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DEVELOPMENT OF ISOLATED ELEMENT METHOD AND ANALYSIS OF UPPER AND LOWER BOUND SOLUTIONS BY A NEW MIXED-HYBRID VARIATIONAL PRINCIPLE

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Abstract. A new discretization analysis method named the isolated element method, that differ from conventional FEM, for solid mechanical problems is proposed. An object to be analyzed is divided into the elements that are separated from each other. A set of displacement functions providing arbitrary number of degrees of freedom is used for each isolated element which expresses the translation and rotation of a rigid body. The extended principle of minimum potential energy is applied to satisfy the continuity of the displacement of isolated elements adjoining to each other. Any node or spring, penalty functions and Lagrange multipliers are not used in this method. The displacement functions of the power series are used to describe the mechanical state of the isolated element and finally, the coefficients of series are determined by a variational principle derived from the extended principle of minimum potential energy. Furthermore, a new mixed and hybrid variational principle which is composed from the potential and the complemental energy functional is proposed. The pair of these energy are constrained by a formula. Using this new principle, in which stress and displacement can be used as independent variables, the stress and displacement are computed at the same time. Besides, upper and lower bounds solutions are analyzed using the new principle and the isolated element method. Some computed examples of the plane stress problems are presented. We show the good convergency of the numerical results, and also present the upper and lower bound results of stress and displacement by the new mixed and hybrid variational principle using the isolated element method.

1 INTRODUCTION

In this paper a new discretization analysis method for solid mechanics problems is proposed. Isolated elements without nodes are used for the new method named the isolated element method. Furthermore, a new mixed and hybrid variational principle which is composed from the potential and the complemental energy functional is proposed.

By facilitating and universalizing the implementation of the new discretization analysis method, we aim to extend the fields for its application and to ensure higher reliability and versatility. Then we consider focussing on two points to be important under the condition that the target region is divided into elements as conventional. One is that the functions, which represent the physical state of elements with arbitrary shape, are relatively free to adopt compared with the collocation methods, and furthermore, boundary conditions can be satisfied automatically. The other one is to estimate the accuracy of stress and displacement at the same time. To finally solve the above two problems, we develop a new discretization analysis method consisting of the isolated elements [1].

In this new method, the solid to be analyzed is divided into multiple separated and isolated elements. In the isolated element method, we use the minimum potential energy principle to guarantee convergence to the correct solution, without using the connecting factors such as nodes, springs, penalty functions etc. A common method that automatically satisfies such as separated boundary conditions is to use the Lagrange multiplier [2]or to identify it and reduce the number of unknowns [1]. A new variational approach is proposed to satisfy the condition of the continuity for displacement without Lagrange multipliers or those identification. Each isolated element is defined as one that can be deformed independently. The continuities of the displacement and stress between the elements are required, therefore the isolated elements need to have the feature of satisfying both of geometrical boundary conditions and mechanical boundary conditions automatically. To give the above features to the isolated elements, we extend the principle of the minimum potential energy. The divided isolated elements are reconstructed into the continuum deformed by natural boundary conditions [3]. By providing a local coordinate system for each isolated element, the displacement functions of power series including the translation and rotation of rigid bodies are used. In numerical analysis, the coefficients of the power series are determined by a variational principle derived from the extended principle of minimum potential energy.

A new mixed and hybrid variational principle, also proposed in this paper, is composed from the potential energy functional and the complemental energy functional that the pair of these energy are constrained by a formula [4]. Using this principle, we can compute the upper and lower bounds results of stress and displacement at the same time.

Some computed examples of the plane stress problems are presented. We show the good convergency of the numerical results, also present the upper and lower bound results of stress and displacement by the new mixed and hybrid variational principle using the isolated element method.

In this paper, the isolated element method is abbreviated as IEM.

2 FEATURES AND VARIATIONAL FORMULATION OF IEM

2.1 Objects discretized by isolated elements

The symbolic of this paper is shown below. The lower index i of the symbol represents the component of the vector, and i,j represent the components of the tensor. Symbols with an upper index $(\bar{\ })$ represent the specified value. The summation convention is used.

