \in (0,1),\ \alpha, \lambda > 0.$$

- Unit Weibull distribution (UWed) by [24]:

$$f(x) = \frac{1}{x}\alpha\beta(-\log x)^{\beta-1} \exp\left[-\alpha(-\log x)^{\beta}\right]; x \in (0,1), \alpha, \beta > 0.$$

- Unit Burr-XII Distribution (UBxD) by [25]:

$$f(x) = \frac{\alpha\theta(-\log(x))^{\alpha-1}\left((-\log(x))^{\alpha}+1\right)^{-\theta-1}}{x}; x \in (0,1),\ \alpha, \theta > 0.$$

- Beta Distribution (BEtD) by [26]:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}; x \in (0,1),\ \alpha, \beta > 0.$$

- Unit Zeghdoudi Distribution (UZeD) by [27]:

$$f(x;\omega) = \frac{\omega^3 x}{(1-x)^4(\omega+2)} e^{-\frac{\omega x}{1-x}}; x \in (0,1), \omega > 0.$$

6.1 First Data

The first dataset, obtained from [28], represents the annual percentage of antimicrobial resistant isolates in Portugal in 2012. It contains 23 proportions ranging between 0.01 and 0.59. The data are as follows: 0.01, 0.01, 0.03, 0.05, 0.08, 0.12, 0.14, 0.15, 0.15, 0.16, 0.19, 0.20, 0.20, 0.23, 0.26, 0.30, 0.32, 0.36, 0.39, 0.43, 0.54, 0.58, 0.59

Table 9 shows the descriptive statistics. The skewness (0.6475) confirms moderate asymmetry, and the kurtosis (2.4630) indicates a light-tailed distribution compared to the normal case. Table 10 provides the parameter estimates and their standard errors for all competing models.

Table 9: Descriptive statistics of the first dataset

n	Q1	Median	Q3	Mean	Kurtosis	Skewness
23	0.1300	0.2000	0.3400	0.2387	2.4630	0.6475

Table 10: Estimates values and SEs for all models applied to the first dataset.

Models	Estimates	SE	Estimates	SE
BUmD ($\hat{\zeta}$)	3.6344	0.5052		
UGoD ($\hat{\alpha}$, $\hat{\beta}$)	0.6793	0.4838	0.4190	0.1609
EPuD ($\hat{\delta}$, $\hat{\theta}$)	0.2411	0.1173	2.5914	1.5607
PUiD ($\hat{\delta}$, $\hat{\beta}$)	0.3434	0.1174	1.3522	0.6645
BWeD ($\hat{\lambda}$, $\hat{\alpha}$)	1.0826	0.3746	0.0002	0.5852
UWeD ($\hat{\alpha}$, $\hat{\beta}$)	0.2695	0.0933	1.7785	0.2753
UBxD ($\hat{\alpha}$, $\hat{\theta}$)	3.8498	0.9567	0.4521	0.1335
BEtD ($\hat{\alpha}$, $\hat{\beta}$)	1.1218	0.2919	3.6501	1.0979
UZeD ($\hat{\beta}$)	5.6156	0.7468		

Table 11 presents the goodness-of-fit statistics. Based on LL, AIC, CAIC, BIC, HQIC, RMSE, and R^2, the BUmD shows superior performance. It records the largest LL value and the smallest values for AIC, CAIC, BIC, and HQIC. It also achieves the highest R^2 (0.9843) and the lowest RMSE (0.0364), both of which confirm a strong fit. The KS, AD, and CVM test results support the same conclusion, since the BUmD has the smallest test values and the largest p-values.

