15th World Congress on Computational Mechanics (WCCM-XV)
8th Asian Pacific Congress on Computational Mechanics (APCOM-VIII)
Virtual Congress: 31 July – 5 August 2022
S. Koshizuka (Ed.)

CRITICAL VELOCITY AND INSTABILITY OF INERTIAL OBJECTS MOVING UNIFORMLY ON LAYERED TRACK MODELS

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Key words: Moving Proximate Masses, Mass-Induced Frequency, Semianalytical Solution, Dynamic Interaction, Instability, Contour Integration

Abstract. *In this contribution, a new form of semianalytical results related to inertial objects that are traversing homogeneous infinite structures, introduced in previous author's work, is used to analyze one-, two- and three-layer models of the railway track. The aim of these analyses is determination of the critical velocity of a moving force and of the onset of instability of moving masses or oscillators. As one of the most important conclusions, it will be shown that in the case of two moving proximate masses, damping can act in the opposite direction than expected and owing to the dynamic interaction, the onset of instability can be shifted deeply into the subcritical velocity range.*

1 INTRODUCTION

The increase in the circulation speed of railway vehicles and capacity of the railway network, led to an increase in dynamic loads on the railway and the consequent acceleration of the degradation rate. Numerical models of the railway track are fundamental tools for the study of their dynamic behaviour. The use of three-dimensional finite element models is common practice, but reduced models are still relevant, due to simplicity of implementation, results interpretation, and low computational cost. According to classification from [1], these models can be named as one-, two- or three-layer models. Extensions to even more layers are also possible. This paper focuses on comparison between these models. It should be noted that the one-layer model is in fact a classic Winkler-Pasternak beam. The comparison will focus on the critical velocity and instability of moving mass or masses. Extensions to moving oscillators is straightforward.

The moving load problems are fundamental problems in structural dynamics. But problems with *moving forces* should be distinguished from *moving masses* or *oscillators*, because there is a fundamental difference in solution methods and undesirable effects that can appear if inertial effects are considered in the moving load or not. With respect to this, the term commonly used as "moving loads" is ambiguous. Under the assumption of linear homogeneous infinite structure, the fundamental differences between moving forces and moving inertial objects are the following: (i) For moving forces the initial instant is not essential because the steady-state regime is readily achieved; therefore, solution methods can neglect the transient vibrations. In addition, superposition is possible, and thus the problem can be solved for one acting force and then superpose the results to get the response for a set of forces. (ii) For moving inertial objects, the initial instant is essential, because without solving for transient vibrations the instability issues would remain completely hidden. Superposition of results is not possible unless the moving objects are quite apart. For proximate objects, the dynamic interaction is of utmost importance because it can significantly affect the onset of instability.

Published works on instability issues are generally dedicated to infinite structures. In [2-3] conditions for instability are determined by D-composition method but full deflection shapes of the beam are not presented. It is assumed that the mass is in permanent contact with the beam. In [4-5] non-linear contact spring is introduced. Deflection shapes are determined numerically, and instability is again analyzed by the D-composition method.

Regarding a sequence of moving masses, double Fourier transform in which the possibility of instability is hidden was implemented in [6]. Sequence of moving oscillators on infinite beam has been dealt with in [7] using Green's function method and D-composition method to determine the conditions for instability. D-composition method is also used in recent work [8]. Whereas in [7] a contact spring was used, in [8] it was again assumed that the moving mass is in permanent contact with the beam.

The problem of dynamic interaction between proximate moving masses requires further attention from an analytical perspective. It is essential to establish the conditions under which superposition is possible and whether the dynamic interaction leads to instability at lower velocity than expected. This can be achieved by exploiting the new form of result presentation published by the author [9-11]. In these approaches the notion of the critical velocity is vital [12-13]. Summary of semianalytical approaches related to such studies can be consulted in [14].

The solution presented in this paper is *conceptually different* from that in [7-8] because the full vibration shapes are determined semianalytically and not numerically, and the instability is identified directly from the so-called *mass-induced frequency*. This term must be distinguished from the natural frequencies of the system because the mass-induced frequency, or simply induced frequency, is induced by the mass movement, and is therefore dependent on its velocity. The final vibrations are presented using dimensionless parameters covering wide range of realistic scenarios. The *safe distance* between proximate masses, for which results superposition is possible, can be easily established, as well as the so-called *critical distance* between masses as the distance for which the lowest value of the imaginary part of an induced frequency is reached. One of the most important conclusions is that the external viscous damping can act oppositely than expected and together with the dynamic interaction shift the onset of instability deeply into the subcritical velocity range.

