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Ramon Codina · Noel Hernández - Silva

Stabilized finite element approximation of the stationary magneto-hydrodynamics equations

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Abstract In this work we present a stabilized finite element method for the stationary magneto-hydrodynamic equations based on a simple algebraic version of the subgrid scale variational concept. The linearization that yields a well posed linear problem is first identified, and for this linear problem the stabilization method is designed. The key point is the correct behavior of the stabilization parameters on which the formulation depends. It is shown that their expression can be obtained only on the basis of having a correct error estimate. For the stabilization parameters chosen, a stability estimate is proved in detail, as well as the convergence of the numerical solution to the continuous one. The method is then extended to nonlinear problems and its performance checked through numerical experiments.

Keywords MHD · Stabilized finite elements · Fixed point iterations

1 Introduction

The objective of this work is to present a stabilized finite element method for the approximation of the stationary magneto-hydrodynamic (MHD) problem. In principle, the unknowns involved are the magnetic field, the fluid velocity and the hydrodynamic pressure. However, to enforce the divergence free condition for the numerical approximation of the magnetic field we introduce a magnetic pressure (whose exact value should be zero). This zero divergence condition is automatically satisfied at the continuous level for the transient problem if the initial magnetic field is solenoidal, but it is convenient to explicitly enforce it in the numerical approximation, especially for stationary problems. With the introduction of the magnetic pressure we are left with a system of four equations with four unknowns. The 'augmented' approach used in this work is discussed for example in [3, 11] for the Maxwell equations. The same approach is used in [13–15] (see also references therein). Other possibilities of enforcing the divergence free condition for the magnetic field are penalty strategies (see for example [1]) or the use of (weakly) divergence free interpolations based on Nédelec-type elements (described for example in [12]). Nevertheless, there is also the possibility of relying on the mathematical structure of the equations and to expect that the original problem will already yield a magnetic field close enough to solenoidal. This is the idea followed in [9], which probably contains the first analysis of a finite element approximation to the MHD problem, and it is also used in [8], among other papers.

Having introduced the magnetic pressure as a new unknown in the problem, its finite element approximation has several difficulties. First, there is the well known compatibility condition between the approximation spaces for the velocity and the pressure, but also for the approximation spaces for the magnetic field and the magnetic pressure. Both conditions can be expressed in a standard inf–sup form [2]. There is also the problem of dealing with situations in which first order derivatives, both in the Navier–Stokes equations and in the equation for the magnetic field, dominate (from the numerical point of view) the second order terms that give an elliptic nature to the system of equations to be solved. These are the classical convection dominated flow problems. Both the compatibility condition between interpolating spaces and the oscillations found in flows dominated by convection can be overcome by using stabilized finite element methods. First approaches in this direction can be found in [8] (without the introduction of the magnetic pressure) and in [13, 14] (where the magnetic pressure is also introduced). However, another particular feature of the MHD problem are the couplings involved. In the magnetic problem, the coupling with the hydrodynamic problem comes from the convective term in the equation for the magnetic field, whereas in the Navier–Stokes equations the coupling with the magnetic

R. Codina (✉)
Universitat Politècnica de Catalunya
Jordi Girona 1-3, Edifici C1
08034 Barcelona, Spain
E-mail: ramon.codina@upc.edu

N. Hernández - Silva
E-mail: noelh@cimne.upc.edu

problem comes from Lorentz's force. Our objective is to design a stabilized finite element method that takes these couplings into account.

The stabilized finite element method presented here is based on the two-scale decomposition of the unknowns into their finite element component and a subscale that cannot be captured by the finite element space. The format that we follow of this idea was introduced in [10]. In particular, the version for systems we employ here was already presented in [4]. The formulation is first designed for linear problems, and therefore our first concern is to devise a linearization technique for the fully coupled problem. For simplicity, we consider a fixed point method. Among the different possibilities, we identify the only one that leads to a linearized problem that is coercive, and thus guarantees existence and uniqueness of solution. It is for this linearized problem that we propose a stabilized finite element method based on the subgrid scale (SGS) concept. The important point is how to approximate the SGS. We use the simplest approach of taking them proportional to the residual of the finite element approximation multiplied by the so called matrix of stabilization parameters. The design of this matrix is solely based on the convergence analysis of the problem. The resulting formulation differs from the one proposed in [13] both in the structure of the stabilizing terms (no attempt is made there to account neither for convection-dominated situations nor for the coupling effects) and in the design of the stabilization parameters. It also differs form the method proposed in [8] in the inclusion of the magnetic pressure and in the design of the stabilization parameters.

The paper is organized as follows. The problem to be solved is presented in the next section, including its strong and its variational form, as well as the linearization we will consider. The core of the work is presented in Sect. 3, where the stabilized finite element formulation is introduced and fully analyzed. Some simple numerical examples are presented in Sect. 4 and some final remarks are made in Sect. 5.

2 Problem description

2.1 Boundary value problem

Let $\boldsymbol{u}$ be the velocity field and p the pressure of a fluid moving in a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3). In this domain we assume that there is a magnetic field $\boldsymbol{B}$ (in fact, a magnetic induction), which must be divergence free. It is well known that from Maxwell's equations it follows that if $\boldsymbol{B}$ is solenoidal at the initial time, it is so for all time. However, we will consider the numerical approximation of the stationary problem, for which it is necessary to enforce explicitly the zero divergence condition on $\boldsymbol{B}$. A possible way to enforce it is by the introduction of a Lagrange multiplier type variable, that we will denote by r and that we will call *magnetic pressure*, simply by analogy with the hydrodynamic pressure p and not related at all to the pressure increment induced by $\boldsymbol{B}$ (often called magnetic pressure in a physical context).

The differential equations we want to consider corresponding to a stationary MHD problem to be solved in Ω are, in dimensionless form:

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{1}{Re}\Delta\boldsymbol{u} + \nabla p + S\boldsymbol{B}\times(\nabla\times\boldsymbol{B}) = \boldsymbol{f}, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2}$$

$$-\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \frac{1}{Rm}\nabla\times(\nabla\times\boldsymbol{B}) + \nabla r = \boldsymbol{g}, \tag{3}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{4}$$

In these equations, $\boldsymbol{f}$ is a the body force, $\boldsymbol{g}$ is a source in the equation for $\boldsymbol{B}$ introduced for generality, Re is the (hydrodynamic) Reynolds number, Rm is the magnetic Reynolds number and S is the coupling number.

We will consider the simplest essential conditions on $\partial\Omega$:

$$\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{n}\times\boldsymbol{B} = \boldsymbol{0}, \quad r = 0 \quad \text{on } \partial\Omega, \tag{5}$$

where $\boldsymbol{n}$ is the unit normal external to $\partial\Omega$. If $\nabla\cdot\boldsymbol{g} = 0$, as we will assume, taking the divergence of (3) yields $\Delta r = 0$, which, together with the boundary condition $r = 0$ on $\partial\Omega$, yields $r = 0$ as exact solution. Another possibility is to take as boundary conditions $\boldsymbol{n}\cdot\boldsymbol{B} = 0$, $\boldsymbol{n}\times\nabla\times\boldsymbol{B} = \boldsymbol{0}$, which in this case would be *natural* in the variational formulation (see below). In this case, (3) implies that $\nabla r = \boldsymbol{0}$, and thus r would be defined up to a constant. All what follows can be easily extended to this case. Nevertheless, for conciseness we will restrict ourselves to the boundary conditions (5).

2.2 Variational problem

Let us group the unknowns of the variational problem to be introduced and the corresponding test functions for the different equations as

$$\boldsymbol{U} := (\boldsymbol{u}, p, \boldsymbol{B}, r) \in W,$$
$$\boldsymbol{V} := (\boldsymbol{v}, q, \boldsymbol{C}, s) \in W,$$

where W is the functional space where the problem is defined, given by

$$W := (H_0^1(\Omega))^d \times L_0^2(\Omega) \times H_0(\mathrm{curl}, \Omega) \times H_0^1(\Omega). \tag{6}$$

The notation involved is standard: $L^2(\Omega)$ is the space of square integrable functions, $H^1(\Omega)$ the space of functions such that they and their derivatives are in $L^2(\Omega)$, $L_0^2(\Omega)$ the subspace of functions in $L^2(\Omega)$ with zero mean, $H_0^1(\Omega)$ the subspace of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$, $H(\mathrm{curl}, \Omega)$ the space of vector functions such that they and their curl are in the space $L^2(\Omega)$, and $H_0(\mathrm{curl}, \Omega)$ the subspace of functions in the space $H(\mathrm{curl}, \Omega)$ with zero tangential component on $\partial\Omega$.

The data of the problem are assumed to be such that $\boldsymbol{f} \in (H^{-1}(\Omega))^d$, the dual of $(H_0^1(\Omega))^d$, and $\boldsymbol{g} \in (L^2(\Omega))^d$ with $\nabla\cdot\boldsymbol{g} = 0$ in $L^2(\Omega)$. The scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and the duality pairing between $(H^{-1}(\Omega))^d$ and $(H_0^1(\Omega))^d$ by $\langle\cdot,\cdot\rangle$.