Table 11: Goodness-of-fit statistics for the first dataset

Model	LL	AIC	CAIC	BIC	HQIC	RMSE	R^2	KS (p-value)	AD (p-value)	CVM (p-value)
BUmD	11.9540	−21.9079	−21.7174	−20.7724	−21.6223	0.0364	0.9843	0.0923 (0.9895)	0.2498 (0.9704)	0.0305 (0.9770)
UGoD	8.3923	−12.7845	−14.5940	−13.6490	−14.4989	0.0673	0.9228	0.1608 (0.5916)	0.8857 (0.4219)	0.1333 (0.4475)
EPuD	9.1092	−14.2184	−13.6185	−11.9475	−13.6473	0.0575	0.9470	0.1424 (0.7397)	0.7083 (0.5503)	0.1003 (0.5875)

(Continued)

Table 11 (continued)

Model	LL	AIC	CAIC	BIC	HQIC	RMSE	R^2	KS (p-value)	AD (p-value)	CVM (p-value)
PUiD	8.5094	−13.0188	−12.4188	−10.7478	−12.4476	0.0657	0.9283	0.1623 (0.5797)	0.8464 (0.4474)	0.1268 (0.4719)
BWeD	10.3068	−16.6137	−18.4232	−17.4782	−18.3281	0.0481	0.9635	0.1145 (0.9238)	0.5047 (0.7402)	0.0753 (0.7245)
UWeD	10.4906	−16.9812	−18.7908	−17.8457	−18.6957	0.0489	0.9679	0.1469 (0.7037)	0.4503 (0.7961)	0.0658 (0.7823)
UBxD	10.3249	−16.6498	−18.4593	−17.5143	−18.3642	0.0598	0.9545	0.1568 (0.6241)	0.4777 (0.7679)	0.0895 (0.6433)
BEtD	11.5364	−19.0728	−20.8824	−19.9374	−20.7873	0.0367	0.9837	0.1102 (0.9428)	0.2546 (0.9677)	0.0326 (0.9698)
UZeD	5.6941	−9.3881	−9.1976	−8.2526	−9.1025	0.1139	0.8880	0.2193 (0.2187)	2.7427 (0.0375)	0.2736 (0.1607)

The graphical comparisons are given in Figs. 11–14. Fig. 11 shows that the BUmD PDF follows the data more closely than the other models. Fig. 12 shows that the BUmD CDF lies nearest to the empirical CDF compared to the competing models. Fig. 13 shows that the survival curve of the BUmD matches better with the empirical survival function. Finally, the P-P plot in Fig. 14 confirms the good fit, as the BUmD points lie closer to the diagonal line than those of the competing models. Overall, the first dataset shows that the BUmD captures the main features of the data better than the other models.

