On Nonlinear Monge-Ampère Equations and Their Symmetry Classifications

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ABSTRACT

The Monge-Ampère equation (MAE) plays a pivotal role across a broad spectrum of theoretical and applied sciences, with its solutions being essential for advancing various fields. This study explores all forms of the fully nonlinear MAE using Lie group transformations to reduce the equation into solvable forms. Analytical solutions are derived using ansatz-based methods, yielding novel and generalized results that enhance the existing body of knowledge. In particular, solutions for cases with diverse source functions and boundary conditions are obtained, addressing gaps in the literature. Stability of the solutions is studied through both analytical and numerical approaches. Comparisons with existing solutions demonstrate the efficiency and generality of the proposed methods. The results presented in this work are poised to impact numerous applications, providing a robust framework for further research on MAEs.

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1 Introduction

The Monge-Ampère equation is a second-order nonlinear partial differential equation with numerous applications in both theoretical and practical domains [1,2]. One prominent use is in

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affine geometry, where it helps describe metrics and curvatures, including the Gauss and Weingarten curvatures. Additionally, the MAE plays a significant role in the study of special Lagrangian submanifolds and complex geometry, especially within the framework of Toric manifolds [3–6]. Beyond its geometric applications, the MAE is integral to problems in optimal mass transportation, dynamic meteorology, mathematical finance, and fluid mechanics, particularly in the context of the semi-geostrophic equations [7,8]. Furthermore, the equations governing one-dimensional gas dynamics with plane waves in Lagrange coordinates are related to the inhomogeneous MAE through the Martin transformation [9]. Another view, results about the large deviation principle for the experimental process of a collection of independent copies of diffusion processes are connected with the Schrödinger problem, which is a special Lagrangian of the large deviation rate function on the Otto-Wasserstein manifold [10].

The most commonly encountered form of the MAE is given by:

$$\det\left(D^2u\right) = u_{tt}u_{xx} - {u_{tx}}^2 = f(x,y), \tag{1}$$

where u_{xx} and u_{tx} stand for the second and mixed partial derivatives of u with respect to x and t, and D^2u is the Hessian of the function u, and u_{tt} denotes the second partial derivative of u with respect to t. This form of the MAE is often used in various applications involving nonlinear wave propagation and optimal transport. Whether the right side of Eq. (1) is strictly positive or touches zero plays an important role in whether solutions are regular or singular. Additionally, multidimensional cases can be seen in the literature [11].

The generalized fully nonlinear Monge-Ampère equation is

$$\left(u_{tt}u_{xx} - {u_{tx}}^2\right) + \sigma_2 u_{tt} + \sigma_3 u_{tx} + \sigma_4 u_{xx} + \sigma_5\left(x, t, u, u_x, u_t\right) = 0, \tag{2}$$

where σ_i $(i = 2, 3, 4)$ are coefficient functions which are generally assumed as constants. Eq. (2) can be seen three types of PDEs according to values of σ_i $(i = 2, 3, 4, 5)$ or in other words whether

$$\Delta = \sigma_5 + \sigma_2\sigma_4 + \frac{{\sigma_3}^2}{4},$$

is less than, greater than or equal to zero like in classical classification of PDEs [12]. When the Hessian matrix D^2u is equivalently or positive definite, when the equation is locally uniformly convex, Eq. (1) becomes an elliptic partial differential equation (PDE) [6], more specifically, a degenerate elliptic equation [13]. The regularity and existence of solutions to this equation were first established by Minkowski [14,15], while the existence of weak solutions was proven by Alexandrov [16–18]. Due to its degenerate elliptic nature, solving this equation becomes significantly more challenging, which has led to an increased focus on numerical solutions [19,20]. Various numerical methods have been explored in the literature, including two-scale finite element approaches [20], eigenproblems [21], spline-based finite element methods [22], and least-squares and augmented Lagrangian techniques [23,24]. Additionally, the MAE is known to possess invariant properties under contact transformations [25,26], and its Lie symmetries have been studied theoretically [27–29], though explicit solutions remain scarce in the literature.

As previously mentioned, the Monge-Ampère equation has notable applications in various fields, including the piston problem [8], optimum investment theory [27], and the propagation of shock waves in thermodynamics. In this work, Eqs. (1), (2), and selected applications of Eq. (2) are systematically examined and reduced to obtain exact analytical solutions through the application of Lie symmetry method. The significance of Lie symmetries in practical research is widely recognized. It is expected that the results presented here will make a substantial contribution to the existing body of literature.

