Solvability of Quadratic Fractional Integral Equations by Family of Measures of Noncompactness in Fréchet Algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$

Hojjatollah Amiri Kayvanloo[1], Maryam Kahrobaeian[1] and Mohammad Mursaleen[2,3,4,*]

[1] Department of Mathematics, Ma.C., Islamic Azad University, Mashhad, 9187147578, Iran

[2] Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai, 602105, India

[3] Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, 40447, Taiwan

[4] Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India

INFORMATION

Keywords:
Fixed point theorem
fractional differential equation
Fréchet algebras
measures of noncompactness

DOI: 10.23967/j.rimni.2025.10.74259

SCIPEDIA
Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería
RIMNI

Solvability of Quadratic Fractional Integral Equations by Family of Measures of Noncompactness in Fréchet Algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$

Hojjatollah Amiri Kayvanloo[1], **Maryam Kahrobaeian**[1] **and Mohammad Mursaleen**[2,3,4,*]

[1]Department of Mathematics, Ma.C., Islamic Azad University, Mashhad, 9187147578, Iran

[2]Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai, 602105, India

[3]Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, 40447, Taiwan

[4]Department of Mathematics, Aligarh Muslim University, Aligarh, 202002, India

ABSTRACT

We define the Fréchet algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ and then define a new family of measures of noncompactness. We prove a fixed point theorem that generalizes the Darbo's fixed point theorem in this space. By applying the technique of measures of noncompactness in conjunction with new fixed point theorem, we investigate the solvability of a certain quadratic fractional integral equations. Then, we state two examples to support our main results.

OPEN ACCESS

Received: 07/10/2025

Accepted: 12/11/2025

Published: 20/03/2026

DOI
10.23967/j.rimni.2025.10.74259

Keywords:
Fixed point theorem
fractional differential equation
Fréchet algebras
measures of noncompactness

1 Introduction and Preliminaries

The first measures of noncompactness (MNC), was defined by Kuratowski [1] for solely topological considerations. Then, Darbo [2] applied this measure to generalize Banach's contraction mapping principle for so-called condensing operators. The MNC theory is a significant field of the nonlinear functional analysis. The technique of MNC in conjunction with Darbo fixed point theorem turned into a tool to checking the solvability of solutions of different classes of integral equations such as Fredholm, Uryson and Volterra type integral equations and fractional differential equations, for example the authors in [3,4] considered the Fréchet spaces and in [5,6] considered the sequence spaces. Also, in references [7–9], the researchers considered the solvability of convolution type integral equations, nonlinear quadratic integral equation of Hammerstein type, Functional integral equations, nonlinear integral equation by technique of MNC.

*Correspondence: Mohammad Mursaleen (mursaleenm@gmail.com).

Fractional differential equations (FDEs) have many novel and interesting applications in real world, boundary value problems supplemented with some types of conditions like periodic, nonlocal, and integral boundary conditions have been considerably analyzed by numerous scholars. Compartmental systems, electrodynamics of complex medium, fluid dynamics, heat conduction in materials with memory, unsteady and aerodynamics are examples of its application. For more details, see [10–14]. Very of physical phenomena can encompass indistinctive memory and heritage properties due to the non-locality nature of the power-law organize. For example, the diffusion of stress waves in viscoelastic media [15–17], the flux of the viscous fluid flowing through the porous medium [18], the viscoelastic responses of hydropolymers [19]. The fractional diffusion-wave equation [16] is a generalization of diffusion and wave equations via time and space fractional derivatives.

Recently, many papers endowed to the nonlinear integral equations considered in Banach algebra see [20–23] and just few were investigated in Fréchet algebra [24,25]. So, it seems that suitable ambience for integral equations on unbounded interval are different Fréchet spaces, which in the case of some types of the product integral equations, naturally lead to Fréchet algebras. In this work we attempt to fill this gap in the theory of the nonlinear integral equations in the Fréchet algebras.

We consider the solvability of a following certain quadratic fractional integral equations in Fréchet algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$

$$q(\wp) = \left(\frac{1}{\Gamma(\varrho_1)}\int_0^{\wp} \frac{f_1(s, q(s))}{(\wp - s)^{1-\varrho_1}} ds\right)\left(g(\wp) + \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp} \frac{f_2(s, q(s))}{(\wp - s)^{1-\varrho_2}} ds\right),$$

where $0 < \varrho_i < 1$, $i = 1, 2$, and $\wp \in \mathbb{R}_+$.

We choose such class of equations, motivated by the Gripenberg integral equations [26–28] studied in the context of epidemic models. This fractional form of the equation under consideration will allow us to show the advantages of the proposed Fréchet algebra and include, as special cases, many previously considered equations (also for a non-quadratic case, cf. [29]). Integral equations considered in Fréchet and Banach algebra have a partly intricate form and the study of such equations needs of the apply of instrumentations different than for non-quadratic ones. The choice of solution space permits us to apply the technique related to MNC and some fixed point theorems in Fréchet algebra, and the resulting solutions are more regular than just continuous. We prove all auxiliary results in this space and it allows us to study some quadratic fractional integral equations.

The manuscript is structured as follows. In Section 2, we define the Fréchet algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ and we introduce a new family of MNC and a fixed point theorem that generalizes the Darbo's fixed point theorem in this space. In Section 3, by using the technique of MNC and new fixed point theorem, we consider the solvability of a certain quadratic fractional integral equations and we state two examples to support our main results.

Let Λ be a Fréchet space and $\varnothing \neq \mathfrak{L} \subset \Lambda$. Then

- $\overline{\mathfrak{L}}$ is closure of $\mathfrak{L}$ and Conv$\mathfrak{L}$ is closed convex hull.
- $\mathfrak{M}_\Lambda \subseteq \Lambda$ is the family of bounded subsets of Λ.
- $\mathfrak{N}_\Lambda \subseteq \Lambda$ is the family of relatively compact subsets of Λ.
- $B(\nu, \sigma)$ is a closed ball in Λ.