Having introduced all this notation, we can write the variational form of the problem as follows: Find $\boldsymbol{U} \in W$ such that

$$\mathcal{N}(\boldsymbol{U}, \boldsymbol{V}) = \mathcal{L}(\boldsymbol{V}) \quad \forall \boldsymbol{V} \in W, \tag{7}$$

where the nonlinear form $\mathcal{N} : W \times W \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned}\mathcal{N}(\boldsymbol{U}, \boldsymbol{V}) := {} & (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) + \frac{1}{Re}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \\ & - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u}) \\ & + S(\boldsymbol{B}, \nabla \times (\boldsymbol{v} \times \boldsymbol{B})) - S(\boldsymbol{C}, \nabla \times (\boldsymbol{u} \times \boldsymbol{B})) \\ & + \frac{S}{Rm}(\nabla \times \boldsymbol{B}, \nabla \times \boldsymbol{C}) + S(\nabla r, \boldsymbol{C}) \\ & - S(\nabla s, \boldsymbol{B}),\end{aligned} \tag{8}$$

and the linear form $\mathcal{L} : W \longrightarrow \mathbb{R}$ by

$$\mathcal{L}(\boldsymbol{V}) := \langle \boldsymbol{f}, \boldsymbol{v} \rangle + S(\boldsymbol{g}, \boldsymbol{C}). \tag{9}$$

It is easily checked that $\mathcal{N}$ and $\mathcal{L}$ are continuous and that $\mathcal{N}$ verifies

$$\mathcal{N}(\boldsymbol{U}, \boldsymbol{U}) = \frac{1}{Re}\|\nabla \boldsymbol{u}\|^2 + \frac{S}{Rm}\|\nabla \times \boldsymbol{B}\|^2, \tag{10}$$

where $\|\cdot\|$ is the norm in $L^2(\Omega)$. This, together with the boundary conditions considered, allows us to control $\boldsymbol{u}$ in $(H_0^1(\Omega))^d$ and $\boldsymbol{B}$ in $H_0(\mathrm{curl}, \Omega)$. However, p and r need to be controlled by invoking the inf–sup conditions

$$\inf_{q \in L_0^2(\Omega)} \sup_{\boldsymbol{v} \in (H_0^1(\Omega))^d} \frac{(q, \nabla \cdot \boldsymbol{v})}{\|q\| \, \|\nabla \boldsymbol{v}\|} \geq \beta_u > 0, \tag{11}$$

$$\inf_{s \in H_0^1(\Omega)} \sup_{\boldsymbol{C} \in H_0(\mathrm{curl}, \Omega)} \frac{(\nabla s, \boldsymbol{C})}{\|\nabla s\| \left(\|\nabla \times \boldsymbol{C}\| + \|\boldsymbol{C}\|\right)} \geq \beta_B > 0, \tag{12}$$

where β_u and β_B are constants. The first of these conditions is the classical inf–sup condition for the velocity and pressure in the incompressible Navier–Stokes equations, known to hold in the continuous setting we are considering now. The second condition is trivially checked in the continuous case: given s, $\boldsymbol{C} = \nabla s$ makes the quotient in the inf–sup condition equal to 1, and thus $\beta_B \geq 1$. However, it is well known that (11) is not inherited automatically by finite element approximations to the problem. Likewise, condition (12) is, in general, not satisfied for arbitrary finite element interpolations of the magnetic field and the magnetic pressure (equal continuous interpolation, for example, does not allow us to take $\boldsymbol{C} = \nabla s$ in the discrete spaces). When approximating problem (7) using finite elements, we can either try to satisfy conditions (11) and (12) or to modify the variational formulation in such a way that control of the discrete counterpart of p and r does not come from these conditions. This is the approach we will follow in this paper.

Apart from the basic stability on the hydrodynamic and magnetic pressures, the formulation we will present in Sect. 3 will allow us to have some control on the terms $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ and $\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$. From (10) it is seen that this can be important when the hydrodynamic and magnetic Reynolds numbers are high, that is to say, situations that can be considered dominated by 'convection'.

In the continuous problem considered so far, it can be proved that (10), (11) and (12) are enough to guarantee that the nonlinear problem (7) has at least one solution. Moreover, if the norm of the data $\boldsymbol{f}$ and $\boldsymbol{g}$ is small enough, this solution is unique [15].

2.3 Linearized problem

Before proceeding to the finite element approximation of problem (7), let us first consider a linearized version for which we will design the stabilized formulation presented in Sect. 3.

The simplest way to linearize (7) is by a fixed point treatment of the quadratic terms. Let us assume we have an estimate for the velocity and the magnetic field at iteration k, $\boldsymbol{u}^k$ and $\boldsymbol{B}^k$, respectively, and we have to compute these fields at iteration $k+1$. The possibilities to approximate the quadratic terms are:

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u}|^{k+1} \approx \begin{cases} \boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^{k+1} & \text{(uu.1)} \\ \boldsymbol{u}^{k+1} \cdot \nabla \boldsymbol{u}^k & \text{(uu.2)} \end{cases}$$

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{B})|^{k+1} \approx \begin{cases} \boldsymbol{B}^k \times (\nabla \times \boldsymbol{B}^{k+1}) & \text{(bb.1)} \\ \boldsymbol{B}^{k+1} \times (\nabla \times \boldsymbol{B}^k) & \text{(bb.2)} \end{cases}$$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{B})|^{k+1} \approx \begin{cases} \nabla \times (\boldsymbol{u}^{k+1} \times \boldsymbol{B}^k) & \text{(ub.1)} \\ \nabla \times (\boldsymbol{u}^k \times \boldsymbol{B}^{k+1}) & \text{(ub.2)} \end{cases}$$

Obviously, we assume that $\nabla \cdot \boldsymbol{u}^k = 0$, $\nabla \cdot \boldsymbol{B}^k = 0$. It is known that for the convective term of the Navier–Stokes equations, only option (uu.1) yields a coercive linearized problem. The reason is that, if we choose (uu.2), there is no way to bound from below $(\boldsymbol{u}^{k+1}, \boldsymbol{u}^{k+1} \cdot \nabla \boldsymbol{u}^k)$ by a nonnegative term, that is what is required to get a stability estimate analogous to (10) for the linearized problem. However, $(\boldsymbol{u}^{k+1}, \boldsymbol{u}^k \cdot \nabla \boldsymbol{u}^{k+1}) = 0$.

For the other two quadratic terms, a simple inspection of the possible combinations reveals that the only way to obtain a coercive linearized problem is to choose (bb.1) and (ub.1). Therefore, calling $\boldsymbol{a} \equiv \boldsymbol{u}^k$, $\boldsymbol{u} \equiv \boldsymbol{u}^{k+1}$, $\boldsymbol{b} \equiv \boldsymbol{B}^k$, $\boldsymbol{B} \equiv \boldsymbol{B}^{k+1}$, the *only* fixed point linearization of problem (7) which is coercive is given by the set of differential equations

$$\boldsymbol{a} \cdot \nabla \boldsymbol{u} - \frac{1}{Re}\Delta \boldsymbol{u} + \nabla p + S\boldsymbol{b} \times (\nabla \times \boldsymbol{B}) = \boldsymbol{f}, \tag{13}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{14}$$

$$-\nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \frac{1}{Rm}\nabla \times (\nabla \times \boldsymbol{B}) + \nabla r = \boldsymbol{g}, \tag{15}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{16}$$

to be solved in Ω with the boundary conditions (5). This is the problem for which the stabilized finite element method will be presented in the next section. Its variational form is: Find $\boldsymbol{U} \in W$ such that

$$\mathcal{B}(\boldsymbol{U}, \boldsymbol{V}) = \mathcal{L}(\boldsymbol{V}) \quad \forall \boldsymbol{V} \in W, \tag{17}$$

where the bilinear form $\mathcal{B}$ is given by

$$\mathcal{B}(\boldsymbol{U},\boldsymbol{V}) = (\boldsymbol{a}\cdot\nabla\boldsymbol{u},\boldsymbol{v}) + \frac{1}{Re}(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) - (p,\nabla\cdot\boldsymbol{v}) + (q,\nabla\cdot\boldsymbol{u}) + S(\boldsymbol{B},\nabla\times(\boldsymbol{v}\times\boldsymbol{b})) - S(\boldsymbol{C},\nabla\times(\boldsymbol{u}\times\boldsymbol{b})) + \frac{S}{Rm}(\nabla\times\boldsymbol{B},\nabla\times\boldsymbol{C}) + S(\nabla r,\boldsymbol{C}) - S(\nabla s,\boldsymbol{B}),$$

and obviously $\mathcal{L}$ is again given by (9). Since $\nabla\cdot\boldsymbol{a} = 0$, $\nabla\cdot\boldsymbol{b} = 0$, $\mathcal{B}$ satisfies the stability estimate (10), that is to say,

$$\mathcal{B}(\boldsymbol{U},\boldsymbol{U}) = \frac{1}{Re}\|\nabla\boldsymbol{u}\|^2 + \frac{S}{Rm}\|\nabla\times\boldsymbol{B}\|^2. \tag{18}$$

This stability estimate and the classical inf–sup conditions between $(H_0^1(\Omega))^d$ and $L_0^2(\Omega)$ and between $H_0(\text{curl},\Omega)$ and $H_0^1(\Omega)$ expressed in (11) and (12) are enough to guarantee that the linearized problem is well posed. Therefore, for each iteration k, given $\boldsymbol{u}^k$ and $\boldsymbol{B}^k$ there is a unique $\boldsymbol{U}^{k+1} = (\boldsymbol{u}^{k+1}, p^{k+1}, \boldsymbol{B}^{k+1}, r^{k+1})$, solution of the linearized problem (17). It can be shown that, under the same condition for which the nonlinear problem (7) has a unique solution (see [15]), the sequence $\{\boldsymbol{U}^k\}_{k\geq 0}$ converges (strongly) to the (unique) solution of the nonlinear problem (7). The proof of this result is technical, but quite simple, and follows the same strategy as for the stationary Navier–Stokes equations without magnetic coupling (see, for example, [7]).

3 Finite element approximation

3.1 Stabilized finite element approximation

In this section we present a stabilized finite element method to approximate problem (17). First of all, we recast it as a system of linear convection-diffusion equations. It is in this general setting that the finite element approximation will be described.

