A mesh-free finite point method for advective-diffusive transport and fluid flow problems

E. Oñate, S. Idelsohn

Abstract The finite point method (FPM) is a gridless numerical procedure based on the combination of weighted least square interpolations on a cloud of points with point collocation for evaluating the approximation integrals. In the paper, details of a procedure for stabilizing the numerical solution for advective-diffusive transport and fluid flow problems using the FPM are given. The method is based on a consistent introduction of the stabilizing terms in the governing differential equations. One example showing the applicability of the FPM is given.

1
Introduction

A number of authors have recently investigated the possibility of deriving numerical methods where meshes are unnecessary. The different methods can be classified as Finite Difference (FD) schemes (Forsythe and Wasow 1960; Perrone and Kao 1975; Liszka and Orkisz 1980; Liszka 1984), Smooth Particle Hydrodynamic (SPH) methods (Moraghan 1988), Diffuse Element (DE) method (Nayroles et al. 1992), Element Free Galerkin (EFG) method (Belytschko et al. 1994 1995; Taylor et al. 1995; Belytschko 1998), Reproducing Kernel Particle (RPK) method (Liu et al. 1995a,b; Liu et al. 1996a,b; Liu et al. 1997), Partition of Unity (PU) based methods (Duarte and Oden 1995; Baduška and Melenk 1995) and Finite Point (FP) method (Oñate et al. 1995a,b; Oñate et al. 1996a,b, Fischer et al. 1995; Fischer 1996). The basic ingredient of all these methods is the existance of an arbitrary collection of points. The points are grouped together in "clouds" where a local interpolation for the problems variables is written. DE, EFG, RPK and PU methods typically use a background grid for computing numerically the integrals

resulting from weighted residual forms of the governing differential equations. On the other hand, FD, SPH and FP methods are truly gridless or mesh-free procedures as the set of algebraic equations is derived via simpler point collocation procedures. A detailed comparison of all these methods can be found in (Oñate et al. 1995).

The work of the authors has been recently focussed in the development of the so called Finite Point Method (FPM). The simpler FPM is based on combining a weighted least square interpolation within each cloud where a weighting function is kept "fixed", with a point collocation scheme to derive the approximation equations. The efficiency of the FPM for solving a range of problems has been reported by Oñate et al. (1995a,b, 1996a,b), Fischer et al. (1995) and Fischer (1996).

In a recent paper (Oñate et al. 1996b) the authors have shown that the successful use of the FPM for solving advective-diffusive transport and fluid flow problems relies on the adequate stabilization of the numerical solution. Here, the standard SUPG, GLS, Taylor-Galerkin, etc. procedures used in FE and FV methods are just not applicable, as the integral form of the equations is not available. In Oñate et al. (1996b) the authors proposed to use an "ad hoc" upwind stabilizing scheme similar to that used in finite difference methods. A new stabilization technique proposed by Oñate (1996, 1998) based on the consistent modification of the governing differential equations originally was successfully applied to a range of advective-diffusive and fluid flow problems using the FPM (Oñate et al. 1996b).

The layout of the paper is as follows. In next section the basis of the FPM are briefly described. Then, the principles of the new stabilization technique for advective-diffusive problems are presented and the corresponding FPM equations are derived for both steady state and transient cases. The concepts are then extended to the solution of compressible and incompressible fluid mechanics equations via the FPM. The paper ends up with one example showing the efficiency and applicability of the stabilized FPM for solving fluid flow problems.

E. Oñate
International Center for Numerical Methods in Engineering (CIMNE) Gran Capitán, s/n, E-08034 Barcelona, Spain

S. Idelsohn
Universidad Nacional del Litoral, Santa Fe, Argentina Visiting Professor, CIMNE

Correspondence to: E. Oñate

2
Basic concept of the FPM

Let us assume a scalar problem governed by a differential equation

$$A(u) = b \qquad \text{in} \qquad \Omega \qquad (1)$$

with Neumann boundary conditions

$$B(u) = t \qquad \text{in} \qquad \Gamma_t \qquad (2)$$

and Dirichlet (essential) boundary conditions

$$u - u_p = 0 \quad \text{in} \quad \Gamma_u \tag{3}$$

to be satisfied in a domain Ω with boundary $\Gamma = \Gamma_t \cup \Gamma_u$. In the above A and B are appropriate differential operators, u is the problem unknown and b and t represent external forces or sources acting over the domain Ω and along the boundary Γ_t, respectively. Finally u_p is the prescribed value of u over the boundary Γ_u.

The most general procedure of solving numerically the above system of differential equations is the weighted residual method in which the unknown function u is approximated by some trial approximation $\hat{u}$ and Eqs. (1) and (2) are replaced by (Zienkiewicz and Taylor 1991)

$$\int_{\Omega} W_i[A(\hat{u}) - b]\mathrm{d}\Omega + \int_{\Gamma_t} \bar{W}_i[B(\hat{u}) - t]\mathrm{d}\Gamma$$
$$+ \int_{\Gamma_u} \bar{\bar{W}}_i[\hat{u} - u_p]\mathrm{d}\Gamma = 0 \; , \tag{4}$$

With the weighting functions W_i, $\bar{W}_i$ and $\bar{\bar{W}}_i$ defined in different ways. FE, FV and FD methods can be considered as particular cases of (4) (Zienkiewicz and Oñate 1991; Oñate et al. 1994; Idelsohn and Oñate 1994) and *indeed so can* all the meshless approximation procedures.

A simple point collocation procedure implies making $W_i = \bar{W}_i = \bar{\bar{W}}_i = \delta_i$, where δ_i is the Dirac delta. This gives the set of equations

$$[A(\hat{u})]_i - b_i = 0 \quad \text{in} \quad \Omega,$$
$$[B(\hat{u})]_i - t_i = 0 \quad \text{on} \quad \Gamma_t, \tag{5}$$
$$\hat{u}_i - u_p = 0 \quad \text{on} \quad \Gamma_u \; .$$

The system of Eqs. (5) can be always written in the form

$$\mathbf{Ka} = \mathbf{f} \; , \tag{6}$$

where $\mathbf{K}$ is the coefficient matrix which symmetry is not generally achieved in point collocation procedures, vector $\mathbf{a}$ contains the problem unknowns and $\mathbf{f}$ is a vector containing contributions from the external sources and forces and the prescribed values u_p.