In [9], the new presentation of the semianalytical solution was derived under the assumption of homogeneous initial conditions. In [10], the solution was extended to non-homogeneous initial conditions, and in [11], further details on the method and analysis of moving one- or two-mass oscillators were presented. These works used Winkler-Pasternak beam and only one moving object: mass or oscillator. Extensions to proximate moving masses is included in [15-16] for Winkler-Pasternak beam and two-layer model of the railway track, respectively. Extension to plane foundation model used in [12-13] was conducted in [17].

In this paper, the main conclusions about proximate moving masses are summarized. In Section 2, the problem to be solved is specified. In Section 3, solution of the problem is derived using integral transforms and methods of contour integration. In Section 4, some results are given, and the models are compared from several points of view. Finally, in Section 5 some conclusions from the studies presented are drawn.

2 PROBLEM STATEMENT

Layered models of the railway track are widely used due to their simplicity and computational efficiency. In these models, the rail, in form of a beam, is supported by linear spring-damper components and discrete masses (Fig. 1).

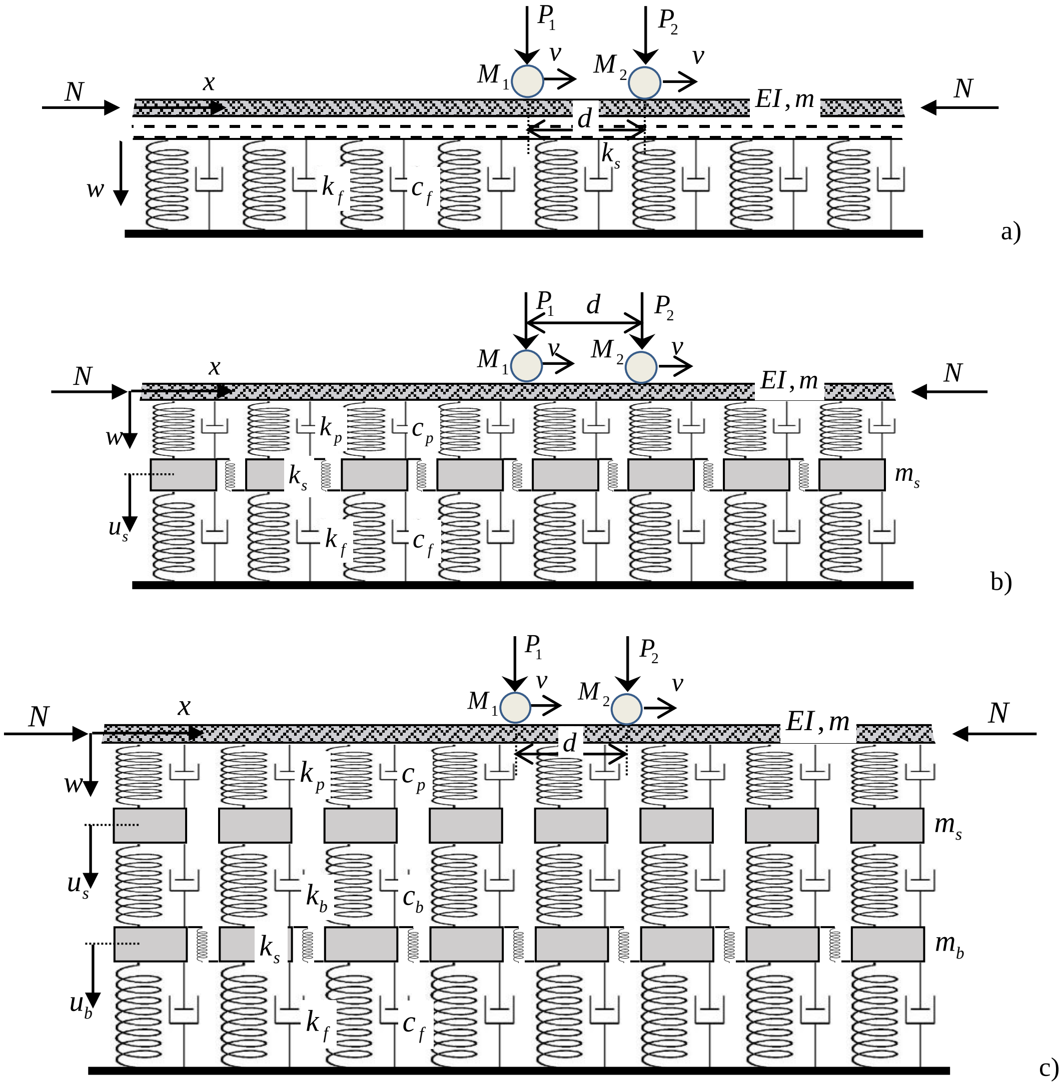


Figure 1: Layered models of the railway track subjected to an axial force and traversing by two proximate masses acted on by vertical forces: a) one-layer; b) two-layer; c) three-layer model.

Realistic models with discrete supports can be replaced by equivalent models with continuous supports because differences in behavior would be noticeable only under high frequencies, particularly around the pinned-to-pinned value. As already mentioned, one-layer model reduces to the classical Winkler-Pasternak beam, but it can also have rigidly connected sleepers.

Layered models subjected to a motion of two constant masses are depicted in Fig. 1, where:

EI bending stiffness of the beam
m mass per unit length of the beam
N axial force acting on the beam axis, considered positive when inducing compression
k_p stiffness of the rail pads
k_b stiffness of the ballast layer