A continuum solid is divided into a set of isolated elements as shown in Fig.1, in which elements and elements or elements and supporting objects are completely separated. Figure 2 shows a loaded activate state in which elastic energy is charged. A local coordinate system is provided to describe the motion of the element and the mechanical state inside the element, and a displacement function is assumed for each element. When each element is completely isolated, each one moves freely because rigid body displacement is not constrained. The displacement function consists of terms representing translation and rotation of rigid body and a power series of one or more orders. Finally rigid body translation and rotation, and undetermined coefficients of power series are solved by variational equations.

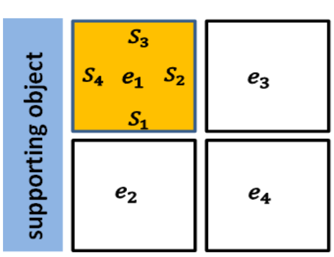


Fig.1Divided isolated elements

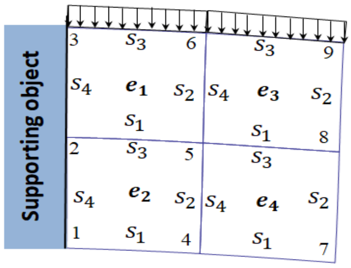


Fig.2 A loaded activate state

2.2 Expression of boundary conditions in IEM

An important point to be considered in the isolated element method is the condition of continuity between the elements. Both mechanical and geometrical boundary conditions are satisfied as natural boundary conditions [3] by the Euler-Lagrange equations.

We express the type of the boundary condition of the element as S_m and S_L attached to the integration symbol. All boundaries of the element are represented by S and $S = S_m + S_L$.

S_m: represents a boundary where the equilibrium conditions of the traction and the continuity conditions of the displacement are imposed.S_L: represents a boundary where the load is applied or a boundary where no load is applied. The sides S_1, and S_2 of the element e_1 in Fig. 1 are S_m, because the boundary condition of the traction between the element e_1 and the adjacent elements, and the continuity condition for the displacement are imposed. The side S_3 of the element e_1 is S_L because a load is applied. The side S_4 of the element e_1 is S_m, because the condition of traction of the element e_1 between the reaction force of the supporting body and the condition of displacement constraint must be satisfied.

2.3 Functional of the isolated elements with boundary potential energy system

In IEM, we use the principle of minimum potential energy to define the functional Eq.(1) same as previous paper [1].

$$\Pi_p(u_i) \coloneqq \int_V u_p(\varepsilon_{ij})\,dV - \int_{S_L} \bar{t}_i u_i dS - \int_V \bar{p}_i u_i\,dV \tag{1}$$

where u_i is displacement, ε_{ij} is Cauchy's strain tensor, $u_p(\varepsilon_{ij})$ is strain energy per unit volume, $\bar{t}_i$ is prescribed traction on the boundary, that is the external load on the S_L and $\bar{p}_i$ is body force in V. The linear displacement-strain relation and linear elastic stress-strain relation are used. Each element has its own rigid body displacement so that it can move freely. In addition, the continuity of the displacement and boundary conditions between the isolated elements are automatically satisfied by the Euler-Lagrange equations.

In this paper, we propose a new variational approach that satisfies the condition of the continuity of displacement between the target element and adjacent element without Lagrange multipliers or those identifications. Equation (2) is the definition of the supplementary function for the continuity of displacement, and we will refer to the integrals on the right side in Eq.(2) as the “boundary potential energy”.

$$I_B := \int_A^B \sigma_{ij} n_j u_i \, dS + \int_{A'}^{B'} \bar{\sigma}_{ij} \bar{n}_j \bar{u}_i \, dS \tag{2}$$

where, σ_{ij} and $\bar{\sigma}_{ij}$ are the stress tensors at point P common to elements e and $\bar{e}$, $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ are the displacement vectors at point P, n_i and $\bar{n}_i$ are the unit normal outward vectors at point P of the surface of elements e and $\bar{e}$ respectively. The first integral on the right side of Eq.(2) represents the boundary potential energy of surface AB of solid e in Fig.3, and the second integral represents the boundary potential energy of surface $A'B'$ of solid $\bar{e}$. In Fig. 3, the boundary potential energy in Eq. (2) is generated by the mechanical and geometrical coupling and matching of e and $\bar{e}$ at each other's boundaries. From Cauchy's formula and $\sigma_{ij} = \bar{\sigma}_{ij}$, $\bar{n}_i = -n_i$ at point P, we have Eq. (3).

$$I_B = \int_A^B t_i u_i \, dS - \int_{A'}^{B'} t_i \bar{u}_i \, dS \tag{3}$$

Figure 4 shows the relationship between the global coordinate system X-Y and the local coordinate systems in terms of position vectors for solids e and $\bar{e}$ respectively.

