Stabilized Finite Element Approximation of the Oseen Equations using Orthogonal Subscales

R.Codina

Publication CIMNE Nº 185, March 2000

Stabilized Finite Element Approximation of the Oseen Equations using Orthogonal Subscales

R. Codina

Publication CIMNE Nº 185, March 2000

International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

STABILIZED FINITE ELEMENT APPROXIMATION OF THE OSEEN EQUATIONS USING ORTHOGONAL SUBSCALES*

RAMON CODINA†

Abstract. In this paper we present a stabilized finite element formulation to solve the Oseen equations as a model problem involving both convection effects and the incompressibility restriction. The need for stabilization techniques to solve this problem arises because of the restriction in the possible choices for the velocity and pressure spaces dictated by the inf–sup condition, as well as the instabilities encountered when convection is dominant. Both can be overcome by resorting from the standard Galerkin method to a stabilized formulation. The one presented here is based on the subgrid scale concept, in which unresolvable scales of the continuous solution are approximately accounted for. In particular, the approach developed herein is based on the assumption that unresolved subscales are orthogonal to the finite element space. The motivation of the method is fully described. It is also shown that this formulation is stable and optimally convergent for an adequate choice of the algorithmic parameters on which the method depends.

Key words. Convection–dominated flows, inf–sup condition, stabilized finite element methods, orthogonal subscales

AMS subject classifications. 65N30, 35Q30

1. Introduction. This paper deals with a finite element formulation to solve second order boundary value problems with two main features: the presence of (dominant) first order terms with the physical meaning of *convection* and the inclusion of constraints in the solution space, in our case *incompressibility*. The simplest linear model that contains both ingredients is the Oseen problem, which consists of finding a pair $[\boldsymbol{u}, p]$ as solution of the equations

$$-\nu\Delta\boldsymbol{u} + \boldsymbol{a}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\boldsymbol{u}$ is the velocity field, p is the pressure, ν is the viscosity, $\boldsymbol{a}$ is the advection velocity, $\boldsymbol{f}$ is the vector of body forces, Ω is the computational domain, assumed to be bounded and polyhedral, and d is the number of space dimensions. For the sake of simplicity, we have considered the simplest Dirichlet condition (1.3). Likewise, several simplifying assumptions will be made for the advection velocity $\boldsymbol{a}$. In particular, we will take it in $\mathcal{C}^0(\bar{\Omega})$, weakly *divergence free* and with derivatives of order up to $k+1$ locally bounded by the maximum of $|\boldsymbol{a}|$ (see assumption H2 in subsection 3.1).

The Oseen problem stated above can be thought of as a linearization of the stationary incompressible Navier–Stokes equations. It also appears as one of the steps of some multilevel methods for these equations (see, e.g., [25]), or may result from a time discretization of the transient Navier–Stokes problem if the advection velocity is treated explicitly. This is why it is often used as a first step towards the analysis of the full nonlinear problem, both to obtain *a priori* (as in [16, 17, 26]) and *a posteriori* (see, e.g., [1]) estimates.

---

*A preliminary version of this paper was presented in a plenary lecture and published in the proceedings of the CEDYA99 Conference, held in Las Palmas de Gran Canaria, Spain, in September 1999, and organized by the Spanish Society of Applied Mathematics (SEMA)

†Universitat Politècnica de Catalunya, Jordi Girona 1–3, Edifici C1, 08034 Barcelona, Spain, E-mail: ramon.codina@upc.es

Let us introduce some standard notation. The space of square integrable functions in a domain ω is denoted by $L^2(\omega)$, and the space of functions whose distributional derivatives of order up to $m \geq 0$ (integer) belong to $L^2(\omega)$ by $H^m(\omega)$. The space $H_0^1(\omega)$ consists of functions in $H^1(\omega)$ vanishing on $\partial\omega$. The topological dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$, and the duality pairing by $\langle\cdot,\cdot\rangle$. A bold character is used to denote the vector counterpart of all these spaces. The L^2 inner product in ω (for scalars, vectors or tensors) is denoted by $(\cdot,\cdot)_\omega$, and the norm in a Banach space X by $\|\cdot\|_X$. This notation is simplified in some cases as follows: $(\cdot,\cdot)_\Omega \equiv (\cdot,\cdot)$, $\|\cdot\|_{L^2(\Omega)} \equiv \|\cdot\|$, for m integer (positive or negative) $\|\cdot\|_{H^m(\Omega)} \equiv \|\cdot\|_m$, and if K is the domain of an element (see below) $\|\cdot\|_{L^2(K)} \equiv \|\cdot\|_K$, $\|\cdot\|_{H^m(K)} \equiv \|\cdot\|_{m,K}$.

Using this notation, the velocity and pressure finite element spaces for the continuous problem are $\mathcal{V}_0 := \boldsymbol{H}_0^1(\Omega)$, $\mathcal{Q}_0 := L^2(\Omega)/\mathbb{R}$, $\mathcal{W}_0 := \mathcal{V}_0 \times \mathcal{Q}_0$. We shall be interested also in the larger spaces $\mathcal{V} := \boldsymbol{H}^1(\Omega)$, $\mathcal{Q} := L^2(\Omega)$, $\mathcal{W} := \mathcal{V} \times \mathcal{Q}$.

Let $\boldsymbol{U} \equiv [\boldsymbol{u}, p] \in \mathcal{W}_0$, $\boldsymbol{V} \equiv [\boldsymbol{v}, q] \in \mathcal{W}_0$. The variational statement for problem (1.1)–(1.2) can be written in terms of the bilinear form defined on $\mathcal{W}_0 \times \mathcal{W}_0$ as

$$B(\boldsymbol{U},\boldsymbol{V}) := \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + (\boldsymbol{a}\cdot\nabla\boldsymbol{u},\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) + (q,\nabla\cdot\boldsymbol{u}). \tag{1.4}$$

Problem (1.1)–(1.2) with the homogeneous Dirichlet condition consists then in finding $\boldsymbol{U} \in \mathcal{W}_0$ such that

$$B(\boldsymbol{U},\boldsymbol{V}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle, \quad \forall \boldsymbol{V} \in \mathcal{W}_0. \tag{1.5}$$

The standard Galerkin approximation of this abstract variational problem is now straightforward. Let $\mathcal{P}_h$ denote a finite element partition of the domain Ω. The diameter of an element domain $K \in \mathcal{P}_h$ is denoted by h_K and the diameter of the finite element partition by $h = \max\{h_K \mid K \in \mathcal{P}_h\}$. For simplicity, we assume that all the element domains are the image of a reference element $\hat{K}$ through a polynomial mapping, affine for simplicial elements, bilinear for quadrilaterals and trilinear for hexahedra. On $\hat{K}$ we define the polynomial spaces $R_k(\hat{K})$ where, as usual, $R_k = P_k$ for simplicial elements and $R_k = Q_k$ for quadrilaterals and hexahedra. From these polynomial spaces we can construct the conforming finite element spaces $\mathcal{V}_h \subset \mathcal{V}$ and $\mathcal{Q}_h \subset \mathcal{Q}$ in the usual manner, as well as the corresponding subspaces $\mathcal{V}_{h,0}$ and $\mathcal{Q}_{h,0}$. In principle, functions in $\mathcal{V}_h$ are continuous, whereas functions in $\mathcal{Q}_h$ not necessarily. Likewise, the orders k of these spaces may be different.

The discrete version of problem (1.5) is: find $\boldsymbol{U}_h \in \mathcal{W}_{h,0}$ such that

$$B(\boldsymbol{U}_h,\boldsymbol{V}_h) = \langle \boldsymbol{f},\boldsymbol{v}_h\rangle, \quad \forall \boldsymbol{V}_h \in \mathcal{W}_{h,0}. \tag{1.6}$$

The well posedness of this problem relies on the ellipticity of the viscous term and the inf–sup or Babuška–Brezzi condition (see [6]), which can be shown to hold for the continuous problem. The first property is automatically inherited by its discrete counterpart. However, the inf–sup condition needs to be explicitly required. This leads to the need of using mixed interpolations, that is, different for $\boldsymbol{u}$ and p, and verifying

$$\inf_{q_h \in \mathcal{Q}_{h,0}} \sup_{\boldsymbol{v}_h \in \mathcal{V}_{h,0}} \frac{(q_h,\nabla\cdot\boldsymbol{v}_h)}{\|q_h\|\,\|\boldsymbol{v}_h\|_1} \geq \beta > 0, \tag{1.7}$$

for a constant β independent of h.

From the computational point of view, and also when equation (1.1) is generalized to include for example zero order terms in $\boldsymbol{u}$, it is convenient to use the same interpolation for the velocity and the pressure. This choice turns out to violate condition

(1.7). This is why many of the so called *stabilized* formulations have been proposed to approximate problem (1.5). The idea is to replace (1.6) by another discrete variational problem in which the bilinear form B is replaced by a possibly mesh dependent bilinear form B_h with enhanced stability properties. Examples of these type of methods are those of Brezzi & Pitkäranta [8], Brezzi & Douglas [5], Douglas & Wang [15], the Galerkin/least–squares (GLS) technique of Hughes, Franca *et al.* [18, 19, 24] and first–order system least–squares methods (see e.g. [3] and references therein).

The second source of instability in the approximation of the Oseen equations arises because of the convective term. When it dominates the viscous one, the stability the latter provides is not enough to have control on the numerical solution and spurious oscillations may appear. Several strategies have been devised to overcome this problem, starting with the classical upwind discretizations. One of the most popular methods to stabilize convection in the finite element context is the so called SUPG method [10]. Variants of this stabilization mechanism, which also allow to use equal velocity–pressure interpolation, can be found in [11, 16, 28, 29]; see also [26, Chapter IV].

In the next section, one of such stabilized formulations is described. It is based on the subgrid scale approach introduced by Hughes in [22, 23] for the scalar convection–diffusion equation (see also [2, 7] for related methods). The particular version analyzed here was presented in [12] and is briefly recalled. The basic idea is to approximate the *effect* of the component of the continuous solution which can not be resolved by the finite element mesh on the discrete finite element solution. An important feature of the formulation developed herein is that the unresolved subscales are assumed to be L^2 orthogonal to the finite element space. It turns out that for the Stokes problem (that it, when convection is absent) this method reduces to the one presented in [13], which was motivated by a completely different reasoning. After having stated two different variants of the proposed formulation, a complete numerical analysis of these is undertaken, showing its stability and convergence properties. Optimal *a priori* convergence estimates are proven for the h-version of the method. A third formulation, which is only intended to stabilize the pressure, is also analyzed.

2. Description of the method.

2.1. The subgrid scale approach. Using Cartesian coordinates $x_1, ..., x_d$, the Oseen equations (1.1)–(1.2) are a particular case of the general system of convection-diffusion-reaction equations

$$\mathcal{L}(\boldsymbol{U}) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(\boldsymbol{A}_i \boldsymbol{U}) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{U}}{\partial x_j}\right) + \boldsymbol{S}\boldsymbol{U} = \boldsymbol{F} \quad \text{in } \Omega,$$

where $\boldsymbol{U} = [\boldsymbol{u}, p]$ and $\boldsymbol{F} = [\boldsymbol{f}, 0]$ are vectors of $D := d+1$ components and $\boldsymbol{A}_i$, $\boldsymbol{K}_{ij}$ and $\boldsymbol{S}$ are $D \times D$ matrices $(i, j = 1, ..., d)$, defined as

$$\boldsymbol{K}_{ii} = \begin{bmatrix} \nu \boldsymbol{I}_d & 0 \\ \boldsymbol{0} & 0 \end{bmatrix}, \quad \boldsymbol{A}_i = \begin{bmatrix} a_i \boldsymbol{I}_d & \boldsymbol{e}_i \\ \boldsymbol{e}_i^{\mathrm{t}} & 0 \end{bmatrix},$$

for $i = 1, ..., d$, and $\boldsymbol{K}_{ij} = \boldsymbol{0}$ for $i \neq j$, $\boldsymbol{S} = \boldsymbol{0}$ (matrix $\boldsymbol{S}$ is not zero when Coriolis forces or permeability effects are introduced). In this expression, $\boldsymbol{I}_d$ is the $d \times d$ identity matrix and $\boldsymbol{e}_i$ is the vector whose j-th component is δ_{ij}.

The bilinear form of the problem in this general case is

$$B(\boldsymbol{U},\boldsymbol{V}) := \sum_{i=1}^{d}\int_\Omega \boldsymbol{V}\cdot\frac{\partial}{\partial x_i}(\boldsymbol{A}_i\boldsymbol{U})\,\mathrm{d}\Omega + \sum_{i,j=1}^{d}\int_\Omega \frac{\partial \boldsymbol{V}}{\partial x_i}\cdot\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{U}}{\partial x_j}\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{V}\cdot\boldsymbol{S}\boldsymbol{U}\,\mathrm{d}\Omega.$$

The contribution from the pressure in the convective term $\boldsymbol{A}_i\boldsymbol{U}$ can be integrated by parts to recover exactly the bilinear form in (1.4), although this is irrelevant when boundary condition (1.3) is used.