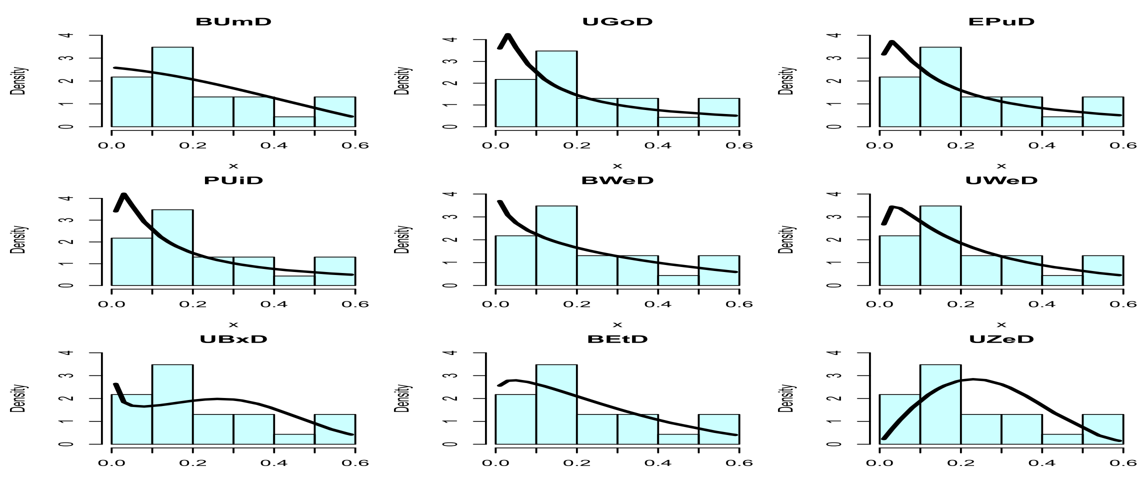

Figure 11: Estimated PDFs for the competing models for the first dataset

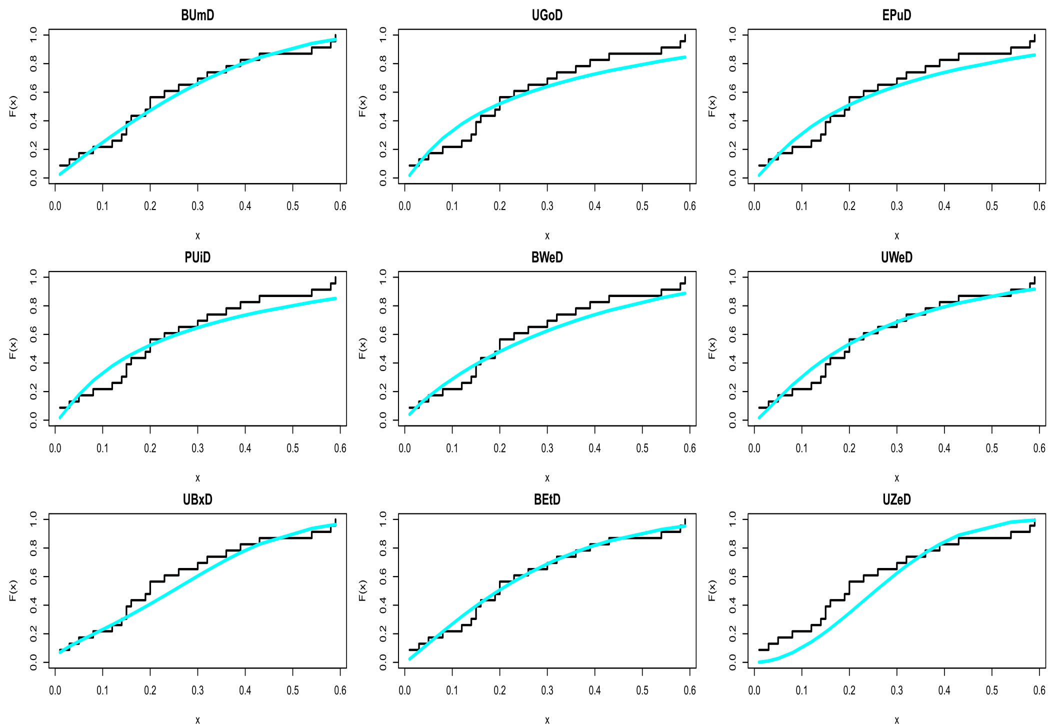

Figure 12: Plots of the empirical CDFs with the fitted CDFs of all models for the first dataset

