

A SIMPLE AND EFFICIENT ELEMENT FOR AXISYMMETRIC SHELLS

O. C. ZIENKIEWICZ*

Department of Civil Engineering, University College Swansea, Swansea, U.K.

J. BAUER

Visiting Fellow from Polish Academy of Sciences, Warsaw, Poland

K. MORGAN†

AND

E. ONATE‡

Department of Civil Engineering, University College Swansea, Swansea, U.K.

SUMMARY

A two noded, straight element which includes shear deformation effects is presented and shown to be extremely efficient in the analysis of axisymmetric shells. A single point of numerical integration is essential for its success when applied to thin shells where the results compare favourably with those achieved with more complex curved elements.

INTRODUCTION

To avoid the introduction of slope continuity in the general displacement analysis of plates and shells a number of workers have used an independent interpolation of slopes and displacements thus allowing for shear deformation effects.^{1,2} Approximations thus achieved worked well for thick situations but were not successful in reproducing as a limit the Kirchhoff type of behaviour encountered in thin sections. A breakthrough was achieved here by introducing a 'reduced', numerical integration procedure^{2,3}—although the full reasons for its success were, at the time, improperly understood. Several such reasons are now known,⁴ the most important being the fact that when the solution involves the minimization of a penalty type functional⁵ in which the second quadratic term is introduced to impose certain constraints $C = 0$

$$\Pi^* = \Pi_1 + \alpha \Pi_2; \quad \Pi_2 = \int_{\Omega} \mathbf{C}^T \mathbf{C} \, d\Omega \quad \alpha \rightarrow \infty \quad (1)$$

leading on discretization to equations of the form

$$(\mathbf{K}_1 + \alpha \mathbf{K}_2) \mathbf{a} = \mathbf{f} \quad (2)$$

then, unless the matrix \mathbf{K}_2 is singular, the only available solution is $\mathbf{a} \rightarrow 0$ as α becomes very large. The use of approximate integration is one of the methods of ensuring such a singularity.

* Professor.

† Lecturer.

‡ Research Student.

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It is easy to show that the introduction of shear strain energy to the predominating bending energy introduces precisely the same situation with the ratio of the appropriate stiffnesses playing the role of the penalty parameter α .

This greater understanding allowed the introduction of many new elements, in addition to the isoparametric parabolic quadrilateral, to the reduced integration family and many old problems have now been successfully reformulated.⁴ In a paper dealing with the problems of plates Hughes *et al.*⁶ show how a single point numerical integration allows an extremely simple beam element to be generated. In this paper an almost identical process is applied to the generation of an equivalent axisymmetric element family with particular attention being given to its lowest, linear member.

Although the formulation permits the use of curvilinear elements the linear segment will be elaborated. Here the examples will show that the additional errors introduced by this approximation to geometry of curvilinear shells are insignificant providing consistent, single point sampling of stresses is reported. In many situations we shall indeed find that the results with this simple element are superior to those reported in the literature using elaborate curvilinear elements.

BACKGROUND THEORY

In a general curvilinear system the Kirchhoff type 'strain' in an axisymmetric shell (under axisymmetric load) can be written using the notation of Figure 1^{7,8} as

$$\epsilon = \begin{Bmatrix} \epsilon_s \\ \epsilon_\theta \\ \chi_s \\ \chi_\theta \end{Bmatrix} = \begin{Bmatrix} \frac{du}{ds} + \frac{w}{R} \\ (w \cos \phi + u \sin \phi)/r \\ -\frac{d^2w}{ds^2} + d(u/R)/ds \\ -\sin \phi \left(\frac{dw}{ds} - \frac{u}{R} \right)/r \end{Bmatrix} \quad (3)$$