Let $(P, \|.\|)$ be a Banach algebra. Then $C(\mathbb{R}_+, P)$ forms a Fréchet algebra [24] by the family of seminorms $(\|.\|_n)_{n\in\mathbb{N}}$ by

$$\|q\|_n = \sup\{\|q(\wp)\| : \wp \in [0, n]\}, q \in C(\mathbb{R}_+, P), n \in \mathbb{N}.$$

$C(\mathbb{R}_+, P)$ have a product by

$$(qw)(\wp) = q(\wp)w(\wp), \quad q, w \in C(\mathbb{R}_+, P), \wp \in \mathbb{R}_+.$$

Also,

(a) The sequence $(q_n) \to q$ in $C(\mathbb{R}_+, P) \Leftrightarrow (q_n)$ is uniformly convergent to q on compact subsets of $[0, \infty)$.

(b) A set $\mathcal{Q} \subset C(\mathbb{R}_+, P)$ is bounded if the set $\{\|q\|_n : q \in \mathcal{Q}\}$ is bounded $\forall\, n \in \mathbb{N}$ and we denote

$$\|\mathcal{Q}\|_n = \sup\{\|q\|_n : q \in \mathcal{Q}\}.$$

(c) A family $\mathcal{Q} \subset C(\mathbb{R}_+, P)$ is relatively compact $\Leftrightarrow$ for each $n > 0$, the restrictions to $[0, n]$ of all functions from $\mathcal{Q}$ form an equicontinuous set and $\mathcal{Q}(\wp)$ is relatively compact in P for each $\wp \in \mathbb{R}_+$.

Definition 1 ([30]): *The family of mappings $\{\mu^n\}_{n\in\mathbb{N}}$, where $\mu^n : \mathfrak{M}_\Lambda \to \mathbb{R}_+$, is a family of (MNC) in the Fréchet space Λ if*

(i) $\mathfrak{N}_\Lambda \supseteq \ker \mu^n = \{\mathcal{Q} \in \mathfrak{M}_\Lambda : \mu^n(\mathcal{Q}) = 0\} \neq \varnothing$.

(ii) *If* $\mathcal{Q} \subset \mathcal{Y}, \Rightarrow \; \mu^n(\mathcal{Q}) \leq \mu^n(\mathcal{Y})$.

(iii) $\mu^n(\mathcal{Q}) = \mu^n(Conv\mathcal{Q}) = \mu^n(\overline{\mathcal{Q}})$

(iv) $\mu^n(\lambda\mathcal{Q} + (1-\lambda)\mathcal{Y}) \leq \lambda\mu^n(\mathcal{Q}) + (1-\lambda)\mu^n(\mathcal{Y})$ *for each* $0 \leq \lambda \leq 1$.

(v) *If for each* $n \in \mathbb{N}$, $\overline{\mathcal{Q}_n} = \mathcal{Q}_n \subseteq \mathfrak{M}_\Lambda$, $\mathcal{Q}_{n+1} \subset \mathcal{Q}_n$ *If* $\lim\limits_{n\to\infty}\mu^n(\mathcal{Q}_n) = 0$, *then* $\varnothing \neq \mathcal{Q}_\infty = \bigcap\limits_{n=1}^{\infty}\mathcal{Q}_n$.

Theorem 1 ([30,31]): *Let $\{\mu^n\}_{n\in\mathbb{N}}$ be a family of MNC and $\varnothing \neq Q = \overline{Q} \subseteq \Lambda$ be bounded and convex of a Fréchet space Λ and $\Upsilon : Q \to Q$ be continuous, if $\varnothing \neq B \subset Q \exists$ a constants $l_n \in [0, 1)$, with the property*

$$\mu^n(\Upsilon(B)) \leq l_n\mu^n(B),$$

then Υ has at least a fixed point in the set Q.

Definition 2 ([24]): *Let Λ be Fréchet algebra family of MNC $\{\mu^n\}_{n\in\mathbb{N}}$ defined on the Λ have property (m) if for arbitrary sets $Q, Y \in \mathfrak{M}_\Lambda$ we have*

$$\mu^n(QY) \leq \|Q\|_n\mu^n(Y) + \|Y\|_n\mu^n(Q)$$

for $n \in \mathbb{N}$.

Theorem 2 ([24]): *Let Λ be Fréchet algebra and $\varnothing \neq \mho = \overline{\mho} \subseteq \Lambda$be convex and bounded, and $G, H, T : \mho \to \Lambda$ that G is compact and $H(\mho)$ and $T(\mho)$ are bounded and $\Psi = G + H.T : \mho \to \mho$. Let H and T fulfill the Darbo condition on the set $\mho$ by the constants k_n and l_n w.r.t the regular family of MNC $\{\mu^n\}_{n\in\mathbb{N}}$ which fulfills the condition (m). So, Ψ fulfills Darbo condition by the constants $(\|H(\mho)\|l_n + \|T(\mho)\|k_n)$ w.r.t the family of MNC $\{\mu^n\}_{n\in\mathbb{N}}$. Especially, if*

$$\|H(\mho)\|l_n + \|T(\mho)\|k_n < 1,$$

for $n \in \mathbb{N}$, then Ψ has at least a fixed point in the set $\mho$.

2 Fréchet Algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$

Let $L^1(\mathbb{R}_+)$ be the Banach space of Lebesgue integrable functions on $\mathbb{R}_+$ by norm

$$\|q\| = \int_0^\infty |q(\wp)|d\wp.$$

Also, $L^1(\mathbb{R}_+)$ is Banach algebra with the product of vectors q and w in $L^1(\mathbb{R}_+)$ by a convolution $q * w$, i.e.,

$$(q * w) = \int_0^\infty q(\wp - s)w(s)ds, \ \wp \in \mathbb{R}_+, \ q, w \in L^1(\mathbb{R}_+). \tag{1}$$

The Banach algebra $L^1(\mathbb{R}_+)$ furnished by weak topology and the product of vectors is defined by (1) (see [24]).

Now, we work in the Fréchet algebra

$$C(\mathbb{R}_+, L^1(\mathbb{R}_+)) = \{q : \mathbb{R}_+ \to L^1(\mathbb{R}_+) : \ q \text{ is continuous}\},$$

by the family of seminorms

$$\|q\|_n = \sup\{\|q(\wp)\| : \ 0 \le \wp \le n, n \in \mathbb{N}\}, \tag{2}$$

$$q \in C(\mathbb{R}_+, L^1(\mathbb{R}_+)).$$

Theorem 3: *Let $\varnothing \neq \mathcal{Q} \subset C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. Then*

(1) *The sequence $(q_n)_{n\in\mathbb{N}} \in \mathcal{Q}$ is convergent to q in $C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \Leftrightarrow (q_n)_{n\in\mathbb{N}}$ is uniformly convergent to q on compact subsets of $[0, \infty)$.*

(2) *The family $\mathcal{Q} \subset C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ is relatively compact $\Leftrightarrow$ for each $n \in \mathbb{N}$ and $0 < \varepsilon$, $\exists\, 0 < \delta$ so that*

$$\int_0^n |q(\wp + h) - q(\wp)|d\wp \le \varepsilon,$$

$\forall q \in \mathcal{Q}$ *and* $|h| < \delta$.