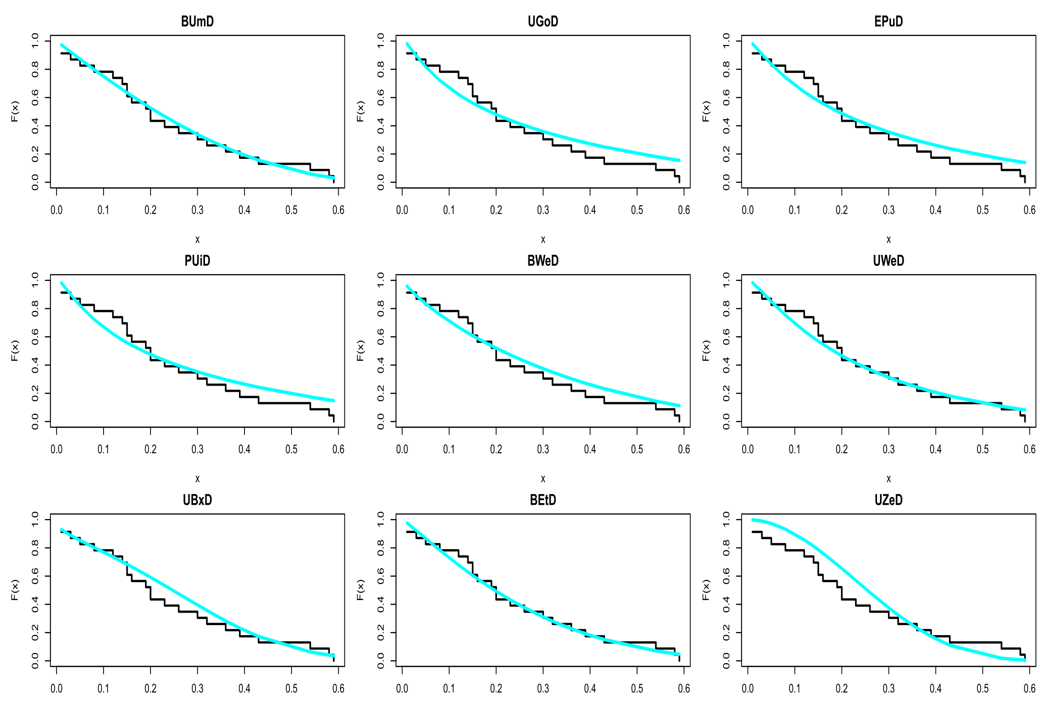

Figure 13: Empirical and survival function curves of the competitor models based on the first dataset