Stabilization for this problem has several goals. The first is to avoid the need to satisfy the discrete counterpart of the inf–sup conditions (11) and (12), which would lead to different interpolation for the variables of the problem. To fix ideas, we will assume equal and continuous interpolation for all the unknowns. The second goal is to obtain error estimates valid in the limit $Re \to \infty$ and $Rm \to \infty$, that is, convection dominated flows (both in the Navier–Stokes equation and the equation for the magnetic field). Finally, the third objective is the account properly for the coupling of the hydrodynamic and the magnetic problems. That these goals are all satisfied will be seen in the error estimate presented at the end of Subsect. 3.2.

3.1.1 Stationary linear convection-diffusion equations

The problem considered can be written as the vector differential equation

$$\mathbf{L}(\boldsymbol{U}) = \boldsymbol{F} \quad \text{in } \Omega, \tag{19}$$

where $\boldsymbol{U} = (\boldsymbol{u}, p, \boldsymbol{B}, r)$, $\boldsymbol{F} = (\boldsymbol{f}, 0, \boldsymbol{g}, 0)$ is a known vector of $n_{\text{unk}} = 2d + 2$ components and the operator $\mathbf{L}$ is given by

$$\mathbf{L}(\boldsymbol{U}) = \begin{bmatrix} \boldsymbol{a}\cdot\nabla\boldsymbol{u} - \frac{1}{Re}\Delta\boldsymbol{u} + \nabla p + S\boldsymbol{b}\times(\nabla\times\boldsymbol{B}) \\ \nabla\cdot\boldsymbol{u} \\ -S\nabla\times(\boldsymbol{u}\times\boldsymbol{b}) + \frac{S}{Rm}\nabla\times(\nabla\times\boldsymbol{B}) + S\nabla r \\ S\nabla\cdot\boldsymbol{B} \end{bmatrix}.$$

This is an operator of the form

$$\mathbf{L}(\boldsymbol{U}) := \boldsymbol{A}_i\frac{\partial\boldsymbol{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\boldsymbol{K}_{ij}\frac{\partial\boldsymbol{U}}{\partial x_j}\right),$$

where $\boldsymbol{A}_i$ and $\boldsymbol{K}_{ij}$ are $n_{\text{unk}} \times n_{\text{unk}}$ matrices ($i, j = 1, ..., d$) whose identification is obvious and will be omitted. Let matrices $\boldsymbol{A}_i$ be split as $\boldsymbol{A}_i = \boldsymbol{A}_i^{\text{c}} + \boldsymbol{A}_i^{\text{f}}$, where $\boldsymbol{A}_i^{\text{c}}$ is the part of the convection matrices which is *not* integrated by parts and $\boldsymbol{A}_i^{\text{f}}$ the part that *is* integrated by parts. In our case, matrices $\boldsymbol{A}_i^{\text{f}}$ come from the first order derivatives of the hydrodynamic pressure p [note that in the variational problem (17) the magnetic pressure is not integrated by parts]. It would be also possible to integrate by parts the first order derivatives corresponding to the terms $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ and $\nabla\times(\boldsymbol{u}\times\boldsymbol{B})$.

The weak form of the problem supplied with the appropriate homogeneous boundary conditions can be written again as the linear variational problem (17), now with the linear form $\mathcal{B}$ defined as

$$\mathcal{B}(\boldsymbol{U},\boldsymbol{V}) := \int_\Omega \boldsymbol{V}^{\text{t}}\boldsymbol{A}_i^{\text{c}}\frac{\partial\boldsymbol{U}}{\partial x_i}\,\text{d}\Omega - \int_\Omega \frac{\partial}{\partial x_i}\left(\boldsymbol{V}^{\text{t}}\boldsymbol{A}_i^{\text{f}}\right)\boldsymbol{U}\,\text{d}\Omega + \int_\Omega \frac{\partial\boldsymbol{V}^{\text{t}}}{\partial x_i}\boldsymbol{K}_{ij}\frac{\partial\boldsymbol{U}}{\partial x_j}\,\text{d}\Omega, \tag{20}$$

and the linear form $\mathcal{L}$ given by

$$\mathcal{L}(\boldsymbol{V}) := \int_\Omega \boldsymbol{V}^{\text{t}}\boldsymbol{F}\,\text{d}\Omega. \tag{21}$$

3.1.2 The SGS approach

The basic idea of the stabilization method proposed here is based on the SGS concept introduced in [10] and that can be also found in different contexts (not only numerical). What follows is a summary of the approach described in [4].

The starting idea is to split the continuous space as $W = W_h \oplus \tilde{W}$, where W_h is the finite element space (and therefore finite dimensional) in which the approximate solution will be sought. We call $\tilde{W}$ the space of subscales or SGS. It is readily checked that the continuous problem can be written as the system of equations:

$$\mathcal{B}(\boldsymbol{U}_h,\boldsymbol{V}_h) + \mathcal{B}(\tilde{\boldsymbol{U}},\boldsymbol{V}_h) = \mathcal{L}(\boldsymbol{V}_h), \tag{22}$$

$$\mathcal{B}(\boldsymbol{U}_h,\tilde{\boldsymbol{V}}) + \mathcal{B}(\tilde{\boldsymbol{U}},\tilde{\boldsymbol{V}}) = \mathcal{L}(\tilde{\boldsymbol{V}}), \tag{23}$$

which must hold for all $\boldsymbol{V}_h \in W_h$ and $\tilde{\boldsymbol{V}} \in \tilde{W}$, and where $\boldsymbol{U} = \boldsymbol{U}_h + \tilde{\boldsymbol{U}}$ and $\boldsymbol{U}_h \in W_h$, $\tilde{\boldsymbol{U}} \in \tilde{W}$.

It is useful for the following to introduce the notation

$$\int_{\Omega'} := \sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e}, \quad \int_{\partial\Omega'} := \sum_{e=1}^{n_{\text{el}}} \int_{\partial\Omega^e},$$

where n_{el} is the number of elements of the finite element partition used to build W_h and Ω^e denotes the domain of element number e.

Integrating by parts all the terms in $\mathcal{B}(\tilde{\boldsymbol{U}}, \boldsymbol{V}_h)$ in (22) and the left-hand-side terms of (23) within each element domain, we get

$$\begin{aligned}
&\mathcal{B}(\boldsymbol{U}_h, \boldsymbol{V}_h) \\
&\quad + \int_{\partial\Omega'} \tilde{\boldsymbol{U}}^{\text{t}} n_i \left(\boldsymbol{K}_{ij} \frac{\partial \boldsymbol{V}_h}{\partial x_j} - \boldsymbol{A}_i^{\text{f}} \boldsymbol{V}_h \right) \text{d}\Gamma \\
&\quad + \int_{\Omega'} \tilde{\boldsymbol{U}}^{\text{t}} \mathsf{L}^*(\boldsymbol{V}_h) \, \text{d}\Omega = \mathcal{L}(\boldsymbol{V}_h),
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\int_{\partial\Omega'} \tilde{\boldsymbol{V}}^{\text{t}} n_i \Big(\boldsymbol{K}_{ij} \frac{\partial}{\partial x_j} (\boldsymbol{U}_h + \tilde{\boldsymbol{U}}) \\
&\quad - \boldsymbol{A}_i^{\text{f}} (\boldsymbol{U}_h + \tilde{\boldsymbol{U}}) \Big) \text{d}\Gamma + \int_{\Omega'} \tilde{\boldsymbol{V}}^{\text{t}} \mathsf{L}(\tilde{\boldsymbol{U}}) \, \text{d}\Omega \\
&\quad = \int_{\Omega'} \tilde{\boldsymbol{V}}^{\text{t}} \left[\boldsymbol{F} - \mathsf{L}(\boldsymbol{U}_h) \right] \text{d}\Omega,
\end{aligned} \tag{25}$$

where n_i is the i-th component of the exterior normal to $\partial\Omega^e$ and L^* is the adjoint operator of L with homogeneous Dirichlet conditions, given by

$$\mathsf{L}^*(\boldsymbol{U}) := -\frac{\partial}{\partial x_i} (\boldsymbol{A}_i^{\text{t}} \boldsymbol{U}) - \frac{\partial}{\partial x_i} \left(\boldsymbol{K}_{ij}^{\text{t}} \frac{\partial \boldsymbol{U}}{\partial x_j} \right).$$

Equation (25) is equivalent to:

$$\mathsf{L}(\tilde{\boldsymbol{U}}) = \boldsymbol{F} - \mathsf{L}(\boldsymbol{U}_h) + \boldsymbol{V}_{h,\text{ort}} \quad \text{in } \Omega^e, \tag{26}$$

$$\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{U}}_{\text{ske}} \quad \text{on } \partial\Omega^e, \tag{27}$$

where $\boldsymbol{V}_{h,\text{ort}}$ is obtained from the condition that $\tilde{\boldsymbol{U}}$ must belong to $\tilde{W}$ (and not to the whole space W) and $\tilde{\boldsymbol{U}}_{\text{ske}}$ is a function defined on the element boundaries and such that

$$\boldsymbol{q}_n := n_i \left(\boldsymbol{K}_{ij} \frac{\partial}{\partial x_j} (\boldsymbol{U}_h + \tilde{\boldsymbol{U}}) - \boldsymbol{A}_i^{\text{f}} (\boldsymbol{U}_h + \tilde{\boldsymbol{U}}) \right)$$

is continuous across interelement boundaries, and therefore the first term in the left-hand-side of (25) vanishes.

3.1.3 Algebraic approximation to the subscales

Different SGS stabilization methods can be devised depending on the way problem (26) and (27) is approximated. The purpose of this paper is not to propose a new methodology, but rather to see how to apply a well established formulation to the incompressible MHD problem. This well known method can be obtained by approximating the subscales by the algebraic expression

$$\tilde{\boldsymbol{U}} \approx \boldsymbol{\tau} \left[\boldsymbol{F} - \mathsf{L}(\boldsymbol{U}_h) \right], \tag{28}$$

where $\boldsymbol{\tau}$ is a $n_{\text{unk}} \times n_{\text{unk}}$ matrix of stabilization parameters, the expression of which is discussed in the following subsection.