In order to keep a local character of the problem (leading to a banded matrix), function u must be approximated by a combination of locally defined functions as

$$u(x) \cong \hat{u}(x) = \sum_{i=1}^{n_p} N_i(x)u_i^h = \mathbf{N}^{\mathrm{T}}(x)\mathbf{u}^h \; , \tag{7}$$

with n_p being the total number of points in the domain and the interpolation functions $N_i(x)$ satisfy

$$N_i(x) \neq 0 \quad \text{if} \quad x \in \Omega_i \; ,$$
$$N_i(x) = 0 \quad \text{if} \quad x \notin \Omega_i \; . \tag{8}$$

Here Ω_i is a subdomain of Ω containing n points, $n \ll n_p$. In (7) u_i^h is the *approximate value of u* at point i such that $u(x_i) \simeq u_i^h$.

The way to define the shape functions N_i is classical in the FEM (Zienkiewicz and Taylor 1991). The derivation of N_i in the FPM will be described next. More details can be found in Oñate et al. (1996a,b).

3
Derivation of interpolation function via fixed weighted least square approximations

Let Ω_i be the interpolation domain (cloud) of a function $u(x)$ and let s_j with $j = 1, 2, \ldots, n$ be a collection of n points with coordinates $x_j \in \Omega_i$. The unknown function u may be approximated within Ω_i by

$$u(x) \cong \hat{u}(x) = \sum_{l=1}^{m} p_l(x)\alpha_l = \mathbf{p}(x)^{\mathrm{T}}\boldsymbol{\alpha} \; , \tag{9}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^{\mathrm{T}}$ and vector $\mathbf{p}(x)$ contains typically monomials, hereafter termed "base interpolating functions", in the space coordinates ensuring that the basis is complete. For a 2D problem we can specify

$$\mathbf{p} = [1, x, y]^{\mathrm{T}} \quad \text{for} \quad m = 3 \tag{10}$$

and

$$\mathbf{p} = \left[1, x, y, x^2, xy, y^2\right]^{\mathrm{T}} \quad \text{for} \quad m = 6 \quad \text{etc.} \tag{11}$$

Function $u(x)$ can now be sampled at the n points belonging to Ω_i giving

$$\mathbf{u}^h = \begin{Bmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_n^h \end{Bmatrix} \cong \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_1^{\mathrm{T}} \\ \mathbf{p}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{p}_n^{\mathrm{T}} \end{Bmatrix} = \boldsymbol{\alpha} = \mathbf{C}\boldsymbol{\alpha} \; , \tag{12}$$

where $u_j^h = u(x_j)$ are the unknown but sought for values of function u at point j, $\hat{u}_j = \hat{u}(x_j)$ are the approximate values, and $\mathbf{p}_j = \mathbf{p}(x_j)$.

In the FE approximation the number of points is chosen so that $m = n$. In this case $\mathbf{C}$ is a square matrix. The procedure leads to the standard shape functions in the FEM (Zienkiewicz and Taylor 1991).

If $n > m$, $\mathbf{C}$ is no longer a square matrix and the approximation can not fit all the u_j^h values. This problem can be simply overcome by determining the $\hat{u}$ values by minimizing the sum of the square distances of the error at each point weighted with a function $\varphi(x)$ as

$$J = \sum_{j=1}^{n} \varphi(x_j)(u_j^h - \hat{u}(x_j))^2$$
$$= \sum_{j=1}^{n} \varphi(x_j)(u_j^h - \mathbf{p}_j^{\mathrm{T}}\boldsymbol{\alpha})^2 \tag{13}$$

with respect to the $\boldsymbol{\alpha}$ parameters. Note that for $\varphi(x) = 1$ the standard least square (LSQ) method is reproduced.

Function $\varphi(x)$ is usually built in such a way that it takes a unit value in the vicinity of the point i typically called "star node" (Liszka 1984) where the function (or its derivatives) are to be computed and vanishes outside a region Ω_i surrounding the point. The region Ω_i can be used to define the number of sampling points n in the interpolation region. A typical choice for $\varphi(x)$ is the normalized Gaussian function. Of course $n \geq m$ is always required in the sampling region and if equality occurs no effect of weighting is present and the interpolation is the same as in the LSQ scheme (Batina 1993).

Standard minimization of Eq. (13) with respect to $\boldsymbol{\alpha}$ gives

$$\boldsymbol{\alpha} = \bar{\mathbf{C}}^{-1}\mathbf{u}^h , \qquad \bar{\mathbf{C}}^{-1} = \mathbf{A}^{-1}\mathbf{B} . \tag{14}$$

The final approximation is obtained substituting the value of $\boldsymbol{\alpha}$ in Eq. (14) into (9) giving

$$\mathbf{A} = \sum_{j=1}^{n} \varphi(x_j)\mathbf{p}(x_j)\mathbf{p}^{\mathrm{T}}(x_j) , \tag{15}$$

$$\mathbf{B} = [\varphi(x_1)\mathbf{p}(x_1), \varphi(x_2)\mathbf{p}(x_2), \ldots, \varphi(x_n)\mathbf{p}(x_n)] . \tag{16}$$

The final approximation is obtained by substituting Eqs. (14) into (9) giving

$$\hat{u}(x) = \mathbf{p}^{\mathrm{T}}\bar{\mathbf{C}}^{-1}\mathbf{u}^h = \mathbf{N}^{\mathrm{T}}\mathbf{u}^h = \sum_{j=1}^{n} N_j^i u_j^h , \tag{17}$$

where the "shape functions" are

$$N_j^i(x) = \sum_{l=1}^{m} p_l(x)\bar{C}_{lj}^{-1} = \mathbf{p}^{\mathrm{T}}(x)\bar{\mathbf{C}}^{-1} . \tag{18}$$

It must be noted that accordingly to the least square character of the approximation

$$u(x_j) \simeq \hat{u}(x_j) \neq u_j^h , \tag{19}$$

i.e. the local values of the approximating function do not fit the nodal unknown values (Fig. 1). Indeed $\hat{u}$ is the true approximation for which we shall seek the satisfaction of the differential equation and boundary conditions and u_j^h are simply the unknown parameters sought.

However if $n = m$ and $\hat{u}(x_j) = u_j^h$ a FEM type approximation is recovered (Zienkiewicz and Taylor 1991).

3.1
Fixed least square approximation (FLS)
The weighted least square approximation described above depends on a great extent on the shape and the way to apply the weighting function. The simplest way is to define a fixed function $\varphi(x)$ for each of the Ω_i interpolation domains (see Fig. 1).