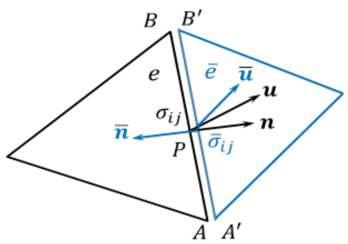


Fig.3 Variables for boundary potential energy

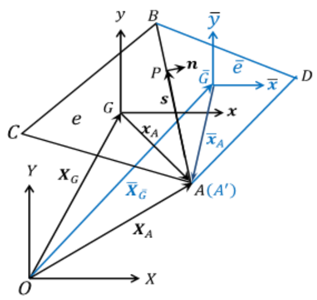


Fig.4 Local and global coordinate systems

Based on the global coordinate system, the starting point A of the integration can be represented in the two local coordinate systems as Eq. (4).

$$\bar{\boldsymbol{X}}_{\boldsymbol{G}} + \bar{\boldsymbol{x}}_{\boldsymbol{A}} = \boldsymbol{X}_{\boldsymbol{G}} + \boldsymbol{x}_{\boldsymbol{A}} \; \rightarrow \; \bar{\boldsymbol{x}}_{\boldsymbol{A}} = \boldsymbol{x}_{\boldsymbol{A}} + \boldsymbol{X}_{\boldsymbol{G}} - \bar{\boldsymbol{X}}_{\boldsymbol{G}} \tag{4}$$

Using Eq. (4), the integration with respect to $\bar{\boldsymbol{x}}$ on the right side of Eq. (3) can be performed in the local coordinate system x-y. Therefore, Eq. (3) can be rewritten as Eq. (5)

$$I_B = sign \int_A^B t_i (u_i - \bar{u}_i) \, dS \tag{5}$$

To maintain convergence to a unique solution, it is important that the positive or negative sign of the terms in the quadratic form of the functional coincide with the “$sign$” of the prefix of the integral in Eq. (5). In this case, the first integral on the right side of Eq. (1) is positive quadratic form because it is strain energy, and the integral of $t_i u_i$ in Eq. (5) is positive for the target element e. Since the inequality $\Pi_p(u_i) + I_B \geq \Pi_p(u_i)$ is hold, "$sign$" in Eq. (5) must be positive. Adding Eq. (5) to the right side of Eq. (1), we obtain the extended functional shown in equation (6).

$$\Pi_I(u_i) \coloneqq \int_V u_p(\varepsilon_{ij})\, dV - \int_{S_L+S_m} \bar{t}_i u_i dS - \int_V \bar{p}_i u_i\, dV + \int_{S_m} t_i\,(u_i - \bar{u}_i) dS \tag{6}$$

where $\Pi_I(u_i)$ is rewrite of $\Pi_p(u_i)$ in Eq. (1).

Equation (6) is the basic functional of the isolated element method.

2.4 Weak form variational equation

From Eq. (6), we can derive the weak and the strong forms of variational equations. We will derive weak form variational equation for numerical analysis. Substituting strain energy function $u_p(\varepsilon_{ij}) \coloneqq \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ into Eq. (6) and setting the first variation of $\Pi_I(u_i)$ as zero, we derive the minimization conditional equation of the functional. In this derivation, we simplify a formula as Eq. (7)

$$\delta \int_{S_m} t_i\,(u_i - \bar{u}_i) dS = \int_{S_m} (u_i - \bar{u}_i)\,\delta t_i dS + \int_{S_m} t_i \delta(u_i - \bar{u}_i)\, dS \;= \int_{S_m} (u_i - \bar{u}_i)\delta t_i\, dS \tag{7}$$

, because $u_i - \bar{u}_i = 0$ by the Euler-Lagrange equation so $\delta(u_i - \bar{u}_i) = 0$. The external force $\bar{t}_i$ not only acts on S_L of the target element, but also the traction of adjacent elements acts on S_m. Therefore, when the first variation of the functional Π_I is set to zero, we have