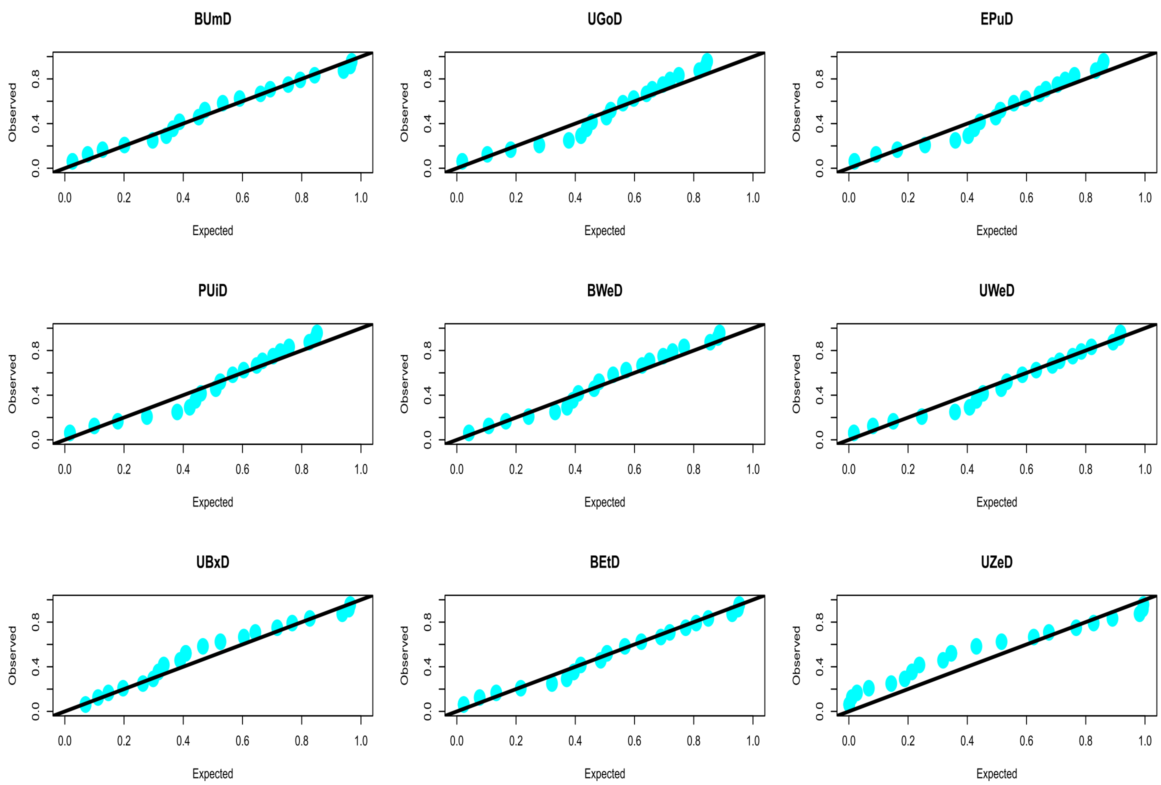

Figure 14: The P-P plots of the fitted models for the first dataset

6.2 Second Data

The second dataset, obtained from [29], describes the relative distribution of individuals with mild difficulties across different age groups in Saudi Arabia. It consists of 17 proportional values ranging from 0.6163 to 0.9168. The data are as follows: 0.8602, 0.8737, 0.8924, 0.9168, 0.9119, 0.8984, 0.9061, 0.9075, 0.8997, 0.8902, 0.8753, 0.8502, 0.8042, 0.7699, 0.7101, 0.6759, 0.6163.

Table 12 gives the descriptive statistics. The skewness (−1.2860) indicates negative asymmetry, and the kurtosis (3.3546) suggests heavier tails compared to the normal case. Parameter estimates and standard errors for all models are reported in Table 13.

Table 14 compares the goodness-of-fit measures. The BUmD achieves the best overall results, with the highest log-likelihood, lowest information criteria, and a higher R^2 than the competitors. The RMSE (0.0798) is also the lowest. The superior fit is further supported by the lowest KS, AD, and CVM values and the largest associated p-values.

Table 12: Descriptive statistics of the second dataset

n	Q1	Median	Q3	Mean	Kurtosis	Skewness
17	0.8042	0.8753	0.8997	0.8388	3.3546	−1.2860

Table 13: Estimates values and SEs for all models applied to the second dataset

Models	Estimates	SE	Estimates	SE
BUmD ($\hat{\zeta}$)	0.5672	0.0669		
UGoD ($\hat{\alpha}$, $\hat{\beta}$)	0.8569	0.7459	3.6592	1.8343
EPuD ($\hat{\delta}$, $\hat{\theta}$)	1.9668	1.3804	3.3343	2.7514
