Interfacing finite element and boundary element discretizations

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Introduction

This paper reviews various methods for coupling a mechanical finite element model to an infinite external acoustic domain discretized using boundary-element techniques. The associated physical problem is that of a three-dimensional structure submerged in an acoustic fluid, and impinged by a pressure shock wave.

Three coupling methods for advancing the dynamic calculations are described: field elimination, simultaneous integration, and partitioned integration. Variants of these techniques have been tried on the case problem over the past seven years.

The three methods are assessed from experience gained, and their advantages and disadvantages noted. Some generalizations to more general FE/BE coupling systems are then offered.

The problem

The specific problem used as a case study in this paper is illustrated in *Figure 1*. A linear or nonlinear three-dimensional structure is submerged in an infinite acoustic fluid. A pressure shock wave propagates through the fluid and impinges on the structure. The structure and fluid are discretized through finite element (FE) and boundary element (BE) methods, respectively.

Before proceeding to the governing equations, two practical considerations of relevance to this problem should be mentioned.

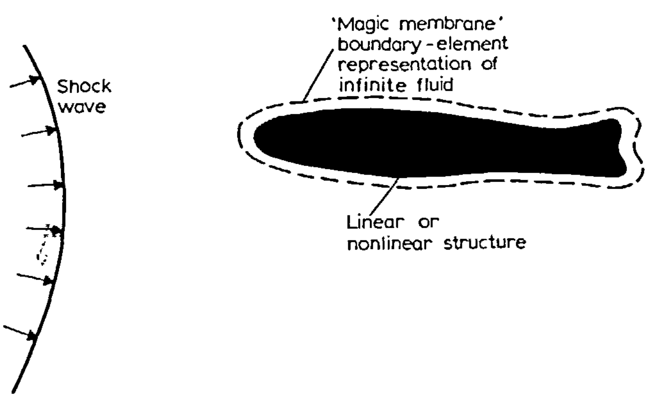

Figure 1 Structure submerged in acoustic field

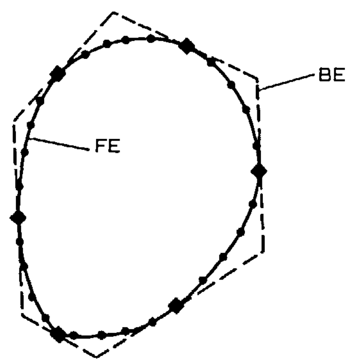

Figure 2 Layout of BE and FE meshes on wet surfaces (♦), BE control points; (•), FE node points

First, the structural response (and most especially the structure's survivability) is of primary concern, whereas what happens in the fluid is of little interest.

Second, the FE and BE meshes on the 'wet surface' are not necessarily in one-to-one correspondence, as illustrated in the two-dimensional sketch of *Figure 2*. Rather, a 'fluid BE' typically overlaps several structural elements. This ties up with the first consideration in the sense that determination of structural deformations and stresses demands a finer subdivision.

Structural response equations

The governing matrix equation of motion for the dynamic response of a discrete structure is:

$$\underset{\sim}{M}_s\ddot{x} + \underset{\sim}{C}_s\dot{x} + \underset{\sim}{K}_s x = f + N \qquad (1)$$

where $x = x(t)$ is the structural displacement vector, $\underset{\sim}{M}_s$, $\underset{\sim}{C}_s$ and $\underset{\sim}{K}_s$ are the structural mass, damping and stiffness matrices, respectively, f is the external force vector, $N = N(x)$ is a nonlinear pseudo-force vector, and a dot denotes temporal differentiation.

For excitation of a submerged structure by an acoustic wave, f is given by

$$f = -GA_f(p_I + p_S) \qquad (2)$$

where p_I and p_S are nodal pressure vectors for the wet-surface fluid mesh pertaining to the (known) incident wave and the (unknown) scattered wave, respectively, $\underset{\sim}{A}_f$ is the diagonal area matrix associated with elements in the fluid mesh, and $\underset{\sim}{G}$ is the transformation matrix that relates the

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structural and fluid nodal forces. Introduction of this matrix takes care of the FE/BE 'mesh-mismatch' noted previously (cf. *Figure 2*).