We define a family of MNC in the $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. Further, fix arbitrarily $\varepsilon > 0$ and a set $\mathcal{Q}$ of the $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. For $q \in \mathcal{Q}$, let us denote

$$\omega^n(q, \varepsilon) = \sup\{\int_0^n |q(\wp + h) - q(\wp)|d\wp : \ |h| < \varepsilon, \ 0 \le \wp \le n, \ n \in \mathbb{N}\}.$$

Moreover,

$$\omega^n(\mathcal{Q}, \varepsilon) = \sup\{\omega^n(q, \varepsilon) : q \in \mathcal{Q}\},$$

$$\mu^n(\mathcal{Q}) = \lim_{\varepsilon \to 0} \omega^n(\mathcal{Q}, \varepsilon). \tag{3}$$

Theorem 4: *The family of mappings $\{\mu^n\}_{n\in\mathbb{N}}$ given by (3) satisfies the conditions $(i) - (v)$ from Definition 1 and condition (m).*

Proof. We show that $\ker\{\mu^n\} \subset \mathfrak{N}_{C(\mathbb{R}_+,L^1(\mathbb{R}_+))}$. Take $\mathcal{Q} \in \mathfrak{M}_{C(\mathbb{R}_+,L^1(\mathbb{R}_+))}$, so that $\mu^n(\mathcal{Q}) = 0$. Fix $\vartheta > 0$ and $\wp \in [0, n]$ we get $\omega^n(\mathcal{Q}, \varepsilon) < \vartheta$, then

$$\sup\{\int_0^n |q(\wp + h) - q(\wp)|d\wp : q \in \mathcal{Q},\ \wp \in [0, n] n \in \mathbb{N},\ |h| < \varepsilon\} < \vartheta.$$

Therefore, from condition (2) of Theorem 3 we get the closure of $\mathcal{Q}$ in $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ is compact and $\ker\{\mu^n\} \subset \mathfrak{N}_{C(\mathbb{R}_+,L^1(\mathbb{R}_+))}$.

The proofs of (*ii*) is standard and is therefore omitted. To prove (*iii*). Let us $\mathcal{Q} \in \mathfrak{M}_{C(\mathbb{R}_+,L^1(\mathbb{R}_+))}$ and $q \in \overline{\mathcal{Q}}$. Thus, sequence $\{q_m\}$ in $\mathcal{Q}$ exists so that $q_m \to q$ in $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. By condition (1) of Theorem 3 for every $\delta > 0$, $\exists m_0 \in \mathbb{N}$ so that for each $m \geq m_0$, $|q_m - q| < \delta$, on compact subsets of $\mathbb{R}_+$. By definition of $\omega^n(\mathcal{Q}, \varepsilon)$, we have

$$\int_0^n |q_m(\wp + h) - q_m(\wp)|d\wp \leq \omega^n(\mathcal{Q}, \varepsilon),$$

for $\wp \in [0, n]$, $n \in \mathbb{N}$ and $|h| < \varepsilon$. Letting $m \to \infty$ we have

$$\int_0^n |q(\wp + h) - q(\wp)|d\wp \leq \omega^n(\mathcal{Q}, \varepsilon),$$

therefore

$$\lim_{\varepsilon\to 0} \omega^n(\overline{\mathcal{Q}}, \varepsilon) \leq \lim_{\varepsilon\to 0} \omega^n(\mathcal{Q}, \varepsilon).$$

Consequently,

$$\mu^n(\overline{\mathcal{Q}}) \leq \mu^n(\mathcal{Q}).$$

Also, $\mathcal{Q} \subset \overline{\mathcal{Q}}$ and by condition (*ii*) of the Definition 1 we deduce

$$\mu^n(\mathcal{Q}) \leq \mu^n(\overline{\mathcal{Q}}).$$

So, we get

$$\mu^n(\mathcal{Q}) = \mu^n(\overline{\mathcal{Q}}). \tag{4}$$

Further, taking into account the convexity of the function q and condition (*ii*) we have

$$\mu^n(Conv\mathcal{Q}) = \mu^n(\mathcal{Q}).$$

To prove (*iv*). Let $0 \leq \lambda \leq 1$ and $0 < n$. So,

$$\int_0^n |\lambda q(\wp) + (1 - \lambda)y(\wp)|d\wp \leq \lambda \int_0^n |q(\wp)|d\wp + (1 - \lambda) \int_0^n |y(\wp)|d\wp.$$

So, by applying the definition of $\omega^n(q, \varepsilon)$, we have

$$\omega^n(\lambda q + (1 - \lambda)y, \varepsilon) \leq \lambda\omega^n(q, \varepsilon) + (1 - \lambda)\omega^n(y, \varepsilon).$$

By taking supremum on $q, y \in \mathcal{Q}$ and letting $\varepsilon \to 0$ in above inequality we infer that

$$\mu^n(\lambda Q + (1 - \lambda)Y) \leq \lambda\mu^n(Q) + (1 - \lambda)\mu^n(Y).$$

Now, we prove condition (v). Suppose that $\emptyset \neq \{\mathcal{Q}_j\} \subset C(\mathbb{R}_+, L^1(\mathbb{R}_+))$, so that $\mathcal{Q}_{j+1} \subset \mathcal{Q}_j$ for $j \in \mathbb{N}$ and $\lim_{j\to\infty} \mu^n(\mathcal{Q}_j) = 0$, for each $n \in \mathbb{N}$. For any $j \in \mathbb{N}$, choose $q_j \in \mathcal{Q}_j$. Claim that $B = \{\overline{q_j}\}$ is compact set in $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. We check the condition (2) of Theorem 3. Let $0 < \varepsilon$ be fixed. By $\lim_{j\to\infty} \mu^n(\mathcal{Q}_j) = 0$, we can find a $i \in \mathbb{N}$ so that for any $n \in \mathbb{N}$,

$$\mu^n(\mathcal{Q}_i) < \varepsilon.$$

Thus, we can find $\delta_1 > 0$ so that

$$\omega^n(\mathcal{Q}_m, \delta_1) < \varepsilon,$$

for $m \geq i$. Therefore, $\forall m \geq i$ we get

$$\sup\{\int_0^n |q_m(\wp + h) - q_m(\wp)|d\wp : \ \wp \in [0, n], \ n \in \mathbb{N} \ wih \ |h| < \delta_1\} < \varepsilon.$$