Let $\varphi_i(x)$ be a weighting functions satisfying

$$\begin{aligned} &\varphi_i(x_i) = 1; \\ &\varphi_i(x) \neq 0, \ x \in \Omega_i; \\ &\varphi_i(x) = 0 \ x \notin \Omega_i . \end{aligned} \tag{20}$$
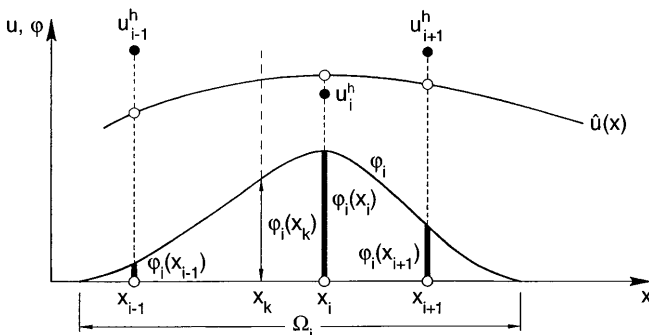

Fig. 1. Fixed weighing least square procedure

Then the minimization square distance becomes

$$J = \sum_{j=1}^{n} \varphi_i(x_j)(u_j^h - \hat{u}(x_j))^2 \qquad \text{minimum} . \tag{21}$$

The expression of matrices $\mathbf{A}$ and $\mathbf{B}$ are now

$$\mathbf{A} = \sum_{j=1}^{n} \varphi_i(x_j)\mathbf{p}(x_j)\mathbf{p}^{\mathrm{T}}(x_j) , \tag{22}$$

$$\mathbf{B} = [\varphi_i(x_1)\mathbf{p}(x_1), \varphi_i(x_2)\mathbf{p}(x_2), \ldots, \varphi_i(x_n)\mathbf{p}(x_n)] . \tag{23}$$

Note that according to (9), the approximate function $\hat{u}(x)$ is defined in each interpolation domain Ω_i. In fact, different interpolation domains can yield different shape functions N_j^i. As a consequence a point belonging to two or more overlapping interpolation domains has different values of the shape functions which means that $N_j^i \neq N_j^k$. The interpolation is now multivalued within Ω_i and, therefore, for any useful approximation a decision must be taken limiting the choice to a single value (Oñate et al. 1996a,b). Indeed, the approximate function $\hat{u}(x)$ will be typically used to provide the values of the unknown function $u(x)$ and its derivatives in only specific regions within each interpolation domain. For instance by using point collocation we may limit the validity of the interpolation to a single point x_i. It is precisely in this context where we have found this meshless method to be more useful for practical purposes.

4
Derivation of the discretized equations
The selection of different weighting functions in the general weighted residual form of Eq. (3) yields different sets of discretized equations. In order to preserve the mesh-free character of the method, the weighting domain must be defined independently of any mesh. All approximation methods of integral type (i.e. Galerkin, area collocation, etc.) necessitate the introduction of complex procedures for integration, i.e. background grid, etc. (Nayroles et al. 1992; Belytschko et al. 1994; Lu et al. 1994; Liu et al. 1995a,b; Liu and Chen 1995; Liu et al. 1996). Some of these procedures are reviewed in Oñate et al. (1995a, 1996a,b). In this paper we shall limit the choice to *point collocation* methods where we feel the advantages of meshless procedures are best realized.

Point collocation implies making $W_i = \bar{W}_i = \bar{\bar{W}}_i = \delta_i$ in Eq. (3) where δ_i is the Dirac delta. This gives the set of equations

$$[A(\hat{u})]_i - b_i = 0 \quad \text{in } \Omega , \tag{24a}$$

$$[B(\hat{u})]_i - t_i = 0 \quad \text{in } \Gamma_t , \tag{24b}$$

$$\hat{u}_i - u_p = 0 \quad \text{in } \Gamma_u . \tag{24c}$$

Any of the previous shape functions may be used now to approximate $\hat{u}$ leading in all cases to the system of equations

$$\mathbf{Ku}^h = \mathbf{f} , \tag{25}$$

with $K_{ij} = [A(N_j)]_i + B(N_j)]_i$ and where the symmetry of the "coefficient" matrix $\mathbf{K}$ is not generally achieved. Vector $\mathbf{u}^h$ contains the problem unknowns, u_i^h, and $\mathbf{f}$ is a vector

containing the contributions from the force terms b and t and the prescribed values u_p.

5
Application of the FPM to advective-diffusive transport problems

Let us consider next the application of the FPM to the standard advection-diffusion equation given by

$$A(\phi) = c\frac{\partial \phi}{\partial t} + \mathbf{u}^{\mathrm{T}}\mathbf{\nabla}\phi - \mathbf{\nabla}^{\mathrm{T}}(k\mathbf{\nabla}\phi) - Q$$
$$= 0 \text{ in } \Omega \ , \tag{26a}$$

$$B(\phi) = \mathbf{n}^{\mathrm{T}}k\mathbf{\nabla}\phi + \bar{q}_n = 0 \text{ in } \Gamma_q \ , \tag{26b}$$

$$\phi - \phi_p = 0 \ in \ \Gamma_\phi \ , \tag{26c}$$

with the initial condition $\phi = \phi_0(x)$ for $t = t_0$.

In (26) $\mathbf{\nabla}$ is the gradient operator, c and k are known physical parameters, $\mathbf{u}$ is the known velocity vector, ϕ the unknown field and Q a source term. $\bar{q}_n$ and ϕ_p are known values of the flux and the known function at the boundaries Γ_q and Γ_ϕ, respectively.

It is widely accepted that the accurate numerical solution of advective-diffusive transport and fluid flow problems using either FD, FV or FE methods requires invariably the introduction of some kind of stabilization procedure. Among the techniques typically used to stabilize FD, FE and FV methods we can list upwind finite difference derivatives, anisotropic balancing diffusion, Petrov-Galerkin weighting functions and characteristic time integration (Zienkiewicz and Taylor 1991). The role of this apparently non-physical stabilization is typically two fold: a) to counter balance the underdiffusive character of most spacial integration schemes (such as the central difference scheme in FD or FV methods, or the equivalent Galerkin form in FEM, etc.), and b) to stabilize the numerical solution in the vecinity of high gradients (such as wave shocks etc.) by smoothing out local oscillations.

Although extensive work has been devoted to the development of stabilization procedures in FD, FV and FE methods, it is acknowledged that most of the existing procedures are based on somewhat heuristic arguments. Indeed it is widely accepted that the origins and precise definition of the stabilization parameters used in numerical computations are in most cases unsolved mysteries, whose solution has motivated much research in recent years mainly within the FEM context (Zienkiewicz and Taylor 1991).