To close the approximation, we neglect the interelement boundary terms in (24), which can be understood as taking $\tilde{\boldsymbol{U}}_{\text{ske}} = \boldsymbol{0}$ on the interelement boundaries. The final problem is: Find $\boldsymbol{U}_h \in W_h$ such that

$$\mathcal{B}(\boldsymbol{U}_h, \boldsymbol{V}_h) + \int_{\Omega'} \tilde{\boldsymbol{U}}^{\text{t}} \mathsf{L}^*(\boldsymbol{V}_h) \, \text{d}\Omega = \mathcal{L}(\boldsymbol{V}_h),$$

for all $\boldsymbol{V}_h \in W_h$, which, upon substitution of the subscales by (28), yields the following discrete problem: Find $\boldsymbol{U}_h \in W_h$ such that

$$\mathcal{B}_{\text{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h) = \mathcal{L}_{\text{stab}}(\boldsymbol{V}_h) \quad \forall \boldsymbol{V}_h \in W_h, \tag{29}$$

where the bilinear form $\mathcal{B}_{\text{stab}}$ and the linear form $\mathcal{L}_{\text{stab}}$ are now given by

$$\begin{aligned}
\mathcal{B}_{\text{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h) &= \mathcal{B}(\boldsymbol{U}_h, \boldsymbol{V}_h) \\
&\quad - \int_{\Omega'} \mathsf{L}^*(\boldsymbol{V}_h)^{\text{t}} \boldsymbol{\tau} \mathsf{L}(\boldsymbol{U}_h) \, \text{d}\Omega,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\mathcal{L}_{\text{stab}}(\boldsymbol{V}_h) &= \mathcal{L}(\boldsymbol{V}_h) \\
&\quad - \int_{\Omega'} \mathsf{L}^*(\boldsymbol{V}_h)^{\text{t}} \boldsymbol{\tau} \boldsymbol{F} \, \text{d}\Omega.
\end{aligned} \tag{31}$$

3.1.4 Stabilized MHD problem

Up to now we have described the *algebraic* version of the SGS stabilization in a general setting. The objective now is to apply it to the MHD problem we are considering. In particular, the operator $\mathsf{L}^*(\boldsymbol{V}_h)$ is now given by

$$\mathsf{L}^*(\boldsymbol{V}) = \begin{bmatrix} -\boldsymbol{a} \cdot \nabla \boldsymbol{v} - \frac{1}{Re} \Delta \boldsymbol{v} - \nabla q - S \boldsymbol{b} \times (\nabla \times \boldsymbol{C}) \\ -\nabla \cdot \boldsymbol{v} \\ S \nabla \times (\boldsymbol{v} \times \boldsymbol{b}) + \frac{S}{Rm} \nabla \times (\nabla \times \boldsymbol{C}) - S \nabla s \\ -S \nabla \cdot \boldsymbol{C} \end{bmatrix}. \tag{32}$$

To define the method for the particular MHD problem, an expression for the matrix of stabilization parameters $\boldsymbol{\tau}$ needs to be proposed. To our knowledge, there is no general way to define it for systems of equations [4]. It must be designed for each particular problem taking into account its stability deficiencies.

In the case we are considering, we will see in the following subsection that stability can be improved maintaining optimal accuracy by taking a simple diagonal expression for $\boldsymbol{\tau}$, with one scalar component for each of the four equations (1), (2), (3) and (4). In the 3D case, we take

$$\boldsymbol{\tau} = \text{diag}(\tau_1, \tau_1, \tau_1, \tau_2, \tau_3, \tau_3, \tau_3, \tau_4). \tag{33}$$

Using this expression and (32), it follows that the stabilized bilinear form that we have to consider in problem (29) is

$$
\begin{aligned}
&\mathcal{B}_{\text{stab}}(\boldsymbol{U}_h, \boldsymbol{V}_h) \\
&= \mathcal{B}(\boldsymbol{U}_h, \boldsymbol{V}_h) - \int_{\Omega'} \mathsf{L}^*(\boldsymbol{V}_h)^{\mathrm{t}} \boldsymbol{\tau} \mathsf{L}(\boldsymbol{U}_h)\, \mathrm{d}\Omega \\
&= (\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}_h) + \frac{1}{Re}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) \\
&\quad -(p_h, \nabla \cdot \boldsymbol{v}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) \\
&\quad + S\,(\boldsymbol{B}_h, \nabla \times (\boldsymbol{v}_h \times \boldsymbol{b})) - S\,(\boldsymbol{C}_h, \nabla \times (\boldsymbol{u}_h \times \boldsymbol{b})) \\
&\quad + \frac{S}{Rm}(\nabla \times \boldsymbol{B}_h, \nabla \times \boldsymbol{C}_h) + S(\nabla r_h, \boldsymbol{C}_h) \\
&\quad - S(\nabla s_h, \boldsymbol{B}_h) \\
&\quad + \Big(X_u(\boldsymbol{v}_h, q_h, \boldsymbol{C}_h) + \frac{1}{Re}\Delta \boldsymbol{v}_h, \\
&\quad \tau_1 (X_u(\boldsymbol{u}_h, p_h, \boldsymbol{B}_h) - \frac{1}{Re}\Delta \boldsymbol{u}_h) \Big)_h \\
&\quad + (\nabla \cdot \boldsymbol{v}_h, \tau_2 (\nabla \cdot \boldsymbol{u}_h))_h \\
&\quad + \Big(X_B(s_h, \boldsymbol{v}_h) - \frac{S}{Rm} \nabla \times (\nabla \times \boldsymbol{C}_h), \\
&\quad \tau_3 (X_B(r_h, \boldsymbol{u}_h) + \frac{S}{Rm} \nabla \times (\nabla \times \boldsymbol{B}_h)) \Big)_h \\
&\quad + S^2\,(\nabla \cdot \boldsymbol{C}_h, \tau_4 (\nabla \cdot \boldsymbol{B}_h))_h \,. \qquad (34)
\end{aligned}
$$

where we have introduced the abbreviations

$$
\begin{aligned}
X_u(\boldsymbol{v}_h, q_h, \boldsymbol{C}_h) &:= \boldsymbol{a} \cdot \nabla \boldsymbol{v}_h + \nabla q_h \\
&\quad + S\boldsymbol{b} \times (\nabla \times \boldsymbol{C}_h), \\
X_B(s_h, \boldsymbol{v}_h) &:= -S \nabla \times (\boldsymbol{v}_h \times \boldsymbol{b}) + S \nabla s_h,
\end{aligned}
$$

and $(\cdot, \cdot)_h$ is defined as

$$
(f, g)_h := \sum_{e=1}^{n_{\text{el}}} \int_{\Omega^e} f\, g\, \mathrm{d}\Omega.
$$

Finally, the right-hand-side of the stabilized problem (31) is now given by

$$
\begin{aligned}
\mathcal{L}_{\text{stab}}(\boldsymbol{V}_h) &= \mathcal{L}(\boldsymbol{V}_h) - \int_{\Omega'} \mathsf{L}^*(\boldsymbol{V}_h)^{\mathrm{t}} \boldsymbol{\tau} \boldsymbol{F}\, \mathrm{d}\Omega \\
&= \langle \boldsymbol{f}, \boldsymbol{v} \rangle + S(\boldsymbol{g}, \boldsymbol{C}) \\
&\quad + \Big(X_u(\boldsymbol{v}_h, q_h, \boldsymbol{C}_h) + \frac{1}{Re}\Delta \boldsymbol{v}_h, \tau_1 \boldsymbol{f} \Big)_h \\
&\quad + \Big(X_B(s_h, \boldsymbol{v}_h) - \frac{S}{Rm} \nabla \times (\nabla \times \boldsymbol{C}_h), \tau_3 \boldsymbol{g} \Big)_h . \qquad (35)
\end{aligned}
$$

The definition of the stabilized finite element method is now complete up to the expression of the stabilization parameters. The expression we propose is the following:

$$
\tau_1 = \left(c_1 \frac{a}{h} + c_2 \frac{1}{Reh^2} + c_3 \frac{Sb}{h} \right)^{-1}, \qquad (36)
$$

$$
\tau_2 = c_4 \frac{h^2}{\tau_1}, \qquad (37)
$$

$$
\tau_3 = \frac{1}{S} \left(c_1 \frac{b}{h} + c_2 \frac{1}{Rmh^2} \right)^{-1}, \qquad (38)
$$

$$
\tau_4 = c_4 \frac{h^2}{S^2 \tau_3}. \qquad (39)
$$

It is understood that these expressions are evaluated element by element. Here, a is the maximum norm of the velocity field $\boldsymbol{a}$ computed in the element under consideration. Likewise, b denotes the maximum norm of $\boldsymbol{b}$ in this element, and h its diameter. In the extension to the nonlinear problem, a and b correspond to a variable velocity and magnetic field and need to be evaluated at each integration point. The constants c_i, $i = 1, 2, 3, 4$, are independent of the physical parameters of the problem and of the mesh discretization. In the numerical examples presented in Sect. 4, obtained using linear elements, we have taken them as $c_1 = 4$, $c_2 = 2$, $c_3 = 1$, $c_4 = 1$.

In the following subsection we justify in detail this choice from the numerical analysis of the problem. We will proceed in a constructive manner, posing conditions on the stabilization parameters obtained from the requirement that the method is stable (coercive) and optimally accurate.