Let $\mathcal{W} = \mathcal{W}_h \oplus \tilde{\mathcal{W}}$, where $\tilde{\mathcal{W}}$ is any space to complete $\mathcal{W}_h$ in $\mathcal{W}$. We may think of $\mathcal{W}$ and $\tilde{\mathcal{W}}$ as finite dimensional, with a large dimension. Once the final method will be formulated, we will see that the resulting space approximating $\mathcal{W}$ can in fact be identified with a vector space of finite dimension (cf. Remark 2.1 below). Likewise, let $\mathcal{W}_0 = \mathcal{W}_{h,0} \oplus \tilde{\mathcal{W}}_0$, with $\tilde{\mathcal{W}}_0$ any complement of $\mathcal{W}_{h,0}$ in $\mathcal{W}_0$. The space $\tilde{\mathcal{W}}_0$ will be called the space of *subgrid scales* or *subscales*.

The continuous problem is equivalent to find $\boldsymbol{U}_h \in \mathcal{W}_{h,0}$ and $\tilde{\boldsymbol{U}} \in \tilde{\mathcal{W}}_0$ such that

$$B(\boldsymbol{U}_h,\boldsymbol{V}_h) + B(\tilde{\boldsymbol{U}},\boldsymbol{V}_h) = \langle \boldsymbol{F},\boldsymbol{V}_h\rangle \quad \forall \boldsymbol{V}_h \in \mathcal{W}_{h,0}, \tag{2.1}$$

$$B(\boldsymbol{U}_h,\tilde{\boldsymbol{V}}) + B(\tilde{\boldsymbol{U}},\tilde{\boldsymbol{V}}) = \langle \boldsymbol{F},\tilde{\boldsymbol{V}}\rangle \quad \forall \tilde{\boldsymbol{V}} \in \tilde{\mathcal{W}}_0. \tag{2.2}$$

where $\langle \boldsymbol{F},\boldsymbol{V}\rangle = \langle \boldsymbol{f},\boldsymbol{v}\rangle$ in the case of the Oseen equations. Integrating by parts within each element in (2.1)–(2.2), it is found that these two equations can be written as

$$B(\boldsymbol{U}_h,\boldsymbol{V}_h) + \sum_{i,j=1}^{d}\sum_K \int_{\partial K} \tilde{\boldsymbol{U}}\cdot n_{i,K}\boldsymbol{K}_{ij}\frac{\partial \boldsymbol{V}_h}{\partial x_j}\,\mathrm{d}\Gamma + \sum_K \int_K \tilde{\boldsymbol{U}}\cdot\mathcal{L}^*(\boldsymbol{V}_h)\,\mathrm{d}\Omega = \langle \boldsymbol{F},\boldsymbol{V}_h\rangle, \tag{2.3}$$

$$\sum_{i,j=1}^{d}\sum_K \int_{\partial K} \tilde{\boldsymbol{V}}\cdot n_{i,K}\boldsymbol{K}_{ij}\frac{\partial}{\partial x_j}(\boldsymbol{U}_h+\tilde{\boldsymbol{U}})\,\mathrm{d}\Gamma + \sum_K \int_K \tilde{\boldsymbol{V}}\cdot\mathcal{L}(\tilde{\boldsymbol{U}})\,\mathrm{d}\Omega = \sum_K \int_K \tilde{\boldsymbol{V}}\cdot[\boldsymbol{F}-\mathcal{L}(\boldsymbol{U}_h)]\,\mathrm{d}\Omega, \tag{2.4}$$

where $\sum_K$ stands for the summation over all $K \in \mathcal{P}_h$, $n_{i,K}$ is the i-th component of the unit normal exterior to ∂K, and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$, given by

$$\mathcal{L}^*(\boldsymbol{V}_h) = -\sum_{i=1}^{d}\boldsymbol{A}_i^{\mathrm{t}}\frac{\partial \boldsymbol{V}_h}{\partial x_i} - \sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(\boldsymbol{K}_{ij}^{\mathrm{t}}\frac{\partial \boldsymbol{V}_h}{\partial x_j}\right) + \boldsymbol{S}^{\mathrm{t}}\boldsymbol{V}_h. \tag{2.5}$$

Since exact diffusive fluxes must be continuous across inter-element boundaries, the first term of (2.4) vanishes. This equation is equivalent to:

$$\mathcal{L}(\tilde{\boldsymbol{U}}) = \boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h) + \boldsymbol{V}_{h,\mathrm{ort}} \quad \text{in } K \in \mathcal{P}_h, \quad \forall \boldsymbol{V}_{h,\mathrm{ort}} \in \tilde{\mathcal{W}}_0^{\perp}, \tag{2.6}$$

which must be satisfied together with boundary conditions on ∂K that are unknown, but who must ensure in particular the continuity of the diffusive fluxes across interior boundaries. It is important to remark that (2.6) holds for any element $\boldsymbol{V}_{h,\mathrm{ort}}$ orthogonal to $\tilde{\mathcal{W}}_0$. Here and below, orthogonality is understood with respect to the L^2 product, unless otherwise specified.

The idea now is to approximate the solution of (2.6) with the appropriate boundary conditions by

$$\tilde{\boldsymbol{U}} = \boldsymbol{\tau}_K \left[\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h)\right] + \boldsymbol{\tau}_K \boldsymbol{V}_{h,\mathrm{ort}} \quad \text{in } K \in \mathcal{P}_h, \tag{2.7}$$

where $\boldsymbol{\tau}_K$ is a matrix of algorithmic parameters depending on K and the coefficients of the operator $\mathcal{L}$. This approximation for $\tilde{\boldsymbol{U}}$ is intended to mimic the effect of the exact subscales in the volume integral of (2.3), whereas the integral over the element faces will be neglected. The lack of analytical knowledge in the design of $\boldsymbol{\tau}_K$ will be substituted by the convergence analysis, which will establish whether a particular form of this matrix is adequate or not.

2.2. Orthogonal subscales. The starting point of our developments have been the decompositions $\mathcal{W} = \mathcal{W}_h \oplus \tilde{\mathcal{W}}$ and $\mathcal{W}_0 = \mathcal{W}_{h,0} \oplus \tilde{\mathcal{W}}_0$. If $\cong$ denotes an isomorphism between two vector spaces, we have that $\tilde{\mathcal{W}} \cong \mathcal{W}_h^{\perp} \cap \mathcal{W}$ and $\tilde{\mathcal{W}}_0 \cong \mathcal{W}_{h,0}^{\perp} \cap \mathcal{W}_0$. Nevertheless, there are many possibilities to choose $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}_0$. *The particular one adopted in this work is to take precisely*

$$\tilde{\mathcal{W}} = \mathcal{W}_h^{\perp} \cap \mathcal{W}. \tag{2.8}$$

To obtain a feasible numerical method we need to introduce some approximations. The first concerns the choice for $\tilde{\mathcal{W}}_0$. First, we assume that functions in $\tilde{\mathcal{W}}$ already vanish on $\partial\Omega$, and thus $\tilde{\mathcal{W}}_0 \approx \tilde{\mathcal{W}}$. Additionally we assume that $\mathcal{W}_h^{\perp} \cap \mathcal{W} \approx \mathcal{W}_h^{\perp}$, which can be thought of as a non-conforming approximation for the subscales. Altogether, this amounts to saying that

$$\tilde{\mathcal{W}}_0 \approx \tilde{\mathcal{W}} \approx \mathcal{W}_h^{\perp}. \tag{2.9}$$

With this approximation, it follows from (2.6) that

$$\boldsymbol{V}_{h,\mathrm{ort}} \in \tilde{\mathcal{W}}_0^{\perp} \approx \mathcal{W}_h, \tag{2.10}$$

$$\tilde{\boldsymbol{U}} \in \tilde{\mathcal{W}}_0 \approx \mathcal{W}_h^{\perp}, \tag{2.11}$$

which means that $\boldsymbol{V}_{h,\mathrm{ort}}$ is a finite element function and therefore *numerically computable.* We refer to this particular choice for the space of $\tilde{\boldsymbol{U}}$, motivated by the election (2.8) and the approximation (2.9), as the *space of orthogonal subscales.*

Imposing condition (2.11) in expression (2.7) for $\tilde{\boldsymbol{U}}$ we have that

$$\begin{aligned} 0 &= (\tilde{\boldsymbol{U}}, \boldsymbol{V}_h) \\ &= \sum_K (\boldsymbol{\tau}_K[\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h)], \boldsymbol{V}_h) + \sum_K (\boldsymbol{\tau}_K \boldsymbol{V}_{h,\mathrm{ort}}, \boldsymbol{V}_h), \quad \forall \boldsymbol{V}_h \in \mathcal{W}_h. \end{aligned} \tag{2.12}$$

Let us assume that *matrices $\boldsymbol{\tau}_K$ are all symmetric and positive-definite.* From (2.12) it follows that $\boldsymbol{V}_{h,\mathrm{ort}}$ *is the projection of the residual $\mathcal{L}(\boldsymbol{U}_h) - \boldsymbol{F}$ onto the finite element space with respect to the L^2 inner product weighted element by element by the matrices of algorithmic parameters $\boldsymbol{\tau}_K$.* We denote this weighted inner product and its associated norm by

$$(\boldsymbol{X}, \boldsymbol{Y})_{\tau} := \sum_K (\boldsymbol{\tau}_K \boldsymbol{X}, \boldsymbol{Y})_K = \sum_K (\boldsymbol{X}, \boldsymbol{\tau}_K \boldsymbol{Y})_K, \tag{2.13}$$

$$\|\boldsymbol{Y}\|_{\tau} := \sqrt{(\boldsymbol{Y}, \boldsymbol{Y})_{\tau}}. \tag{2.14}$$

In these expressions, the functions $\boldsymbol{X}$ are $\boldsymbol{Y}$ need not being continuous for the local L^2 products to make sense. The inner product (2.13) will play an essential role in the analysis of the following section.

Equation (2.12) now becomes

$$(\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h), \boldsymbol{V}_h)_\tau + (\boldsymbol{V}_{h,\mathrm{ort}}, \boldsymbol{V}_h)_\tau = 0, \quad \forall \boldsymbol{V}_h \in \mathcal{W}_h. \tag{2.15}$$

If we call Π_τ the projection onto $\mathcal{W}_h$ associated to the inner product (2.13), hereafter referred to as *τ-projection*, we see that

$$\boldsymbol{V}_{h,\mathrm{ort}} = -\Pi_\tau[\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h)], \tag{2.16}$$

Likewise, we will denote by $\Pi_{\tau,0}$ the τ-projection onto $\mathcal{W}_{h,0}$ and $\Pi_\tau^\perp := I - \Pi_\tau$, where I is the identity in $\mathcal{W}_h$.

From (2.7) and (2.16) it follows that

$$\tilde{\boldsymbol{U}} = \boldsymbol{\tau}_K \Pi_\tau^\perp[\boldsymbol{F} - \mathcal{L}(\boldsymbol{U}_h)] \quad \text{in } K \in \mathcal{P}_h.$$

If this expression is now introduced in (2.3) and, as already mentioned, the integrals over the interelement boundaries are neglected, we finally obtain the modified discrete problem: find $\boldsymbol{U}_h \in \mathcal{W}_{h,0}$ such that

$$B_h(\boldsymbol{U}_h, \boldsymbol{V}_h) = \langle \boldsymbol{F}, \boldsymbol{V}_h \rangle - (\Pi_\tau^\perp(\boldsymbol{F}), \mathcal{L}^*(\boldsymbol{V}_h))_\tau, \quad \forall \boldsymbol{V}_h \in \mathcal{W}_{h,0}. \tag{2.17}$$

where the stabilized bilinear form B_h is

$$B_h(\boldsymbol{U}_h, \boldsymbol{V}_h) = B(\boldsymbol{U}_h, \boldsymbol{V}_h) - (\Pi_\tau^\perp[\mathcal{L}(\boldsymbol{U}_h)], \mathcal{L}^*(\boldsymbol{V}_h))_\tau. \tag{2.18}$$

The hope is that the stability properties of (2.17) are much better than those of the original discrete problem (1.6).

Remark 2.1. Equation (2.7), together with (2.10) and (2.11), indirectly determine the approximation to the space $\mathcal{W}$ in which the discrete solution is sought. This space is $\mathcal{W}_h$ enlarged with piecewise discontinuous functions generated by functions in $\mathcal{W}_h$ as indicated by (2.7). We could have started the developments by identifying $\mathcal{W}$ with this finite dimensional vector space.