k_f stiffness of the foundation (Winkler's modulus in a one-layer model)
k_s shear stiffness (Pasternak's modulus in a one-layer model)
c_p viscous damping coefficient of the rail pad
c_b viscous damping coefficient of the ballast layer
c_f viscous damping coefficient of the foundation
m_s half sleeper mass
m_b ballast mass
P_j moving force (j=1,2)
M_j moving mass (j=1,2)
d distance between forces/masses
v velocity

Assumptions and simplifications for the analysis are very similar for different models and can be consulted in previous works, [9-11,15-17].

The equations of motion for the one-layer model are

$$EIw_{,xxxx} + \left(N - k_s\right) w_{,xx} + mw_{,tt} + c_f w_{,t} + k_f w = p\left(x,t\right) \tag{1}$$

for the two-layer model one more equation must be added

$$EIw_{,xxxx} + Nw_{,xx} + mw_{,tt} + c_p\left(w_{,t} - u_{s,t}\right) + k_p\left(w - u_s\right) = p\left(x,t\right) \tag{2}$$

$$m_s u_{,tt} - c_p\left(w_{,t} - u_{s,t}\right) - k_p\left(w - u_s\right) + k_f u_s - k_s u_{s,xx} + c_f u_{s,t} = 0 \tag{3}$$

and for the three-layer model two more equations are needed, but the first equation is the same as before, meaning it is equivalent to Eq. (2)

$$m_s u_{s,tt} - c_p\left(w_{,t} - u_{s,t}\right) - k_p\left(w - u_s\right) + k_b\left(u_s - u_b\right) + c_b\left(u_{s,t} - u_{b,t}\right) = 0 \tag{4}$$

$$m_b u_{b,tt} - k_b\left(u_s - u_b\right) - c_b\left(u_{s,t} - u_{b,t}\right) + k_f u_b + c_f u_{b,t} - k_s u_{b,xx} = 0 \tag{5}$$

The loading terms is defined as

$$\begin{aligned} p\left(x,t\right) = &\left(P_{C_1} + P_{A_1} \mathrm{e}^{\mathrm{i}\left(\omega_{f1} t + \varphi_{f1} + 3\pi/2\right)} - M_1 w_{01,tt}\left(t\right)\right)\delta\left(x - vt\right) \\ &+ \left(P_{C_2} + P_{A_2} \mathrm{e}^{\mathrm{i}\left(\omega_{f2} t + \varphi_{f2} + 3\pi/2\right)} - M_2 w_{02,tt}\left(t\right)\right)\delta\left(x - vt - d\right) \end{aligned} \tag{6}$$

or equivalently as

$$p(x,t) = \left(P_{C_1} + P_{A_1}\sin\left(\omega_{f_1} t + \varphi_{f_1}\right) - M_1 w_{01,tt}(t)\right)\delta(x - vt)$$
$$+\left(P_{C_2} + P_{A_2}\sin\left(\omega_{f_2} t + \varphi_{f_2}\right) - M_2 w_{02,tt}(t)\right)\delta(x - vt - d) \tag{7}$$

where:

P_{Cj} moving constant force (j=1,2)
P_{Aj} amplitude of moving harmonic force (j=1,2)
ω_{f1} external frequency of moving harmonic force (j=1,2)
φ_{fj} phase angle of external moving harmonic force (j=1,2)

and the unknown displacement fields are: $w(x,t)$, $u_s(x,t)$ and $u_b(x,t)$, at the beam, sleeper and ballast level, respectively. x is the spatial coordinate and t is the time. It is of note that in Eqs. (1-5) all parameters are already considered in its distributed form. Eq. (7) is suitable for analysis of an equivalent finite beam, which is important for validation, while Eq. (6) is more adequate for analysis of infinite beams.

3 PROBLEM SOLUTION

Steps to be followed in the solution method are as follows. At first, masses displacements must be expressed using the beam displacement field, and thus, the chain rule has to be applied on the second derivative with respect to the time. Then, fixed coordinates are switched to moving ones; it was chosen to associate the origin of the moving spatial coordinate with the rear mass. After that, dimensionless parameters can be introduced. The aim is to keep as much as possible similarity between the models.

Following the solution method from previous works, the Laplace transform is applied first to catch correctly the initial instant necessary for transient vibrations. Then the Fourier transform is applied, and the problem is solved in the transformed space analytically. The inverse Fourier transform is still fully analytical, but the inverse Laplace one must be helped by the methods of contour integration which require numerical determination of the poles. Solution is then expressed as a sum of residues. Unfortunately, there are some discontinuities in the function to be integrated which has to be eliminated by branch cuts and the integration along them is numerical. It has been demonstrated in previous works that this contribution, named as the *truly transient part* is in most cases insignificant. The other part of the transient solution is dominant, it corresponds to residues of finite number of poles and is described by harmonic function. This part is named as the *unsteady harmonic part*. Then there is obviously the steady-state part of the solution. The steady-state part with the unsteady harmonic part is named as the *harmonic solution*. It thus holds for all models:

$$\widehat{w}(\xi,\tau) = \sum res\left(q, \mathrm{i}\tilde{W}(\xi,q)\mathrm{e}^{\mathrm{i}q\tau}\right) + \widehat{w}_{tr}(\xi,\tau) \tag{8}$$

where $\widehat{w}$ is the dimensionless beam displacement and $\tilde{W}$ is its Fourier image. q is the frequency and ξ and τ are dimensionless moving spatial coordinate and time. The sum of residues defines the harmonic part (steady as well as unsteady) and $\widehat{w}_{tr}$ is the truly transient part.

After expressing the residues, it can be confirmed that there is no interaction term in the steady-state part of the solution, only in the harmonic unsteady part. The interaction terms are then responsible for differences in the onset of instability between one and more moving masses. Crucial function in all cases is a function *K*, which is proportional to the equivalent flexibility of the models.

$$K(\xi,q)=\int_{-\infty}^{\infty}\frac{e^{ip\xi}dp}{D(p,q)} \tag{9}$$

The difference between the models is then expressed by increasing complexity of $D(p,q)$, where *p* is the Fourier counterpart of ξ.

Deep analysis of function *K* is important for at least two reasons. First of all, the characteristic equation for all models, given by

$$\det=\left(\pi-2\eta_M q^2 K(0,q)\right)^2-4\eta_M^2 q^4 K\left(\hat{d},q\right)K\left(-\hat{d},q\right) \tag{10}$$

contains the *K*-function and serves for the determination of poles (so called induced frequencies), identifying the harmonic unsteady part of the solution. Evolution of these frequencies as a function of velocity is named as *frequency lines*. Secondly, *K*-function discontinuity defines the so-called *discontinuity lines* in complex *q*-plane, where the frequency lines are cut.

All residues can be treated semianalytically, in closed forms; integrals are evaluated exactly using the methods of contour integration and analytical expressions for single and double pole residues. The main effort has to be spent on the determination of the induced frequencies.

4 RESULTS AND DISCUSSION

By analysing several sets of input data, [19], possible range of dimensional parameters can be identified. Within these ranges, there are significant differences between the models. These differences are reflected in the critical velocity, but also in the onset of instability for one or two moving masses.