The broad applicability and significance of the Monge-Ampère equation serve as the primary motivation for this study, particularly in diverse scientific fields such as thermodynamics, optimal investment theory, and shock wave propagation. While the nonhomogeneous form of the MA Eq. (1) is commonly encountered in the literature, this study focuses on the generalized fully nonlinear form of the MA Eq. (2), which is less explored. The significance of Eq. (2) lies in its classification into elliptic, parabolic, and hyperbolic types, each corresponding to distinct and varied applications. By leveraging Lie symmetries, this work reduces Eq. (2) to a more manageable form and solves it using ansatz-based methods. This method introduces a novel approach by employing Lie symmetries to derive analytical solutions, expanding the scope of solvable Monge-Ampère problems and offering fresh insights into their applications. The results presented are expected to enhance the understanding and application of the generalized MA equation across various disciplines.

The structure of this paper is as follows: Section 2 describes the methodology and the primary approach used in the study. Section 3 investigates the symmetries of the Monge-Ampère equation (MAE) when a source term is present. Section 4 analyzes the stability of the solutions. In Section 5, the Lie symmetries of the generalized form of the MAE are derived. Finally, Section 6 provides the conclusion of the study, summarizing the obtained results.

2 Methodology

The study of analytical solutions for nonlinear equations is a topic of significant interest across various application areas and has been extensively explored in the literature [30–34]. Among the methods employed, the Group Analysis method, commonly known as the Lie symmetry method, has proven to be particularly effective. In physics, engineering, and the natural sciences, this method is widely utilized to establish conservation laws by incorporating inherent physical properties [35–38]. Lie transformations are particularly effective in reducing the order or dimensionality of differential equations or transforming them into solvable forms. The ability to obtain exact solutions in explicit form is crucial for both the theoretical analysis and practical applications of nonlinear systems in science [39–42]. A single parameter group G of transformations is considered

$$\tilde{x}^i = f^i\left(x, \varepsilon\right), i = 1.2\ldots, n, \tag{3}$$

here, ε represents a canonical parameter, and $f^i\left(x, \varepsilon\right)$ can be expressed as a Taylor series expansion around $\varepsilon = 0$, retaining only terms up to a specific desired order while omitting higher-order terms like $O\left(\varepsilon\right)$.

$$\tilde{\chi}^i \approx \chi^i + \varepsilon\xi_i(x), i = 1, 2, \ldots, n, \quad \xi_i(x) = \left(\frac{\partial f^i\left(x, \varepsilon\right)}{\partial \varepsilon}\right)\bigg|_{\varepsilon=0}. \tag{4}$$

Eq. (4) determines the tangent vector at point x.

$$\xi(x) = (\xi_1(x), \ldots, \xi_n(x)). \tag{5}$$

The infinitesimal generator for the one-parameter group (3) is given by

$$X = \sum_{i=1}^{n} \xi_i\left(x\right)\frac{\partial}{\partial x^i},$$

in which ξ_i denotes the tangent vector field.

3 The Symmetries of MAE with Source Terms

Most seen case of MAE is nonhomogeneous form i.e. the values of $\sigma_i = 0\,(i = 2, 3, 4)$ and $\sigma_5\,(x, t, u, u_x, u_t) = f\,(x, y)$ and the theoretical study for the problem is studied by [25].

In this section, analytical solutions of the most common type of the MAE, which is a particular case of Eq. (1) according to

$$u_{xx}u_{tt} - {u_{xt}}^2 = f\,(x, t)\,, \tag{6}$$

is investigated. The exact solution is given by [23] for the following source terms,

$$f\,(x, t) = \begin{cases} \left(1 + x^2 + t^2\right) e^{\left(x^2+t^2\right)} \\ \dfrac{1}{\sqrt{x^2 + t^2}}, \\ \\ \dfrac{2}{\left(2 - x^2 - t^2\right)^2}. \end{cases}$$

The considered cases of $f\,(x, t)$ correspond to radial solution that has application to the noisy data, which the effects of adding high-frequency sinusoidal noise to the data are examined, blow-up solution, which is a complex but important phenomenon that reflects the interplay between the problem's geometry, boundary conditions, and the nonlinear nature of the equation, and surface of a ball, which highlights the important role of convexity and geometry in solution behavior, respectively.