It has to be remarked that the design of the stabilization parameters from the convergence analysis is based on the assumption that matrix $\boldsymbol{\tau}$ is diagonal. It would be possible also to start from more generals forms of this matrix and to study what the numerical analysis dictates in this case.

3.2 Numerical analysis and design of the stabilization parameters

In this subsection we proceed to analyze the formulation introduced above and, in particular, to justify the choice (36), (37), (38) and (39). For the sake of simplicity in the notation, we will assume that a and b are constant. Likewise, we will assume that the finite element meshes are quasi-uniform. In this case, h in (36), (37), (38) and (39) can be taken the same for all the elements (the maximum element diameter), and therefore τ_i, $i = 1, 2, 3, 4$, are also constant. Moreover, for quasi-uniform meshes the following inverse estimates hold:

$$
\|\nabla v_h\| \leq \frac{C_{\text{inv}}}{h} \|v_h\|, \quad \|\nabla \nabla v_h\| \leq \frac{C_{\text{inv}}}{h} \|\nabla v_h\|, \qquad (40)
$$

for any function v_h in the finite element space and for a certain constant C_{inv}.

The stability and convergence analysis will be made using the following mesh-dependent norm:

$$
\begin{aligned}
|\!|\!| \boldsymbol{U}_h |\!|\!|^2 &:= \frac{1}{Re} \|\nabla \boldsymbol{u}_h\|^2 + \frac{S}{Rm} \|\nabla \times \boldsymbol{B}_h\|^2 \\
&\quad + \tau_1 \|\boldsymbol{a} \cdot \nabla \boldsymbol{u}_h + \nabla p_h + S\boldsymbol{b} \times (\nabla \times \boldsymbol{B}_h)\|^2 \\
&\quad + \tau_2 \|\nabla \cdot \boldsymbol{u}_h\|^2 \\
&\quad + \tau_3 S^2 \| - \nabla \times (\boldsymbol{u}_h \times \boldsymbol{b}) + \nabla r_h\|^2 \\
&\quad + \tau_4 S^2 \|\nabla \cdot \boldsymbol{B}_h\|^2 \\
&\equiv \frac{1}{Re} \|\nabla \boldsymbol{u}_h\|^2 + \frac{S}{Rm} \|\nabla \times \boldsymbol{B}_h\|^2 \\
&\quad + \tau_1 \|X_u(\boldsymbol{u}_h, p_h, \boldsymbol{B}_h)\|^2 + \tau_2 \|\nabla \cdot \boldsymbol{u}_h\|^2 \\
&\quad + \tau_3 \|X_B(r_h, \boldsymbol{u}_h)\|^2 + \tau_4 S^2 \|\nabla \cdot \boldsymbol{B}_h\|^2. \qquad (41)
\end{aligned}
$$

In all what follows, C will denote a positive constant, not necessarily the same at different appearances.

3.2.1 Coercivity

Let us start by proving stability in the form of coercivity of the bilinear form (34). It is immediately checked that

$$\begin{aligned}\mathcal{B}_{\text{stab}}(\boldsymbol{U}_h,\boldsymbol{U}_h) &= \mathcal{B}(\boldsymbol{U}_h,\boldsymbol{U}_h)\\ &\quad -\int_{\Omega'} \mathsf{L}^*(\boldsymbol{U}_h)^{\text{t}}\boldsymbol{\tau}\mathsf{L}(\boldsymbol{U}_h)\,\mathrm{d}\Omega\\ &= \frac{1}{Re}\|\nabla\boldsymbol{u}\|^2+\frac{S}{Rm}\|\nabla\times\boldsymbol{B}\|^2\\ &\quad +\tau_1\|X_u(\boldsymbol{u}_h,p_h,\boldsymbol{B}_h)\|^2-\tau_1\frac{1}{Re^2}\|\Delta\boldsymbol{u}_h\|^2\\ &\quad +\tau_2\|\nabla\cdot\boldsymbol{u}_h\|^2\\ &\quad +\tau_3\|X_B(r_h,\boldsymbol{u}_h)\|^2-\tau_3\frac{S^2}{Rm^2}\|\nabla\times\nabla\times\boldsymbol{B}\|^2\\ &\quad +\tau_4 S^2\|\nabla\cdot\boldsymbol{B}_h\|^2.\end{aligned}$$

Using the second inverse estimate in (40), it is clear that the necessary and sufficient condition for $\mathcal{B}_{\text{stab}}$ to be coercive is that

$$\frac{1}{Re}-\tau_1\frac{1}{Re^2}\frac{C_{\text{inv}}^2}{h^2}\geq\alpha\frac{1}{Re}$$
$$\Longleftrightarrow\quad \tau_1\leq(1-\alpha)Re\frac{h^2}{C_{\text{inv}}^2},\qquad(42)$$
$$\frac{S}{Rm}-\tau_3\frac{S^2}{Rm^2}\frac{C_{\text{inv}}^2}{h^2}\geq\alpha\frac{S}{Rm}$$
$$\Longleftrightarrow\quad \tau_3\leq(1-\alpha)\frac{Rm}{S}\frac{h^2}{C_{\text{inv}}^2},\qquad(43)$$

with $0<\alpha<1$. Conditions (42) and (43) yield

$$\mathcal{B}_{\text{stab}}(\boldsymbol{U}_h,\boldsymbol{U}_h)\geq C\|\boldsymbol{U}_h\|^2,\qquad(44)$$

for a constant C independent of the discretization and of the physical parameters (it depends only on the constants of the stabilization parameters).

3.2.2 Optimal accuracy

We have obtained conditions (42) and (43) on the stabilization parameters by requiring stability. The rest of conditions will be obtained by imposing that the stabilized method proposed is optimally accurate, which will lead to optimal convergence.

For a function v, let $\pi_h(v)$ be its optimal finite element approximation. We assume that the following interpolation estimates hold:

$$\begin{aligned}\varepsilon_i(v) &:= \|v-\pi_h(v)\|_{H^i(\Omega)}\\ &\leq Ch^{k+1-i}\|v\|_{H^{k+1}(\Omega)},\quad i=0,1,\end{aligned}\qquad(45)$$

where $\|v\|_{H^q(\Omega)}$ is the $H^q(\Omega)$-norm of v, that is, the sum of the $L^2(\Omega)$-norm of the derivatives of v up to degree q (and thus the $H^0(\Omega)$-norm coincides with the $L^2(\Omega)$-norm), and k is the degree of the finite element approximation.

We will prove in the following that the error function of the formulation is

$$\begin{aligned}E(h) &:= \tau_1^{-1/2}\varepsilon_0(\boldsymbol{u})+\tau_2^{-1/2}\varepsilon_0(p)\\ &\quad +\tau_3^{-1/2}\varepsilon_0(\boldsymbol{B})+\tau_4^{-1/2}\varepsilon_0(r).\end{aligned}\qquad(46)$$

The conditions on the stabilization parameters we will obtain will in fact show that this is indeed the error function and that this error function is optimal.

Let $\boldsymbol{U}$ be the solution of the continuous problem and $\pi_h(\boldsymbol{U})$ its optimal finite element approximation. The accuracy estimate that will be needed to prove convergence later on is

$$\mathcal{B}_{\text{stab}}(\boldsymbol{U}-\pi_h(\boldsymbol{U}),\boldsymbol{V}_h)\leq CE(h)\|\boldsymbol{V}_h\|,\qquad(47)$$

for any finite element function $\boldsymbol{V}_h$.

Let us prove this by showing that both the Galerkin and the stabilization terms in $\mathcal{B}_{\text{stab}}$ satisfy estimate (47). Starting with the Galerkin contribution we have that

$$\begin{aligned}&\mathcal{B}(\boldsymbol{U}-\pi_h(\boldsymbol{U}),\boldsymbol{V}_h)\\ &= -(\boldsymbol{u}-\pi_h(\boldsymbol{u}),\boldsymbol{a}\cdot\nabla\boldsymbol{v}_h)\\ &\quad +\frac{1}{Re}(\nabla(\boldsymbol{u}-\pi_h(\boldsymbol{u})),\nabla\boldsymbol{v}_h)\\ &\quad -(p-\pi_h(p),\nabla\cdot\boldsymbol{v}_h)-(\nabla q_h,\boldsymbol{u}-\pi_h(\boldsymbol{u}))\\ &\quad -S(\boldsymbol{u}-\pi_h(\boldsymbol{u}),\boldsymbol{b}\times\nabla\times\boldsymbol{C}_h)\\ &\quad +S(\boldsymbol{B}-\pi_h(\boldsymbol{B}),\nabla\times(\boldsymbol{v}_h\times\boldsymbol{b}))\\ &\quad +\frac{S}{Rm}(\nabla\times\boldsymbol{C}_h,\nabla\times(\boldsymbol{B}-\pi_h(\boldsymbol{B})))\\ &\quad -S(r-\pi_h(r),\nabla\cdot\boldsymbol{C}_h)\\ &\quad -S(\nabla s_h,\boldsymbol{B}-\pi_h(\boldsymbol{B}))\\ &\leq C\Big(\varepsilon_0(\boldsymbol{u})\tau_1^{-1/2}\tau_1^{1/2}\|X_u(\boldsymbol{v}_h,q_h,\boldsymbol{C}_h)\|\\ &\quad +\frac{1}{Re^{1/2}}\varepsilon_1(\boldsymbol{u})\frac{1}{Re^{1/2}}\|\nabla\boldsymbol{v}_h\|\\ &\quad +\varepsilon_0(p)\tau_2^{-1/2}\tau_2^{1/2}\|\nabla\cdot\boldsymbol{v}_h\|\\ &\quad +\varepsilon_0(\boldsymbol{B})\tau_3^{-1/2}\tau_3^{1/2}\|X_B(s_h,\boldsymbol{v}_h)\|\\ &\quad +\frac{S^{1/2}}{Rm^{1/2}}\varepsilon_1(\boldsymbol{B})\frac{S^{1/2}}{Rm^{1/2}}\|\nabla\times\boldsymbol{C}_h\|\\ &\quad +\varepsilon_0(s)\tau_4^{-1/2}\tau_4^{1/2}S\|\nabla\cdot\boldsymbol{C}_h\|\Big).\end{aligned}\qquad(48)$$