In a quite recent work Oñate (1996, 1998) has shown that the stabilization terms emerge naturally in the governing differential equations of the problem, once the concept of flow balance (or equilibrium) over a "finite" domain is accepted. This allows to reinterpret the stabilization terms as an intrinsic and natural contribution to the original differential equations, instead of an "extrinsic" correction term introduced at the discretization level, as it is usually understood by most FD, FV and FE practitioners. Moreover, by exploiting this natural stabilization concept, the standard forms of the balancing terms appearing in many well known stabilized schemes for ad-

vective-diffusive and fluid flow problems such as direct artificial diffusion, upwinding and Petrov-Galerkin weighting, Streamline Upwind Petrov-Galerkin (SUPG), Subgrid Scale (SS), Galerkin Least Squares (GLS), Lax-Wendroff, Characteristic Galerkin, Laplacian pressure operator etc. are easily recognized and can be reinterpreted in a more physical manner. An additional important benefit from the approach proposed is that it provides an expression for computing numerically the stabilization parameter (Oñate 1996, 1998, Oñate et al 1997).

5.1
Stabilized form of the governing equation for advective- diffusive transport

The key to the successful development of a stabilized numerical FP scheme for analysis of advective type problems is the derivation of an "intrinsic" stabilized form of the differential equations governing the balance of fluxes in a finite domain.

For this purpose let us consider a finite rectangular domain of dimensions h_x and h_y in directions x and y, respectively. Both the advective and diffusive fluxes are assumed to vary linearly along the four sides (Fig. 2). The flux balance equation will be obtained using the following Taylor expansions:

diffusive term: third order expansion
advective term: third order expansion
source term: second order expansion

The balance of flux across the four sides of the rectangular domain of Fig. 2 gives after some algebra (Oñate 1996, 1998)

$$r - \tfrac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{\nabla}r = 0 \ , \tag{27}$$

$$r = -c\mathbf{\nabla}^{\mathrm{T}}\mathbf{f} + \mathbf{\nabla}^{\mathrm{T}}(\mathbf{D}\mathbf{\nabla}\phi) + Q \ , \tag{28}$$

where

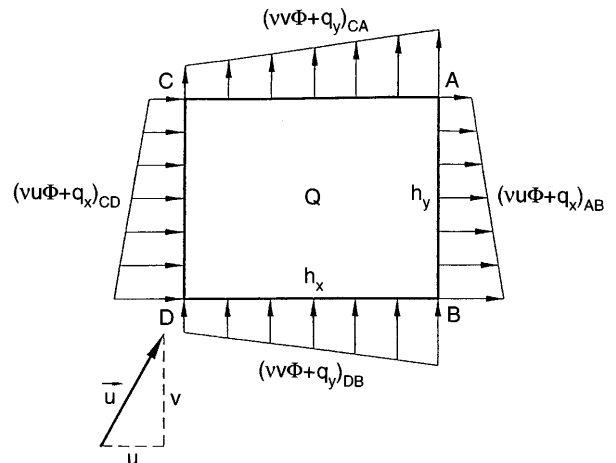$$\mathbf{f} = [u\phi, v\phi]^{\mathrm{T}} \ , \tag{29}$$

Fig. 2. Balance domain for 2D advection-diffusion problem. Advective and diffusive fluxes are assumed to vary linearly along the sides

$$\mathbf{D} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \qquad (30)$$

is the conductivity matrix

$$\mathbf{V} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]^{\mathrm{T}} \quad \text{and} \quad \mathbf{h} = [h_x, h_y]^{\mathrm{T}} \ . \qquad (31)$$

Vector $\mathbf{h}$ is called the "characteristic length vector" and $h = |\mathbf{h}|$ is known as "characteristic length". Usually $h = \alpha l^{(e)}$, where α is the sought stabilization parameter and $l^{(e)}$ is an adequate element dimension (i.e. the element length in 1D problems) (Oñate 1996).

The boundary conditions are written as

$$\phi - \bar{\phi} = 0 \qquad \text{on } \Gamma_\phi \ , \qquad (32)$$

where Γ_ϕ is the Dirichlet boundary, where the variable is prescribed, and

$$\mathbf{n}^{\mathrm{T}}\mathbf{D}\mathbf{V}\phi + \bar{q}_n - \tfrac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{n}r = 0 \qquad \text{on } \Gamma_q \ , \qquad (33)$$

where $\bar{q}_n$ is the prescribed diffusive flux across the Neumann boundary Γ_q with $\Gamma = \Gamma_\phi \cup \Gamma_q$ and $\mathbf{n} = [n_x, n_y]^{\mathrm{T}}$ is the normal vector.

Equation (33) has been obtained by balance of fluxes in the boundary domain of Fig. 3 using higher order approximation for the advective and diffusive fluxes. The full derivation can be found in Oñate (1996).

The stabilized transient form of Eq. (26a) is obtained using the same previous arguments as

$$c\frac{\partial \phi}{\partial t} - r - \frac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{V}\left(c\frac{\partial \phi}{\partial t} - r\right) = 0 \ , \qquad (34)$$

with the initial condition $\phi = \phi_0$ at time $t = t_0$.

Note that the standard differential equations (26) are simply obtained neglecting the stabilizing terms in Eqs. (27)–(34) (i.e making $\mathbf{h} = 0$).

5.2
The concept of intrinsic time

It can be further assumed that $h_x = h\cos\alpha$ and $h_y = h\sin\alpha$, where α is the angle which the velocity vector
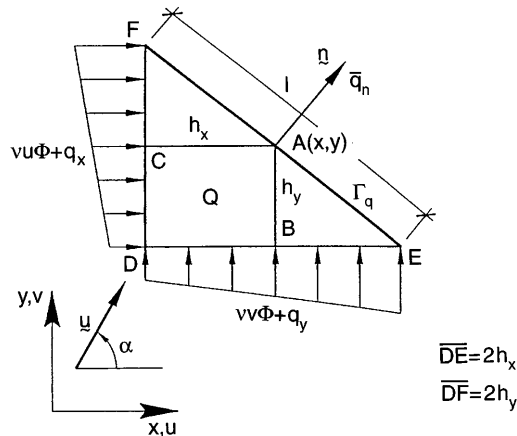

Fig. 3. Balance of fluxes in a two-dimensional domain next to a Neumann boundary

$\mathbf{u} = [u, v]^{\mathrm{T}}$ forms with the global x axis. The stabilized governing equations (27) and (34) can now be written as:

Steady state

$$r - \frac{h}{2|\mathbf{u}|}\mathbf{u}^{\mathrm{T}}\mathbf{V}r = 0 \ ; \qquad (35a)$$

Transient

$$c\frac{\partial \phi}{\partial t} - r - \frac{h}{2|\mathbf{u}|}\mathbf{u}^{\mathrm{T}}\mathbf{V}\left(c\frac{\partial \phi}{\partial t} - r\right) = 0 \ . \qquad (35b)$$

In the derivation of above expressions, use of the identities $\sin\alpha = u/|\mathbf{u}|$ and $\cos\alpha = v/|\mathbf{u}|$ has been made.