Generally, there are several resonant velocities that induce infinite displacements in the steady-state regime. The lowest value corresponds to the critical velocity. While for the one- and two- layer models the resonant velocities are well-defined by a double real pole and their number is 1 or 3, respectively, in the three-layer model their number depends on parameter values and can be 1, 3 or 5. If one compares the resonant values with the analysis of equivalent long finite beam, [18], it can be concluded that even positions of resonant velocities are weakly marked in parametric analysis and do not have the typical properties (there is not the typical jump to zero displacement at the force position), because they correspond to a position of a local maximum in a mode number and not to a minimum, as usual. The problem is causing the three-layer model, because in the absence of some values analytically well-defined, there are *pseudo-critical velocities* [13], but only some of them lead to excessive displacement increase and only some of them have a typical relation to the onset of instability of one moving mass. In Fig. 2, this is exemplified for the two-layer model and in Fig. 3 for the three-layer one.

In Fig. 2, a case of the two-layer model is shown. As already written, in such a case, all three resonant values are well-defined. The odd values clearly mark the typical jump to zero displacement in the force position. The even value is hardly seen in this graph, even if the step of 0.001 in α was used. α is the velocity ratio, μ_s and κ_p are the mass and stiffness ratios with respect to the sleeper and rail pad, respectively. “max”, “min” and “force” in the legend relate to the maximum and minimum displacement value of the steady-state regime along the beam, and displacement value at the force position, respectively.

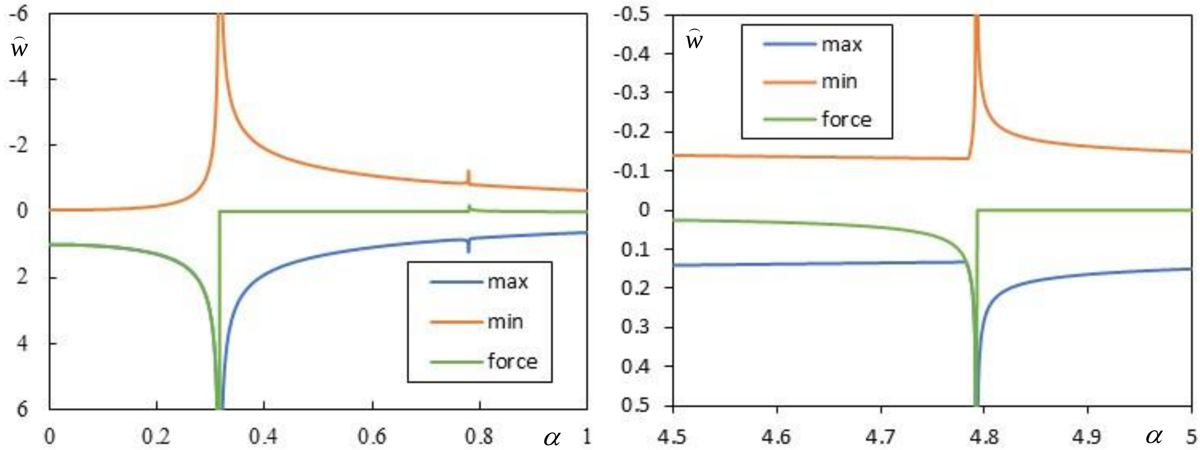


Figure 2: Critical velocity and the other two resonant velocities in two-layer model and $\mu_s = 9$ and $\kappa_p = 500$.