To reduce Eq. (6), the transformation $u\,(x, t) = F\,(\zeta)\,, \zeta = x^2 + t^2$ is obtained because of Lie procedure. Hence the reduced equation is hold:

$$8\zeta F''(\zeta)\,F'\,(\zeta) + 4\,(F'\,(\zeta))^2 = f\,(\zeta)\,. \tag{7}$$

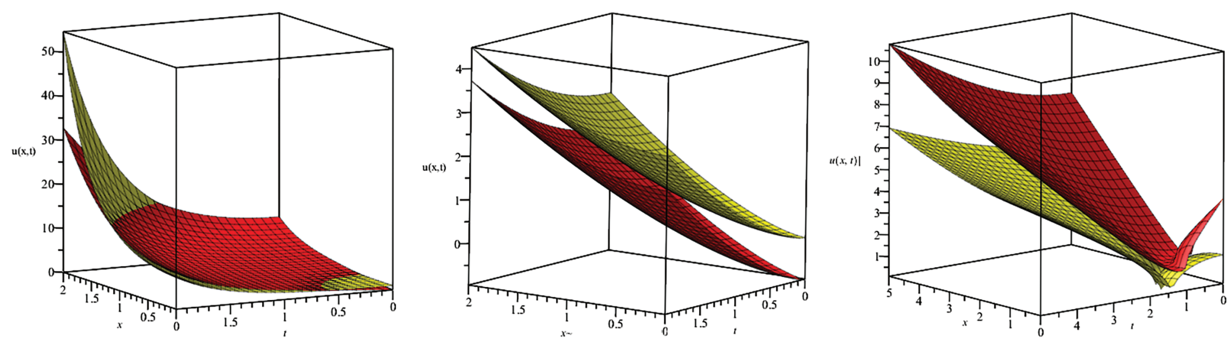
Table 1 provides a comparison between the exact solutions obtained in this study and those found in the literature. In this context, $C_i\,(i = 1, 2)$ are constants that are determined based on the initial or boundary conditions. The comparison highlights how the solutions derived in this investigation align with or differ from the known solutions presented in [23]. The graphical representation of these comparisons is depicted in Fig. 1 for specific parameter values. The results demonstrate strong agreement with the known solutions, validating the accuracy and reliability of the employed methodology. Specifically, in the case of $C_1 = C_2 = 0$, the derived solution aligns precisely with the existing solution in the literature, as shown in Table 1. This finding confirms that the proposed approach accurately reproduces known results, reinforcing the robustness of the method. Moreover, the study introduces additional new solutions that have not been reported in previous works. These solutions can be regarded as general exact solutions, where the constants C_i $(i = 1, 2)$ can be adapted to fit specific initial or boundary conditions. This flexibility highlights the potential of these solutions to model various physical processes and yield novel outcomes.

The comparative plots in Fig. 1 further emphasize the validity of the new solutions. For instance:

- In case (i), with $C_1 = 1$ and $C_2 = 1$, the proposed solution exhibits similar behavior to the radial solutions in the literature.
- In case (ii), for the blow-up scenario with $C_1 = 0.1$ and $C_2 = -1$, the solution coincides with the literature results.
- In case (iii), involving a surface of a ball with $C_1 = 0.8$ and $C_2 = 1$, the solution demonstrates distinct characteristics, showcasing the adaptability of the method.

Table 1: The exact solutions of Eq. (6) for each case

Cases	The source $\boldsymbol{f}(\boldsymbol{\zeta})$ $(\boldsymbol{\zeta} = \boldsymbol{x}^2 + \boldsymbol{t}^2)$	The obtained exact solutions $(\boldsymbol{\zeta} = \boldsymbol{x}^2 + \boldsymbol{t}^2)$	The given exact solution [19]
1. radial solution	$(\zeta + 1)\exp(\zeta)$	$F(\zeta) = \mp\frac{1}{2}\int\sqrt{\exp(\zeta) + \frac{4}{\zeta}C_1 d\zeta} + C_2$	$u(x,t) = \exp\left(\frac{x^2+t^2}{2}\right)$
2. blow-up solution	$\frac{1}{\sqrt{\zeta}}$	$F(\zeta) = \pm\frac{2\sqrt{2}\sqrt{\zeta\left(\sqrt{\zeta}+C_1\right)}\left(\sqrt{\zeta}+C_1\right)}{3\sqrt{\zeta}} + C_2$	$u(x,t) = \frac{2\sqrt{2}}{3}\left(x^2+t^2\right)^{3/4}$
3. surface of a ball	$\frac{2}{(2-\zeta)^2}$	$F(\zeta) = \pm\frac{\sqrt{2}}{2}\int\frac{\sqrt{(\zeta-1)\zeta(-1-2C_1+2C_1\zeta)}}{(\zeta-1)\zeta}d\zeta + C_2$	$u(x,t) = -\sqrt{2-x^2-t^2}$