Since the set $\{q_1, q_2 \dots q_{i-1}\}$ is compact, so for each $j \in \{1, 2 \dots i-1\} \exists \delta_2 > 0$ so that

$$\sup\{\int_0^n |q_m(\wp + h) - q_m(\wp)|d\wp : \ \wp \in [0, n], \ n \in \mathbb{N} \ with \ |h| < \delta_2\} < \varepsilon.$$

Therefore, we obtain $\omega^n(B, \delta) < \varepsilon$, for $\delta < \min\{\delta_1, \delta_2\}$ and $\forall n > 0$. We have

$$\mu^n(B) = \lim_{\delta\to 0} \omega^n(B, \delta).$$

Then, $\mu^n(B) = 0$, i.e., B is compact. Then a subsequence $\{q_{nj}\}$ and $q_0 \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ exists such that $\{q_{nj}\} \to q_0$. Since $q_j \in \mathcal{Q}_j, \ \mathcal{Q}_j = \overline{\mathcal{Q}_j}$ and $\mathcal{Q}_{j+1} \subset \mathcal{Q}_j$ for any $j \in \mathbb{N}$, we have

$$q_0 \in \bigcap_{j=1}^{\infty} \mathcal{Q}_j = \mathcal{Q}_\infty.$$

To prove condition (m). Take arbitrary $\mathcal{Q}, \mathcal{W} \in \mathfrak{M}_{C(\mathbb{R}_+, L^1(\mathbb{R}_+))}$, so that functions belonging to these sets are nonnegative on $[0, n]$. Further, fix arbitrarily $q \in \mathcal{Q}$, $w \in \mathcal{W}$ and take $\wp \in [0, n]$ and $h \in \mathbb{R}_+$ by $|h| < \delta$. So, we obtain

$$\begin{aligned}
\omega^n(q * w, \varepsilon) &= \sup\{\int_0^n |(q * w)(\wp + h) - (q * w)(\wp)|d\wp\} \\
&= \sup\{\int_0^n \int_0^n |q(\wp + h - s)w(s) - q(\wp + h)w(s)|dsd\wp\} \\
&\leq \int_0^n |w(s)|ds \int_0^n |q(\wp + h - s) - q(\wp + h)|d\wp \\
&\leq \|w\|_n \omega^n(q, \varepsilon),
\end{aligned}$$

and similarly

$$\omega^n(w * q, \varepsilon) \leq \|q\|_n \omega^n(w, \varepsilon).$$

Furthermore,

$$\omega^n(q * w, \varepsilon) = \frac{1}{2}\omega^n(q * w, \varepsilon) + \frac{1}{2}\omega^n(q * w, \varepsilon).$$

Taking supremum on $q \in \mathcal{Q}, w \in \mathcal{W}$ and letting $\varepsilon \to 0$ we infer that

$$\mu^n(\mathcal{Q}\mathcal{W}) \le \frac{1}{2}\|\mathcal{Q}\|_n\mu^n(\mathcal{W}) + \frac{1}{2}\|\mathcal{W}\|_n\mu^n(\mathcal{Q}).$$

So, μ^n satisfies the condition (m). □

Theorem 5: *Let $\varnothing \ne \mho = \overline{\mho} \subseteq C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ be bounded, convex and the operators $H, L : \mho \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ be continuous in such way that $H(\mho)$ and $L(\mho)$ are bounded and the operator $S = H.L : \mho \to \mho$. If H and L fulfill on the set $\mathcal{Q}$ the Darbo condition w.r.t the family of MNC $\{\mu^n\}_{n\in\mathbb{N}}$ with the constants k_n and l_n, respectively, then S fulfill on $\mho$ the Darbo condition by constant $\|H(\mathcal{Q})\|_n l_2 + \|L(\mathcal{Q})\|_n k_n$. Particularly, if:*

$$\|H(\mathcal{Q})\|_n l_n + \|L(\mathcal{Q})\|_n k_n < 1, \tag{5}$$

for $n \in \mathbb{N}$. Then S has at least a fixed point in $\mho$.

Proof. Let $\varnothing \ne \mathcal{Q} \subset \mho$ be fixed and $n \in \mathbb{N}$. By assumption that the $\{\mu^n\}_{n\in\mathbb{N}}$ satisfies the condition (m) so we get

$$\begin{aligned}\mu^n\big(S(\mathcal{Q})\big) &= \mu^n\big(H(Q)L(Q)\big)\\ &\le \frac{1}{2}\|H(\mathcal{Q})\|_n\mu^n\big(L(\mathcal{Q})\big) + \frac{1}{2}\|L(\mathcal{Q})\|_n\mu^n\big(H(\mathcal{Q})\big)\\ &\le \|H(\mathcal{Q})\|_n l_n\mu^n(\mathcal{Q}) + \|L(\mathcal{Q})\|_n k_n\mu^n(\mathcal{Q})\\ &= (\|H(\mathcal{Q})\|_n l_n + \|L(\mathcal{Q})\|_n k_n)\mu^n(\mathcal{Q}).\end{aligned}$$

So, S satisfies the Darbo condition by the constant $\|H(\mathcal{Q})\|_n l_n + \|L(\mathcal{Q})\|_n k_n$, $n \in \mathbb{N}$. By (5) and Theorem 1, we deduced that S has at least a fixed point on the set $\mho$. □

3 Application

Now, we state an existence result for a quadratic fractional integral equations on the Fréchet algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. Eventually, to indicate the fruitfulness of our result we offer two genuine examples.