2.3. Application to the Oseen equations. The previous developments are applicable to any linear system of convection-diffusion-reaction equations. Our purpose now is to apply it to the particular case of the Oseen equations. First of all, note that the adjoint (2.5) in this case is given by:

$$\mathcal{L}^*(\boldsymbol{V}_h) = \begin{bmatrix} -\nu\Delta \boldsymbol{v}_h - \boldsymbol{a}\cdot\nabla \boldsymbol{v}_h - \nabla q_h \\ -\nabla\cdot \boldsymbol{v}_h \end{bmatrix}.$$

The following assumptions will lead to the first of the methods proposed in this paper, the analysis of which is presented in the following section:

1. The matrix of stabilization parameters $\boldsymbol{\tau}_K$ is taken within each element domain $K \in \mathcal{P}_h$ as

$$\boldsymbol{\tau}_K = \mathrm{diag}(\boldsymbol{\tau}_{1,K}, \tau_{2,K}), \quad \boldsymbol{\tau}_{1,K} = \tau_{1,K}\boldsymbol{I}_d, \tag{2.19}$$

$$\tau_{1,K} = \left[\frac{c_1\nu}{h_K^2} + \frac{c_2|\boldsymbol{a}|_{\infty,K}}{h_K}\right]^{-1}, \tag{2.20}$$

$$\tau_{2,K} = c_3\nu + c_4|\boldsymbol{a}|_{\infty,K}h_K, \tag{2.21}$$

where c_i are constants ($i = 1, 2, 3, 4$), on which precise conditions will be given later on, and $|\boldsymbol{a}|_{\infty,K}$ is the maximum of the Euclidian norm of $\boldsymbol{a}$ in the element domain K. Obviously, matrix (2.19) is symmetric and positive–definite, a requirement needed for (2.13) to be an inner product.

2. $\Pi_\tau^\perp(\boldsymbol{F}) = \boldsymbol{0}$, which means that the force vector belongs to the finite element space $\mathcal{W}_h$ (or it is approximated by an element of this space).

3. Second order derivatives of finite element functions within element interiors will be neglected. They are exactly zero for linear elements and for higher order interpolations disregarding them leads to a method which is still consistent (in a sense explained later; cf. Remark 3.1).

Under these conditions, the second term in the RHS of (2.17) vanishes and the stabilized bilinear form (2.18) reduces to

$$\begin{aligned} B_I(\boldsymbol{U}_h, \boldsymbol{V}_h) &= B(\boldsymbol{U}_h, \boldsymbol{V}_h) + (\Pi_{\tau_1}^\perp(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h + \nabla p_h), \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h)_{\tau_1} \\ &\quad + (\Pi_{\tau_2}^\perp(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h)_{\tau_2}, \end{aligned} \tag{2.22}$$

where B is defined in (1.4). Here and in what follows, the symbols Π_{τ_i}, $\Pi_{\tau_i,0}$ and $\Pi_{\tau_i}^\perp$ are used for the projections onto $\mathcal{V}_h$, $\mathcal{V}_{h,0}$ and $\mathcal{V}_h^\perp$, for $i = 1$, and onto $\mathcal{Q}_h$, $\mathcal{Q}_{h,0}$ and $\mathcal{Q}_h^\perp$, for $i = 2$. These projections are associated to the inner products $(\cdot, \cdot)_{\tau_i}$, defined as in (2.13) but using the elementwise value of the scalar algorithmic parameters τ_i ($i = 1, 2$) instead of matrix $\boldsymbol{\tau}_K$. We will use the term *τ_i-orthogonality* to refer to the orthogonality with respect to $(\cdot, \cdot)_{\tau_i}$.

Once arrived to (2.22) it is observed that what the present method provides with respect to the standard Galerkin method is a *least–squares control on the component of the terms $\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h + \nabla p_h$ and $\nabla \cdot \boldsymbol{u}_h$ orthogonal to the corresponding finite element spaces* with respect to the appropriate inner product.

There is a simple modification of the bilinear form (2.22) which leads to another stabilized method with slightly better stability properties. The idea is to control separately the components of $\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h$ and ∇p_h τ_1-orthogonal to $\mathcal{V}_h$. The bilinear form associated to this method is

$$\begin{aligned} B_{II}(\boldsymbol{U}_h, \boldsymbol{V}_h) &= B(\boldsymbol{U}_h, \boldsymbol{V}_h) + (\Pi_{\tau_1}^\perp(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h), \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h)_{\tau_1} \\ &\quad + (\Pi_{\tau_1}^\perp(\nabla p_h), \nabla q_h)_{\tau_1} + (\Pi_{\tau_2}^\perp(\nabla \cdot \boldsymbol{u}_h), \nabla \cdot \boldsymbol{v}_h)_{\tau_2}. \end{aligned} \tag{2.23}$$

Dropping the orthogonal projections $\Pi_{\tau_1}^\perp$ and $\Pi_{\tau_2}^\perp$, the method reduces to a general version of that analyzed in [11], which has a consistency error that makes it only applicable with P_1 elements.

Remark 2.2. Both methods I and II could be slightly modified by projecting onto $\mathcal{W}_{h,0}$ in (2.15) instead of projecting onto $\mathcal{W}_h$. This would simplify the analysis presented in the following section, since the stability condition (3.8) stated there would not be needed, and all the results to be presented carry over to this case. However, even though the global convergence is optimal, projecting onto $\mathcal{W}_{h,0}$ leads to spurious numerical boundary layers, similar to those found for the pressure in classical fractional step schemes for the transient problem (see for example [21]). Further discussion about this point can be found in [13].

2.4. Matrix form of the discrete problem. In order to highlight the modifications of the stabilized methods I and II (associated to the bilinear forms B_I and B_{II}, respectively) with respect to the standard Galerkin method, we consider here the matrix form of all these formulations.

The matrix form of the Galerkin method is

$$\begin{bmatrix} \mathsf{K}+\mathsf{A} & \mathsf{G} \\ \mathsf{D} & 0 \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{P} \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ 0 \end{bmatrix},$$

where U and P are the arrays of nodal velocities and pressures, respectively, K is the matrix arising from the viscous term, A from the advection term, G from the pressure gradient, D from the velocity divergence and F is the resulting vector of nodal forces. Here and in the following we assume that the modifications on the first equation to account for the boundary conditions have *not yet* been performed.

Let us consider now Method I, for simplicity with $\tau_{1,K} \equiv \tau_1$ constant for all the elements and $\tau_{2,K} = 0$. The practical way to compute the orthogonal τ_1-projection $\Pi_{\tau_1}^{\perp}$ is to compute Π_{τ_1} and then use $\Pi_{\tau_1}^{\perp} = I - \Pi_{\tau_1}$. Therefore, if we call $\boldsymbol{\xi}_h$ the τ_1-projection of $\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h$ onto $\mathcal{V}_h$ (which for τ_1 constant is equal to the L^2-projection) Method I consists in fact of three discrete variational equations which allow to find $[\boldsymbol{u}_h, p_h, \boldsymbol{\xi}_h] \in \mathcal{V}_{h,0}\times\mathcal{Q}_{h,0}\times\mathcal{V}_h$, namely,

$$\begin{aligned} &\nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + (\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h) + (\nabla p_h,\boldsymbol{v}_h) \\ &\qquad +\tau_1(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \boldsymbol{\xi}_h, \boldsymbol{a}\cdot\nabla\boldsymbol{v}_h) = \langle \boldsymbol{f},\boldsymbol{v}_h\rangle, \\ &(q_h,\nabla\cdot\boldsymbol{u}_h) + \tau_1(\nabla q_h, \boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h - \boldsymbol{\xi}_h) = 0, \\ &(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h, \boldsymbol{\eta}_h) - (\boldsymbol{\xi}_h,\boldsymbol{\eta}_h) = 0, \end{aligned}$$

which must hold for all $[\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h] \in \mathcal{V}_{h,0}\times\mathcal{Q}_{h,0}\times\mathcal{V}_h$. If we denote by a subscript a the matrices arising from terms weighted by $\boldsymbol{a}\cdot\nabla\boldsymbol{v}_h$ (which suggests 'derivative with respect to $\boldsymbol{a}$') and by subscript d the matrices arising from terms weighted by $-\nabla q_h$ (suggesting the 'divergence'), it is easy to see that the matrix version of the previous equations is

$$\begin{bmatrix} \mathsf{K}+\mathsf{A}+\tau_1\mathsf{A}_a & \mathsf{G}+\tau_1\mathsf{G}_a & -\tau_1\mathsf{M}_a \\ \mathsf{D}-\tau_1\mathsf{A}_d & -\tau_1\mathsf{G}_d & \tau_1\mathsf{M}_d \\ \mathsf{A} & \mathsf{G} & -\mathsf{M} \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{P} \\ \Xi \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ 0 \\ 0 \end{bmatrix}, \tag{2.24}$$

where M is the Gramm matrix of the finite element interpolation, and thus M_a and M_d the matrices obtained by replacing the test function $\boldsymbol{\eta}_h$ by $\boldsymbol{a}\cdot\nabla\boldsymbol{v}_h$ and $-\nabla q_h$, respectively.

The algebraic problem (2.24) can be effectively solved by using a block iteration algorithm segregating the calculation of Ξ from that of U and P (see below). Moreover, if M is approximated by a diagonal matrix (using for instance a nodal quadrature rule) it is numerically feasible to condense Ξ from (2.24), yielding the system

$$\begin{bmatrix} \mathsf{K}+\mathsf{A}+\tau_1(\mathsf{A}_a - \mathsf{M}_a\mathsf{M}^{-1}\mathsf{A}) & \mathsf{G}+\tau_1(\mathsf{G}_a-\mathsf{M}_a\mathsf{M}^{-1}\mathsf{G}) \\ \mathsf{D}+\tau_1(\mathsf{M}_d\mathsf{M}^{-1}\mathsf{A}-\mathsf{A}_d) & \tau_1(\mathsf{M}_d\mathsf{M}^{-1}\mathsf{G}-\mathsf{G}_d) \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{P} \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ 0 \end{bmatrix}. \tag{2.25}$$

Let us consider now Method II, which consists of finding $[\boldsymbol{u}_h, p_h, \boldsymbol{\xi}_{h,1}, \boldsymbol{\xi}_{h,2}] \in \mathcal{V}_{h,0}\times\mathcal{Q}_{h,0}\times\mathcal{V}_h\times\mathcal{V}_h$ such that

$$\begin{aligned} &\nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + (\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h,\boldsymbol{v}_h) + (\nabla p_h,\boldsymbol{v}_h) \\ &\qquad +\tau_1(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h - \boldsymbol{\xi}_{h,1}, \boldsymbol{a}\cdot\nabla\boldsymbol{v}_h) = \langle \boldsymbol{f},\boldsymbol{v}_h\rangle, \\ &(q_h,\nabla\cdot\boldsymbol{u}_h) + \tau_1(\nabla q_h, \nabla p_h - \boldsymbol{\xi}_{h,2}) = 0, \\ &(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h, \boldsymbol{\eta}_{h,1}) - (\boldsymbol{\xi}_{h,1},\boldsymbol{\eta}_{h,1}) = 0, \\ &(\nabla p_h, \boldsymbol{\eta}_{h,2}) - (\boldsymbol{\xi}_{h,2},\boldsymbol{\eta}_{h,2}) = 0, \end{aligned}$$

for all $[v_h, q_h, \eta_{h,1}, \eta_{h,2}] \in \mathcal{V}_{h,0} \times \mathcal{Q}_{h,0} \times \mathcal{V}_h \times \mathcal{V}_h$. The matrix version of this discrete variational problem is

$$\begin{bmatrix} \mathsf{K}+\mathsf{A}+\tau_1\mathsf{A}_a & \mathsf{G} & -\tau_1\mathsf{M}_a & 0 \\ \mathsf{D} & -\tau_1\mathsf{G}_d & 0 & \tau_1\mathsf{M}_d \\ \mathsf{A} & 0 & -\mathsf{M} & 0 \\ 0 & \mathsf{G} & 0 & -\mathsf{M} \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{P} \\ \Xi_1 \\ \Xi_2 \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.26}$$

and the condensed counterpart is

$$\begin{bmatrix} \mathsf{K}+\mathsf{A}+\tau_1(\mathsf{A}_a - \mathsf{M}_a\mathsf{M}^{-1}\mathsf{A}) & \mathsf{G} \\ \mathsf{D} & \tau_1(\mathsf{M}_d\mathsf{M}^{-1}\mathsf{G} - \mathsf{G}_d) \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{P} \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ 0 \end{bmatrix}. \tag{2.27}$$

The difference in the terms introduced by methods I and II is clearly observed by comparing (2.24) and (2.26) or (2.25) and (2.27). It is seen that Method II introduces less terms, but two projections onto $\mathcal{V}_h$ need to be performed.