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV - \int_{S_L} \bar{t}_i \delta u_i dS - \int_V \bar{p}_i \delta u_i\, dV + \int_{S_m} (u_i - \bar{u}_i)\,\delta t_i dS - \int_{S_m} \bar{t}_i \delta u_i\, dS = 0 \tag{8}$$

,in which

$$\begin{aligned}\int_V \sigma_{ij}\delta\varepsilon_{ij}\, dV &= \int_S \sigma_{ij} n_j\, \delta u_i dS - \int_V \sigma_{ij,j}\delta u_i\, dV \\ &= \int_{S_L} t_i\, \delta u_i dS + \int_{S_m} t_i\, \delta u_i dS - \int_V \sigma_{ij,j}\delta u_i\, dV\end{aligned} \tag{9}$$

where n_j represents the outward unit normal vector on the boundary surface S of the element.

Then, substituting Eq. (9) into the first integral term of Eq. (8), we have the so-called a strong form variational equation, and the equations of Euler-Lagrange can be derived [1].

3 ASSUMPTION OF DISPLACEMENT FUNCTIONS

We use a displacement function based on M.A.Biot's displacement-strain theory [5], which is effective in stabilizing numerical calculations for IEM [1]. We assume a displacement function consisted of the only strains of 2-dimensional as Eq. (10),

$$\left.\begin{aligned} u_e &\coloneqq \varepsilon_{x0}x + \varepsilon_{xy0}y + a_1x^2 + a_2xy + a_3y^2 + \cdots \\ v_e &\coloneqq \varepsilon_{y0}y + \varepsilon_{xy0}x + b_1x^2 + b_2xy + b_3y^2 + \cdots \end{aligned}\right\} \tag{10}$$

where $\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0}$ are unknown constant strains. Adding a rigid displacement to the displacement-strain relationship devised by Biot [5], we have

$$\left.\begin{aligned} u &= u_0 - \omega_z y + e_{xx}x + e_{xy}y \\ &\quad = u_0 - \omega_z y + \frac{\partial u_e}{\partial x}x + \frac{1}{2}\left(\frac{\partial u_e}{\partial y} + \frac{\partial v_e}{\partial x}\right)y \\ v &= v_0 + \omega_z x + e_{yy}y + e_{xy}x \\ &\quad = v_0 + \omega_z x + \frac{\partial v_e}{\partial y}y + \frac{1}{2}\left(\frac{\partial u_e}{\partial y} + \frac{\partial v_e}{\partial x}\right)x \end{aligned}\right\} \tag{11}$$

, where $u_0, v_0 \; and \; \omega_z$ are rigid body translation and rotation respectively.

4 STABILIZATION OF RIGID BODY DISPLACEMENT

There is essentially a rigid body displacement in the isolate element, we explicitly describe what is necessary for rigid body displacements from the viewpoint of stabilizing numerical analysis same as previous paper [1].

The displacement function u_i of the isolated target element is represented by the sum of the rigid body displacement component u_i^G and the strain-causing component u_i^ε, that is $u_i = u_i^G + u_i^\varepsilon$. Substituting this equation into the integration Eq. (12)

$$\int_{S_m} \frac{1}{2}(u_i - \bar{u}_i)^2 dS \tag{12}$$

, in which $u_i, \bar{u}_i$ represent the displacement of the target element and the adjacent element respectively, and applying a small variation only to u_i^G of the target element, we have

$$\begin{aligned} \delta \int_{S_m} \frac{1}{2}(u_i - \bar{u}_i)^2 dS &= \int_{S_m} (u_i - \bar{u}_i) \frac{\partial (u_i - \bar{u}_i)}{\partial u_i^G} \delta u_i^G dS \\ &\cong \int_{S_m} (u_i - \bar{u}_i) \delta u_i^G dS \end{aligned} \tag{13}$$

In the case of plane stress, Eq. (13) is expressed by the following equations.

$$\int_{S_m} (u - \bar{u}) \delta u_0 dS \,, \; \int_{S_m} (v - \bar{v}) \delta v_0 dS \qquad \text{(14A), (14B)}$$

To make the element matrix positive-definite, in the case of plane stress analysis, it is necessary to superimpose Eq.(14A) and Eq.(14B) on the corresponding row of the element matrix. When using low-order power functions of displacement, the boundary conditions for each side of the element may not be fully satisfied due to the small number of degrees of freedom. Then, an unbalanced force or an unbalanced moment is generated, and the rigid body displacement becomes unstable. We used a method to minimize these unnecessary forces and moments.