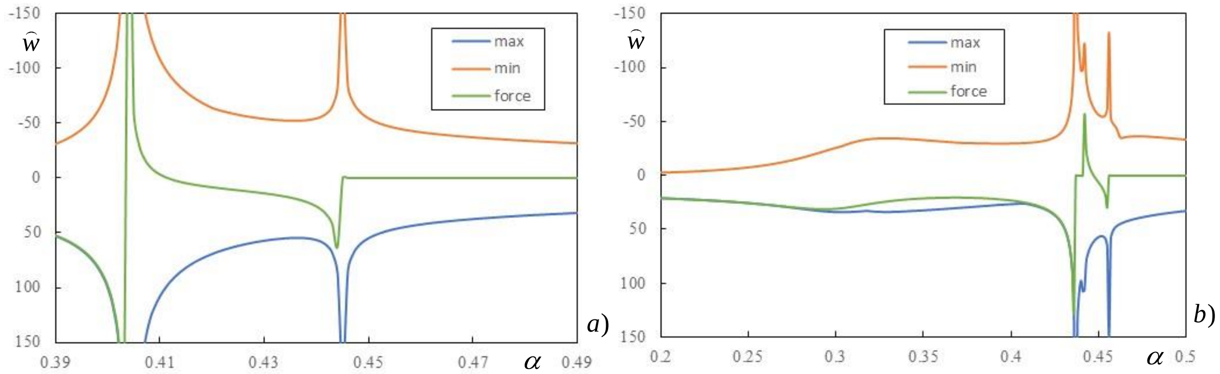


Figure 3: Pseudo-critical and other resonant values in three-layer model for: a) $\mu_s = 6$, $\mu_b = 5$, $\kappa_p = 0.03$ and $\kappa_b = 3$; b) $\mu_s = 3$, $\mu_b = 10$, $\kappa_p = 0.03$ and $\kappa_p = 0.1$.

In Fig.3, two cases with pseudocritical velocity in the three-layer model are shown. The situation on the left has the pseudo-critical value well marked, nevertheless, displacement values do not reach infinity. Also, the typical jump to zero value is missing. The α -position can only be identified by a parametric analysis. Nevertheless, this value is so strong that for one moving mass works in the same way as the critical velocity. On the other hand, the situation on the right has the pseudo-critical value only with a small displacement increase and as such, does not have the same properties as the previous one. Next values still visible in the graph have the typical properties, meaning the first and the third positions exhibit the typical jump to zero value. Both cases in Fig. 3 have only three resonant values well-defined by double real pole. Regarding the additional parameters used, μ_b and κ_b are the mass and stiffness ratios with respect to the ballast, respectively.

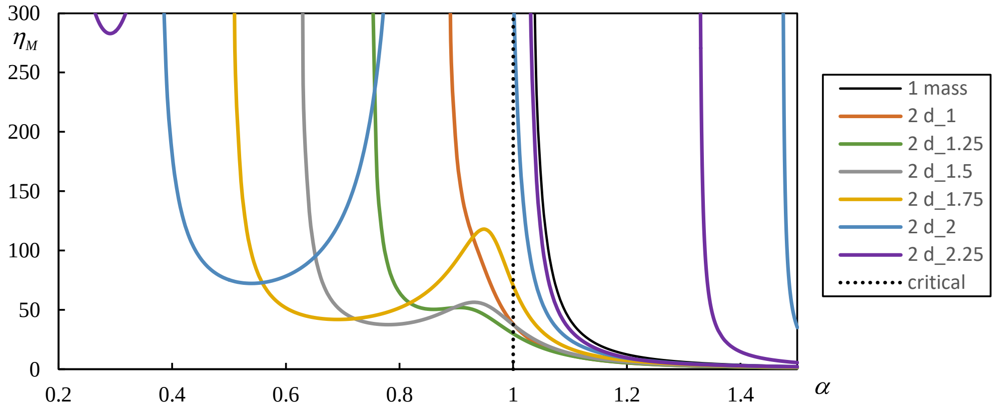
Regarding the onset of instability, there are also significant differences, not only between the models but also between the cases with one or more moving masses. For the one-layer model, instability of one moving mass has regular behaviour and occurs always in the supercritical velocity range when damping is present and at the critical velocity in case of no damping. Two moving proximate masses already introduce severe alterations, because in damped case the dynamic interaction can shift the onset of instability deeply into the subcritical velocity range. The other models introduce further irregularities, even for one moving mass.

The onset of instability can be tracked as a function of the moving mass ratio, η_M . Some cases of one- and two-layer models are selected, and their specification is given in Table 1. For the sake of simplicity, masses are considered of the same value. η_p and η_f correspond to the representative damping ratio of the rail pad and of the foundation, respectively. In the legend of the following graphs, $\hat{d}$ stands for the dimensionless distance between two moving masses.

Table 1: Cases definition, Case 1 and 2 are for one-layer model and 3 and 4 for two-layer one.