Although it is not the purpose of this paper to discuss in detail the implementation aspects of the methods analyzed here, let us briefly comment about this point and consider again the matrix form of Method I given by (2.24). The first possibility is to treat this algebraic system in a monolithic fashion. This of course is practically unaffordable in large 3D calculations and the condensed form of the algorithm (2.25) is to be preferred. However, as it has been already mentioned, dealing with M^{-1} is only possible if M is approximated by a diagonal matrix or an iterative scheme is used to solve (2.25), case in which a system of the form $\mathsf{MY} = \mathsf{Z}$ needs to be solved every time a matrix–vector product has to be computed. In either case the computational cost is high. Perhaps the best option is use an iterative coupling for dealing with (2.24), replacing this system by

$$\begin{bmatrix} \mathsf{K}+\mathsf{A}+\tau_1\mathsf{A}_a & \mathsf{G}+\tau_1\mathsf{G}_a & 0 \\ \mathsf{D}-\tau_1\mathsf{A}_d & -\tau_1\mathsf{G}_d & 0 \\ \mathsf{A} & \mathsf{G} & -\mathsf{M} \end{bmatrix} \begin{bmatrix} \mathsf{U}^n \\ \mathsf{P}^n \\ \Xi^n \end{bmatrix} = \begin{bmatrix} \mathsf{F}+\tau_1\mathsf{M}_a\Xi^{n-1} \\ -\tau_1\mathsf{M}_d\Xi^{n-1} \\ 0 \end{bmatrix},$$

where the superscript refers to the iteration counter. Observe that the system matrix remains unaltered during the iterative process. We have found this option very efficient in both 2D and 3D calculations. The cost induced by the need to iterate is small compared to the overall cost of the calculation.

Similar considerations apply to Method II and the method analyzed in subsection 3.4.

3. Numerical analysis.

3.1. Preliminaries. In this section we prove that methods I and II are stable and optimally convergent. We will consider also a slight modification of these methods that is only intended to stabilize the pressure, and therefore with poor stability properties for convection dominated flows. However, this method allows us to prove convergence in a finer norm than for methods I and II.

Let us state now some properties of the family of finite element partitions $\mathcal{F} := \{\mathcal{P}_h \mid h > 0\}$ that we will use. First, we assume that $\mathcal{F}$ is non-degenerate, and therefore the inverse estimate

$$\|\nabla v_h\|_K \leq \frac{C_{\text{inv}}}{h_K}\|v_h\|_K, \quad K \in \mathcal{P}_h, \tag{3.1}$$

holds for any finite element function v_h (see, e.g., [4]).

The precise conditions we will need for the constants c_i in (2.20) and (2.21) can be written in terms of the constant $C_{\rm inv}$ in the inverse estimate (3.1). These conditions are

$$c_1 = \alpha^2 C_{\rm inv}^2, \quad c_2 = \alpha C_{\rm inv}, \quad \text{with} \quad \alpha > 1, \tag{3.2}$$

$$c_3 = \sigma, \quad c_4 = \frac{\sigma}{\alpha C_{\rm inv}}, \quad \text{with} \quad 0 < \sigma \leq 1. \tag{3.3}$$

We shall restrict our attention to interpolations of degree k *for both the velocity and the pressure*, although the extension to different velocity–pressure interpolations offers no difficulty, *provided the pressure interpolation is continuous.* We will need the standard approximation property, namely, for any function v in $H^{k+1}(\Omega)$ there exists a finite element interpolant $\hat{v}_h$ such that

$$\|v - \hat{v}_h\|_{m,K} \leq C_I h_K^{n-m} \|v\|_{n,K}, \quad 0 \leq n \leq k+1, \quad 0 \leq m \leq n, \quad K \in \mathcal{P}_h,$$

where C_I is a positive constant.

We will also need a rather technical condition on the family $\mathcal{F}$, also encountered in [14]. For each $h > 0$, let $\mathcal{N}_h$ be the set of nodal points of the partition $\mathcal{P}_h$. Scalar finite element functions are uniquely determined by its values at the nodes in $\mathcal{N}_h$. Likewise, we denote by $\mathcal{N}_K$ the set of nodes in an element domain $K \in \mathcal{P}_h$. We say that $\mathcal{F}$ is *continuously graded* if there exists a mapping $\{h_K \mid K \in \mathcal{P}_h\} \mapsto \{\bar{h}_a \mid a \in \mathcal{N}_h\}$ such that

$$\max_{K \in \mathcal{P}_h} \left(\max_{a \in \mathcal{N}_K} \left| \frac{\bar{h}_a}{h_K} - 1 \right| \right) \leq \varphi(h), \quad \text{with} \quad \lim_{h \to 0} \varphi(h) = 0. \tag{3.4}$$

A natural way to construct the mapping from the set of element sizes to obtain nodal values from them is the following simple weighted average. Let M_a be the macroelement obtained from the union of the elements to which a node a belongs. Then, we may define

$$\bar{h}_a = \frac{1}{\text{meas}(M_a)} \sum_{K \subset M_a} \text{meas}(K) h_K.$$

The parameters $\bar{h}_a$ constructed this way do not necessarily satisfy condition (3.4), unless some mild conditions hold for $\mathcal{F}$. For example, local mesh refinement is permitted, but provided the ratio between the sizes of the elements sharing a fixed nodal point decreases as the mesh is refined.

In our analysis, we will always assume that $\mathcal{F}$ is continuously graded.

From the set of nodal values $\{\bar{h}_a \mid a \in \mathcal{N}_h\}$ we may construct as well a set of algorithmic parameters $\{\bar{\tau}_{i,a} \mid a \in \mathcal{N}_h, \ i = 1, 2\}$, where $\tau_{i,a}$ is simply defined by replacing in (2.20)–(2.21) the element size h_K by $\bar{h}_a$ and $|\boldsymbol{a}|_{\infty,K}$ by the Euclidian norm of $\boldsymbol{a}$ evaluated at node $a \in \mathcal{N}_h$.

To prove stability (cf. Theorems 3.2, 3.6, 3.8), we will need in particular to take the velocity test function close to $\tau_{1,K} \Pi_{\tau_1,0}(\boldsymbol{\xi}_h)$ within each element domain K (for certain $\boldsymbol{\xi}_h$). Unfortunately, these functions are discontinuous, and therefore we will need to approximate them by continuous functions, belonging to the finite element space. We construct these approximations as follows. Let $N_a(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$, be the standard shape (basis) function associated to node $a \in \mathcal{N}_h$. A finite element function v_h can be thus written as

$$v_h(\boldsymbol{x})\big|_K = \sum_{a \in \mathcal{N}_K} N_a(\boldsymbol{x})\big|_K v^a, \quad K \in \mathcal{P}_h,$$

where $\{v^a \mid a \in \mathcal{N}_K\}$ is the set of element nodal parameters of v_h. From $\{\bar{\tau}_{i,a} \mid a \in \mathcal{N}_h,\ i = 1, 2\}$ and $\{\tau_{i,K} \mid K \in \mathcal{P}_h,\ i = 1, 2\}$ we define

$$\tau \circ v_h \quad \text{by} \quad \tau \circ v_h(\boldsymbol{x})\big|_K := \tau_K v_h(\boldsymbol{x})\big|_K, \tag{3.5}$$

$$\tau \diamond v_h \quad \text{by} \quad \tau \diamond v_h(\boldsymbol{x})\big|_K := \sum_{a \in \mathcal{N}_K} N_a(\boldsymbol{x})\big|_K \bar{\tau}_a v^a. \tag{3.6}$$

Here and in the following result τ may be either τ_1 or τ_2:

LEMMA 3.1. *Assume that the family $\mathcal{F}$ of finite element partitions is continuously graded. Then, for any finite element function v_h, the functions $\tau \circ v_h$ and $\tau \diamond v_h$ defined in (3.5) and (3.6), respectively, satisfy*

$$\|\tau \circ v_h - \tau \diamond v_h\|_K \le \tau_K \psi(h) \|v_h\|_K, \tag{3.7}$$

where $\psi(h) \to 0$ as $h \to 0$.

Proof. For any piecewise continuous finite element function w_h we have that

$$C_1 h_K^{d/2} \|w_h\|_{L^\infty(K)} \le \|w_h\|_{L^2(K)} \le C_2 h_K^{d/2} \|w_h\|_{L^\infty(K)},$$

where C_1 and C_2 are positive constants. The first inequality is an inverse estimate valid for non-degenerate $\mathcal{F}$ (see [4]) and the second is obvious. Using this we obtain

$$\begin{aligned}
C_2^{-1} h_K^{-d/2} &\|\tau \circ v_h - \tau \diamond v_h\|_{L^2(K)} \le \|\tau \circ v_h - \tau \diamond v_h\|_{L^\infty(K)} \\
&= \Big\| \sum_{a \in \mathcal{N}_K} (\tau_K - \bar{\tau}_a) N_a(\boldsymbol{x}) v^a \Big\|_{L^\infty(K)} \\
&\le \tau_K \max_{a \in \mathcal{N}_K} \left| \frac{\bar{\tau}_a}{\tau_K} - 1 \right| \Big\| \sum_{a \in \mathcal{N}_K} |N_a(\boldsymbol{x}) v^a| \Big\|_{L^\infty(K)} \\
&\le C \tau_K \max_{a \in \mathcal{N}_K} \left| \frac{\bar{\tau}_a}{\tau_K} - 1 \right| \|v_h\|_{L^\infty(K)} \\
&\le C \tau_K \max_{a \in \mathcal{N}_K} \left| \frac{\bar{\tau}_a}{\tau_K} - 1 \right| C_1^{-1} h_K^{-d/2} \|v_h\|_{L^2(K)},
\end{aligned}$$

for a positive constant C. From the continuity assumed for the advection velocity $\boldsymbol{a}$ and the assumption that $\mathcal{F}$ is continuously graded, we have that $\bar{\tau}_a \to \tau_K$ as $h \to 0$. The result follows taking $\psi(h) = C C_1^{-1} C_2 \max_{K \in \mathcal{P}_h} (\max_{a \in \mathcal{N}_K} |1 - \bar{\tau}_a / \tau_K|)$. □

This result and definitions (3.5) and (3.6) are also valid when v_h is a vector function. We have thus constructed a continuous function $\tau \diamond v_h$ that approximates $\tau \circ v_h$ when the mesh diameter goes to zero.

The final assumption is the most delicate one. We will assume that there is a constant $\beta_0 > 0$ such that

$$\|z_h\|_{\tau_1} \le \frac{1}{\beta_0} \|\Pi_{\tau_1,0}(z_h) + \Pi_{\tau_1}^{\perp}(z_h)\|_{\tau_1}, \quad z_h \equiv \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h, \tag{3.8}$$

for all $[\boldsymbol{v}_h, q_h] \in \mathcal{V}_{h,0} \times \mathcal{Q}_{h,0}$. This condition means that a bound for the norms of $\Pi_{\tau_1,0}(z_h)$ and $\Pi_{\tau_1}^{\perp}(z_h)$ is enough to bound the whole norm of vector z_h, which in turn implies that the component of z_h in $\mathcal{V}_h$ which is τ_1-orthogonal to $\mathcal{V}_{h,0}$ is not independent of the other two components, $\Pi_{\tau_1,0}(z_h)$ and $\Pi_{\tau_1}^{\perp}(z_h)$.

We will not pursue a detailed study of this condition in this paper. The assumption that it holds will be one of the hypothesis of the analysis presented below.

Nevertheless, let us mention that exactly the same analysis as in [13] can be applied here. This is based on a generalization of the macroelement technique to check the classical inf–sup condition, which is described for example in [27], and leads to a condition that can be effectively checked for a given family of finite element partitions $\mathcal{F}$: Assume that for every h the set of macroelements $\{M_a \mid a \in \mathcal{N}_h\}$, with M_a defined as above, is a covering of Ω. Then, if for all z_h *continuous* on M_a, the assertion

$$\sum_{K\subset M_a} \tau_{1,K} \int_K z_h \cdot w_h \, \mathrm{d}\Omega = 0 \quad \forall w_h \in \mathcal{V}_h \text{ vanishing on } \partial M_a \implies z_h = \mathbf{0} \tag{3.9}$$

holds true, $\mathcal{F}$ satisfies (3.8). The way to prove this fact consists basically of two steps—first it is shown that (3.9) implies that a condition analogous to (3.8) holds with the integrals restricted to the macroelements; then it is shown that this is in fact enough for (3.8) to hold.

To give an idea of how to check condition (3.9), let us consider the case of P_1 interpolations. In this case, ∇v_h and ∇q_h are piecewise constant within each macroelement M_a under consideration. The condition that z_h is continuous implies that in fact both ∇v_h and ∇q_h must be constant over M_a if h is sufficiently small, and thus $z_h \equiv z$ is a continuous function which does not depend on h. It is clear then that (3.9) holds true if h is small enough.

For $z_h = \nabla \cdot v_h$ (now a scalar), and replacing $\tau_{1,K}$ by $\tau_{2,K}$, it is trivially verified that (3.8) always holds with $\beta_0 = 1$, since $v_h = \mathbf{0}$ on $\partial\Omega$ implies that $\nabla \cdot v_h$ has zero mean, and therefore $\Pi_{\tau_2,0}(\nabla \cdot v_h) = \Pi_{\tau_2}(\nabla \cdot v_h)$. However, to keep the notation compact, we will use also (3.8) in this case.

To close the discussion on this condition, let us mention that it would not be needed introducing the modification discussed in Remark 2.2.

The stability condition (3.8) completes the set of assumptions that will be used in the following. For future reference, let us collect them:

H1. The advection velocity a is in $\mathcal{C}^0(\bar{\Omega})$ and weakly divergence free.