5 A MIXED-HYBRID VARIATIONAL PRINCIPLE AND UPPER AND LOWER BOUND SOLUTIONS

Lower bound solutions can be analyzed by FEM which is based on the displacement method, and by IEM in which if the displacement function is used. However, the quantitative accuracy of the solutions is unknown only by the lower bound analysis. We propose a new mixed-hybrid method. The accuracy of the solutions of stress and displacement can be quantitatively calculated in the linear elastic problems by upper and lower solutions using the new mixed-hybrid method with IEM.

5.1 Variational principle of the mixed-hybrid energy functional

In the local coordinate system, the displacement u_i of the independent variable and the stress σ_{ij} of the independent variable are assumed by polynomials. The generalized parameters, which are undetermined coefficients, of the displacement polynomial are represented by $\alpha_k(k=1,2\cdots m)$, and of the stress polynomial are represented by $\beta_k(k=1,2\cdots n)$. Displacement $u_i(\alpha_k)$ and stress $\sigma_{ij}(\beta_k)$ are defined as independent variables respectively. The $\tau_{ij}(\alpha_k)$ is the stress derived from α_k , t_i (α_k) and $t_i(\beta_k)$ are the tractions derived from α_k and β_k respectively. Strain energy $u_p(\alpha_k)$ is defined as a function of α_k, and complementary energy $u_c(\beta_k)$ is defined as a function of β_k.

We define the functional Π_m for the mixed method based as

$$\Pi_m(\alpha_k,\beta_k)=I_p+I_c+I_B+H|_{sign=-1} \tag{15}$$

The functional $I_p(\alpha_k)$ of the principle of minimum potential energy is expressed as Eq. (16)

$$I_p(\alpha_k)=\int_V u_p(\varepsilon_{ij})\,dV-\int_{S_L+S_m}\bar{t}_i u_i dS-\int_V \bar{p}_i u_i\,dV \tag{16}$$

$$u_i-\bar{u}_i=0 \tag{17}$$

in which $\bar{p}_i$ is a body force, and $\bar{t}_i$ is the traction of an adjacent element acting on the boundary S_m. The auxiliary conditions for this functional is Eq. (17) . The functional $I_c(\beta_k)$ of the principle of minimum complementary energy is expressed as Eq. (18), and the auxiliary conditions are the Eqs. (19) and (20).

$$I_c(\beta_k)=\int_V u_c(\beta_k)dV-\int_{S_m}\bar{u}_i t_i dS \tag{18}$$

$$\sigma_{ij,j}+\bar{p}_i=0 \tag{19}$$

$$t_i-\bar{t}_i=0 \tag{20}$$

The functional I_B is Eq. (5) with *sign* = (-1). The variational equation of the functional extended by I_p and I_B to satisfy Euler-Lagrange equations is Eq. (21). From the respective integrals on the left-hand side of Eq. (21), the auxiliary conditions for the functionals I_p, I_B, namely Eqs. (17),(19),(20), are satisfied as Euler-Lagrange equations.

$$\int_{S_L}(t_i-\bar{t}_i)\,\delta u_i dS+\int_{S_m}(t_i-\bar{t}_i)\,\delta u_i dS-\int_{V}\left(\sigma_{ij,j}+\bar{p}_i\right)\delta u_i\,dV+\int_{S_m}(u_i-\bar{u}_i)\,\delta t_i dS=0 \tag{21}$$

We described about a fomula of energy constraint condition, that express the coupled condition related to displacement and stress. Figure 5 show that even if u_p and u_c change independently, the boundary between u_p and u_c has no gap or overlap, and the sum of u_p and u_c are always rectangular. We called this state a constraint condition of energy functions. Eq. (22) is defined to express the relationship between u_p and u_c.

$$H(\alpha_k,\beta_k)=sign\int_V\left(\sigma_{ij}\varepsilon_{ij}-u_p-u_c\right)dV \tag{22}$$

From the requirement that the element matrix, be in positive-valued quadratic form, *sign* must be (-1) [4]. Using $\delta u_p=\tau_{ij}\delta\varepsilon_{ij}$, $\delta u_c=e_{ij}\delta\sigma_{ij}$, from the variational expression in Eq. (22), Eq. (23) can be obtained as a constraint on the energy inside the element.