The issue of our study is the following certain quadratic equation of fractional order

$$q(\wp) = \left(\frac{1}{\Gamma(\varrho_1)}\int_0^{\wp}\frac{f_1(s, q(s))}{(\wp - s)^{1-\varrho_1}}ds\right)\left(g(\wp) + \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp}\frac{f_2(s, q(s))}{(\wp - s)^{1-\varrho_2}}ds\right), \tag{6}$$

where $0 < \varrho_i < 1$, $i = 1, 2$, and $\wp \in \mathbb{R}_+$. Assume that:

- (a) $g \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ is uniformly continuous function on $\mathbb{R}_+$.
- (b) The functions $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous. Also, for each $s \in \mathbb{R}_+$, $q, w \in \mathbb{R}$, we have

  $|f_i(s, q) - f_i(s, w)| \le |q - w|.$
- (c) $f_i(s, 0) \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ for $i = 1, 2$ and $s \in \mathbb{R}_+$.

(d) $\exists$ a positive function $r : \mathbb{R}_+ \to \mathbb{R}_+$ of the inequality

$$(r(\wp) + \|f(s,0)\|_n)\left(\frac{n^{\varrho_1+1}}{(\varrho_1+1)\Gamma(\varrho_1+1)} + \frac{n^{\varrho_2+1}}{(\varrho_2+1)\Gamma(\varrho_2+1)}\right) + \|g\|_n \le r(\wp).$$

Theorem 6: *Let* $(a)-(d)$ *hold, then the Eq. (6) has at least one solution* q *in* $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$.

Proof. Define the operators $P, G, H : C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ for $q \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ and $\wp \in \mathbb{R}_+$ by

$$P(q)(\wp) = \frac{1}{\Gamma(\varrho_1)}\int_0^{\wp} \frac{f_1(s,q(s))}{(\wp - s)^{1-\varrho_1}} ds,$$

$$G(q)(\wp) = g(\wp) + \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp} \frac{f_2(s,q(s))}{(\wp - s)^{1-\varrho_2}} ds,$$

and

$$H(q)(\wp) = P(q)(\wp)G(q)(\wp).$$

By conditions (a) and (b) we infer that $P, G : C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ are continuous. So, $H = PG$ is continuous.

Now, we show that H is bounded on

$$\mathbb{B}_r(C(\mathbb{R}_+, L^1(\mathbb{R}_+))) = \{q \in C(\mathbb{R}_+, L^1(\mathbb{R}_+)) : \|q\|_n \le r(\wp),\ \wp \in [0,n],\ n > 0\}.$$

Let $q \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$, $\wp, s \in [0,n]$ with $\wp > s$ and $n > 0$, then we get

$$\begin{aligned}\int_0^n |G(q)(\wp)| d\wp &= \int_0^n |g(\wp) + \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp} \frac{f_2(s,q(s))}{(\wp - s)^{1-\varrho_2}} ds| d\wp \\ &\le \int_0^n |g(\wp)| d\wp + \int_0^n \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp} \frac{(|f_2(s,q(s)) - f_2(s,0)| + |f_2(s,0)|)}{(\wp - s)^{1-\varrho_2}} ds d\wp \\ &\le \|g\|_n + \int_0^n \frac{1}{\Gamma(\varrho_2)}(|q| + |f_2(s,0)|)\frac{\wp^{\varrho_2}}{\varrho_2} d\wp \\ &\le \|g\|_n + \frac{(\|q\|_n + \|f_2(s,0)\|_n) n^{\varrho_2+1}}{(\varrho_2+1)\Gamma(\varrho_2+1)}.\end{aligned}$$

Furthermore,

$$\|Gq\|_n \le \|g\|_n + \frac{(\|q\|_n + \|f_2(s,0)\|_n) n^{\varrho_2+1}}{(\varrho_2+1)\Gamma(\varrho_2+1)}. \tag{7}$$

So, the function $G(q)$ is bounded on $\mathbb{R}_+$ and by the continuity of $G(q)$ on $\mathbb{R}_+$ we get $G : C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. Similarly, we have

$$\|Pq\|_n \le \frac{(\|q\|_n + \|f_1(s,0)\|_n) n^{\varrho_1+1}}{(\varrho_1+1)\Gamma(\varrho_1+1)}. \tag{8}$$

It means that $P(q)$ is bounded on $\mathbb{R}_+$ and $P : C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. By (7) and (8) we get

$$\|Hq\|_n \leq (\|q\|_n + \|f(s,0)\|_n)\left(\frac{n^{\varrho_1+1}}{(\varrho_1+1)\Gamma(\varrho_1+1)} + \frac{n^{\varrho_2+1}}{(\varrho_2+1)\Gamma(\varrho_2+1)}\right) + \|g\|_n.$$

where $\|f(s,0)\|_n = \sup\{\|f_1(s,0)\|_n, \|f_2(s,0)\|_n,\ s \in \mathbb{R}_+\}$. So, $H : C(\mathbb{R}_+, L^1(\mathbb{R}_+)) \to C(\mathbb{R}_+, L^1(\mathbb{R}_+))$. Hence, by (d) we have $H : \mathbb{B}_r(C(\mathbb{R}_+, L^1(\mathbb{R}_+))) \to \mathbb{B}_r(C(\mathbb{R}_+, L^1(\mathbb{R}_+)))$ is continuous.

Obviously, the ball $\varnothing \neq \mathbb{B}_r(C(\mathbb{R}_+, L^1(\mathbb{R}_+))) = \overline{\mathbb{B}_r}(C(\mathbb{R}_+, L^1(\mathbb{R}_+)))$ is bounded and convex. Now, we show that the function Hq is continuous. Assume that $\wp_1, \wp_2 \in [0,n]$ and $n > 0$ by $\wp_1 < \wp_2$ with $|\wp_2 - \wp_1| < \varepsilon$ and $\Omega \subset \mathbb{B}_r(C(\mathbb{R}_+, L^1(\mathbb{R}_+)))$ and $0 < \varepsilon$. Let us choose $q \in \Omega$, then by our conditions we can write