Conditions (42) and (43) and the expression of the interpolation errors imply

$$\frac{1}{Re^{1/2}}\varepsilon_1(\boldsymbol{u})\leq C\varepsilon_0(\boldsymbol{u})\tau_1^{-1/2},$$
$$\frac{S^{1/2}}{Rm^{1/2}}\varepsilon_1(\boldsymbol{B})\leq C\varepsilon_0(\boldsymbol{B})\tau_3^{-1/2},$$

and therefore from (48) it follows that the Galerkin contribution $\mathcal{B}(\boldsymbol{U}-\pi_h(\boldsymbol{U}),\boldsymbol{V}_h)$ can be bounded as indicated in (47). It remains to prove that also the stabilization terms can be bounded in this way:

$$
\begin{aligned}
&-\int_{\Omega'} \mathsf{L}^*(\boldsymbol{V}_h)^{\mathrm{t}} \boldsymbol{\tau} \mathsf{L}(\boldsymbol{U} - \pi_h(\boldsymbol{U}))\, \mathrm{d}\Omega \\
&= \Big(X_u(\boldsymbol{u} - \pi_h(\boldsymbol{u}), p - \pi_h(p), \boldsymbol{B} - \pi_h(\boldsymbol{B})) \\
&\quad - \frac{1}{Re} \Delta(\boldsymbol{u} - \pi_h(\boldsymbol{u})), \\
&\quad \tau_1 (X_u(\boldsymbol{v}_h, q_h, \boldsymbol{C}_h) + \frac{1}{Re} \Delta \boldsymbol{v}_h) \Big)_h \\
&\quad + (\nabla \cdot (\boldsymbol{u} - \pi_h(\boldsymbol{u})), \tau_2 \nabla \cdot \boldsymbol{v}_h)_h \\
&\quad + \Big(X_B(r - \pi_h(r), \boldsymbol{u} - \pi_h(\boldsymbol{u})) \\
&\quad + \frac{S}{Rm} \nabla \times \nabla \times (\boldsymbol{B} - \pi_h(\boldsymbol{B})), \\
&\quad \tau_3 (X_B(s_h, \boldsymbol{v}_h) - \frac{S}{Rm} \nabla \times \nabla \times \boldsymbol{C}_h) \Big)_h \\
&\quad + S^2 (\nabla \cdot (\boldsymbol{B} - \pi_h(\boldsymbol{B})), \tau_4 \nabla \cdot \boldsymbol{C}_h)_h \\
&\leq C \Big(\tau_1^{1/2} \| X_u(\boldsymbol{u} - \pi_h(\boldsymbol{u}), p - \pi_h(p), \\
&\quad \boldsymbol{B} - \pi_h(\boldsymbol{B})) \| + \tau_1^{1/2} \frac{1}{Re} \| \Delta(\boldsymbol{u} - \pi_h(\boldsymbol{u})) \| \Big) \\
&\quad \times \Big(\| \boldsymbol{V}_h \| + \tau_1^{1/2} \frac{1}{Re} \| \Delta \boldsymbol{v}_h \| \Big) + C \tau_2^{1/2} \varepsilon_1(\boldsymbol{u}) \| \boldsymbol{V}_h \| \\
&\quad + C \Big(\tau_3^{1/2} \| X_B(r - \pi_h(r), \boldsymbol{u} - \pi_h(\boldsymbol{u})) \| \\
&\quad + \tau_3^{1/2} \frac{S}{Rm} \| \nabla \times \nabla \times (\boldsymbol{B} - \pi_h(\boldsymbol{B})) \| \Big) \\
&\quad \times \Big(\| \boldsymbol{V}_h \| + \tau_3^{1/2} \frac{S}{Rm} \| \nabla \times \nabla \times \boldsymbol{C}_h \| \Big) \\
&\quad + C \tau_4^{1/2} S\, \varepsilon_1(\boldsymbol{B}) \| \boldsymbol{V}_h \|.
\end{aligned} \tag{49}
$$

Using once again conditions (42) and (43) and the inverse estimates (40) we have that

$$
\begin{aligned}
&\tau_1^{1/2} \frac{1}{Re} \| \Delta(\boldsymbol{u} - \pi_h(\boldsymbol{u})) \| \\
&\quad \leq \tau_1^{1/2} \frac{C}{Reh^2} \varepsilon_0(\boldsymbol{u}) \\
&\quad \leq C \tau_1^{-1/2} \varepsilon_0(\boldsymbol{u}) \leq C E(h),
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\tau_3^{1/2} \frac{S}{Rm} \| \nabla \times \nabla \times (\boldsymbol{B} - \pi_h(\boldsymbol{B})) \| \\
&\quad \leq \tau_3^{1/2} \frac{C\, S}{Rmh^2} \varepsilon_0(\boldsymbol{B}) \\
&\quad \leq C \tau_3^{-1/2} \varepsilon_0(\boldsymbol{B}) \leq C E(h),
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
&\tau_1^{1/2} \frac{1}{Re} \| \Delta \boldsymbol{v}_h \| \\
&\quad \leq C \tau_1^{1/2} \frac{1}{Re^{1/2}} \frac{C_{\text{inv}}}{h} \frac{1}{Re^{1/2}} \| \nabla \boldsymbol{v}_h \| \leq C \| \boldsymbol{V}_h \|,
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&\tau_3^{1/2} \frac{S}{Rm} \| \nabla \times \nabla \times \boldsymbol{C}_h \| \\
&\quad \leq C \tau_3^{1/2} \frac{S^{1/2}}{Rm^{1/2}} \frac{C_{\text{inv}}}{h} \frac{S^{1/2}}{Rm^{1/2}} \| \nabla \times \boldsymbol{C}_h \| \\
&\quad \leq C \| \boldsymbol{V}_h \|.
\end{aligned} \tag{53}
$$

So far, we have not posed any additional conditions on the stabilization parameters other than (42) and (43), found from the requirement of coercivity. The rest of conditions will come from the requirement of optimal accuracy. We have that

$$
\begin{aligned}
&\tau_1^{1/2} \| X_u(\boldsymbol{u} - \pi_h(\boldsymbol{u}), p - \pi_h(p), \boldsymbol{B} - \pi_h(\boldsymbol{B})) \| \\
&\quad \leq C \tau_1^{1/2} \left(\frac{a}{h} \varepsilon_0(\boldsymbol{u}) + \frac{1}{h} \varepsilon_0(p) + \frac{b\, S}{h} \varepsilon_0(\boldsymbol{B}) \right),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
&\tau_3^{1/2} \| X_B(r - \pi_h(r), \boldsymbol{u} - \pi_h(\boldsymbol{u})) \| \\
&\quad \leq C \tau_3^{1/2} \left(\frac{S}{h} \varepsilon_0(r) + \frac{b\, S}{h} \varepsilon_0(\boldsymbol{u}) \right).
\end{aligned} \tag{55}
$$

Both terms are bounded by $C E(h)$ *if the following conditions hold*:

$$
\begin{aligned}
&\tau_1 \leq C \frac{h}{a}, \quad \tau_1 \leq C \frac{h}{b\, S}, \quad \tau_2 \leq C \frac{h^2}{\tau_1}, \\
&\tau_3 \leq C \frac{h}{b\, S}, \quad \tau_4 \leq C \frac{h^2}{S^2 \tau_3}.
\end{aligned} \tag{56}
$$

Using these conditions in (54) and (55) it follows that these terms are bounded by $C E(h)$, which, combined with (50), (51), (52) and (53) implies that (49) can also be bounded by the right-hand-side of (47). This is precisely what we wished to prove.

The important point now is that the stabilization parameters given by (36), (37), (38) and (39) satisfy conditions (42), (43) and (56), and therefore the stabilized finite element method proposed is stable [cf. (44)] and optimally accurate [cf. (47)]

3.2.3 Convergence

As a trivial consequence of the properties of stability and accuracy in the sense of (47), it is trivial to show that the method is optimally convergent. From the orthogonality property $\mathcal{B}_{\text{stab}}(\boldsymbol{U} - \boldsymbol{U}_h, \boldsymbol{V}_h) = 0$ for any finite element function $\boldsymbol{V}_h$ (a direct consequence of the consistency of the method), we have that

$$
\begin{aligned}
&C \| \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h \|^2 \\
&\quad \leq \mathcal{B}_{\text{stab}}(\pi_h(\boldsymbol{U}) - \boldsymbol{U}_h, \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h) \\
&\quad \leq \mathcal{B}_{\text{stab}}(\pi_h(\boldsymbol{U}) - \boldsymbol{U}, \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h) \\
&\quad \leq C E(h) \| \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h \|,
\end{aligned}
$$

from where $\| \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h \| \leq C E(h)$. Now the triangle inequality implies

$$
\begin{aligned}
\| \boldsymbol{U} - \boldsymbol{U}_h \| &\leq \| \boldsymbol{U} - \pi_h \boldsymbol{U} \| + \| \pi_h(\boldsymbol{U}) - \boldsymbol{U}_h \| \\
&\leq \| \boldsymbol{U} - \pi_h \boldsymbol{U} \| + C E(h).
\end{aligned}
$$

A trivial check using the expression of the norm $\|\cdot\|$ given by (41), the interpolation estimates (45) and the stabilization parameters (36), (37), (38) and (39) shows that $\| \boldsymbol{U} - \pi_h \boldsymbol{U} \| \leq C E(h)$, from where

$$
\| \boldsymbol{U} - \boldsymbol{U}_h \| \leq C E(h). \tag{57}
$$

The fact that this error estimate is exactly the same as the estimate for the interpolation error $\| \boldsymbol{U} - \pi_h \boldsymbol{U} \| \leq C E(h)$

justifies why it has to be considered 'optimal'. Moreover, a simple inspection of what happens in the limit of dominant second order terms shows that in this case the error estimate reduces to the estimate that could be found using the Galerkin method using finite element spaces satisfying the discrete form of (11) and (12), but now, however, using equal interpolation for all the variables. Likewise, in the limit $Re \to \infty$ and $Rm \to \infty$, the error estimate (57) does not blow up and the result can also be considered optimal (see [5] for a similar discussion).