	η_p	η_f	α_{cr}	κ_p	μ_s	$\hat{d}$
Case 1	---	0.05	1	---	---	1:0.25:2.25
Case 2	---	0.3	1	---	---	1:0.25:2.25
Case 3	0.05	0.05	0.707	300	1	1:0.25:2.25
Case 4	0.05	0.3	0.707	300	1	1:0.25:2.25

In Fig. 4, the onset lines for the cases from Table 1 are plotted. It can be seen that there are several onset lines covering the whole range of velocities, contrary to one moving mass case, where there is only one onset line in supercritical range of velocities. There are some general tendencies, but exact determination is essential to avoid instability in subcritical velocity range.



a)

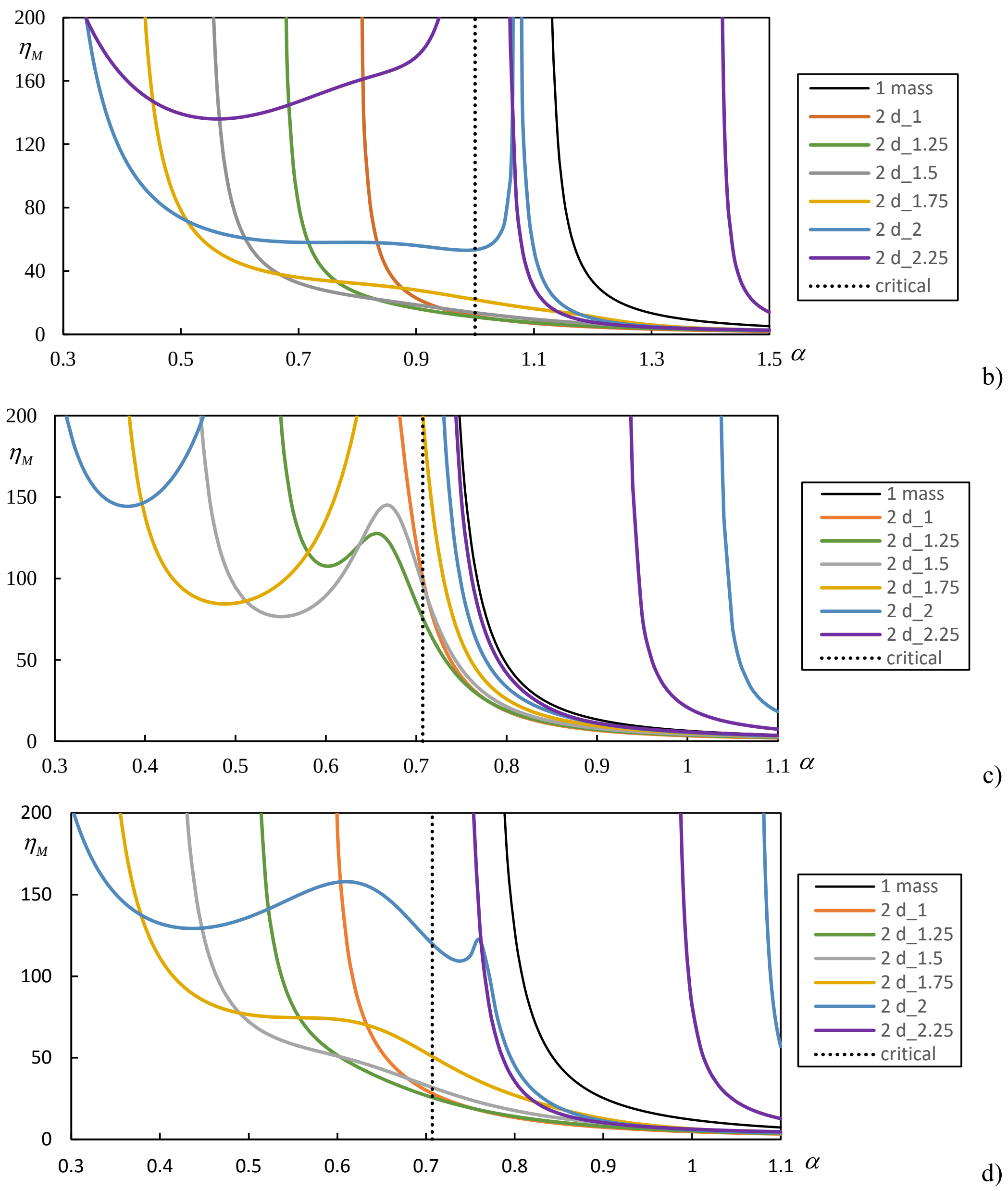


Figure 4: Onset lines: a) Case 1; b) Case 2; c) Case 3; d) Case 4.