$$\begin{aligned}
&|(Gq)(\wp_2) - (Gq)(\wp_1)| \\
&\leq \left|g(\wp_2) - g(\wp_1)\right| + \left|\frac{1}{\Gamma(\varrho_2)}\int_0^{\wp_2} \frac{f_2(s,q(s))}{(\wp_2 - s)^{1-\varrho_2}}ds - \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp_1} \frac{f_2(s,q(s))}{(\wp_1 - s)^{1-\varrho_2}}ds\right| \\
&= \left|g(\wp_2) - g(\wp_1)\right| + \left[\left|\frac{1}{\Gamma(\varrho_2)}\int_0^{\wp_1} \frac{f_2(s,q(s))}{(\wp_2 - s)^{1-\varrho_2}}ds + \frac{1}{\Gamma(\varrho_2)}\int_{\wp_1}^{\wp_2} \frac{f_2(s,q(s))}{(\wp_2 - s)^{1-\varrho_2}}ds\right.\right. \\
&\left.\left. - \frac{1}{\Gamma(\varrho_2)}\int_0^{\wp_1} \frac{f_2(s,q(s))}{(\wp_1 - s)^{1-\varrho_2}}ds\right|\right] \\
&= |g(\wp_2) - g(\wp_1)| + \left[\left|\frac{1}{\Gamma(\varrho_2)}\left(\int_0^{\wp_1} \frac{1}{(\wp_2 - s)^{1-\varrho_2}} - \frac{1}{(\wp_1 - s)^{1-\varrho_2}}(f_2(s,q(s)) - f_2(s,0) + f_2(s,0))\right)ds\right.\right. \\
&\left.\left. + \frac{1}{\Gamma(\varrho_2)}\int_{\wp_1}^{\wp_2} \frac{1}{(\wp_2 - s)^{1-\varrho_2}}(f_2(s,q(s)) - f_2(s,0) + f_2(s,0))ds\right|\right] \\
&\leq \left|g(\wp_2) - g(\wp_1)\right| + \frac{1}{\Gamma(\varrho_2)}\left(\int_0^{\wp_1} \frac{1}{(\wp_1 - s)^{1-\varrho_2}} - \frac{1}{(\wp_2 - s)^{1-\varrho_2}}(|f_2(s,q(s)) - f_2(s,0)| + |f_2(s,0)|)\right)ds \\
&+ \frac{1}{\Gamma(\varrho_2)}\int_{\wp_1}^{\wp_2} \frac{1}{(\wp_2 - s)^{1-\varrho_2}}(|f_2(s,q(s)) - f_2(s,0)| + |f_2(s,0)|)ds \\
&\leq \left|g(\wp_2) - g(\wp_1)\right| + \frac{1}{\Gamma(\varrho_2)}(|q| + |f_2(s,0)|)\left(\frac{2(\wp_2 - \wp_1)^{\varrho_2}}{\varrho_2} + \frac{\wp_1^{\varrho_2}}{\varrho_2} - \frac{\wp_2^{\varrho_2}}{\varrho_2}\right).
\end{aligned}$$

Since, $g \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ is uniformly continuous and according $|\wp_2 - \wp_1| < \varepsilon$ and $\frac{\wp_1^{\varrho_2}}{\varrho_2} - \frac{\wp_2^{\varrho_2}}{\varrho_2} \leq 0$ (since $\wp_1 < \wp_2$) then we have (Gq) is continuous. Similarly, we get (Pq) is continuous. So $H(q)(\wp) = P(q)(\wp)G(q)(\wp)$ is continuous.

Finally, let us fix a $\varnothing \neq \Omega \subset \mathcal{Q}$, and let us fix arbitrarily $0 < \varepsilon < 1$ and $\wp, h \in [0,n]$, $n \in \mathbb{N}$, by $|h| < \varepsilon$ and $q \in \Omega$ then we have

$$\begin{aligned}
&\omega^n((Gq)(\wp), \varepsilon) \\
&\leq \sup\left\{\int_0^n |g(\wp + h) - g(\wp)|d\wp + \int_0^n \frac{1}{\Gamma(\varrho_2)}\left(\left|\int_0^{\wp+h} \frac{f_2(s,q(s))}{(\wp + h - s)^{1-\varrho_2}}ds - \int_0^{\wp} \frac{f_2(s,q(s))}{(\wp - s)^{1-\varrho_2}}ds\right|\right)d\wp\right\}
\end{aligned}$$

$$\leq \sup\left\{\|\tau_h g-g\|_n+\int_0^n\frac{1}{\Gamma(\varrho_2)}\int_0^{\wp}\left(\frac{1}{(\wp+h-s)^{1-\varrho_2}}-\frac{1}{(\wp-s)^{1-\varrho_2}}\right)(|f_2(s,q(s))-f_2(s,0)|+|f_2(s,0)|)ds\right.$$
$$\left.+\int_{\wp}^{\wp+h}\frac{1}{(\wp+h-s)^{1-\varrho_2}}(|f_2(s,q(s))-f_2(s,0)|+|f_2(s,0)|)ds\Big)d\wp\right\}$$
$$\leq \sup\left\{\|\tau_h g-g\|_n+(\|q\|_n+\|f_2(s,0)\|_n)\left(\frac{1}{\Gamma(\varrho_2)}\left(\frac{\wp^{\varrho_2}}{\varrho_2}-\frac{(\wp+h)^{\varrho_2}}{\varrho_2}-\frac{2h^{\varrho_2}}{\varrho_2}\right)\right)\right\},$$

where $\tau_h g=g(\wp+h)$. Since $0<\varepsilon<1$ is arbitrary and g is compact set in $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ by taking $\varepsilon\to 0$ we get $\|\tau_h g-g\|_n\to 0$ and $h\to 0$, then we have

$$\mu^n(G\Omega)\leq 0. \tag{9}$$

Similarly, we get

$$\mu^n(P\Omega)\leq 0. \tag{10}$$

The relations (9) and (10) are equivalently by

$$\mu^n(G(\Omega))\leq k_n\mu^n(\Omega),\ \text{and}\ \mu^n(P(\Omega))\leq l_n\mu^n(\Omega),$$

where $k_n=0$ and $l_n=0$. By using the condition (m) and two above inequalities, we have

$$\begin{aligned}\mu^n(H(\Omega)) &\leq \mu^n(G(\Omega)P(\Omega))\\ &\leq \|P(\Omega)\|_n\mu^n(G(\Omega))+\|G(\Omega)\|_n\mu^n(P(\Omega))\\ &\leq \|P(\Omega)\|_n k_n\mu^n(\Omega)+\|G(\Omega)\|_n l_n\mu^n(\Omega)\\ &=\big(\|P(\Omega)\|_n k_n+\|G(\Omega)\|_n l_n\big)\mu^n(\Omega)\\ &=0.\end{aligned}$$