H2. There is a constant C_D such that the $k+1$ derivatives of a within element K are bounded above by $C_D|a|_{\infty,K}$, $K \in \mathcal{P}_h$.

H3. The family $\mathcal{F}$ of finite element partitions is continuously graded.

H4. The stability condition (3.8) holds.

H5. The algorithmic parameters $\tau_{1,K}$ and $\tau_{2,K}$ are given by (2.20) and (2.21), respectively, with the constants c_i given by (3.2)–(3.3).

H6. The exact velocity components are in $H^{k+1}(\Omega)$ and the exact pressure in $H^k(\Omega)$.

Assumption H6 will only be needed to prove that convergence is optimal when finite element interpolations of degree k are used. We will call

$$\varepsilon(h) := \sum_K \Big(\tau_{1,K}^{-1/2} h_K^{k+1} \|u\|_{H^{k+1}(K)} + \tau_{2,K}^{-1/2} h_K^{k} \|p\|_{H^k(K)} \Big). \tag{3.10}$$

The ultimate purpose of the analysis below is to show that this is the error function (in norms to be defined) of the different methods considered.

In what follows, C denotes a positive constant, independent of the mesh size h and of the coefficients of the differential equation. The value of C may vary in its different appearances.

3.2. Method I. The problem in this case is to find $U_h \in \mathcal{W}_{h,0}$ such that $B_I(U_h, V_h) = \langle f, v_h \rangle$ for all $V_h \in \mathcal{W}_{h,0}$, with B_I defined in (2.22). We prove now that this method is stable and convergent in the mesh dependent norm

$$\|\!|\boldsymbol{V}_h\|\!|_I \equiv \|\!|[\boldsymbol{v}_h, q_h]\|\!|_I := \nu^{1/2}\|\nabla \boldsymbol{v}_h\| + \|\boldsymbol{a}\cdot\nabla \boldsymbol{v}_h + \nabla q_h\|_{\tau_1} + \|\nabla\cdot \boldsymbol{v}_h\|_{\tau_2}. \tag{3.11}$$

THEOREM 3.2 (Stability of Method I). *Under assumptions H1, H3, H4 and H5, there is a constant $\beta_I > 0$ such that, for h sufficiently small and α in (3.2)–(3.3) large enough,*

$$\inf_{\boldsymbol{U}_h\in\mathcal{W}_{h,0}} \sup_{\boldsymbol{V}_h\in\mathcal{W}_{h,0}} \frac{B_I(\boldsymbol{U}_h,\boldsymbol{V}_h)}{\|\!|\boldsymbol{U}_h\|\!|_I\,\|\!|\boldsymbol{V}_h\|\!|_I} \geq \beta_I. \tag{3.12}$$

Proof. Fix $\boldsymbol{U}_h \equiv [\boldsymbol{u}_h, p_h] \in \mathcal{W}_{h,0}$, arbitrary, and let us introduce the abbreviations $\boldsymbol{\xi}_h \equiv \boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h$, $\delta_h \equiv \nabla\cdot\boldsymbol{u}_h$. From the definition of B_I it follows that

$$B_I([\boldsymbol{u}_h,p_h],[\boldsymbol{u}_h,p_h]) = \nu\|\nabla\boldsymbol{u}_h\|^2 + \|\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h)\|^2_{\tau_1} + \|\Pi^{\perp}_{\tau_2}(\delta_h)\|^2_{\tau_2}. \tag{3.13}$$

Clearly, B_I is not coercive in the norm (3.11). All we can expect is stability in the form given by (3.12). If now we take $[\boldsymbol{v}_h, q_h] = [\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]$ it is found that

$$\begin{aligned}
B_I([\boldsymbol{u}_h,p_h],[\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h),\tau_2\diamond\Pi_{\tau_2,0}(\delta_h)]) &= \nu(\nabla\boldsymbol{u}_h,\nabla[\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)])\\
&+(\boldsymbol{\xi}_h,\tau_1\circ\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)) + (\boldsymbol{\xi}_h,\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h) - \tau_1\circ\Pi_{\tau_1,0}(\boldsymbol{\xi}_h))\\
&+(\tau_2\circ\Pi_{\tau_2,0}(\delta_h),\delta_h) + (\tau_2\diamond\Pi_{\tau_2,0}(\delta_h) - \tau_2\circ\Pi_{\tau_2,0}(\delta_h),\delta_h)\\
&+(\boldsymbol{a}\cdot\nabla[\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)],\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h))_{\tau_1} + (\nabla[\tau_2\diamond\Pi_{\tau_2,0}(\delta_h)],\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h))_{\tau_1}\\
&+(\nabla\cdot[\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)],\Pi^{\perp}_{\tau_2}(\delta_h))_{\tau_2}.
\end{aligned} \tag{3.14}$$

Note that

$$(\boldsymbol{\xi}_h,\tau_1\circ\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)) = \sum_K \tau_{1,K}(\boldsymbol{\xi}_h,\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)) = (\boldsymbol{\xi}_h,\Pi_{\tau_1,0}(\boldsymbol{\xi}_h))_{\tau_1} = \|\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|^2_{\tau_1},$$

and, similarly, $(\tau_2\circ\Pi_{\tau_2,0}(\delta_h),\delta_h) = \|\Pi_{\tau_2,0}(\delta_h)\|^2_{\tau_2}$. Using this, Schwarz's inequality and the inverse estimate (3.1) in (3.14) we get

$$\begin{aligned}
B_I([\boldsymbol{u}_h,p_h],[\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h),\tau_2\diamond\Pi_{\tau_2,0}(\delta_h)]) &\geq -\sum_K \frac{C_{\text{inv}}}{h_K}\nu\|\nabla\boldsymbol{u}_h\|_K\|\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_K\\
&+\|\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|^2_{\tau_1} - \sum_K\|\boldsymbol{\xi}_h\|_K\|\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h) - \tau_1\circ\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_K\\
&+\|\Pi_{\tau_2,0}(\delta_h)\|^2_{\tau_2} - \sum_K\|\delta_h\|_K\|\tau_2\diamond\Pi_{\tau_2,0}(\delta_h) - \tau_2\circ\Pi_{\tau_2,0}(\delta_h)\|_K\\
&-\sum_K\frac{C_{\text{inv}}}{h_K}|\boldsymbol{a}|_{\infty,K}\tau_{1,K}\|\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_K\|\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h)\|_K\\
&-\sum_K\frac{C_{\text{inv}}}{h_K}\tau_{1,K}\|\tau_2\diamond\Pi_{\tau_2,0}(\delta_h)\|_K\|\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h)\|_K\\
&-\sum_K\frac{C_{\text{inv}}}{h_K}\tau_{2,K}\|\tau_1\diamond\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_K\|\Pi^{\perp}_{\tau_2}(\delta_h)\|_K.
\end{aligned} \tag{3.15}$$

The bounds (3.2)–(3.3) assumed for the constants c_i in the definition of $\tau_{1,K}$ and $\tau_{2,K}$ imply

$$\frac{C_{\text{inv}}}{h_K}\nu^{1/2} \leq \frac{1}{\alpha}\tau_{1,K}^{-1/2},\qquad \frac{C_{\text{inv}}}{h_K}|\boldsymbol{a}|_{\infty,K} \leq \frac{1}{\alpha}\tau_{1,K}^{-1},\qquad \frac{C_{\text{inv}}}{h_K}\tau_{1,K}^{1/2}\tau_{2,K}^{1/2} \leq \frac{1}{\alpha}.$$

Using these inequalities, Lemma 3.1 (which in particular implies that $\|\tau \diamond v_h\|_K \le \tau_K(1+\psi(h))\|v_h\|_K$), and the arithmetic-geometric inequality, it can be readily checked that

$$\begin{aligned} B_I([\boldsymbol{u}_h, p_h], [\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]) &\ge -\frac{1}{2\alpha}[1+\psi(h)]\nu\|\nabla \boldsymbol{u}_h\|^2 \\ &+\Big[1 - \frac{3}{2\alpha}[1+\psi(h)] - \frac{1}{2}\psi(h)\Big]\|\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|^2_{\tau_1} \\ &-\frac{1}{\alpha}[1+\psi(h)]\|\Pi^{\perp}_{\tau_1}(\boldsymbol{\xi}_h)\|^2_{\tau_1} - \frac{1}{2}\psi(h)\|\boldsymbol{\xi}_h\|^2_{\tau_1} \\ &+\Big[1 - \frac{1}{2\alpha}[1+\psi(h)] - \frac{1}{2}\psi(h)\Big]\|\Pi_{\tau_2,0}(\delta_h)\|^2_{\tau_2} \\ &-\frac{1}{2\alpha}[1+\psi(h)]\|\Pi^{\perp}_{\tau_2}(\delta_h)\|^2_{\tau_1} - \frac{1}{2}\psi(h)\|\delta_h\|^2_{\tau_2} \end{aligned} \tag{3.16}$$

Let us call $\boldsymbol{v}_h^0 \equiv \boldsymbol{u}_h + \tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h)$, $q_h^0 \equiv q_h + \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)$. Adding up (3.13) and (3.16), taking h small enough and α large enough ($\alpha > 3/2$) so that

$$1 - \frac{3}{2\alpha}[1+\psi(h)] - \frac{1}{2}\psi(h) \ge C_1 > 0,$$

and using the stability condition (3.8), we obtain

$$B_I([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h^0, q_h^0]) \ge C_1\nu\|\nabla \boldsymbol{u}_h\|^2 + \Big[\beta_0^2 C_1 - \frac{1}{2}\psi(h)\Big]\Big[\|\boldsymbol{\xi}_h\|^2_{\tau_1} + \|\delta_h\|^2_{\tau_2}\Big],$$

and therefore, if h is small enough,

$$B_I([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h^0, q_h^0]) \ge C|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|^2_I. \tag{3.17}$$

On the other hand, using repeatedly the inverse estimate (3.1), the definition of τ_1 and τ_2, Lemma 3.1 and the fact that the norm of projection operators is ≤ 1, it follows that

$$\begin{aligned} &|\!|\!|[\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]|\!|\!|^2_I \\ &\le \sum_K C\frac{C^2_{\rm inv}}{h_K^2}[1+\psi(h)]^2\Big[\Big(\nu\tau^2_{1,K} + 2|\boldsymbol{a}|^2_{\infty,K}\tau^3_{1,K} + \tau^2_{1,K}\tau_{2,K}\Big)\|\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|^2_K \\ &\quad +2\tau_{1,K}\tau^2_{2,K}\|\Pi_{\tau_2,0}(\delta_h)\|^2_K\Big] \\ &\le C\Big(\|\boldsymbol{\xi}_h\|^2_{\tau_1} + \|\delta_h\|^2_{\tau_2}\Big) \\ &\le C|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|^2_I, \end{aligned}$$

and therefore,

$$|\!|\!|[\boldsymbol{v}_h^0, q_h^0]|\!|\!|_I \le |\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_I + |\!|\!|[\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]|\!|\!|_I \le C|\!|\!|[\boldsymbol{u}_h, p_h]|\!|\!|_I.$$

The theorem follows using this in (3.17). □

Let $\boldsymbol{U}$ be the solution of the continuous problem. It verifies $B(\boldsymbol{U}, \boldsymbol{V}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle$ for all $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$, and therefore

$$\begin{aligned} B_I(\boldsymbol{U}, \boldsymbol{V}_h) &= \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle + (\Pi^{\perp}_{\tau_1}(\boldsymbol{a}\cdot\nabla\boldsymbol{u} + \nabla p), \boldsymbol{a}\cdot\nabla\boldsymbol{v}_h + \nabla q_h)_{\tau_1} \\ &+ (\Pi^{\perp}_{\tau_2}(\nabla\cdot\boldsymbol{u}), \nabla\cdot\boldsymbol{v}_h)_{\tau_2}. \end{aligned}$$

Since $B_I(\boldsymbol{U}_h, \boldsymbol{V}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$ it follows that

$$
\begin{aligned}
B_I(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) = {}& (\Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p), \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h)_{\tau_1} \\
& + (\Pi_{\tau_2}^{\perp}(\nabla \cdot \boldsymbol{u}), \nabla \cdot \boldsymbol{v}_h)_{\tau_2},
\end{aligned} \tag{3.18}
$$

from where we see that the method is *not consistent* in the classical variational sense, since the RHS of (3.18) is not zero. However, the consistency error can be bounded as follows:

LEMMA 3.3 (Bound for the consistency error of Method I). *Suppose that hypothesis H2, H5 and H6 hold. Then, there is a constant C such that*

$$
B_I(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) \leq C\varepsilon(h) \|\!|\boldsymbol{V}_h\|\!|_I \tag{3.19}
$$

for all $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$.