Fig. 4 is illustrating many cases with reasonable η_M and instability at unexpectedly low velocity, which must be avoided. One would expect more interaction for lower distances, but in fact, until the distance 1.75, the situation is quite regular. As regards the analyzed cases, most irregularities happen for distances of 2 and 2.25.

Finally, to check that none of the branches of the onset lines are missing, it is possible to identify the asymptotic tendency of these lines. For the cases selected, asymptotic values within the range of analyzed velocities are summarized in Table 2. Some of the branches are not visualized in Fig. 4, because the range was limited by η_M =200 and 300, respectively, which is already a very high, mostly academic value.

Table 2: Asymptotic velocities of the onset lines.

	max α	$\hat{d}$	$\alpha_{1,as}$	$\alpha_{2,as}$	$\alpha_{3,as}$	$\alpha_{4,as}$
Case 1	1.5	1	0.876	---	---	---
		1.25	0.743	---	---	---
		1.5	0.618	---	---	---
		1.75	0.497	---	---	---
		2	0.365	0.834	0.978	1.472
		2.25	0.186	0.414	1.016	1.326
Case 2	1.5	1	0.796	---	---	---
		1.25	0.664	---	---	---
		1.5	0.535	---	---	---
		1.75	0.405	---	---	---
		2	0.263	1.065	1.068	---
		2.25	0.093	0.998	1.050	1.414
Case 3	1.1	1	0.622	---	---	---
		1.25	0.528	---	---	---
		1.5	0.440	---	---	---
		1.75	0.354	---	---	---
		2	0.261	0.578	0.704	1.031
		2.25	0.136	0.287	0.716	0.929
Case 4	1.1	1	0.583	---	---	---
		1.25	0.493	---	---	---
		1.5	0.404	---	---	---
		1.75	0.313	---	---	---
		2	0.213	1.073	---	---
		2.25	0.084	0.717	0.739	0.978

For a three-layer model, one would expect something similar, but this is not true. When there are pseudo-critical velocities, then they can mark the limit for the onset of instability of one moving mass or not, and when this is not true, then the limit is attributed to the critical velocity. It is not clear how these two cases should be distinguished a priori, without performing the parametric analyses.

5 CONCLUSIONS

In this paper, a detailed analysis of layered models of the railway track was presented. The analysis addressed several issues, but the main emphasis was on the critical velocity and the onset of instability of one mass, or two moving proximate masses. It has been shown that external damping, coupled with the dynamic interaction between the moving masses can shift the onset of instability into a subcritical range of velocities, i.e., instability can occur at lower velocity than expected, which is a real danger and should be taken into account when designing the railway. It should be noted that homogeneous initial conditions have been implemented, but conclusions regarding instability remain unchanged, because as shown in [10] inhomogeneous initial conditions do not change the induced frequencies.

The onset of instability has been explained in detail and several illustrative examples have been shown. It can be concluded that for one- and two-layer model, one moving mass exhibits relatively regular behaviour, because the onset of instability always occurs in the supercritical range of velocities and external damping helps to shift such an onset to higher velocity values. This is, however, not the case of two moving proximate masses. Superposition of results is not possible in such a case, and if used, can lead to completely wrong results and conclusions. For several cases, the so-called onset lines were derived, showing the irregularity of the dynamic interaction. It has been demonstrated that instability can occur at very low velocities, therefore damping actually worsens the situation, not improves it, as is generally acknowledged. External damping causes high irregularity in the onset lines, which may have several branches. There are cases indicating closed intervals of velocities where instability occurs. Thus, instability may occur at a certain velocity, but for higher velocities, stability is restored and after that lost again. In the undamped case, the onset of instability always matches the critical velocity for one or more moving masses.

For a three-layer model none of these conclusions are valid. This is mainly due to the fact that the resonant velocities are well-defined with all values only in some cases. There are cases with only one or three values, where the missing values are compensated by pseudo-critical velocities. These values can play the role of the critical velocity or not and such distinction is difficult to make a priori.

ACKNOWLEDGEMENTS

This work was supported by the Portuguese Foundation for Science and Technology (FCT), through IDMEC, under LAETA, project UIDB/50022/2020.

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