PUiD ($\hat{\delta}$, $\hat{\beta}$)	3.0365	1.3576	1.5964	0.9763
BWeD ($\hat{\lambda}$, $\hat{\alpha}$)	10.9791	1.6851	0.0004	0.4361
UWeD ($\hat{\alpha}$, $\hat{\beta}$)	15.6125	6.9696	1.7430	0.2982
UBxD ($\hat{\alpha}$, $\hat{\theta}$)	1.7956	0.2953	17.8428	7.8247
BEtD ($\hat{\alpha}$, $\hat{\beta}$)	17.3247	1.2107	3.3621	0.4432
UZeD ($\hat{\beta}$)	0.4200	0.0591		

Table 14: Goodness-of-fit statistics for the second dataset

Model	LL	AIC	CAIC	BIC	HQIC	RMSE	R^2	KS (p-value)	AD (p-value)	CVM (p-value)
BUmD	22.4502	−42.9004	−42.6337	−42.0672	−42.8176	0.0798	0.9241	0.1691 (0.6553)	0.6431 (0.6058)	0.1132 (0.5288)
UGoD	16.8513	−29.7026	−31.4360	−30.8695	−31.6199	0.1094	0.7808	0.2743 (0.1273)	1.3679 (0.2113)	0.2442 (0.1963)
EPuD	17.5127	−31.0254	−30.1682	−29.3590	−30.8597	0.1039	0.8108	0.2672 (0.1462)	1.2700 (0.2419)	0.2205 (0.2319)
PUiD	16.9192	−29.8385	−28.9813	−28.1721	−29.6728	0.1091	0.7881	0.2677 (0.1447)	1.3522 (0.2159)	0.2414 (0.2003)
BWeD	18.9545	−33.9089	−35.6422	−35.0757	−35.8261	0.0936	0.8483	0.2473 (0.2112)	1.0772 (0.3182)	0.1833 (0.3042)
UWeD	18.7707	−33.5415	−35.2748	−34.7083	−35.4586	0.1070	0.8417	0.2248 (0.3088)	1.1985 (0.2675)	0.2162 (0.2391)
UBxD	18.9320	−33.8641	−35.5974	−35.0308	−35.7812	0.1046	0.8490	0.2182 (0.3420)	1.1568 (0.2838)	0.2070 (0.2555)
BEtD	20.0348	−36.0696	−37.8029	−37.2364	−37.9867	0.1081	0.8607	0.2194 (0.3359)	1.1603 (0.2824)	0.2081 (0.2535)
UZeD	22.0924	−42.1848	−41.9182	−41.3516	−42.1020	0.0847	0.9034	0.1836 (0.5544)	0.7473 (0.5184)	0.1368 (0.4361)