3.3 Iterative scheme for the nonlinear problem

In view of the linearization described previously, it is clear which is the iterative scheme that needs to be used for the nonlinear problem. For each iteration k, given $\boldsymbol{u}_h^k$ and $\boldsymbol{B}_h^k$, the unknown $\boldsymbol{U}_h^{k+1} = (\boldsymbol{u}_h^{k+1}, p_h^{k+1}, \boldsymbol{B}_h^{k+1}, r_h^{k+1})$ will be obtained by solving the discrete stabilized problem (29), with $\mathcal{B}_{\text{stab}}$ given by (34) identifying $\boldsymbol{a} \equiv \boldsymbol{u}_h^k$, $\boldsymbol{b} \equiv \boldsymbol{B}_h^k$ and $\boldsymbol{U} \equiv \boldsymbol{U}_h^{k+1}$, and with $\mathcal{L}_{\text{stab}}$ given by (35), again with the same identification.

Let us make two remarks concerning the linearized problem. The first is that, obviously, $\mathcal{B}_{\text{stab}}$ and $\mathcal{L}_{\text{stab}}$ depend on the iteration through $\boldsymbol{a} \equiv \boldsymbol{u}_h^k$ and $\boldsymbol{b} \equiv \boldsymbol{B}_h^k$. This affects not only the terms with first order derivatives, but also the expression of the stabilization parameters (36), (37), (38) and (39), that need to be recomputed at each iteration. The second remark is that (44) guarantees that the linearized problem has a unique solution. As for the continuous case, it is not difficult (but certainly technical) to show that, under the conditions for which the continuous problem has a unique solution, so does the nonlinear *discrete* problem, and the sequence $\{\boldsymbol{U}_h^k\}_{k\geq 0}$ converges precisely to this solution.

4 Numerical examples

In this section we present some simple numerical examples whose aim is to show that the finite element formulation proposed is optimally convergent and free of spurious oscillations using equal interpolation for all the variables. Rather than considering cases with high values of Re, Rm or S, we will show the behavior of the solution in terms of the so called Hartmann number, defined as

$$Ha = \sqrt{Re\, Rm\, S}.$$

For the nonlinear problem, it is not possible to consider very high values of Ha, since they are associated to complex flow features and, in general, to situations in which the solution of the flow equations is not unique.

4.1 Convergence test

In order to analyze the convergence properties of the stabilized finite element approximation presented, a two dimensional problem in the square domain $\Omega =]0, 1[\times]0, 1[$ is considered. This problem possesses a closed form analytical solution. The components of the body forces $\boldsymbol{f} = (f_x, f_y)$ and $\boldsymbol{g} = (g_x, g_y)$ are prescribed as:

$$\begin{aligned}
f_x = {} & f_1(x)(d_1'(y))^2 f_1'(x) \\
& - f_1'(x) d_1(y) f_1(x) d_1''(y) \\
& - \frac{1}{Re}\left[f_1''(x)d_1'(y) + f_1(x)d_1'''(y)\right] \\
& + S\left[f_2'(x)d_2(y)(f_2''(x)d_2(y) + f_2(x)d_2''(y))\right], \\
f_y = {} & - f_1(x)d_1'(y) f_1''(x) d_1(y) \\
& + (f_1'(x))^2 d_1(y) d_1'(y) \\
& + \frac{1}{Re}\left[d_1(y) f_1'''(x) + f_1'(x) d_1''(y)\right] \\
& + S\left[f_2(x)d_2'(y)\left(f_2''(x)d_2(y) + f_2(x)d_2''(y)\right)\right], \\
g_x = {} & f_1(x) f_2'(x)\left[d_1''(y)d_2(y) + d_1'(y)d_2'(y)\right] \\
& - f_1'(x) f_2(x)\left[d_1'(y)d_2'(y) + d_1(y)\, d_2''(y)\right] \\
& - \frac{1}{Rm}\left[f_2''(x)d_2'(y) + f_2(x)d_2'''(y)\right], \\
g_y = {} & - d_1'(y)d_2(y)\left[f_1'(x) f_2'(x) + f_1(x)\, f_2''(x)\right] \\
& + d_1(y)d_2'(y)\left[f_1''(x) f_2(x) + f_1'(x) f_2'(x)\right] \\
& + \frac{1}{Rm}\left[f_2'''(x)d_2(y) + f_2'(x)d_2''(y)\right],
\end{aligned}$$

where the prime denotes differentiation. Endowed with this body forces the 2D problem has an exact solution for the velocity given by $\boldsymbol{u} = (u_x, u_y)$, where

$$\begin{aligned}
u_x(x, y) &= f_1(x)d_1'(y), \\
u_y(x, y) &= -f_1'(x)d_1(y).
\end{aligned}$$

The analytical solution for the magnetic field is $\boldsymbol{B} = (B_x, B_y)$, now with

$$\begin{aligned}
B_x(x, y) &= f_2(x)d_2'(y), \\
B_y(x, y) &= -f_2'(x)d_2(y).
\end{aligned}$$

In this particular example, the functions $f_1(x)$, $f_2(x)$, $d_1(y)$ and $d_2(y)$ are chosen as

$$\begin{aligned}
f_1(x) &= x^2(1-x)^2, \quad & f_2(x) &= x^2(1-x)^2, \\
d_1(y) &= y^2(1-y)^2, \quad & d_2(y) &= y^2(1-y)^2.
\end{aligned}$$

The square domain Ω has been discretized with five different uniform meshes of $2 \times 25 \times 25$, $2 \times 50 \times 50$, $2 \times 75 \times 75$, $2 \times 100 \times 100$ and $2 \times 125 \times 125$ triangular linear elements. The characteristic length of the meshes are $h = 1/25, 1/50, 1/75, 1/100$ and $1/125$.

The convergence plots measured in the discrete $L^2(\Omega)$-norm for the velocity and the magnetic field are shown in Figs. 1 and 2, respectively. The slope of the convergence curve has to be compared with the line of slope two also shown in the figures. It is observed that the numerical convergence has also approximately slope two (1.93 for the velocity and 2.03 for the magnetic field), which is optimal for the linear elements employed in the calculation.

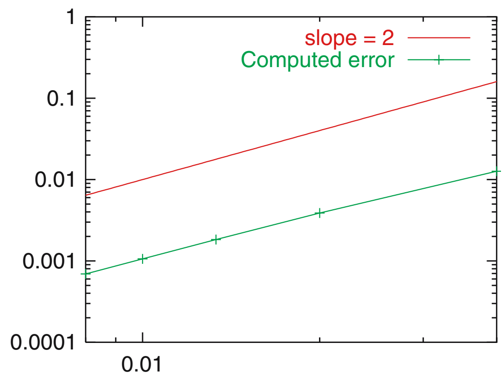

Fig. 1 Velocity error versus element size

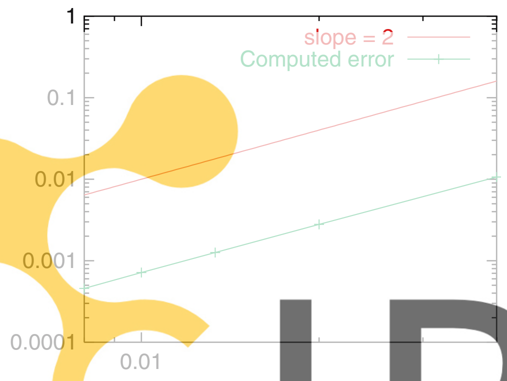

Fig. 2 Magnetic field error versus element size

4.2 Hartmann problem

Hartmann flow is a typical test in MHD. It consists in the flow of a liquid metal through a channel, in this particular case two-dimensional. The liquid flows in the x-direction by the influence of a prescribed pressure gradient G. A uniform magnetic field (0, 1) is applied on the boundaries, as shown in Fig. 3. The Reynolds number is taken as $Re = 10^2$, the magnetic Reynolds number as $Rm = 10^{-7}$ and different values of G are chosen in such a way that the Hartmann numbers considered are $Ha = 0$, 10, 50 and 100. The problem definition employed here is the same as in [1].

The flow of the liquid induces a perturbation in the magnetic field in the x-direction. The width of the channel is 2,

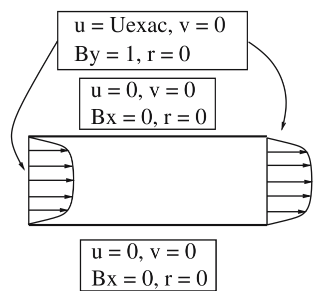

Fig. 3 Hartmann problem: problem setting

and we assume that the upper and lower walls are located at $y = 1$ and $y = -1$, respectively. In this case, there is an exact solution to the problem, given by

$$\begin{aligned}
\boldsymbol{u} &= (u_x(y), 0),\\
u_x(y) &= -\frac{G}{Ha\,\tanh(Ha)}\left[1 - \frac{\cosh(Ha\,y)}{\cosh(Ha)}\right],\\
\boldsymbol{B} &= (b_x(y), 1),\\
b_x(y) &= -\frac{Ha\,G}{Re\,S}\left[\frac{\sinh(Ha\,y)}{\sinh(Ha)} - y\right].
\end{aligned}$$

This exact velocity profile is prescribed at the inlet and at the outlet of the computational domain, as it is shown in Fig. 3, where the exact profile has been labeled ‘Uexac’.