Fluid equations

The response of the fluid is modelled by the Doubly Asymptotic Approximation (DAA) of Geers[1,2]

$$\underset{\sim}{M}_f \dot{p}_S + \rho c \underset{\sim}{A}_f p_S = \rho c \underset{\sim}{M}_f \dot{u}_S \quad (3)$$

where u_S is the vector of scattered-wave fluid-particle velocity normal to the structure's wet surface, ρ and c are the density and sound velocity of the fluid, respectively, and $\underset{\sim}{M}_f$ is the symmetric fluid mass matrix for the wet-surface fluid mesh. This matrix is produced by a boundary element treatment of Laplace's equation for the irrotational flow generated in an infinite, inviscid, incompressible fluid by motions of the structure's wet surface; it is fully populated with nonzero matrix elements. When transformed into structural coordinates, the fluid mass matrix yields the added mass matrix, which, when combined with the structural mass matrix, yields the virtual mass matrix for motions of a structure submerged in an incompressible fluid. Details of the calculation procedure may be found in DeRuntz and Geers.[3]

The approximate pressure–velocity relation (3) is called 'doubly asymptotic' because it approaches exactness in both the high-frequency (early-time) and low-frequency (late-time) limits. For high-frequency motions, $|\dot{p}_S| \gg |p_S|$, so that (3) approaches the relation $p_S = \rho c u_S$, which is the correct limit for short acoustic wavelengths. For low-frequency motions, $|\dot{p}_S| \ll |p_S|$, so that (3) approaches the incompressible-flow relation $\underset{\sim}{A}_f p_S = \underset{\sim}{M}_f \dot{u}_S$, which is the correct limit for long acoustic wavelengths.

For excitation by an incident acoustic wave, u is related to structural response by the kinematic compatibility relation:

$$\underset{\sim}{G}' \dot{x} = u_I + u_S \quad (4)$$

where the prime superscript denotes matrix transposition. Equation (4) expresses the constraint that normal fluid-particle velocity match normal structural velocity on the wet surface of the structure. The fact that the transformation matrix relating these velocities is $\underset{\sim}{G}'$ follows from the invariance of virtual work with respect to either of the wet-surface coordinate systems.

Generally, $\underset{\sim}{G}$ is a rectangular matrix whose height greatly exceeds its width, inasmuch as the number of structural DOF usually considerably exceeds the number of fluid DOF, as noted previously. Typical numbers: 5000 structural DOFs and 160 fluid DOFs.

Interaction equations

The introduction of equation (2) into (1) and (4) into (3) yields the interaction equations:

$$\begin{aligned} \underset{\sim}{M}_s \ddot{x} + \underset{\sim}{C}_s \dot{x} + \underset{\sim}{K}_s x - N &= -\underset{\sim}{G}\underset{\sim}{A}_f (p_I + p_S) \\ \underset{\sim}{M}_f \dot{p}_S + \rho c \underset{\sim}{A}_f p_S &= \rho c \underset{\sim}{M}_f (\underset{\sim}{G}' \ddot{x} - \dot{u}_I) \end{aligned} \quad (5)$$

The computational structure of these coupled systems is very different. As can be expected, the FE (structural) system is usually large but sparse. The BE (fluid) system is typically small but dense. It is therefore of interest to design solution methods that exploit these attributes to maximum advantage

Solution approaches

Three approaches to solving the coupled FE/BE system (4) are reviewed here. They are presented in chronological order, i.e., in roughly the same sequence as they have been tried and evaluated over the past seven years.

Field elimination

The first approach tried is now known as 'field elimination', where the term 'field' refers to one of the physical components of the coupled system (structural and fluid in our case). As noted previously, the structural response is of primary interest. It is therefore natural to think of eliminating the scattered-pressure vector p_S from the coupled equations of motion (5). If $\underset{\sim}{G}$ is the identity matrix, this results in the following third-order ODE system for the structural displacements x:

$$\begin{aligned} &\underset{\sim}{M}_s \dddot{x} + [\underset{\sim}{C}_s + \underset{\sim}{A}_f \underset{\sim}{M}_f^{-1} \underset{\sim}{M}_s] \ddot{x} + \underset{\sim}{K}_s \dot{x} + \underset{\sim}{A}_f \underset{\sim}{M}_f^{-1} \underset{\sim}{K}_s x \\ &= r(\underset{\sim}{A}_f, \underset{\sim}{M}_f, p_I, \dot{u}_I, N) \end{aligned} \quad (6)$$

where the right hand side vector r accommodates incident pressure and incident fluid-particle velocity boundary conditions, and nonlinear effects. If $\underset{\sim}{G}$ is not the identity, as invariably happens in realistic three-dimensional problems, the coefficient matrices in equation (6) become considerably more complex, because the generalized inverse of $\underset{\sim}{G}$ enters the elimination process.

The structural response $x(t)$ can now be determined by numerically integrating equation (6).

This was in fact the first approach attempted to solve the time-integration of the coupled system; cf. Felippa *et al.*[4] Although moderately successful for the first problem series (submerged shells of revolution, linear structural behaviour), from the current perspective it can be properly characterized as a poor strategy that eventually leads to a 'computational horror show' for more general problems, for the following reasons:

(1) The order of the reduced differential system is raised (in this example, from two to three). The appearance of higher derivatives can be the source of many difficulties, the worst of which is noted next.
(2) Proper treatment of initial conditions is complicated by the increased ODE order. In our case study, it turned out that (6) had to be integrated once (yielding an integro-differential system) so as to regularize the treatment of wavefront-induced singularities. Time integrals of forcing terms had then to be carried along in the calculations – a considerable burden.
(3) Sparseness and symmetry attributes of the original matrices are adversely affected by the elimination process, as can be observed in (6) for an identity $\underset{\sim}{G}$. For a general transformation matrix, all left hand matrices become unsymmetric and dense.
(4) The development of specialized software is required. For example, available software for dealing with the uncoupled problems (structural dynamics and acoustic shocks) separately is not likely to be of much use in solving the reduced system (6).

Simultaneous integration

In this approach equations (5) are viewed as a single second-order system:

$$\begin{bmatrix} \underset{\sim}{M}_s & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{p} \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{C}_s & \underset{\sim}{G}\underset{\sim}{A}_f \\ \rho c \underset{\sim}{M}_f \underset{\sim}{G}' & \underset{\sim}{M}_f \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{p}_S \end{Bmatrix} + \begin{bmatrix} \underset{\sim}{K}_s & \underset{\sim}{0} \\ \underset{\sim}{0} & \rho c \underset{\sim}{A}_f \end{bmatrix} \begin{Bmatrix} x \\ p_S \end{Bmatrix} = \begin{Bmatrix} -\underset{\sim}{G}\underset{\sim}{A}_f p_I + N \\ -\rho c \underset{\sim}{M}_f u_I \end{Bmatrix} \qquad (7)$$

This approach removes many of the objections raised ıgainst the field elimination technique. Inasmuch as the ODE order is not raised, initial condition difficulties do not ırise and better use can be made of existing software for dealing with second order ODE systems.

The computational burden for realistic three-dimensional problems, however, can be prohibitive. Note that the coefficient matrix of higher-derivative terms is singular, which means that the implicit integration is required to construct a marching scheme in time.

But the assembly and formation of the implicit coefficient matrix was found to pose enormous computational demands because of the presence of BE/FE coupling terms that can extend across thousands of equations. For example, it was estimated that the factorization of that matrix for a 5000-DOF problem would require 3 hours on a Cyber 175 computer. Carrying out a nonlinear transient response analysis of a realistic model was then judged to be infeasible.

Partitioned integration

In the partitioned integration approach, the solution state is advanced over each of the two subsystems: FE structural model and BE fluid model, in a staged fashion. Interaction terms are treated as 'forcing' actions that have to be judiciously extrapolated.

What is now called the staggered solution procedure is a specific partitioned-integration method originally formulated for the system (5) by Park *et al.*[5] A version of this procedure was implemented in a production-level computer program described by DeRuntz *et al.*[6]

The success of this method led to further applications and eventually the development of a general theory of partitioned time integration; see Park,[7] Park and Felippa.[8] A state-of-the-art review of formulation aspects has been provided in a recent survey by Felippa and Park.[9]

The staggered solution procedure was found to offer two important advantages: enhanced software modularity and computational efficiency.