Finally, applying Theorem 5 completes the proof. □

Example 1: *Consider the quadratic fractional integral equations*

$$q(\wp)=\left(\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\wp}\frac{e^{-2s}\ln(1+|q(s)|)}{(\wp-s)^{\frac{1}{2}}}ds\right)\left(\frac{\cos\wp}{3}+\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\wp}\frac{\frac{\sin(q(s))}{(s+1)}}{(\wp-s)^{\frac{1}{2}}}ds\right) \tag{11}$$

where $n=\varrho_1=\varrho_2=\frac{1}{2}$, $g(\wp)=\frac{\cos\wp}{3}$, $f_1(s,q)=e^{-2s}\ln(1+|q|)$, $f_2(s,q)=\frac{\sin(q)}{(s+1)}$. *Then we observe that*

$$\begin{aligned}|f_1(s,q)-f_1(s,w)| &= |(\ln(1+q)-\ln(1+w))e^{-2s}|\\ &\leq |\ln\left(\frac{1+q}{1+w}\right)|\\ &\leq |\ln\left(\frac{1+q-w+w}{1+w}\right)|\end{aligned}$$

$$
\begin{aligned}
&= |\ln\left(1+\frac{q-w}{1+w}\right)| \\
&\leq |\frac{q-w}{1+w}| \leq |q-w|.
\end{aligned}
$$

and

$$
\begin{aligned}
|f_2(s,q) - f_2(s,w)| &= |\frac{1}{s+1}(\sin(q)-\sin(w))| \\
&\leq |(\sin(q)-\sin(w))| \\
&\leq |q-w|.
\end{aligned}
$$

So, condition (b) holds. Also, $g(\wp) = \dfrac{\cos\wp}{3} \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ *and* $f_1(s,0) = e^{-2s}\ln 1 = 0$, $f_2(s,0) = \dfrac{1}{s+1} sin(0) = 0$ *so conditions (a) and (c) hold. Moreover, we have*

$$
(r(\wp)+0)\left(\frac{\frac{1}{2}^{\frac{3}{2}}}{(\frac{3}{2})\Gamma(\frac{3}{2})} + \frac{\frac{1}{2}^{\frac{3}{2}}}{\frac{3}{2}\Gamma(\frac{3}{2})}\right) + \frac{1}{3} \leq r(\wp).
$$

Thus, $r(\wp) \geq 0.71$. *So, all the conditions of Theorem 6 are fulfilled. So the Eq. (11) has a solution* $q(\wp) \in \mathbb{B}_{r(\wp)}$.

Example 2: *Consider the certain quadratic equation of fractional order*

$$
q(\wp) = \left(\frac{1}{\Gamma(\frac{1}{3})}\int_0^{\wp} \frac{\frac{1}{s^2+1}\cos(q(s))}{(\wp-s)^{\frac{2}{3}}}ds\right)\left(\frac{\sin\wp}{\sqrt{7}} + \frac{1}{\Gamma(\frac{1}{3})}\int_0^{\wp}\frac{e^{-5s}\arctan(q(s))}{(\wp-s)^{\frac{2}{3}}}ds\right) \tag{12}
$$

where $n = \varrho_1 = \varrho_2 = \dfrac{1}{3}$, $g(\wp) = \dfrac{\sin\wp}{\sqrt{7}}$, $f_1(s,q) = \dfrac{1}{s^2+1}\cos(q)$, $f_2(s,q) = e^{-5s}\arctan(q)$. *Then we observe that*

$$
\begin{aligned}
|f_1(s,q) - f_1(s,w)| &= |\frac{1}{s^2+1}(\cos(q)-\cos(w))| \\
&\leq |(\cos(q)-\cos(w))| \\
&\leq |q-w|.
\end{aligned}
$$

and

$$
\begin{aligned}
|f_2(s,q) - f_2(s,w)| &= |e^{-5s}(\arctan(q)-\arctan(w))| \\
&\leq |\arctan(q)-\arctan(w)| \\
&\leq |q-w|.
\end{aligned}
$$

So, condition (b) *holds. Also,* $g(\wp) = \frac{\sin \wp}{\sqrt{7}} \in C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ *and* $f_1(s, 0) = \frac{1}{s^2+1}\cos 0 = \frac{1}{s^2+1}$, $f_2(s, 0) = e^{-5s}\arctan(0) = 0$ *so conditions* (a) *and* (c) *hold. Moreover, we have*

$$(r(\wp)+1)\left(\frac{\frac{1}{3}^{\frac{4}{3}}}{(\frac{4}{3})\Gamma(\frac{4}{3})} + \frac{\frac{1}{3}^{\frac{4}{3}}}{\frac{4}{3}\Gamma(\frac{4}{3})}\right) + \frac{1}{\sqrt{7}} \leq r(\wp).$$

Thus, $r(\wp) \geq 1.23$. *So, all the conditions of Theorem 6 fulfilled. So the Eq. (12) has a solution* $q(\wp) \in \mathbb{B}_{r(\wp)}$.

4 Conclusions

In this paper, first we introduced the Fréchet algebra $C(\mathbb{R}_+, L^1(\mathbb{R}_+))$ and then defined a new family of MNC on this space. A fixed point theorem in this space is established that generalizes the well-known Darbo's fixed point theorem. Further, we investigated the solvability of a certain quadratic fractional integral equation by applying the technique of MNC in conjunction with the new fixed point theorem. Finally, the applicability of our main result is shown by constructing two examples.

Acknowledgement: Not applicable.

Funding Statement: The authors received no specific funding for this study.

Author Contributions: The authors confirm contribution to the paper as follows: Hojjatollah Amiri Kayvanloo: writing—original draft, data curation, investigation, resources. Maryam Kahrobaeian: writing—review and editing. Mohammad Mursaleen: supervision, validation, final editing. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: Not applicable.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Kuratowski K. Sur les espaces complets. Fund Math. 1930;15:301–9.
2. Darbo G. Punti uniti in trasformazioni a codominio non compatto. Rend Sem Mat Uni Padova. 1955;24: 84–92.
3. Das A, Hazarika B, Mursaleen M. Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in $\ell_p(1 \leq p < \infty)$. Rev R Acad Cienc Exactas Fis Nat Ser A Mat RACSAM. 2019;113(1):31–40. doi:10.1007/s13398-017-0452-1.
4. Mehravaran H, Amiri Kayvanloo H, Allahyari R. Measures of noncompactness in the space of regulated functions $R(J, \mathbb{R}^\infty)$) and its application to some nonlinear infinite systems of fractional differential equations. Math Sci. 2023;17(3):223–32. doi:10.1007/s40096-022-00464-2.
5. Amiri Kayvanloo H, Herawati E, Mursaleen M. New class of n-order fractional differential equations and solvability in the double sequence space $m^2(\Delta_v^u, \phi, p)$. Carpathian J Math. 2024;41(1):157–69.
6. Amiri Kayvanloo H, Mehravaran H, Mursaleen M, Allahyari R, Allahyari A. Solvability of infinite systems of Caputo-Hadamard fractional differential equations in the triple sequence space $c^3(\Delta)$. J Pseudo-Differ Oper Appl. 2024;15(2):26. doi:10.1007/s11868-024-00601-6.