The *intrinsic time* in 2D advective-diffusive problems is defined now as (Hughes and Mallet 1986)

$$\tau = \frac{h}{2|\mathbf{u}|} \ . \qquad (36)$$

With this definition Eqs. (35a,b) can be rewritten as:

Steady state

$$r - \tau\mathbf{u}^{\mathrm{T}}\mathbf{V}r = 0 \qquad \text{on } \Omega \ ; \qquad (37a)$$

Transient

$$c\frac{\partial \phi}{\partial t} - r - \tau\mathbf{u}^{\mathrm{T}}\mathbf{V}\left(c\frac{\partial \phi}{\partial t} - r\right) = 0 \qquad \text{on } \Omega \ . \qquad (37b)$$

The stabilized Neumann boundary condition (33) can also be expressed in terms of the intrinsic time as

$$\mathbf{n}^{\mathrm{T}}\mathbf{D}\mathbf{V}\phi + \bar{q}_n - \tau\mathbf{u}^{\mathrm{T}}\mathbf{n}r = 0 \qquad \text{on } \Gamma_q \ . \qquad (38)$$

5.3
Transient solution using the FPM

Let us consider the stabilized transient equation (37b) neglecting for simplicity the effect of the time derivative in the stabilization term. A simple explicit integration from t^n to $t^{n+1} = t^n + \Delta t$ gives

$$\Delta\phi = \frac{\Delta t}{c}\left[r - \tau\mathbf{u}^{\mathrm{T}}\mathbf{V}r\right]^n \ . \qquad (39)$$

Equation (39) can be found to be *identical* to that obtained using a *characteristic approximation* as described in Zienkiewicz and Codina (1995) and Zienkiewicz et al. (1995), if the intrinsic time is taken to coincide with $\Delta t/2$. Indeed Eq. (39) provides a more meaningful explanation of the stabilization term within the brackets.

Equation (39) can now be discretized in space using the FPM by substituting the interpolation (17) for the unknown ϕ. Choosing a point collocation procedure leads to an explicit system of equations from which the values of $\Delta\phi$ can be obtained and the solution advanced in time in the usual manner. Indeed the value of the time step increment Δt in (39) must be adequately chosen to ensure stability of the time integration scheme. Local time stepping can be used if only the steady state solution is needed.

Note that for the full success of the numerical scheme the stabilized form of the Neuman boundary conditions (38) must be used (Oñate et al. 1996b).

The expression of the stabilized term arising in Eq. (39) can be simplified if a linear or quadratic approximation is used as the cubic derivatives of ϕ in the diffusive term are

zero. A further simplification arises if the source term is constant and the stabilizing term is then simply given by

$$\tau \mathbf{u}^{\mathrm{T}}(\mathbf{u}^{\mathrm{T}} \nabla \phi) \ . \tag{40}$$

Note that the balancing term may be interpreted now as an additional diffusion as typically occurs in the FE literature (Zienkiewicz and Taylor 1981).

The value of the optimal intrinsic time τ can be found by using the same arguments of standard upwinding and Petrov-Galerkin FE procedures, i.e. by searching exact nodal values for the simple 1D problem with $Q = 0$. Application of this concept to the FPM gives (Oñate et al. 1995a,b, 1996a,b):

$$\tau = \begin{cases} \frac{\alpha h}{4|\mathbf{u}|} & \text{for linear interpolations } (m = 2), \\ \frac{\alpha h}{8|\mathbf{u}|} & \text{for quadratic interpolations } (m = 3) \ , \end{cases} \tag{41}$$

where

$$\alpha = \coth |\mathrm{Pe}| - \frac{1}{|\mathrm{Pe}|} \ , \tag{42}$$

with the Peclet number defined as $\mathrm{Pe} = |\mathbf{u}|h/2k$.

In Eq. (42) h is half the distance measured along the streamline between the end points for a particular cloud as shown in Fig. 4.

An alternative procedure for computing the stabilization parameter based on the ideas presented in Oñate (1996, 1998) and Oñate, Garcia, Idelsohn (1997) can also be used.

6
Extension of the FPM to fluid flow problems

6.1
Compressible fluid flow
The approach proposed in previous section will be generalized now to solve two dimensional compressible fluid mechanics problems governed by the Navier-Stokes equations written in conservation form as (Zienkiewicz and Taylor 1991)

$$\mathbf{A}(\mathbf{v}) = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{f}_i}{\partial x_i} + \frac{\partial \mathbf{g}_i}{\partial x_i} + \mathbf{q} = 0 \quad i = 1, 2 \ , \tag{43}$$

where

$$\mathbf{v} = [\rho, \rho u_1, \rho u_2, \rho u_3, \rho E]^{\mathrm{T}} \ , \tag{44}$$

$$\begin{aligned} \mathbf{f}_i = [\rho u_i, \rho u_1 u_i + \delta_{1i} p, \rho u_2 u_i \\ + \delta_{2i} p, \rho u_3 u_i + \delta_{3i} p, u_i(\rho E + p)]^{\mathrm{T}} \end{aligned} \tag{45}$$
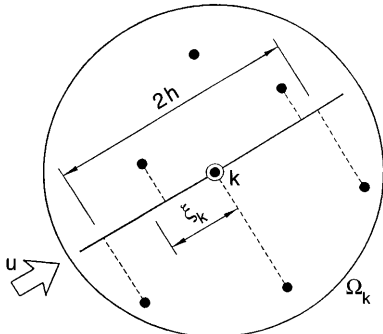

Fig. 4. Definition of the characteristic length h

defines the convective flux vector and

$$\mathbf{g}_i = \left[0, -\tau_{1i}, -\tau_{2i}, -\tau_{3i}, -k\frac{\partial T}{\partial x_i} - \tau_{ij}u_j\right]^{\mathrm{T}} \tag{46}$$

defines the diffusion flows. Finally

$$\mathbf{q} = [0, g_1, g_2, g_3, 0]^{\mathrm{T}} \tag{47}$$

gives the source terms due to (minus) the gravity acceleration. In above ρ is the density, u_k is the velocity in the kth direction p is the pressure and E is the energy. The stress components τ_{ij} are related to velocity gradients by

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \ , \tag{48}$$

where μ is the viscosity. The equations are completed by the universal gas law

$$p = \rho R T \ , \tag{49}$$

where R is the gas constant and T the temperature.