Proof. From (3.18) we have that

$$
\begin{aligned}
B_I(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) \leq {}& C \|\boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h\|_{\tau_1} \|(\boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p) - \Pi_{\tau_1}(\boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p)\|_{\tau_1} \\
& + C \|\nabla \cdot \boldsymbol{v}_h\|_{\tau_2} \|(\nabla \cdot \boldsymbol{u}) - \Pi_{\tau_2}(\nabla \cdot \boldsymbol{u})\|_{\tau_2} \\
\leq {}& C \|\![\boldsymbol{v}_h, q_h]\|\!|_I \left(\|\boldsymbol{a} \cdot \nabla \boldsymbol{u} - \Pi_{\tau_1}(\boldsymbol{a} \cdot \nabla \boldsymbol{u})\|_{\tau_1} + \|\nabla p - \Pi_{\tau_1}(\nabla p)\|_{\tau_1} \right. \\
& \left. + \|(\nabla \cdot \boldsymbol{u}) - \Pi_{\tau_2}(\nabla \cdot \boldsymbol{u})\|_{\tau_2} \right),
\end{aligned}
$$

for all $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$. Let $v \in H^s(\Omega)$, $0 \leq s \leq k+1$, and let $\hat{v}_h$ be its finite element interpolant. Due to the best approximation property of the τ-projection Π_τ ($\tau = \tau_1$ or τ_2) with respect to the norm $\|\cdot\|_\tau$, we have that

$$
\|v - \Pi_\tau(v)\|_\tau \leq \|v - \hat{v}_h\|_\tau \leq C \sum_K \tau_K^{1/2} h_K^s \|v\|_{H^s(K)}. \tag{3.20}
$$

The result follows now from this, the boundedness of the derivatives of $\boldsymbol{a}$ and the bounds (3.2)–(3.3) assumed for the constants c_i, which imply that $\tau_{1,K}$ behaves as $h_K^2 \tau_{2,K}^{-1}$. □

Remark 3.1. There is a possible way to formulate the present method in a manner that it can be viewed as *consistent*. Indeed, if we introduce

$$
\begin{aligned}
B_I^\star([\boldsymbol{u}_h, p_h, \boldsymbol{\xi}_h, \delta_h], [\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h, \gamma_h]) := {}& B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\
& + (\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h + \nabla p_h - \boldsymbol{\xi}_h, \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h - \boldsymbol{\eta}_h)_{\tau_1} + (\nabla \cdot \boldsymbol{u}_h - \delta_h, \nabla \cdot \boldsymbol{v}_h - \gamma_h)_{\tau_2},
\end{aligned}
$$

the discrete problem is equivalent to find $[\boldsymbol{u}_h, p_h, \boldsymbol{\xi}_h, \delta_h] \in \mathcal{V}_{h,0} \times \mathcal{Q}_{h,0} \times \mathcal{V}_h \times \mathcal{Q}_h$ such that $B_I^\star([\boldsymbol{u}_h, p_h, \boldsymbol{\xi}_h, \delta_h], [\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h, \gamma_h]) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$ for all $[\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h, \gamma_h] \in \mathcal{V}_{h,0} \times \mathcal{Q}_{h,0} \times \mathcal{V}_h \times \mathcal{Q}_h$. This problem is consistent in the sense that $B_I^\star([\boldsymbol{u}, p, \boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p, \nabla \cdot \boldsymbol{u}], [\boldsymbol{v}_h, q_h, \boldsymbol{\eta}_h, \gamma_h]) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$ for smooth enough solutions $[\boldsymbol{u}, p]$ of the continuous problem.

Remark 3.2. Apart from the fact that the use of the weighted L^2 inner product defined by (2.13) and its associated projection arises naturally from the orthogonality condition (2.12), it turns out to be essential to establish the best approximation property used in (3.20).

The final result we need prior to proving convergence is:

LEMMA 3.4 (Estimates for the interpolation error of Method I). *Let $\boldsymbol{U} = [\boldsymbol{u}, p]$ be the solution of the continuous problem and $\hat{\boldsymbol{U}}_h = [\hat{\boldsymbol{u}}_h, \hat{p}_h]$ the finite element interpolant of $\boldsymbol{U}$, and assume that H1, H5 and H6 hold. Then*

$$
B_I(\boldsymbol{U} - \hat{\boldsymbol{U}}_h, \boldsymbol{V}_h) \leq C\varepsilon(h) \|\!|\boldsymbol{V}_h\|\!|_I, \quad \forall \boldsymbol{V}_h \in \mathcal{W}_{h,0}, \tag{3.21}
$$

$$
\|\!|\boldsymbol{U} - \hat{\boldsymbol{U}}_h\|\!|_I \leq C\varepsilon(h). \tag{3.22}
$$

Proof. Let $\hat{e}_u := u - \hat{u}_h$ and $\hat{e}_p := p - \hat{p}_h$ be the finite element interpolation errors for the velocity and the pressure, respectively, which for all K satisfy $\|\hat{e}_u\|_K \leq Ch_K^{k+1}\|u\|_{H^{k+1}(K)}$ and $\|\hat{e}_p\|_K \leq Ch_K^k\|p\|_{H^k(K)}$. From the definition (2.22) of B_I we have that

$$\begin{aligned}B_I(\boldsymbol{U} - \hat{\boldsymbol{U}}_h, \boldsymbol{V}_h) &= \nu(\nabla\hat{e}_u, \nabla v_h) + (a \cdot \nabla\hat{e}_u, v_h) - (\hat{e}_p, \nabla \cdot v_h) + (q_h, \nabla \cdot \hat{e}_u) \\ &+ (\Pi_{\tau_1}^{\perp}(a \cdot \nabla\hat{e}_u + \nabla\hat{e}_p), a \cdot \nabla v_h + \nabla q_h)_{\tau_1} + (\Pi_{\tau_2}^{\perp}(\nabla \cdot \hat{e}_u), \nabla \cdot v_h)_{\tau_2}.\end{aligned}$$

Let us check that each of these terms satisfies estimate (3.21). For the first we have

$$\nu(\nabla\hat{e}_u, \nabla v_h) \leq \nu^{1/2}\|\nabla v_h\|\sum_K \nu^{1/2}\frac{C_{\text{inv}}}{h_K}\|\hat{e}_u\|_K \leq C\|\boldsymbol{V}_h\|_I \sum_K \tau_{1,K}^{-1/2}\|\hat{e}_u\|_K.$$

Adding up the second and the fourth terms and integrating by parts we get

$$\begin{aligned}&(a \cdot \nabla\hat{e}_u, v_h) + (q_h, \nabla \cdot \hat{e}_u) = -(\hat{e}_u, a \cdot \nabla v_h + \nabla q_h) \\ &\leq C\sum_K \tau_{1,K}^{1/2}\|a \cdot \nabla v_h + \nabla q_h\|_K \tau_{1,K}^{-1/2}\|\hat{e}_u\|_K \leq C\|\boldsymbol{V}_h\|_I \sum_K \tau_{1,K}^{-1/2}\|\hat{e}_u\|_K.\end{aligned}$$

The third term can be bounded as

$$-(\hat{e}_p, \nabla \cdot v_h) \leq C\sum_K \tau_{2,K}^{1/2}\|\nabla \cdot v_h\|_K \tau_{2,K}^{-1/2}\|\hat{e}_p\|_K \leq C\|\boldsymbol{V}_h\|_I \sum_K \tau_{2,K}^{-1/2}\|\hat{e}_p\|_K.$$

Using the fact that the norm of projection operators is ≤ 1 we obtain the following bound for the fifth term

$$\begin{aligned}&(\Pi_{\tau_1}^{\perp}(a \cdot \nabla\hat{e}_u + \nabla\hat{e}_p), a \cdot \nabla v_h + \nabla q_h)_{\tau_1} \\ &\leq C\|a \cdot \nabla v_h + \nabla q_h\|_{\tau_1}\sum_K \tau_{1,K}^{1/2}\left(|a|_{\infty,K}\frac{C_{\text{inv}}}{h_K}\|\hat{e}_u\|_K + \frac{C_{\text{inv}}}{h_K}\|\hat{e}_p\|_K\right) \\ &\leq \|\boldsymbol{V}_h\|_I \sum_K \left(\tau_{1,K}^{-1/2}\|\hat{e}_u\|_K + \tau_{2,K}^{-1/2}\|\hat{e}_p\|_K\right).\end{aligned}$$

Likewise

$$(\Pi_{\tau_2}^{\perp}(\nabla \cdot \hat{e}_u), \nabla \cdot v_h)_{\tau_2} \leq \|\nabla \cdot v_h\|_{\tau_2}\sum_K \tau_{2,K}^{1/2}\frac{C_{\text{inv}}}{h_K}\|\hat{e}_u\|_K \leq C\|\boldsymbol{V}_h\|_I \sum_K \tau_{1,K}^{-1/2}\|\hat{e}_u\|_K,$$

which completes the proof of (3.21). Estimate (3.22) can be proved in a similar manner, using the inverse estimate (3.1) and the expressions of $\tau_{1,K}$ and $\tau_{2,K}$. □

Now we are ready to prove the convergence result:

THEOREM 3.5 (Convergence of Method I). *Under assumptions H1 to H6, for h small enough there is a constant C such that*

$$\|\boldsymbol{U} - \boldsymbol{U}_h\|_I \leq C\varepsilon(h).$$

Proof. The proof is standard: from Theorem 3.2 and using Lemmas 3.3 and 3.4 (estimate (3.21)), there exists $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$ such that

$$\beta_I\|\hat{\boldsymbol{U}}_h - \boldsymbol{U}_h\|_I\|\boldsymbol{V}_h\|_I \leq B_I(\hat{\boldsymbol{U}}_h - \boldsymbol{U}, \boldsymbol{V}_h) + B_I(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) \leq C\varepsilon(h)\|\boldsymbol{V}_h\|_I,$$

and therefore $\|\hat{\boldsymbol{U}}_h - \boldsymbol{U}_h\|_I \leq C\varepsilon(h)$. The result follows now from Lemma 3.4 (estimate (3.22)) and the triangle inequality. □

3.3. Method II. Now the problem consists of finding $\boldsymbol{U}_h \in \mathcal{W}_{h,0}$ such that $B_{II}(\boldsymbol{U}_h, \boldsymbol{V}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle$ for all $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$, with B_{II} defined in (2.23). The norm in which we will prove stability and convergence is now

$$\text{(3.23)} \qquad |\!|\!| \boldsymbol{V}_h |\!|\!|_{II} \equiv |\!|\!| [\boldsymbol{v}_h, q_h] |\!|\!|_{II} := |\!|\!| [\boldsymbol{v}_h, q_h] |\!|\!|_{I} + \| \Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{v}_h) \|_{\tau_1} + \| \Pi_{\tau_1}^{\perp}(\nabla q_h) \|_{\tau_1}.$$

It is observed that this norm is slightly finer than $|\!|\!| \cdot |\!|\!|_I$. Now we will have control over the orthogonal component of both the convective derivative of the velocity and the pressure gradient. However, we still do not have control over all the components of these two vectors separately (see the following subsection).

THEOREM 3.6 (Stability of Method II). *Under the same assumptions as in Theorem 3.2, there is a constant $\beta_{II} > 0$ such that*

$$\inf_{\boldsymbol{U}_h \in \mathcal{W}_{h,0}} \sup_{\boldsymbol{V}_h \in \mathcal{W}_{h,0}} \frac{B_{II}(\boldsymbol{U}_h, \boldsymbol{V}_h)}{|\!|\!| \boldsymbol{U}_h |\!|\!|_{II} \, |\!|\!| \boldsymbol{V}_h |\!|\!|_{II}} \geq \beta_{II}.$$

Proof. Let us proceed exactly as in the proof of Theorem 3.2. Using the inequality $a^2 + b^2 \geq (a^2 + b^2)/3 + (a + b)^2/3$, it is found that instead of (3.13) we now have

$$\begin{aligned} B_{II}([\boldsymbol{u}_h, p_h], [\boldsymbol{u}_h, p_h]) &\geq \nu \|\nabla \boldsymbol{u}_h\|^2 + \frac{1}{3} \|\Pi_{\tau_1}^{\perp}(\boldsymbol{\xi}_h)\|_{\tau_1}^2 + \|\Pi_{\tau_2}^{\perp}(\delta_h)\|_{\tau_2}^2 \\ &+ \frac{1}{3} \|\Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h)\|_{\tau_1}^2 + \frac{1}{3} \|\Pi_{\tau_1}^{\perp}(\nabla p_h)\|_{\tau_1}^2. \end{aligned}$$

Once again, the bilinear form B_{II} is not coercive in the norm (3.23). If now we take $[\boldsymbol{v}_h, q_h] = [\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]$, an expression similar to (3.14) is found. Only the sixth and seventh terms of the RHS of this inequality are different. They and their bounds in (3.15) have to be replaced by

$$\begin{aligned} &(\boldsymbol{a} \cdot \nabla [\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h)], \Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h))_{\tau_1} + (\nabla [\tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)], \Pi_{\tau_1}^{\perp}(\nabla p_h))_{\tau_1} \\ &\geq -\sum_K \frac{C_{\text{inv}}}{h_K} |\boldsymbol{a}|_{\infty,K} \tau_{1,K} \|\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_K \|\Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h)\|_K \\ &\quad -\sum_K \frac{C_{\text{inv}}}{h_K} \tau_{1,K} \|\tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)\|_K \|\Pi_{\tau_1}^{\perp}(\nabla p_h)\|_K. \end{aligned}$$