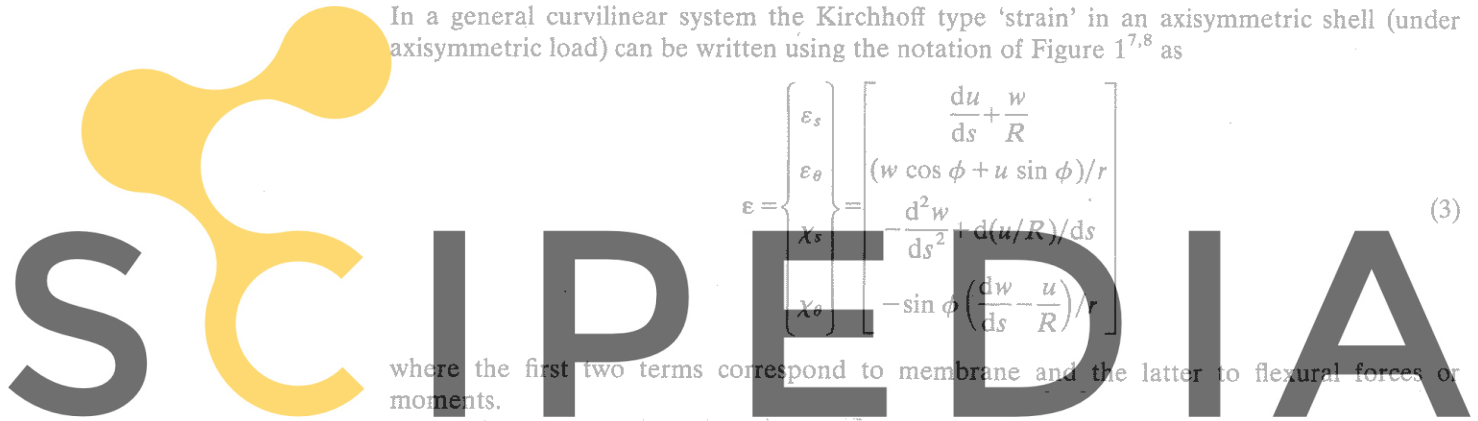
where the first two terms correspond to membrane and the latter to flexural forces or moments.

Introducing now the shear strain γ defined as

$$\gamma = \frac{dw}{ds} - \beta \quad (4)$$

where β is the section rotation, and rewriting appropriately the flexural strains we can redefine the 'strains' as

$$\epsilon = \begin{Bmatrix} \epsilon_s \\ \epsilon_\theta \\ \chi_s \\ \chi_\theta \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \frac{du}{ds} + \frac{w}{R} \\ (w \cos \phi + u \sin \phi)/r \\ -\frac{d\beta}{ds} + d(u/R)ds \\ -\sin \phi \left(\beta - \frac{u}{R} \right)/r \\ \frac{dw}{ds} - \beta \end{Bmatrix} \quad (5)$$



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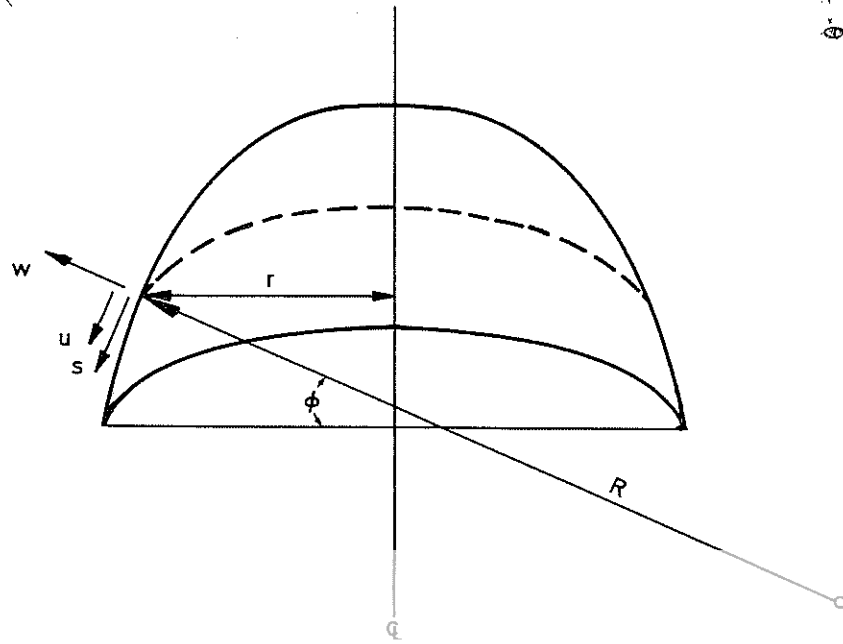


Figure 1. Axi-symmetric shell definition

We can define the appropriate elasticity matrix D as⁹

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$$D = \begin{bmatrix} D^1 & & \\ & D^2 & \\ & & D^3 \end{bmatrix} = \begin{bmatrix} \hat{D} \\ & D^3 \end{bmatrix} \quad (6)$$

where the appropriate matrices D^1 , D^2 and the scalar D^3 stand respectively for axial, flexural and shear rigidities.