Figs. 15–18 show the fitted plots (PDF, CDF, survival, and P-P). The estimated PDF of the BUmD in Fig. 15 gives the closest match to the empirical distribution. Fig. 16 shows that the CDF of the BUmD lies nearest to the empirical CDF compared to the other models. Fig. 17 shows that the survival curve of the BUmD matches better with the empirical survival function. Finally, the P-P plot in Fig. 18 confirms that the BUmD provides the best fit among the competing models, as its points lie closest to

the diagonal line. Overall, the second dataset shows that the BUmD captures the main features of the data better than the other models.

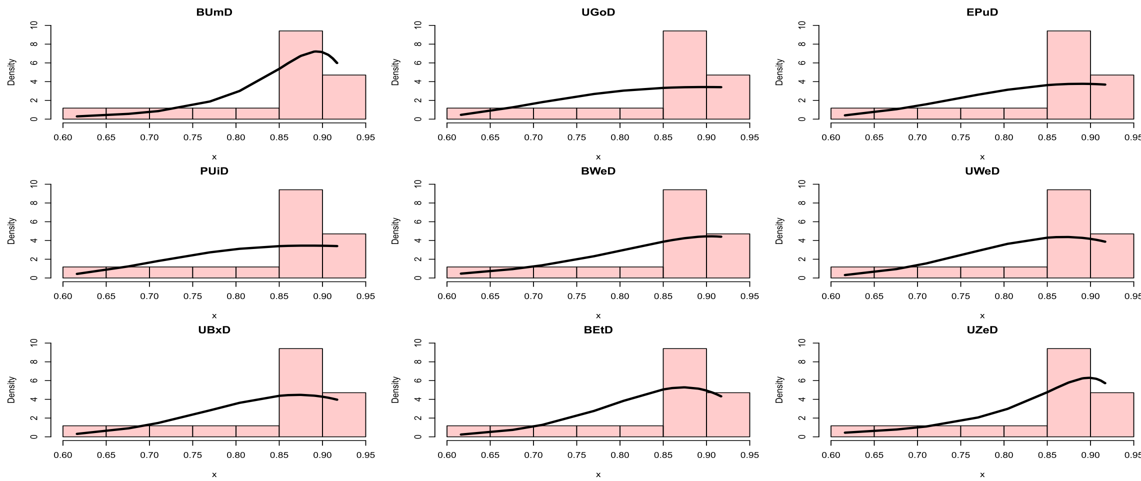

Figure 15: Estimated PDFs for the competing models for the second dataset

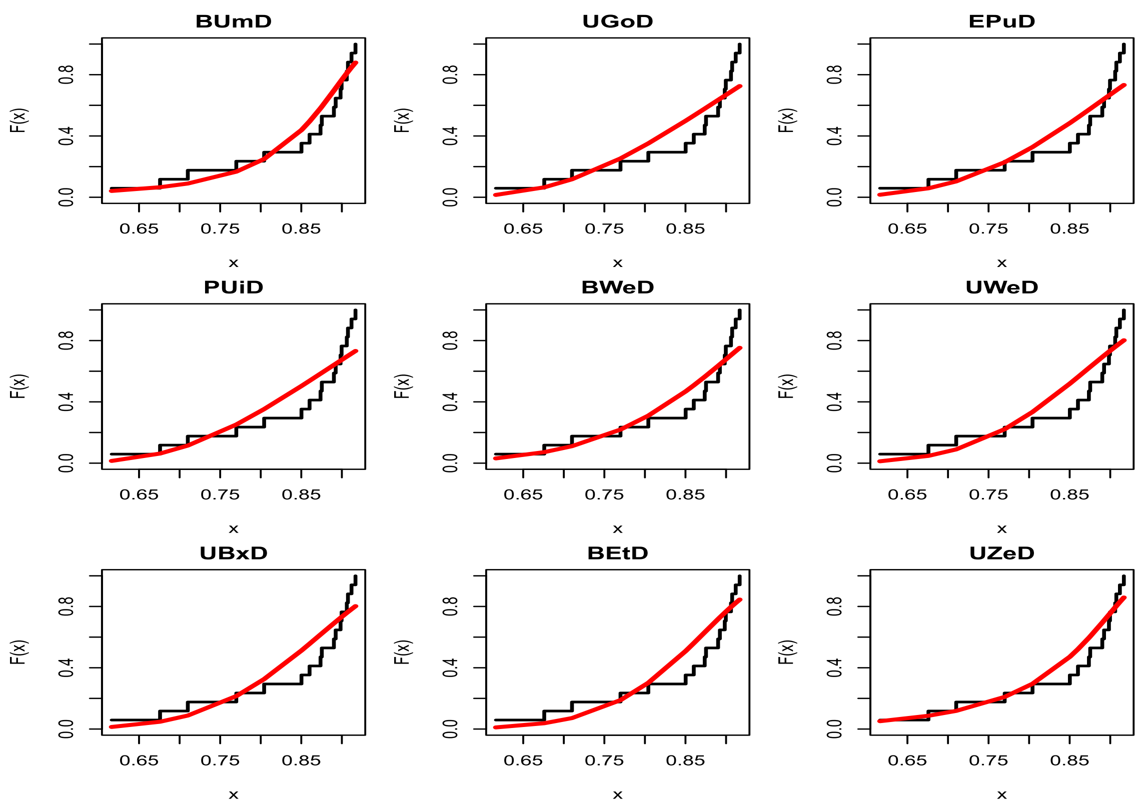

Figure 16: Plots of the empirical CDFs with the fitted CDFs of all models for the second dataset