6.2
Stabilization of compressible flow equations
Following similar arguments as in previous section the following stabilized form of the compressible flow equations can be found (Oñate 1996)

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r} - \frac{1}{2}\mathbf{H}^k \frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r}\right) = 0, \quad k = 1, 2, 3 \ , \tag{50}$$

where as usual

$$\mathbf{r} = -\frac{\partial \mathbf{f}_i}{\partial x_i} - \frac{\partial \mathbf{g}_i}{\partial x_i} - \mathbf{q} \ . \tag{51}$$

Matrix $\mathbf{H}^k$ appearing in the stabilizing term of Eq. (50) can be obtained as

$$\mathbf{H}^k = \begin{bmatrix} h_\rho^k & & & & 0 \\ & h_u^k & & & \\ & & h_u^k & & \\ & & & h_u^k & \\ 0 & & & & h_e^k \end{bmatrix}, \quad k = 1, 2, 3 \ . \tag{52}$$

In above $h_\rho^k, h_u^k, h_e^k, k = 1, 2, 3$ are the dimensions of the right prism where mass, momentum and energy balances are enforced, respectively. Note that for $\mathbf{H}^k = 0$ the standard conservation form of the compressible flow equation is recovered.

6.3
Definition of characteristic lengths, intrinsic times and stabilized traction boundary conditions
The following *characteristic lengths* for each of the five equations governing mass, momentum and energy balances can be now defined as

$$h_\rho = |\mathbf{h}_\rho| \quad \text{with} \quad \mathbf{h}_\rho = \left[h_\rho^1, h_\rho^2, h_\rho^3\right]^{\mathrm{T}} \ , \tag{53a}$$

$$h_u = |\mathbf{h}_u| \quad \text{with} \quad \mathbf{h}_u = \left[h_u^1, h_u^2, h_u^3\right]^{\mathrm{T}} \ , \tag{53b}$$

$$h_e = |\mathbf{h}_e| \quad \text{with} \quad \mathbf{h}_e = \left[h_e^1, h_e^2, h_e^3\right]^{\mathrm{T}} \ . \tag{53c}$$

It will now be assumed that the characteristic length vectors $\mathbf{h}_\rho$ $\mathbf{h}_u$ and $\mathbf{h}_e$ have all the direction of the velocity vector $\mathbf{u} = [u_1, u_2, u_3]^\mathrm{T}$. This allows us to express matrix $\mathbf{H}^k$ as

$$\mathbf{H}^k = \frac{u_k}{|\mathbf{u}|}\mathbf{H} \quad \text{with} \quad \mathbf{H} = \begin{bmatrix} h_\rho & & & & 0 \\ & h_u & & & \\ & & h_u & & \\ & & & h_u & \\ 0 & & & & h_e \end{bmatrix} \quad (54)$$

and the stabilized equations have now the following form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r} - \frac{u_k}{2|\mathbf{u}|}\mathbf{H}\frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r}\right) = 0 \ . \quad (55)$$

The intrinsic time for each of the balancing equations can be defined as

$$\tau_\rho = \frac{h_\rho}{2|\mathbf{u}|}, \quad \tau_u = \frac{h_u}{2|\mathbf{u}|}, \quad \tau_e = \frac{h_e}{2|\mathbf{u}|} \ . \quad (56)$$

Equation (55) can be rewritten using (56) as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r} - [\boldsymbol{\tau}]u_k\frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r}\right) = 0 \ , \quad (57)$$

where the intrinsic time matrix is

$$[\boldsymbol{\tau}] = \begin{bmatrix} \tau_\rho & & & & 0 \\ & \tau_u & & & \\ & & \tau_u & & \\ & & & \tau_u & \\ 0 & & & & \tau_e \end{bmatrix} \ . \quad (58)$$

Similar higher order balance concepts can be applied to a finite domain next to a Neumann boundary Γ_t where tractions $\bar{\mathbf{t}}$ are specified (see Fig. 5). The resulting stabilized form of the traction boundary condition can be written as (Oñate 1996, 1998)

$$\mathbf{M}\boldsymbol{\sigma} - \bar{\mathbf{t}} - \frac{h_u}{2|\mathbf{u}|}\mathbf{n}^\mathrm{T}\mathbf{u}\mathbf{r}_m = 0 \ , \quad (59)$$

where $\mathbf{n}$ is the normal vector

$$\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]^\mathrm{T} \quad (60)$$
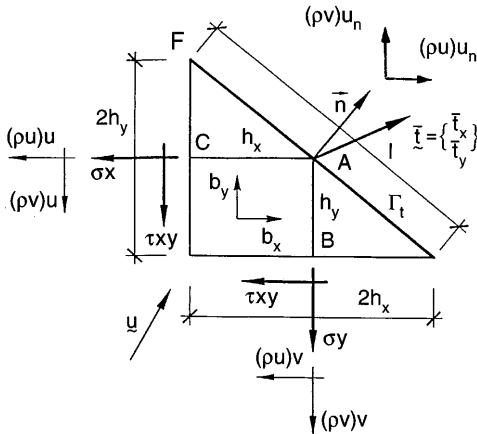

Fig. 5. Flow domain next to a Neumann boundary where surface tractions are prescribed

with

$$\sigma_{ij} = \tau_{ij} - p\delta_{ij} \ , \quad (61)$$

$$\mathbf{M} = \begin{bmatrix} n_x & 0 & 0 & n_y & n_z & 0 \\ 0 & n_y & 0 & n_x & 0 & n_z \\ 0 & 0 & n_z & 0 & n_x & n_y \end{bmatrix} \quad (62)$$

and $\mathbf{r}_m$ contains the momentum equations, i.e. the rows 2, 3, and 4 of vector $\mathbf{r}$ of Eq. (51).

Indeed for boundaries where the outgoing heat flux is specified the stabilized form given by Eq. (38) should be used.