(a) (b) (c)

Figure 1: Comparison of exact solutions: (**i**) radial, (**ii**) blow-up, and (**iii**) ball surface. Red surfaces represent proposed solutions; yellow surfaces depict solutions from [19]. (**a**) $C_1 = 1$, $C_2 = 1$. (**b**) $C_1 = 0.1$, $C_2 = -1$. (**c**) $C_1 = 0.8$, $C_2 = 1$

Fig. 1 illustrates these cases by comparing the proposed new solutions (represented by red surfaces) with the exact solutions from the literature (depicted in yellow). The strong agreement between these solutions confirms the precision of the methodology. Additionally, the discovery of new solutions enriches the understanding of the problem and provides opportunities for further exploration in various physical contexts. The proposed methodology effectively reproduces known solutions while enabling the derivation of new, generalized solutions. These results highlight the capability of the approach to address a broader range of scenarios, offering valuable insights into the underlying physical processes.

4 Stability Analysis

Stability is a crucial aspect in the study of nonlinear partial differential equations, ensuring that small perturbations in initial conditions or numerical approximations do not lead to unbounded solutions. This section investigates the stability of the Monge-Ampère equation from both analytical and numerical perspectives. First, the stability of analytical solutions is examined by introducing small perturbations and analyzing their evolution. Then, the numerical stability of the discretized MAE is assessed using von Neumann analysis, a standard technique for investigating the stability of numerical schemes for nonlinear differential equations [39,40,42–44].

4.1 Stability of Analytical Solutions

To analyze the stability of the obtained analytical solutions, we introduce a small perturbation to the solution $u(x,t)$

$$\tilde{u}(x,t) = u(x,t) + \epsilon v(x,t),$$

where ϵ is a small parameter and $v(x,t)$ represents the perturbation. Substituting $\tilde{u}(x,t)$ into the Eq. (6), we obtain

$$(u_{xx} + \epsilon v_{xx})(u_{tt} + \epsilon v_{tt}) - (u_{xt} + \epsilon v_{xt})^2 = f(x,t).$$

Expanding and neglecting higher-order terms in ϵ, we obtain the linearized perturbation equation:

$$u_{xx}v_{tt} + u_{tt}v_{xx} - 2u_{xt}v_{xt} = 0.$$

The perturbation $v(x,t)$ remains bounded if the homogeneous equation

$$u_{xx}v_{tt} + u_{tt}v_{xx} - 2u_{xt}v_{xt} = 0$$

admits solutions that do not grow indefinitely with time. A Lyapunov function $V(x,t)$ can also be introduced, ensuring that

$$\frac{dV}{dt} \leq 0.$$

If such a function exists, the solution is asymptotically stable. Otherwise, if perturbations grow exponentially, the solution is unstable.

4.2 Numerical Stability of Discretized MAE

To validate our analytical results numerically, we discretize the MAE using a finite difference scheme. Using central difference approximations, we obtain

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} \cdot \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta t^2} - \left(\frac{U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}}{4\Delta x \Delta t}\right)^2 = f_{i,j}.$$

To assess stability, we perform **von Neumann stability analysis:**

1. Assume Fourier mode solutions of the form

$$U_{i,j} = \hat{U} e^{ikx} e^{i\omega t}.$$

2. Substituting into the discretized equation leads to a characteristic equation for the **amplification factor** G:

$$G = \frac{1 - \lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2},$$

where $\lambda_1 = \dfrac{4\sin^2\left(\dfrac{k\Delta x}{2}\right)}{\Delta x^2}$ and $\lambda_2 = \dfrac{4\sin^2\left(\dfrac{\omega\Delta t}{2}\right)}{\Delta t^2}$.

3. Stability requires $|G| \leq 1$, expanding the expression for G, stability is ensured if:

$$\frac{4\sin^2\left(\frac{k\Delta x}{2}\right)}{\Delta x^2} \cdot \frac{4\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\Delta t^2} - \left(\frac{\sin(k\Delta x)\sin(\omega\Delta t)}{\Delta x\Delta t}\right)^2 \leq 1.$$

Additionally, for iterative solvers, we analyze the **condition number** of the discretized system matrix. A large condition number indicates numerical instability, requiring stabilization techniques such as **preconditioning** or **implicit schemes**.