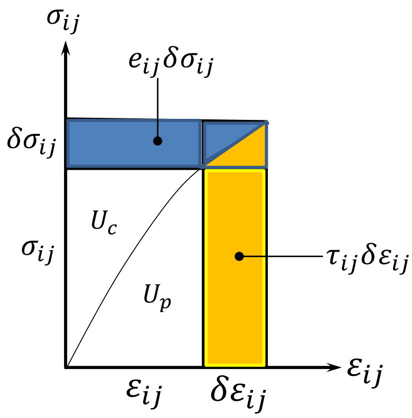


Fig5 A constraint condition of energy

$$\int_V\delta\{\sigma_{ij}\varepsilon_{ij}-\left(u_p+u_c\right)\}dV=\int_V\left(\sigma_{ij}-\tau_{ij}\right)\delta\varepsilon_{ij}dV+\int_V\left(\varepsilon_{ij}-e_{ij}\right)\delta\sigma_{ij}dV=0 \tag{23}$$

, where ε_{ij} is the strain derived from the u_i and e_{ij} is the strain derived from the σ_{ij}.

Substituting Eqs. (5), (16), (18) and (22) into Eq. (15), we have Eq. (24)

$$\Pi_m(\alpha_k,\beta_k)=\int_V u_p dV+\int_V u_c dV-\int_{S_m}\bar{t}_i u_i dS-\int_V \bar{p}_i u_i\,dV-\int_{S_L}\bar{t}_i u_i dS-\int_{S_m}\bar{u}_i t_i dS$$
$$+\int_{S_m}t_i\,(u_i-\bar{u}_i)dS-\int_V\left(\sigma_{ij}\varepsilon_{ij}-u_p-u_c\right)dV \tag{24}$$

In linear solid mechanics problems, Eq. (24) is established due to the uniqueness of the solution, and in the principle of minimum complementary energy.

Substituting $\delta u_p=\tau_{ij}\delta\varepsilon_{ij}$, $\delta u_c=e_{ij}\delta\sigma_{ij}$, and Eq. (23) into Eq. (24), with the first variation of the functional Π_m as zero, we find the variational equation Eq. (25)

$$\int_V\tau_{ij}\delta\varepsilon_{ij}dV+\int_V e_{ij}\delta\sigma_{ij}dV+\int_{S_m}(u_i-\bar{u}_i)\,\delta t_i dS-\int_{S_m}\bar{t}_i\delta u_i dS-\int_V\bar{p}_i\delta u_i\,dV-$$
$$\int_{S_m}\bar{u}_i\delta t_i dS-\int_{S_L}\bar{t}_i\delta u_i dS-\int_V\left(\sigma_{ij}-\tau_{ij}\right)\delta\varepsilon_{ij}dV-\int_V\left(\varepsilon_{ij}-e_{ij}\right)\delta\sigma_{ij}dV=0 \tag{25}$$

5.2 Assumption of the admissible functions for the mixed method

The three types of admissible functions [4] are described for computing of plane stress, as

1) Self-equilibrium function assuming stress.
   Using a function in which some variables are deleted so that Eq. (19) is satisfied.
2) Self-equilibrium function assuming displacement.
   Using the M.A. Biot type displacement function, Eq. (11) in 2-dimentional.
3) Airy's stress function.
   Using the stress function Φ must be satisfied the bi-harmonic equation.

6 NUMERICAL EXAMPLES

6.1 Verification of convergency of IEM

To verify the convergency of the numerical solutions by IEM, the first quadrant area in Fig.6 is analyzed using the biaxial symmetry of the plane stress problem and applying a distributed load of $\bar{t}_x = S(1 - y^2/b^2)$ on both sides, and S is the magnitude of the basis load at point A. The description of the element mesh in Fig.6 is a typical example.