Calling again $\boldsymbol{v}_h^0 \equiv \boldsymbol{u}_h + \tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h)$, $q_h^0 \equiv q_h + \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)$, it is found now that

$$\begin{aligned} B_I([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h^0, q_h^0]) &\geq \Big[1 - \frac{1}{2\alpha}[1 + \psi(h)]\Big] \nu \|\nabla \boldsymbol{u}_h\|^2 \\ &+ \Big[1 - \frac{3}{2\alpha}[1 + \psi(h)] - \frac{1}{2}\psi(h)\Big] \|\Pi_{\tau_1,0}(\boldsymbol{\xi}_h)\|_{\tau_1}^2 + \frac{1}{3}\|\Pi_{\tau_1}^{\perp}(\boldsymbol{\xi}_h)\|_{\tau_1}^2 - \frac{1}{2}\psi(h)\|\boldsymbol{\xi}_h\|_{\tau_1}^2 \\ &+ \Big[\frac{1}{3} - \frac{1}{2\alpha}[1 + \psi(h)]\Big] \|\Pi_{\tau_1}^{\perp}(\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h)\|_{\tau_1}^2 + \Big[\frac{1}{3} - \frac{1}{2\alpha}[1 + \psi(h)]\Big] \|\Pi_{\tau_1}^{\perp}(\nabla p_h)\|_{\tau_1}^2 \\ &+ \Big[1 - \frac{1}{2\alpha}[1 + \psi(h)] - \frac{1}{2}\psi(h)\Big] \|\Pi_{\tau_2,0}(\delta_h)\|_{\tau_2}^2 \\ &+ \Big[1 - \frac{1}{2\alpha}[1 + \psi(h)]\Big] \|\Pi_{\tau_2}^{\perp}(\delta_h)\|_{\tau_1}^2 - \frac{1}{2}\psi(h)\|\delta_h\|_{\tau_2}^2 \end{aligned}$$

From this and (3.8) it follows that for h small enough there is an $\alpha > 1$ for which

$$B_{II}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h^0, q_h^0]) \geq C |\!|\!| [\boldsymbol{u}_h, p_h] |\!|\!|_{II}^2.$$

Similar bounds to those employed in Theorem 3.2 yield

$$|||[\tau_1 \diamond \Pi_{\tau_1,0}(\boldsymbol{\xi}_h), \tau_2 \diamond \Pi_{\tau_2,0}(\delta_h)]|||_{II}^2 \leq C\left(\|\boldsymbol{\xi}_h\|_{\tau_1}^2 + \|\delta_h\|_{\tau_2}^2\right) \leq C|||[\boldsymbol{u}_h, p_h]|||_{II}^2,$$

and the proof concludes as in Theorem 3.2. □

The consistency error of Method II is

$$B_{II}(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) = (\Pi_{\tau_1}^{\perp}(\boldsymbol{a}\cdot\nabla\boldsymbol{u}), \boldsymbol{a}\cdot\nabla\boldsymbol{v}_h)_{\tau_1} + (\Pi_{\tau_1}^{\perp}(\nabla p), \nabla q_h)_{\tau_1} + (\Pi_{\tau_2}^{\perp}(\nabla\cdot\boldsymbol{u}), \nabla\cdot\boldsymbol{v}_h)_{\tau_2}.$$

The bound (3.19) of Lemma 3.3 also holds for this case, as well as the estimates for the interpolation error given in (3.21) and (3.22). The proof of all these facts follows the same lines as for Method I, only with minor modifications. We give directly the convergence result, whose proof is also straightforward:

THEOREM 3.7 (Convergence of Method II). *Under the same assumptions as in Theorem 3.5, there is a constant C such that*

$$|||\boldsymbol{U} - \boldsymbol{U}_h|||_{II} \leq C\varepsilon(h),$$

where $\varepsilon(h)$ is the same error function as for Method I, given by (3.10).

3.4. Viscous dominated case. Both in methods I and II the stability result obtained shows that $\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h$ is under control. However, we do not have explicit bounds over these two terms (and their errors) separately. Nevertheless, there is the possibility of bounding the pressure gradient making use of the control over the viscous term, since

$$\begin{aligned} \tau_{1,K}\|\nabla p_h\|_K^2 &\leq \tau_{1,K}\|\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h\|_K^2 + \tau_{1,K}\|\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h\|_K^2 \\ &\leq \tau_{1,K}\|\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h + \nabla p_h\|_K^2 + C\left(\frac{|\boldsymbol{a}|_{\infty,K} h_K}{\nu}\right)\nu\|\nabla\boldsymbol{u}_h\|_K^2. \end{aligned} \tag{3.24}$$

Let us introduce the dimensionless quantities

$$\mathrm{Re} := \frac{|\boldsymbol{a}|_\infty L}{\nu}, \quad \mathrm{Re}_K := \frac{|\boldsymbol{a}|_{\infty,K} h_K}{\nu}, \quad \mathrm{Re}_h := \max\{\mathrm{Re}_K \mid K \in \mathcal{P}_h\},$$

where L is a characteristic length of Ω. These numbers may be called the *global*, *cell* and *mesh Reynolds numbers*, respectively.

From (3.24) it is seen that we have control over $\tau_{1,K}\|\nabla p_h\|_K^2$, but with a constant depending on the inverse of Re_K. Therefore, this estimate is numerically meaningful only for small values of Re_K. However, if we allow our stability and error estimates to depend on this parameter, it is not necessary to use neither Method I nor Method II, but rather a simplified form of these which does not include the stabilizing term for the velocity streamline derivative. This method consists of finding $\boldsymbol{U}_h \in \mathcal{W}_{h,0}$ such that $B_\nu(\boldsymbol{U}_h, \boldsymbol{V}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle$ for all $\boldsymbol{V}_h \in \mathcal{W}_{h,0}$, with B_ν defined as

$$B_\nu(\boldsymbol{U}_h, \boldsymbol{V}_h) = B(\boldsymbol{U}_h, \boldsymbol{V}_h) + (\Pi_\tau^{\perp}(\nabla p_h), \nabla q_h)_\tau, \tag{3.25}$$

and with B given in (1.4). Clearly, the only purpose of this method is to stabilize the pressure. The behavior in convective dominated situations will be similar to that of the standard Galerkin method using div–stable velocity–pressure interpolations.

Except for the presence of the convective term in B, this formulation is the same as that introduced in [13]. We will present here a different stability proof which furthermore will show the dependence of the stability and error estimates on Re_h and Re. For that, let us introduce the norm

$$\|\!|\boldsymbol{V}_h|\!\|_\nu \equiv \|\!|[\boldsymbol{v}_h, q_h]|\!\|_\nu := \nu^{1/2}\|\nabla \boldsymbol{v}_h\| + \frac{1}{1+\mathrm{Re}_h}\|\nabla q_h\|_\tau + \frac{1}{1+\mathrm{Re}}\frac{1}{\nu^{1/2}}\|q_h\|, \tag{3.26}$$

in which the analysis of the method will be performed. Now the parameters τ_K, which correspond to $\tau_{1,K}$ of the previous methods, can be taken as

$$\tau_K = \frac{h_K^2}{\alpha^2 C_{\mathrm{inv}}^2 \nu}, \tag{3.27}$$

and $\tau_{2,K}$ simply set to zero.

THEOREM 3.8 (Stability of the viscous dominated case). *Assume that H3 and H4 hold, and the parameters τ_K are given by (3.27). Then, for h sufficiently small there is a constant $\beta_\nu > 0$ such that*

$$\inf_{\boldsymbol{U}_h \in \mathcal{W}_{h,0}} \sup_{\boldsymbol{V}_h \in \mathcal{W}_{h,0}} \frac{B_\nu(\boldsymbol{U}_h, \boldsymbol{V}_h)}{\|\!|\boldsymbol{U}_h|\!\|_\nu \, \|\!|\boldsymbol{V}_h|\!\|_\nu} \geq \beta_\nu. \tag{3.28}$$

Proof. The proof of this result is similar to the proofs of Theorems 3.2 and 3.6, except for the presence of the L^2 norm of q_h in the definition (3.26). Now we have that

$$B_\nu([\boldsymbol{u}_h, p_h], [\boldsymbol{u}_h, p_h]) = \nu\|\nabla \boldsymbol{u}_h\|^2 + \|\Pi_\tau^\perp(\nabla p_h)\|_\tau^2, \tag{3.29}$$

and, using the same strategy as in Theorem 3.2,

$$\begin{aligned}
B_\nu([\boldsymbol{u}_h, p_h], [\tau \diamond \Pi_{\tau,0}(\nabla p_h), 0]) &= \nu(\nabla \boldsymbol{u}_h, \nabla[\tau \diamond \Pi_{\tau,0}(\nabla p_h)]) \\
&\quad + (\boldsymbol{a}\cdot\nabla \boldsymbol{u}_h, \tau \diamond \Pi_{\tau,0}(\nabla p_h)) + (\nabla p_h, \tau \circ \Pi_{\tau,0}(\nabla p_h)) \\
&\quad + (\nabla p_h, \tau \diamond \Pi_{\tau,0}(\nabla p_h) - \tau \circ \Pi_{\tau,0}(\nabla p_h)) \\
&\geq -\sum_K \nu \frac{C_{\mathrm{inv}}}{h_K}\|\nabla \boldsymbol{u}_h\|_K \|\tau \diamond \Pi_{\tau,0}(\nabla p_h)\|_K - \sum_K |\boldsymbol{a}|_{\infty,K}\|\nabla \boldsymbol{u}_h\|_K \|\tau \diamond \Pi_{\tau,0}(\nabla p_h)\|_K \\
&\quad + \|\Pi_{\tau,0}(\nabla p_h)\|_\tau^2 - \sum_K \|\nabla p_h\|_K \|\tau \diamond \Pi_{\tau,0}(\nabla p_h) - \tau \circ \Pi_{\tau,0}(\nabla p_h)\|_K.
\end{aligned}$$

Noting that $\tau_K|\boldsymbol{a}|_{\infty,K}^2 = \mathrm{Re}_K^2\nu/(\alpha^2 C_{\mathrm{inv}}^2) \leq C\mathrm{Re}_h^2$ and using Lemma 3.1, it is not difficult to see that this last inequality can be written as

$$\begin{aligned}
&B_\nu([\boldsymbol{u}_h, p_h], [\tau \diamond \Pi_{\tau,0}(\nabla p_h), 0]) \\
&\qquad \geq C_1\|\Pi_{\tau,0}(\nabla p_h)\|_\tau^2 - C_2\psi(h)\|\nabla p_h\|_\tau^2 - C_3(1+\mathrm{Re}_h^2)\nu\|\nabla \boldsymbol{u}_h\|^2,
\end{aligned} \tag{3.30}$$

where the constants C_i, $i = 1, 2, 3$, do not depend neither on Re_h nor on Re.

To introduce the L^2 norm of p_h, let us invoke the inf–sup condition for the continuous problem, namely, the continuous counterpart of condition (1.7). Since p_h belongs to $L^2(\Omega)$, there exists a function $\boldsymbol{v} \in \mathcal{V}_0$ such that

$$\beta\|p_h\|\|\nabla \boldsymbol{v}\| \leq |(p_h, \nabla\cdot\boldsymbol{v})|.$$

We have used the L^2 norm of $\nabla \boldsymbol{v}$ in the LHS since due to the Poincaré–Friedrics inequality it is equivalent to the H^1 norm of $\boldsymbol{v}$. We may thus normalize $\boldsymbol{v}$ so that

$\|\nabla \boldsymbol{v}\| = \|p_h\|/\nu$. Let now $\hat{\boldsymbol{v}}_h$ be the finite element interpolant of $\boldsymbol{v}$. Using the fact that $|a-b| \geq |a| - |b|$, we have:

$$\begin{aligned} B_\nu([\boldsymbol{u}_h, p_h], [\hat{\boldsymbol{v}}_h, 0]) &= |\nu(\nabla \boldsymbol{u}_h, \nabla \hat{\boldsymbol{v}}_h) + (\boldsymbol{a}\cdot\nabla \boldsymbol{u}_h, \hat{\boldsymbol{v}}_h) - (p_h, \nabla\cdot\hat{\boldsymbol{v}}_h)| \\ &\geq |(p_h, \nabla\cdot \boldsymbol{v})| - |(p_h, \nabla\cdot(\boldsymbol{v}-\hat{\boldsymbol{v}}_h))| - |\nu(\nabla \boldsymbol{u}_h, \nabla\hat{\boldsymbol{v}}_h)| - |(\boldsymbol{a}\cdot\nabla\boldsymbol{u}_h, \hat{\boldsymbol{v}}_h)|. \end{aligned} \tag{3.31}$$