7. Banaś J, Nalepa R. On a measure of noncompactness in the space of functions with tempered increments. J Math Anal Appl. 2016;435(2):1634–51. doi:10.1016/j.jmaa.2015.11.033.
8. Benhamouche L, Djebali S. Solvability of functional integral equations in the Fréchet space $C(\Omega)$. Mediterr J Math. 2016;13(6):4805–17.
9. Saiedinezhad S. On a measure of noncompactness in the Holder space $Ck, \gamma(\Omega)$ and its application. J Comput Applied Math. 2019;346(1):566–71. doi:10.1016/j.cam.2018.07.030.
10. Akram M, Muhammad G, Ahmad D. Analytical solution of the Atangana-Baleanu–Caputo fractional differential equations using Pythagorean fuzzy sets. Granul Comput. 2023;8(4):667–87. doi:10.1007/s41066-023-00364-3.
11. Burqan A, Saadeh R, Qazz A, Momani S. ARA-residual power series method for solving partial fractional differential equations. Alex Eng J. 2023;62(6):47–62. doi:10.1016/j.aej.2022.07.022.
12. Li G, Zhang Y, Guan Y, Li W. Stability analysis of multi-point boundary conditions for fractional differential equation with non-instantaneous integral impulse. Math Biosci Eng. 2023;20(4):7020–41. doi:10.3934/mbe.2023303.
13. Mahdy AM. Stability, existence, and uniqueness for solving fractional glioblastoma multiforme using a Caputo-Fabrizio derivative. Math Methods Appl Sci. 2025;48(7):7360–77. doi:10.1002/mma.9038.
14. Mahdy AM, Abdou MA, Mohamed DS. Computational methods and dynamical analysis for studying (1 + 1) dimensional functional equations of mixed integro-differential type. Boundary Value Probl. 2025;(1):108. doi:10.1186/s13661-025-02078-8.
15. Podlubny I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Vol. 198. Amsterdam, The Netherlands: Elsevier; 1998. doi:10.1016/s0076-5392(99)80025-3.
16. Hanyga A. Fractional diffusion and wave equations, series on advances in mathematics for applied sciences. Math Models Methods For Smart Mater. 2002;62:171–88. doi:10.1142/9789812776273_0017.
17. Matteo AD, Pirrotta A. Generalized differential transform method for nonlinear boundary value problem of fractional order. Commun Nonlin Sci Numer Simul. 2015;29(1–3):88–101. doi:10.1016/j.cnsns.2015.04.017.
18. Coussot C, Kalyanam S, Yapp R, Insana M. Fractional derivative models for ultrasonic characterization of polymer and breast tissue viscoelasticity. IEEE Trans Ultrason Ferroelectr Freq Control. 2009;56(4):715–25. doi:10.1109/TUFFC.2009.1094.
19. Butera S, Paola MD. A physically based connection between fractional calculus and fractal geometry. Ann Phys. 2014;350(5):146–58. doi:10.1016/j.aop.2014.07.008.
20. Cichon M, Metwali MMA. On the Banach algebra of integral-variation type Hölder spaces and quadratic fractional integral equations. Banach J Math Anal. 2022;16(2):34. doi:10.1007/s43037-022-00188-4.
21. Deep A, Deepmala, Tunç C. The existence of solutions of some non-linear functional integral equations in Banach algebra with applications. Arab J Basic Appl Sci. 2020;27(1):279–86. doi:10.1080/25765299.2020.1796199.
22. Dudek S. Measures of noncompactness in a Banach algebra and their applications. J Math Appl. 2017;40:69–84. doi:10.7862/rf.2017.5.
23. Srivastava HM, Das A, Hazarika B, Mohiuddine SA. Existence of solution for nonlinear functional integral equations of two variables in Banach algebra. Symmetry. 2019;11(5):674. doi:10.3390/sym11050674.
24. Dudek S. Fixed point theorems in Fréchet algebras and Fréchet spaces and applications to nonlinear integral equations. Appl Anal Discrete Math. 2017;11(2):340–57. doi:10.2298/AADM1702340D.
25. Hung LK. Fixed point theorems in locally convex algebras and applications to nonlinear integral equations. J Fixed Point Theory Appl. 2015;2015(1):60. doi:10.1186/s13663-015-0310-9.
26. Gripenberg G. On some epidemic models. Q Appl Math. 1981;39(3):317–27. doi:10.1090/qam/636238.

27. Brestovanska E. Qualitative behaviour of an integral equation related to some epidemic model. Demonstr Math. 2003;36(3):603–10. doi:10.1515/dema-2003-0312.
28. Brestovanska E, Medved M. Fixed point theorems of the Banach and Krasnosels type for mappings on m -tuple Cartesian product of Banach algebras and systems of generalized Gripenbergs equations. Acta Univ. Palacki. Olomuc. Math. 2012;51(2):27–39.
29. Webb JRL. Initial value problems for Caputo fractional equations with singular nonlinearities. Electron J Differ Equ. 2019;117(1):1–32. doi:10.1186/1687-1847-2014-102.
30. Olszowy L. Fixed point theorem in the Fréchet space $C(\mathbb{R}_+)$ and functional integral equaions on an unbounded interval. App Math Comput. 2012;218(18):9066–74. doi:10.1016/j.amc.2012.03.044.
31. Olszowy L, Dudek S. On generalizaign of Darbo-Sadovski type fixed point theorems for iterated mapping in Fréchet spaces. J Fixed Point Theory Appl. 2018;20(4):146. doi:10.1007/s11784-018-0628-0.