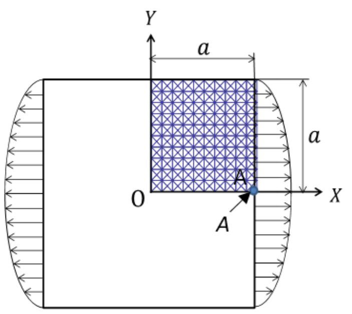


Fig.6 Rectangular flat plate subjected to parabolic tensile load

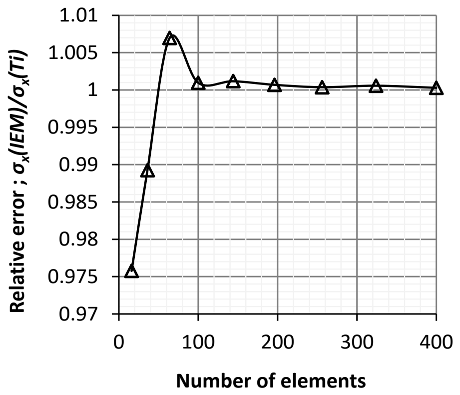


Fig.7 Convergency of stress by IEM

Biot's displacement function including a constant is used. Triangular element with 20 unknowns of cubic displacement function (T3) is used. Timoshenko's theoretical solution [7] sets the power series of the stress function to the eighth order including a constant. The specifications are as follows, $a = 500$mm, Young's modulus = 200GPa, Poisson's ratio = 0.3 and unit thickness. S = 1N / mm^2 , $\sigma_x^{(T)}$= 0.858832MPa. In Fig.7, $\sigma_x^{(T)}$ is Timoshenko's theoretical value and $\sigma_x^{(IEM)}$ is the stress of the same point analyzed by IEM at origin O. The ratio of $\sigma_x^{(T)}$ and $\sigma_x^{(IEM)}$ is shown in Fig.7 as a relative error.

6.2 Verification of upper and lower bound analysis by the new mixed-hybrid method with IEM

First, Fig.6 is analyzed for upper and lower bound solutions by the new mixed-hybrid method with IEM, and compare with the results by FEM. Airy's quadratic stress function and 1-4th order Biot's displacement function are used with rectangular elements in IEM, and triangular constant strain elements are used in FEM. Figure 8 shows the relative error of the Mises stress at the origin in Fig.6, the criterion of relative error is Timoshenko's theoretical solution. One block in on the horizontal axis in Fig.8, which is a rectangular elementin in IEM, shows four elements in FEM. Figure 8 shows that the convergence curve of $\sigma_M(\beta_k)$ is a monotonically increasing lower bound, and the convergence curve of $\tau_M(\alpha_k)$ is upper bound, and the FEM (σ_M) is lower bound.

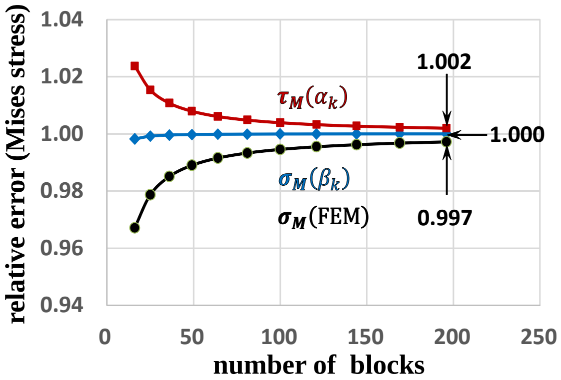


Fig.8 Upper and lower bound solutions of stress by IEM , compared to FEM

Next, a plane stress concentration problem of the square plate with a circular hole shown in Fig. 9 is analyzed, using the mixed–hybrid method with IEM. We applied the weak formulation Eq. (25) to this analysis. The 2nd-4th order Airy stress function and the 1st-4th order Biot's displacement function are used. Distributed load $f_x = 1(\frac{\text{N}}{\text{mm}})$, $a = 500\text{m}$, $d = 40\text{mm}$, Young's modulus=200GPa, Poisson's ratio = 0.3 .