5 Lie Symmetries of the Generalized MAE

In this section, we explore the Lie symmetries of the generalized Monge–Ampère equation (MAE), which is less commonly discussed than its classical form [11,12,27–29]. By systematically applying the method outlined in Section 2, we derive the corresponding infinitesimal generator for this extended model.

The generalized MAE is

$$\left(u_{tt}u_{xx} - u_{tx}{}^2\right) + \sigma_2 u_{tt} + \sigma_3 u_{tx} + \sigma_4 u_{xx} + \sigma_5(x, t, u, u_x u_t) = 0, \tag{8}$$

where σ_i $(i = 2, 3, 4)$ coefficient functions and $\sigma_5\,(x, t, u, u_x, u_t) = \sigma_5$ are constants. This equation is not seen so frequently as Eq. (1). Applying the given procedure in Section 2, the infinitesimal generator is obtained:

$$X = (C_1 x + C_2)\frac{\partial}{\partial x} + (C_1 t + C_3)\frac{\partial}{\partial t} + (2C_1 u + C_6 x + C_4 t + C_5)\frac{\partial}{\partial u}. \tag{9}$$

As a result, six one-parameter groups with infinitesimal transformation are obtained which include not only trivial transformations also new transformations. In the case $C_1 = 1$ and others are zero, the infinitesimal generator $X_1 = x\dfrac{\partial}{\partial x} + t\dfrac{\partial}{\partial t} + 2u\dfrac{\partial}{\partial u}$ and the transformation is

$$u\,(x, t) = x^2 F\,(\zeta)\,, \zeta = \frac{t}{x}.$$

Consequently, the reduced equation is given below:

$$\left(\zeta^2\sigma_4 - \sigma_3\zeta - 2F + \sigma_2\right)F'' + (F')^2 + (-2\zeta\sigma_4 + \sigma_3)\,F' + 2F\sigma_4 + \sigma_5 = 0. \tag{10}$$

Eq. (10) represents a nonlinear ordinary differential equation that can be solved using the Bernoulli approximation method or the ansatz technique. The ansatz is determined by balancing principle that $F\,(\zeta) = g_0 + g_1 z\,(\zeta)\,, g_0, g_1$ are constants and $z\,(\zeta)$ is the solution of the Bernoulli differential equation $z'\,(\zeta) = P\,(\zeta)\,z\,(\zeta) + Q\,(\zeta)\,z^2\,(\zeta)$, $P\,(\zeta)$ and $Q\,(\zeta)$ will be determined by the system. Applying the considered method's procedure,

$$P\,(\zeta) = \frac{8\sigma_4\left(\sigma_4\,(\zeta - 2C_1) - \frac{1}{4}\sigma_3\right)}{4\zeta\sigma_4{}^2\,(\zeta - 4C_1) + \sigma_4\,(\sigma_3\,(-2\zeta + 8C_1) - 4\sigma_2) + \sigma_3{}^2 - 4\sigma_5}, \tag{11}$$

where

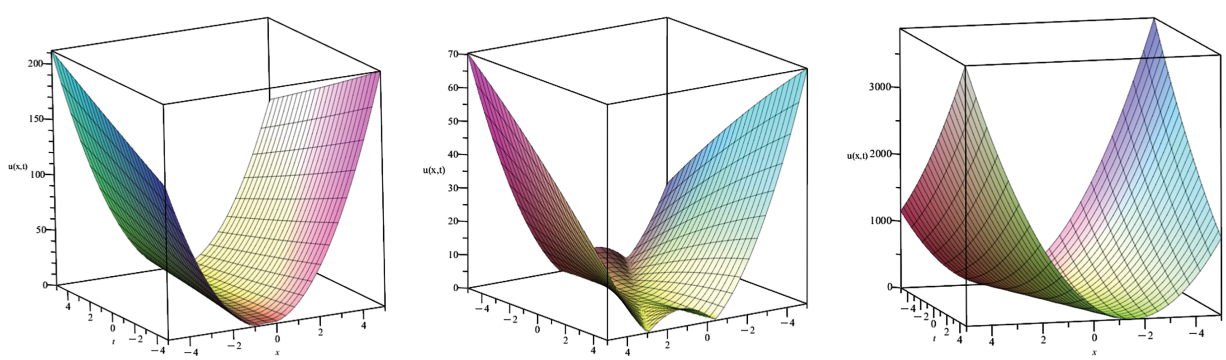
$$g_0 = -\frac{\sigma_5}{2\sigma_4} C_1 = \frac{\sigma_3 \pm 2\sqrt{-4\sigma_2\sigma_4 + \sigma_3{}^2 - 4\sigma_5}}{8\sigma_4},\ Q(\zeta) = 0.$$