The equivalent stabilized equations (57) are the basis for the numerical solution of compressible flow problems using the FPM. As usual the time integration can be performed prior to the application of the spatial discretization scheme. Neglecting the time derivative in the stabilization term and using a simple forward integration gives the following expression

$$\Delta \mathbf{v} = \Delta t\left[\mathbf{r} - [\boldsymbol{\tau}]u_k\frac{\partial \mathbf{r}}{\partial x_k}\right]^n \ , \quad (63)$$

where $\Delta \mathbf{v} = \mathbf{v}^{n+1} - \mathbf{v}^n$.

6.4
Equivalence with the characteristic approximation
Note that Eq. (57) involves a different intrinsic time scale for each of the governing differential equations. A simpler form can be derived assuming $\tau_\rho = \tau_u = \tau_e = \tau$, i.e. the balancing domain is taken to have equal dimensions for each of the governing equations. The equivalent stabilized equations can be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r} - \tau u_k\frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{r}\right) = 0 \ , \quad (64)$$

where

$$h = |\mathbf{h}| \quad \text{with} \quad \mathbf{h} = [h^1, h^2, h^3]^\mathrm{T} \ ,$$

$$\bar{\mathbf{H}}^k = h^k\mathbf{I}_5 \quad \text{and} \quad \tau = \frac{h}{2|\mathbf{u}|} \ , \quad (65)$$

where $\mathbf{I}_5$ is the 5×5 unit matrix.

The explicit integration of Eq. (64) gives (neglecting the time derivative in the stabilization term)

$$\Delta \mathbf{v} = \Delta t\left[\mathbf{r} - \tau u_k\frac{\partial \mathbf{r}}{\partial x_k}\right]^n \ . \quad (66)$$

This expression is found again to be analogous as that derived via characteristic Galerkin procedures if the intrinsic time is taken to be equal to $\Delta t/2$ (Zienkiewicz and Codina 1995; Zienkiewicz et al. 1995).

Equation (66) can now be taken as the basis for the spatial discretization using the FPM. The procedure follows precisely the same steps as described in a previous section for advective-diffusive problems.

6.5
Stabilized equations for incompressible flows
Assuming the flow to be isothermal and the density to be constant in Eqs. (50) gives the following stabilized form of the incompressible low equations:

Continuity equation

$$\frac{\partial u_i}{\partial x_i} - \frac{h_p^k}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_i}\right) = 0 \quad ; \tag{67}$$

Momentum equations

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{r}_m - \frac{1}{2}\mathbf{H}_m^k\frac{\partial}{\partial x_k}\left(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{r}_m\right) = 0 \quad , \tag{68}$$

where

$$\mathbf{u} = [u_1, u_2, u_3]^{\mathrm{T}}, \quad \mathbf{H}_m^k = h_u^k\mathbf{I}_3;$$

$$\mathbf{r}_m = -\left[\nabla \mathbf{u}^{\mathrm{T}}\right]\mathbf{u} - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma} + \mathbf{b} \quad , \tag{69}$$

with

$$\mathbf{b} = [f_1, f_2, f_3]^{\mathrm{T}}, \quad \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^{\mathrm{T}} \quad ; \tag{70}$$

$$\mathbf{L}^{\mathrm{T}} = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & \partial/\partial z & 0 \\ 0 & \partial/\partial y & 0 & \partial/\partial x & 0 & \partial/\partial z \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial x & \partial/\partial y \end{bmatrix} \tag{71}$$

and the expression of $\boldsymbol{\sigma}$ is given by Eq. (60).

Equations (67) and (68) can be rewritten in terms of the characteristic lengths and the intrinsic times by using the following expressions

$$\begin{aligned} h_\rho^k &= \frac{h_\rho}{2|\mathbf{u}|}u_k = \tau_\rho u_k, \\ h_u^k &= \frac{h_\rho}{2|\mathbf{u}|}u_k = \tau_u u_k \quad . \end{aligned} \tag{72}$$

Equation (68) can now be integrated in time using an adequate implicit scheme. Alternatively, any of the well known fractional step approaches can be chosen (Zienkiewicz and Codina 1995). The application of the spatial discretization using the FPM as described in previous sections follows precisely the steps explained for advective-diffusive problems.

7
Example: Subsonic viscous compressible flow past a 0012 NACA profile ($M_\infty = 0.5$, $Re = 5000, \alpha = 0°$)

The grid of points (Fig. 6) that was used to resolve the test problem consisted of 14106 points which were expanded in structured form away from the boundary. The clouds of points exhibit a strong stretching close to the boundary and in the wake.

The computation using 5 points per cloud defined using a quadrant searching technique (Fisher 1996). A linear approximation ($m = 3$) was chosen. The Mach number contours (Fig. 7) around the airfoil or velocity vectors along the trailing edge (Fig. 8) show good quality of the solution. The pressure coefficient (Fig. 9), the skin friction coefficient (Fig. 10) and the separation point at 84.0% chord are within reasonable limits for this difficult test case. The pressure drag coefficient of 0.025 shows a very good agreement compared to the value 0.0228 reported by
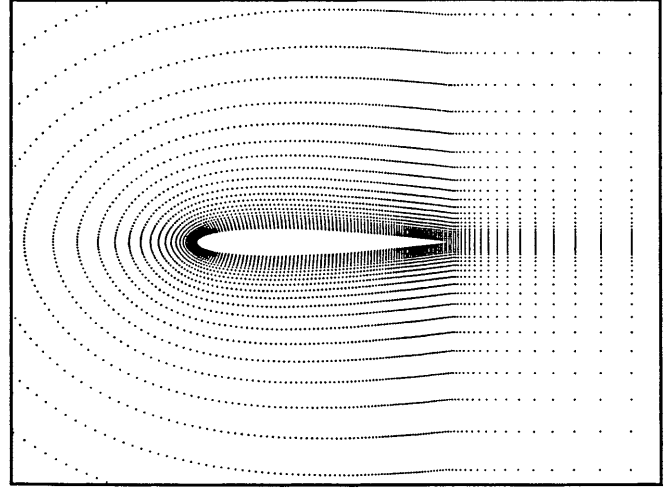