Several meshes have been used in the calculation. In Fig. 4, the profiles of $u_x(y)$ and $b_x(y)$ for a mesh of 320 elements along the width are shown. For such a fine mesh, the numerical and the analytical solution virtually coincide. However, even for coarse meshes, the solution is smooth, completely free of numerical oscillations.

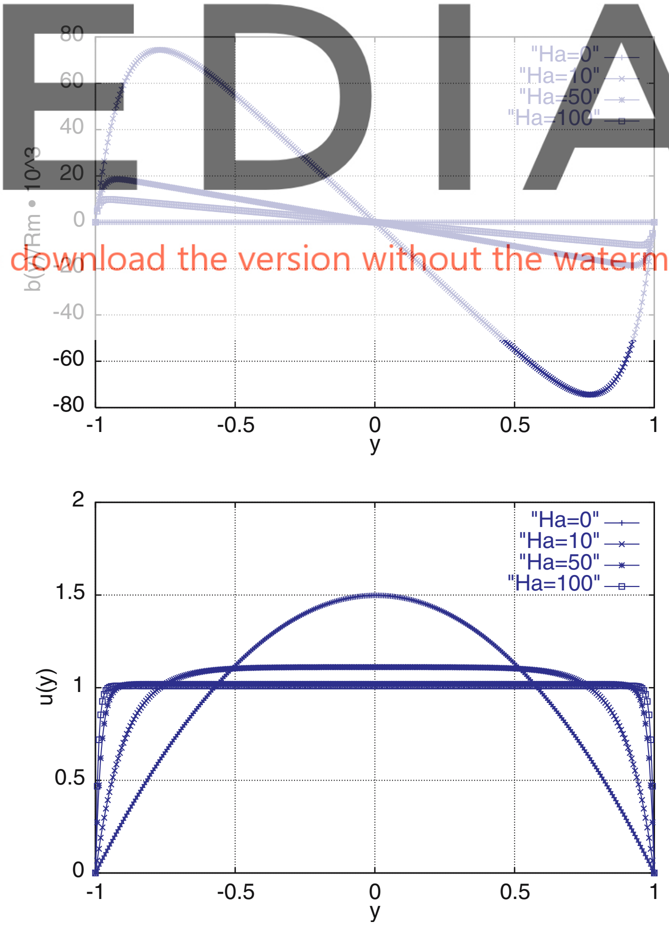

Fig. 4 Hartmann problem. *Top*: Profiles of the magnetic field. *Bottom*: Velocity profiles

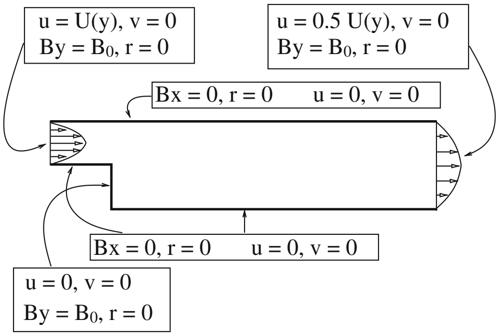

Fig. 5 Flow over a backward facing step: problem setting

4.3 Flow over a backward facing step

We solve here the classical problem of the flow over a backward facing step in the presence of a magnetic field roughly orthogonal to the flow. This problem was already introduced in [8], and serves to understand physically the effect of the magnetic field in damping the vortex after the inverted step. Its interest is merely to observe that the numerical method reproduces this effect without any oscillation in the numerical solution.

The walls of the pipe are considered perfect conductors. The velocity is prescribed at the inlet and outlet to a Poiseuille profile. The complete set of boundary conditions is depicted in Fig. 5. The Reynolds number based on the maximum of the velocity profile and the step height is fixed to $Re = 100$, and the magnetic Reynolds number to $Rm = 1$. The coupling number computed with B_0 is modified to obtain different values of Ha.

In Fig. 6 it is shown how the vortex after the step disappears as the Hartmann number increases, that is to say, as the effects of the magnetic field on the flow become more important. This is a well known physical effect exploited in several industrial applications. It is also seen in Fig. 6 that the velocity field obtained is perfectly smooth, as it is the magnetic field and the hydrodynamic and magnetic pressures (see Figs. 7 and 8).

5 Final remarks

In this paper we have presented a stabilized finite element method to approximate the MHD equations when a magnetic pressure is introduced to enforce the zero divergence condition on the magnetic field. Its main features are:

- It is based on an algebraic expression for the subscales.
- Design of the stabilization parameters from the numerical analysis.
- Optimal stability and convergence.
- Good numerical performance.

The important point, and what is the main contribution of this work, is the design of the stabilization parameters from the numerical analysis of the formulation. This design

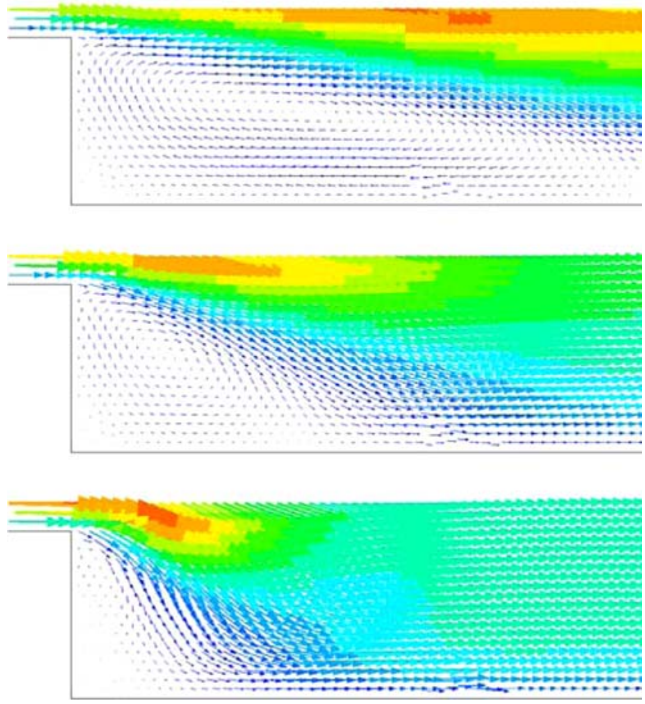

Fig. 6 Velocity field for the flow over a backward facing step. From the *top* to the *bottom*: Ha = 0, Ha = 5 and Ha = 10

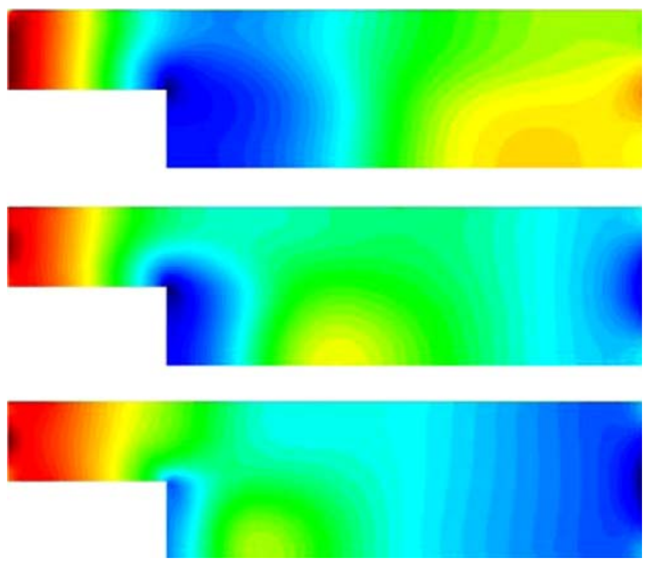

Fig. 7 Hydrodynamic pressure for the flow over a backward facing step. From the *top* to the *bottom* Ha = 0, Ha = 5 and Ha = 10

accounts properly not only for the interpolation of variables (compatibility conditions) and convection dominated situations, but also for the coupling of the magnetic and hydrodynamic problems.

Once the analysis has revealed how the stabilization terms have to act, there are several possible extensions to the formulation presented here, not only to broaden the physical model (to transient cases, for example), but also at the purely

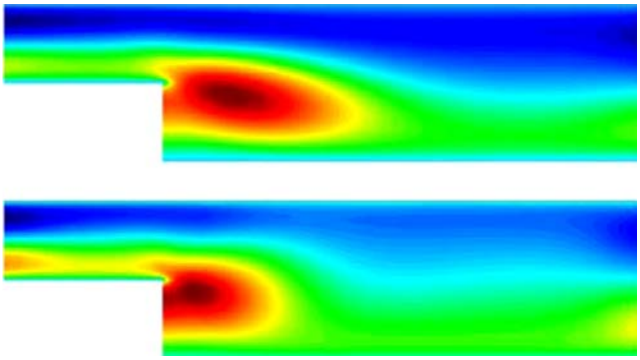

Fig. 8 x-component of the magnetic field for the flow over a backward facing step. From the *top* to the *bottom*: Ha = 5 and Ha = 10

numerical point of view. In particular, our intention is to analyze aspects such as nondiagonal expressions for matrix $\boldsymbol{\tau}$, the error analysis for the nonlinear problem (in branches of nonsingular solutions) or the introduction of stabilization terms accounting for the subscales on the element edges. Likewise, we also would like to explore strategies such as the tracking of subscales along the nonlinear iterative process or taking the subscales orthogonal to the finite element space, which we have already employed for the Navier-Stokes equations (see [6]).

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