The parameters derived for Eq. (11) and the corresponding transformation are formulated in the ansatz, leading to the general solution of Eq. (8), as the group transformation remains invariant:

$$u(x,t) = x^2 \left(-\frac{\sigma_5}{2\sigma_4} - \frac{g_1}{C_2} \left(\frac{\dfrac{2\sigma_4 t\left(\sigma_3 + 2\sqrt{-4\sigma_4\sigma_2 + \sigma_3{}^2 - 4\sigma_5}\right)}{x}}{-\dfrac{4t^2\sigma_4{}^2}{x^2} + \dfrac{2t\sigma_4\sigma_3}{x} + 4\sigma_4\sigma_2 - \sigma_3{}^2 + 4\sigma_5 - \sigma_3\left(\sigma_3 + 2\sqrt{-4\sigma_4\sigma_2 + \sigma_3{}^2 - 4\sigma_5}\right)} \right) \right). \tag{12}$$

The classification of Eq. (9) into elliptic, parabolic, and hyperbolic forms based on the coefficients is illustrated in Fig. 2. These plots were generated with the parameters $g_1 = 2$ and $C_2 = 1$ for different cases:

a) Elliptic form: $\sigma_2 = 0.1,\ \sigma_3 = -1,\ \sigma_4 = -0.1,\ \sigma_5 = 1,\ \Delta > 0,$
b) Hyperbolic form: $\sigma_2 = 0.1,\ \sigma_3 = -1,\ \sigma_4 = -0.1,\ \sigma_5 = -1,\ \Delta < 0,$
c) Parabolic form: $\sigma_2 = 1,\ \sigma_3 = \sqrt{8},\ \sigma_4 = -1,\ \sigma_5 = -1,\ \Delta = 0.$



(a) (b) (c)

Figure 2: Solution behavior of (**a**) elliptic, (**b**) hyperbolic, and (**c**) parabolic forms of MAE with specified parameters. (**a**) $\sigma_2 = 0.1,\ \sigma_3 = -1,\ \sigma_4 = -0.1,\ \sigma_5 = 1,\ \Delta > 0$. (**b**) $\sigma_2 = 0.1,\ \sigma_3 = -1,\ \sigma_4 = -0.1,\ \sigma_5 = -1,\ \Delta < 0$. (**c**) $\sigma_2 = 1,\ \sigma_3 = \sqrt{8},\ \sigma_4 = -1,\ \sigma_5 = -1,\ \Delta = 0$

Fig. 2 demonstrates the distinct solution behaviors of these forms, providing valuable insights into their underlying dynamics. The elliptic form (a) exhibits bounded solutions, the hyperbolic form (b) shows unbounded wave-like behavior, and the parabolic form (c) presents a transitional state. In existing literature, exact and numerical solutions are typically obtained under the assumption that the coefficient functions $\sigma_i = 0$ for $i = 2, 3, 4$. However, to the best of our knowledge, no exact solutions for Eq. (8) have been reported. The approach in this study generalizes the modified algebraic equation (MAE) by including all terms with nonzero coefficient functions. This comprehensive consideration of coefficients bridges the gap in existing research, yielding exact solutions that account for all terms in the equation.

The proposed solutions not only fill this gap but also demonstrate the robustness of the method in addressing the generalized MAE. These solutions offer flexibility for analyzing a variety of physical scenarios where all terms contribute significantly to the equation's behavior. Thus, the methodology provides a broader framework for both theoretical analysis and practical applications.

6 Conclusion

This work examines the fully nonlinear MAE and its variations with diverse source functions. Using Lie group transformations, the equations are simplified into solvable forms, and exact solutions are derived through ansatz-based methods, some of which are novel. The solutions are presented in a generalized form, with constants determined by initial or boundary conditions. Stability of the solutions is rigorously studied, ensuring their validity both analytically and numerically. Comparisons with existing results for equivalent parameter values highlight the robustness and reliability of the proposed solutions. This study fills critical gaps in the literature and introduces general solutions, backed by stability analysis, that have potential for widespread applications.

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