The first advantage results from the fact that relatively few modifications to programs available for processing the uncoupled systems are necessary. Given current costs in software development, augmentation and maintenance, this is an important virtue of this approach. For our specific problem, a BE fluid analysis module was written, and data-coupled to existing large-scale structural analysis codes such as NASTRAN, SPAR and STAGS.

An obvious advantage of 'plug-in' modularity is the freedom afforded the analyst as regards the selection of a structural analysis code that best fits the problem at hand; for example, the nonlinear analyser STAGS when plasticity or finite displacements had to be considered. Moreover, if there is a choice among structural analysers that can do almost the same thing, the user can select the one he or she is most confortable with.

As regards computational efficiency, the cost per time step is roughly the same as adding up those incurred in processing the FE and BE models as isolated entities. This is because the overhead introduced by the flow of information (which consists primarily of computational vectors) among the two analysis modules becomes comparatively insignificant, in large-scale problems. It follows that the staggered integration procedure appears as economically attractive should time stepsize considerations be excluded from consideration.

Unfortunately, the latter assumption was not easy to realize in practice. The high computational efficiency per time step is counteracted by the fact that satisfactory numerical stability properties are hard to achieve; in fact, the practical feasibility of this technique hinges almost entirely on the stability analysis. The reader is referred to the cited sources for additional details. Suffice it to say that a specific integration algorithm of unconditional stability was found for (7) when the BE equations were suitably modified through an 'augmentation' technique.

Conclusions

The case study deals with a dynamic problem as we have used boundary-element techniques primarily for transient response analysis of coupled mechanical systems. In particular, wave propagation, solid-fluid impact and re-entry studies. This has provided a body of experience complementary to that gathered by investigators dealing with static problems.

It should be noted that the computer implementation of dynamic and nonlinear-static analysis shares many common facets. In fact, using the dynamic relaxation concept, nonlinear static analysis can be viewed as finding steady-state solutions of pseudo-dynamical systems, and much of the discussion on time-marching algorithms applies when nonlinearity is confined to the finite element mesh.

Our experience has been that BE techniques are primarily useful for discretizing unbounded homogeneous domains governed by linear equations. For linear problems in bounded homogeneous domains, they have not proved to be competitive with properly written finite element codes, and the latter are far easier to extend to nonhomogeneous regions, transient dynamics, and nonlinear behaviour. Advertised reductions in mesh preparation efforts are largely illusory in these days of powerful pre-processors complemented by inexpensive interactive graphics.

But boundary-element methods come on their own for treating unbounded linear media, most particularly when their internal response is of little interest. The model-description effort is greatly reduced, and the construction of 'quiet boundaries' is simplified. Interfacing boundary and finite element discretization for complex interior-exterior problems becomes quite attractive.

The computer implementation of the interface should be as 'loose' (in the software sense) as possible. This goes along with the philosophy of maximizing software modularity. A facet of this philosophy says that the analyst ought to have the freedom of selecting FE and BE meshes independently according to the physics of the problem and the response-resolution requirements. In our case study, for instance, it would have been inadvisable to have the BE fluid mesh constrained by the presence of internal structural stiffeners.

The analysis of fluid-solid impact and similar problems requiring moving and sliding interfaces provide further ammunition on the argument for a high degree of independence in FE/BE mesh definitions.

Nontrivial FE and BE programs tend to be fairly complex beasts even when taken separately. And the complexity of a monolithic marriage can easily escape anybody's control. There are many things that can go wrong and (true to Murphy's law) will: modelling, numerics, machine problems, data management, result interpretation. Nonlinear dynamics problems, for example, are particularly vulnerable to many trouble sources. Experience has shown that keeping modular interfaces not only reduces the chances for trouble, but also makes their resolution more prompt.

For our problem, modularity and computational efficiency requirements demanded the development of a new solution approach: partitioned analysis procedures. The success or failure of this approach, however, is contingent upon implementation details of the time advancing process. This has been the main theme of two recent survey papers.[9,10]

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