If C_L denotes the constant of the Poincaré–Friedrics inequality and C_I the constant in the standard interpolation estimates, we have that

$$\begin{aligned} &\|\boldsymbol{v}-\hat{\boldsymbol{v}}_h\|_K \leq C_I h_K \|\nabla \boldsymbol{v}\|_K, \\ &\|\nabla\hat{\boldsymbol{v}}_h\| \leq \|\nabla \boldsymbol{v} - \nabla\hat{\boldsymbol{v}}_h\| + \|\nabla\boldsymbol{v}\| \leq (C_I+1)\|\nabla \boldsymbol{v}\| = (C_I+1)\frac{1}{\nu}\|p_h\|, \\ &\|\hat{\boldsymbol{v}}_h\| \leq C_L\|\nabla\hat{\boldsymbol{v}}_h\| \leq C_L(C_I+1)\frac{1}{\nu}\|p_h\|. \end{aligned}$$

Integrating by parts the second term in (3.31) and using these bounds we obtain

$$\begin{aligned} &B_\nu([\boldsymbol{u}_h, p_h], [\hat{\boldsymbol{v}}_h, 0]) \\ &\geq \beta\frac{1}{\nu}\|p_h\|^2 - \sum_K \|\nabla p_h\|_K \|\boldsymbol{v}-\hat{\boldsymbol{v}}_h\|_K - \nu\|\nabla\boldsymbol{u}_h\|\|\nabla\hat{\boldsymbol{v}}_h\| - |\boldsymbol{a}|_\infty \|\nabla \boldsymbol{u}_h\|\|\hat{\boldsymbol{v}}_h\| \\ &\geq \beta\frac{1}{\nu}\|p_h\|^2 - C_I\sum_K h_K\|\nabla p_h\|_K\|\nabla\boldsymbol{v}\|_K - (C_I+1)\|\nabla\boldsymbol{u}_h\|\|p_h\| \\ &\quad - |\boldsymbol{a}|_\infty C_L(C_I+1)\frac{1}{\nu}\|\nabla\boldsymbol{u}_h\|\|p_h\|. \end{aligned} \tag{3.32}$$

On the other hand, from Young's inequality we have that

$$\sum_K h_K\|\nabla p_h\|_K\|\nabla\boldsymbol{v}\|_K \leq \sum_K\Big[\frac{h_K^2}{2\nu\epsilon}\|\nabla p_h\|_K^2 + \frac{\nu\epsilon}{2}\|\nabla\boldsymbol{v}\|_K^2\Big] \leq \frac{C}{\epsilon}\|\nabla p_h\|_\tau^2 + \frac{\epsilon}{2\nu}\|p_h\|^2,$$

for all $\epsilon > 0$. Using a similar inequality for the last two terms of (3.32), taking ϵ small enough and noting that since C_L is proportional to L, $|\boldsymbol{a}|_\infty C_L/\nu$ is proportional to Re, we obtain

$$B_\nu([\boldsymbol{u}_h, p_h], [\hat{\boldsymbol{v}}_h, 0]) \geq C_4\frac{1}{\nu}\|p_h\|^2 - C_5\|\nabla p_h\|_\tau^2 - C_6(1+\mathrm{Re}^2)\nu\|\nabla\boldsymbol{u}_h\|^2, \tag{3.33}$$

for constants C_4, C_5 and C_6 independent of Re_h and Re. If now we take

$$\begin{aligned} \boldsymbol{v}_h^0 &\equiv \boldsymbol{u}_h + \frac{A_1}{1+\mathrm{Re}_h^2}\tau\diamond\Pi_{\tau,0}(\nabla p_h) + \frac{A_2}{1+\mathrm{Re}^2}\hat{\boldsymbol{v}}_h, \\ q_h^0 &\equiv p_h, \end{aligned} \tag{3.34}$$

and add up (3.29), (3.30) and (3.33) multiplied by the corresponding coefficients, we obtain

$$\begin{aligned} &B_\nu([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h^0, q_h^0]) \geq \Big[1 - C_3A_1 - C_6A_2\Big]\nu\|\nabla\boldsymbol{u}_h\|^2 \\ &+\Big[1 - \frac{C_2\psi(h)}{1+\mathrm{Re}_h^2}\frac{A_1}{\beta_0} - \frac{C_5}{1+\mathrm{Re}^2}\frac{A_2}{\beta_0}\Big]\|\Pi_\tau^\perp(\nabla p_h)\|_\tau^2 \\ &+\Big[\frac{C_1A_1}{1+\mathrm{Re}_h^2} - \frac{C_2\psi(h)}{1+\mathrm{Re}_h^2}\frac{A_1}{\beta_0} - \frac{C_5}{1+\mathrm{Re}^2}\frac{A_2}{\beta_0}\Big]\|\Pi_{\tau,0}(\nabla p_h)\|_\tau^2 + \Big[\frac{C_4}{\nu}\frac{A_2}{1+\mathrm{Re}^2}\Big]\|p_h\|^2 \end{aligned}$$

where we have made use of the stability condition (3.8) (now with $z_h = \nabla p_h$). From this, it follows that there are values of the constants A_1 and A_2 for which

$$B_\nu([u_h, p_h], [v_h^0, q_h^0]) \geq C \|[u_h, p_h]\|_\nu^2.$$

The theorem now follows after checking that

$$\|[v_h^0, q_h^0]\|_\nu \leq C \|[u_h, p_h]\|_\nu,$$

which is easily verified from the definition (3.34) of $[v_h^0, q_h^0]$ and noting that $(1+x^2)^{-1} \leq 2(1+x)^{-1}$ for all $x > 0$. □

The same strategy as for methods I and II can now be followed to prove convergence. We omit the intermediate steps and simply state the final result:

THEOREM 3.9 (Convergence of the viscous dominated case). *If assumptions H1 to H5 hold and the parameters τ_K are given by (3.27), for h small enough there is a constant C such that*

$$\|U - U_h\|_\nu \leq C(1 + \mathrm{Re}_h) \sum_K \left(\tau_K^{-1/2} h_K^{k+1} \|u\|_{H^{k+1}(K)} + \nu^{-1/2} h_K^k \|p\|_{H^k(K)} \right).$$

This convergence estimate, as well as the stability estimate (3.28), deteriorates as ν decreases. Due to the dependence on Re_h and Re *explicitly displayed* by (3.26), it is seen that control over the L^2 norm of the pressure is rapidly lost as $\nu \to 0$, since in this case $\mathrm{Re} \to \infty$. However, a somewhat stronger control is obtained on $\|\nabla q_h\|_\tau$. We may consider that the finite element mesh is sufficiently refined so as to maintain Re_h (relatively) small. These results are similar to those obtained in [29] for the nonlinear Navier–Stokes equations, even though the method analyzed in this reference is also intended to stabilize convection.

Remark 3.3. In the absence of convection, the norm (3.26) in which stability and convergence has been proven is even finer than for the Galerkin method using div–stable velocity–pressure interpolations. This in particular allows to extend this pressure stabilized method to the nonlinear Navier–Stokes equations and obtain *exactly* the same results as for the Galerkin method (see [9, 20]). This extension is analyzed in [14].

4. Concluding remarks. Three different stabilized finite element formulations for the Oseen problem have been presented in this paper. Their main features are:

1. The original method (referred to as Method I in the paper) is directly based on the subgrid scale concept, assuming that the subscales are orthogonal to the finite element space. After some simple approximations, a stabilized formulation is obtained with two major benefits with respect to the original Galerkin method: it allows the use of equal velocity–pressure interpolations and it provides optimal control on the streamline derivative of the velocity field.

2. The second method (Method II) is somewhat simpler, since it introduces less coupling in the discrete velocity–pressure equations (although one more projection needs to be performed). Furthermore, stability and error estimates have been shown to hold in a norm finer than for Method I, since now it is possible to control the orthogonal components of the convective term and the pressure gradient.

3. If only the pressure interpolation is to be stabilized, a simplification of methods I and II has been proposed and analyzed. The norm in which stability and convergence has been proven depends explicitly on the mesh Reynolds number and the global Reynolds number.

Finally, even though no numerical examples have been presented, let us mention that the accuracy of the formulation presented here is one of its most salient features, both for Method I as for Method II (which yield very similar results). The order of convergence is the same as in other stabilized formulations, such as the Galerkin/least–squares technique [16, 17], but it is less diffusive. This in particular leads to a better treatment of the pressure near boundaries. See [12, 13] for further discussion about this point and some numerical results.

REFERENCES

[1] M. Ainsworth and J. Oden, *A posteriori error estimators for the Stokes and Oseen equations*, SIAM J. Numer. Anal., 34 (1997), pp. 228–245.

[2] C. Baiocchi, F. Brezzi, and L. Franca, *Virtual bubbles and Galerkin/least-squares type methods (Ga.L.S)*, Comput. Methods Appl. Mech. Engrg., 105 (1993), pp. 125–141.

[3] P. Bochev, Z. Cai, T. Manteuffel, and S. McCormick, *Analysis of velocity-flux first-order system least-squares principles for the Navier-Stokes equations: Part I*, SIAM J. Numer. Anal., 35 (1998), pp. 990–1009.

[4] S. Brenner and L. Scott, *The mathematical theory of finite element methods*, Springer–Verlag, 1994.

[5] F. Brezzi and J. Douglas, *Stabilized mixed methods for the Stokes problem*, Numer. Math., 53 (1988), pp. 225–235.

[6] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Verlag, 1991.

[7] F. Brezzi, L. Franca, T. Hughes, and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., 145 (1997), pp. 329–339.

[8] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximation of the Stokes problem, in *Efficient Solution of Elliptic Systems*, W. Hackbush (Ed.), Vieweg, Braunschweig, 1984, pp. 11–19.

[9] F. Brezzi, J. Rappaz, and P. Raviart, *Finite dimensional approximation of nonlinear problems. Part I: Branches of non-singular solutions*, Numer. Math., 36 (1981), pp. 1–25.

[10] A. Brooks and T. Hughes, *Streamline upwind / Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equation*, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199–259.

[11] T. Chacón, *A term by term stabilization algorithm for the finite element solution of incompressible flow problems*, Numer. Math., 79 (1998), pp. 283–319.

[12] R. Codina, *Stabilization of incompressibility and convection through orthogonal sub-scales in finite element methods*, Comput. Methods Appl. Mech. Engrg., (to appear).

[13] R. Codina and J. Blasco, *A finite element formulation for the Stokes problem allowing equal velocity-pressure interpolation*, Comput. Methods Appl. Mech. Engrg., 143 (1997), pp. 373–391.

[14] ———, *Analysis of a pressure-stabilized finite element approximation of the stationary Navier-Stokes equations*, Numer. Math., (to appear).

[15] J. Douglas and J. Wang, *An absolutely stabilized finite element method for the Stokes problem*, Math. Comp., 52 (1989), pp. 495–508.

[16] L. Franca and S. Frey, *Stabilized finite element methods: II. The incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 99 (1992), pp. 209–233.

[17] L. Franca and T. Hughes, *Convergence analyses of Galerkin least-squares methods for advective-diffusive forms of the Stokes and incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 105 (1993), pp. 285–298.

[18] L. Franca, T. Hughes, A. Loula, and I. Miranda, *A new family of stable elements for nearly incompressible elasticity based on a mixed Petrov-Galerkin finite element formulation*, Numer. Math., 53 (1988), pp. 123–141.

[19] L. Franca and R. Stenberg, *Error analysis of some Galerkin least-squares methods for the elasticity equations*, SIAM J. Numer. Anal., 28 (1991), pp. 1680–1697.

[20] V. Girault and P. Raviart, *Finite element methods for Navier-Stokes equations*, Springer–Verlag, 1986.

[21] P. Gresho, *On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via a finite element method that also introduces a nearly consistent mass matrix. Part I: Theory*, Internat. J. Numer. Methods Fluids, 11 (1990), pp. 587–620.

[22] T. Hughes, *Multiscale phenomena: Green's function, the Dirichlet-to-Neumann formulation,*

subgrid scale models, bubbles and the origins of stabilized formulations, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.

[23] T. Hughes, G. Feijóo, L. Mazzei, and J. Quincy, *The variational multiscale method–a paradigm for computational mechanics*, Comput. Methods Appl. Mech. Engrg., 166 (1998), pp. 3–24.

[24] T. Hughes, L. Franca, and M. Balestra, *A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation for the Stokes problem accommodating equal-order interpolations*, Comput. Methods Appl. Mech. Engrg., 59 (1986), pp. 85–99.

[25] W. Layton and L. Tobiska, *A two-level method with backtracking for the Navier-Stokes equations*, SIAM J. Numer. Anal., 35 (1998), pp. 2035–2054.

[26] H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations—Convection-Diffusion and Flow Problems*, Springer–Verlag, 1996.

[27] R. Stenberg, *Analysis of mixed finite element methods for the Stokes problem: a unified approach*, Math. Comp., 42 (1984), pp. 9–23.

[28] L. Tobiska and G. Lube, *A modified streamline-diffusion method for solving the stationary Navier-Stokes equations*, Numer. Math., 59 (1991), pp. 13–29.

[29] L. Tobiska and R. Verfürth, *Analysis of a streamline diffusion finite element method for the Stokes and Navier-Stokes equations*, SIAM J. Numer. Anal., 33 (1996), pp. 107–127.