Fig. 6. Grid of 14106 points for the solution of viscous flow around a NACA0012 airfoil ($n = 5$, $m = 3$)
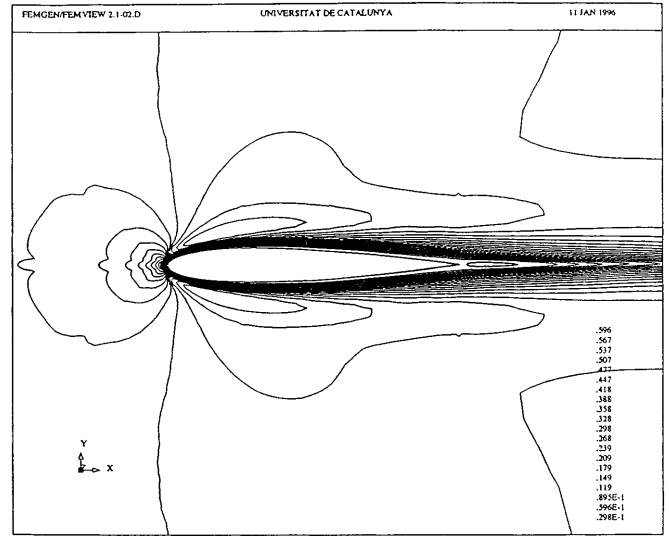

Fig. 7. Mach number contours for the solution of viscous flow around a NACA0012 airfoil (Re = 5000, $M_\infty = 0.5$, $\alpha = 0°$) using 141106 points

Mavripilis et al. (1989). Figure 11 shows the pressure contours on the wake region showing the expected parallel distribution of lines and Fig. 12 shows an acceptable rate of convergence of the results.

8
Conclusions

In the paper details of the derivation of a stabilized finite point method (FPM) for analysis of advective-diffusive transport and compressible/incompressible fluid flow problems have been presented. Recent applications of the FPM indicate that it is a promising "meshless" numerical procedure for analysis of a wide range of CFD problems. Further testing is in progress, including the necessary extension of solving three dimensional problems where the efficiency of the FPM should become more apparent.

Fig. 8. Velocity vectors at the trailing edge for the solution of viscous flow around a NACA0012 airfoil. Note the recirculation at tip of the trailing edge
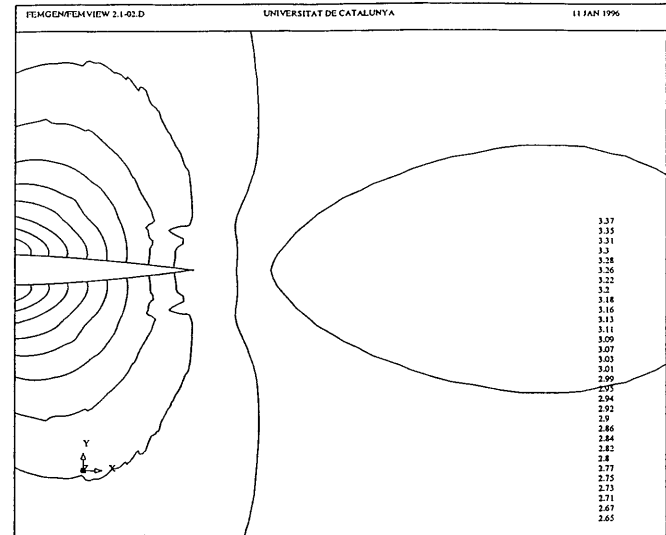

Fig. 11. Pressure lines in the wake for the solution of viscous flow around a NACA0012 airfoil. Note their parallel position to each other in the wake
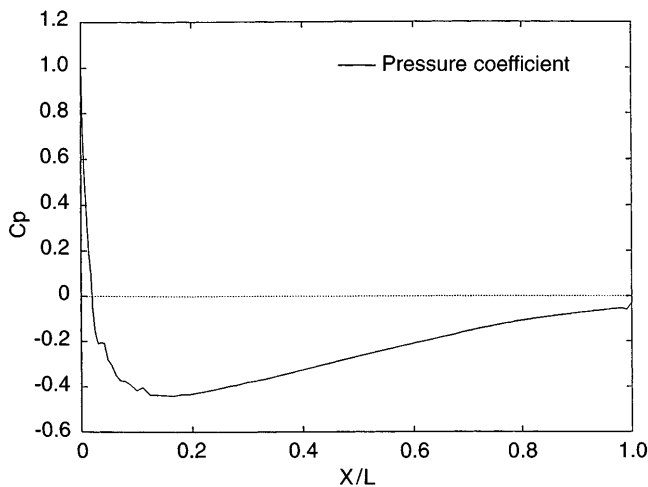
Fig. 9. Pressure coefficient along the airfoil for the solution of viscous flow around a NACA0012 airfoil
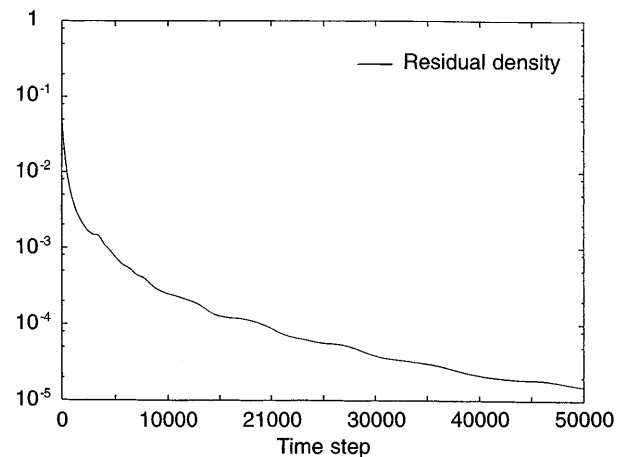

Fig. 12. Convergence history of the density residual for the solution of viscous flow around a NACA0012 airfoil (Re = 5000, $M_\infty = 0.5$, $\alpha = 0°$)
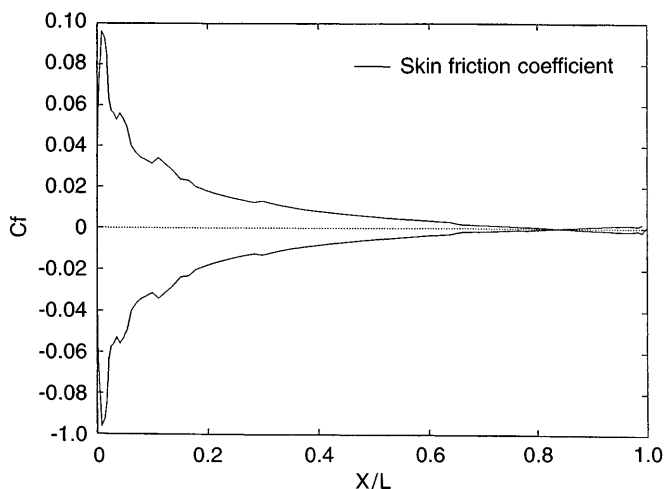

Fig. 10. Skin friction coefficient along the airfoil for the solution of viscous flow around a NACA0012 airfoil

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