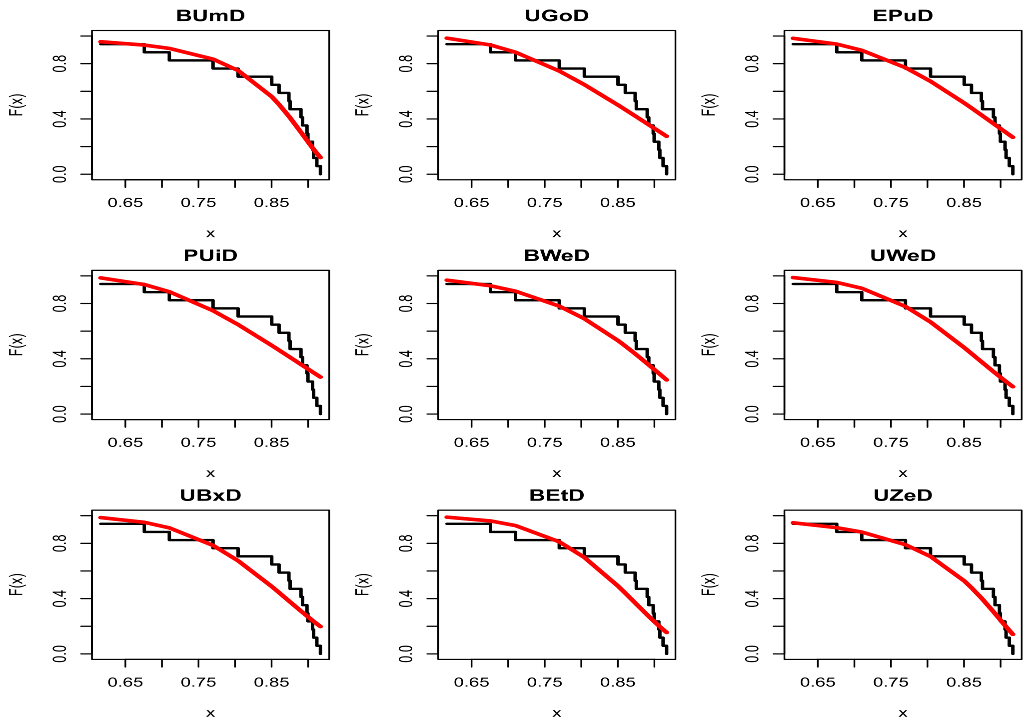

Figure 17: Empirical and survival function curves of the competitor models based on the second dataset

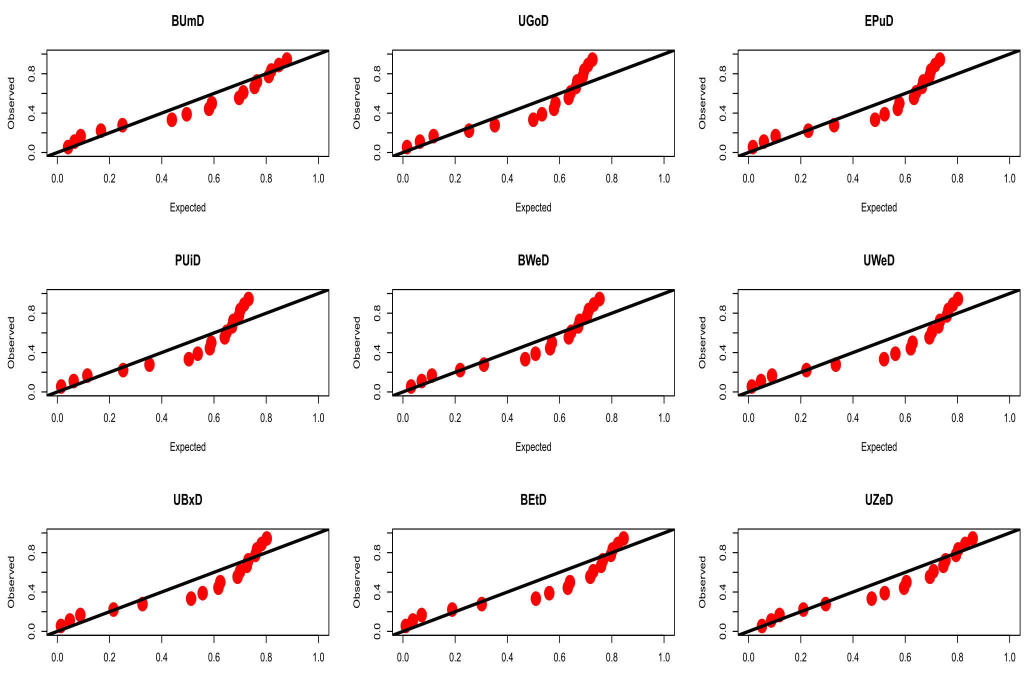

Figure 18: The P-P plots of the fitted models for the second dataset

7 Conclusion

We created and examined the BUmD, an innovative bounded probability model derived from the Uma distribution. Theoretical features, encompassing moments, quantile function, extropy, and reliability measures, were produced and examined, underscoring the adaptability of the BUmD in simulating a range of behaviors, including different hazard rate shapes and skewness. Sixteen traditional and contemporary estimating methods were employed, and their efficacy was thoroughly evaluated through comprehensive Monte Carlo simulations. The findings indicated that the MLE and the MPSE consistently yielded the most efficient parameter estimates. Moreover, applications to real epidemiology and public health datasets revealed the practical superiority of the BUmD over rival models in terms of goodness-of-fit and adaptability. The BUmD serves as a strong and adaptable distribution for modeling bounded data, providing significant insights and practical resources for researchers in statistics, epidemiology, reliability engineering, and associated disciplines.

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Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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