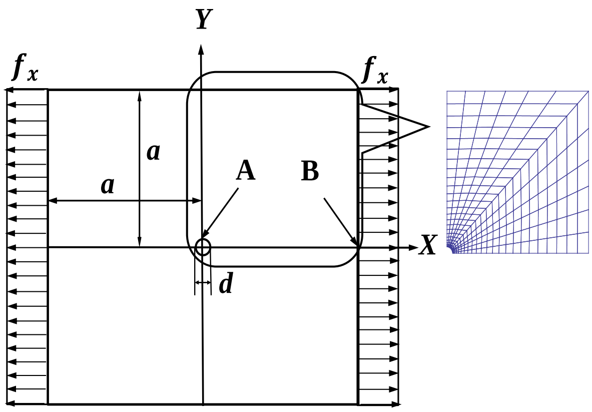
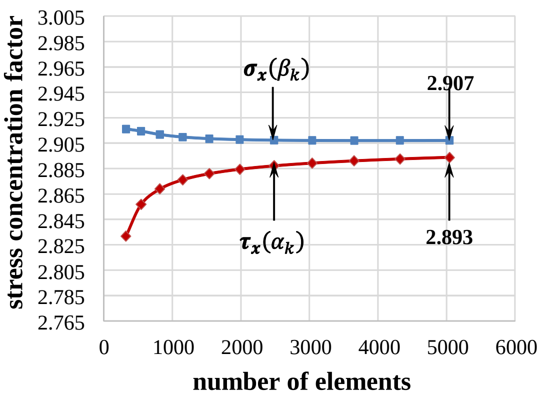


Fig.9 Stress concentration problem of square flat plate with circular holes

Figure 10 shows the convergence of the stress concentration factor at point A in Fig.9. The convergence curve of $\tau_x(\alpha_k)$ is lower bound and the $\sigma_x(\beta_k)$ is upper bound.

Figure 11 shows the convergence of the horizontal displacement of point B in Fig.9, $u^{(\alpha)}$ and $u^{(\beta)}$ represents the displacements derived from α_k and β_k respectively. The curve $u^{(\alpha)}$shows a monotonically increasing lower bound precisely, and the curve of $u^{(\beta)}$ shows an upper bound in



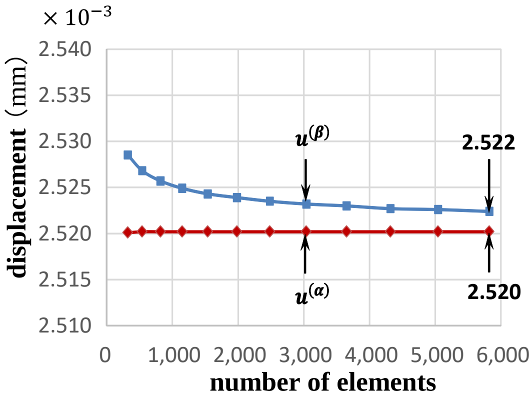


Fig.10 The upper and lower bound of the stress concentration factor at point A

Fig.11 The upper and lower bounds of the horizontal displacement at point B

7 CONCLUSIONS

Based on the principle of extended minimum potential energy, a new discrete analysis method, named the isolated element method (IEM), of solid mechanics is proposed without using the connecting factors such as nodes, springs, penalty functions and Lagrange multipliers. It is clarified that the geometrical boundary condition, the mechanical boundary condition and each equilibrium condition inside the element are automatically satisfied by the Euler-Lagrange equation in each of the divided isolated elements, and the condition of the continuity for displacement is valid by the new variational approach proposed in this paper. In the isolated element method, it is possible to use any polygonal or polyhedron element with the functions providing arbitrary number of degrees of freedom. We show a basic numerical example for verification of convergency of this method.

The new mixed-hybrid method is proposed. Displacement and stress are defined as independent variables respectively. We treat the mixed method as a coupling problem. Using the coupled condition related to displacement and stress by the constraint condition formula of strain energy and complementary energy, we devise an approach to analyze the coupling problem of two types of generalized parameters by the variational method. It is possible to analyze upper and lower bound solutions of stress and displacement by the new mixed-hybrid method with IEM. We show the numerical examples of the upper and lower bound solutions of stress and displacements.

By the isolated element method, it is expected that analysis of assembly structure, discontinuous object, contact problem etc. can be applied more flexibly than with the nodal point method. The mixed-hybrid method is expected to be applied to various kinds of problems.

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