In isotropic materials these are

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$$D^1 = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \quad D^3 = k \frac{Et}{2(1+\nu)} = kGt \quad (7)$$

$$D^2 = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \quad k = 5/6$$

but in sandwich type structures much more general values can obviously be used.

The above simple theory is, indeed, identical with that derived directly by Naghdi.¹⁰

If, now, the energy of the system is written as

$$\Pi^* = \frac{1}{2} \int \epsilon^T D \epsilon 2\pi r ds - \int q^T u 2\pi r ds \quad u = \begin{Bmatrix} u \\ w \\ \beta \end{Bmatrix} \quad (8)$$

we note that the energy can be expressed in the 'form' of equation (1) where

$$\Pi_1 = \frac{1}{2} \int \hat{\boldsymbol{\epsilon}}^T \hat{\mathbf{D}} \hat{\boldsymbol{\epsilon}} 2\pi r \, ds - \int \mathbf{q}^T \mathbf{u} 2\pi r \, ds \quad (9a)$$

$$\alpha \Pi_2 = \frac{1}{2} \int \gamma D^3 \gamma 2\pi r \, ds \quad (9b)$$

The first of these includes only axial and flexural strain energy whilst the second involves the shear strain energy terms only. For this shell theory D^3 tends to become very large and can be identified simply as 2α —the penalty number of equation (1) introduced to enforce the Kirchhoff constraint ($\gamma = 0$).

With the above formulation we note that immediately an interpolation involving only C^0 continuity is required as only first derivatives of u , w and β occur. Obviously any of the numerous isoparametric formulations are possible.

With an interpolation of the type

$$\mathbf{u} = \begin{Bmatrix} u \\ w \\ \beta \end{Bmatrix} = \sum N_i \mathbf{a}_i \quad \mathbf{a}_i = \begin{Bmatrix} u_i \\ w_i \\ \beta_i \end{Bmatrix} \quad (10)$$

the form of equation (2) will be obtained in a fairly straightforward manner, involving appropriate integrals with respect to s . Such integrals can be evaluated by an approximate numerical integration which, if convergence order is to be maintained, must be capable *at least* of exactly integrating polynomials of order $2(p-1)$ where p is complete polynomial order contained in the shape function N_i .

To ensure singularity it is now necessary that in the establishment of the overall stiffness matrix K_2 the total number of degrees-of-freedom is less than the number of independent relations provided at the integration points.⁴ For this matrix as only a single such relation is given at each point this matter is easy.

For a straight element the simplest interpolation can be written as

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$$N_i = (1 + \eta_i \eta) / 2 \quad \eta = 2s/L \quad \eta_i = 2s_i/L \quad (11)$$

for an element with nodes at $\eta = \pm 1$.

Standard manipulation with respect to local nodal co-ordinates defined in equation (10) gives the element matrices in a form

$$\mathbf{K}_{ij} = \int_{-1}^1 \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j 2\pi r \, ds \, d\eta$$

$$\mathbf{f}_i = \int N_i \mathbf{q} 2\pi r L \, d\eta \quad (12)$$

where \mathbf{q} is the vector of forces in the direction of u , w , β per unit length of the shell.

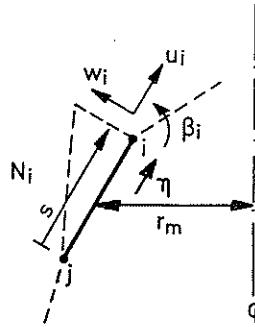


Figure 2. A typical linear element

In the above B_i is the standard strain matrix obtained from equation (5) putting $R = \infty$ as the element is straight, viz

$$B_i = \begin{bmatrix} \frac{dN_i}{ds} & 0 & 0 \\ N_i \frac{\sin \phi}{r} & \frac{N_i \cos \phi}{r} & 0 \\ 0 & 0 & -\frac{dN_i}{ds} \\ 0 & 0 & -\frac{N_i \sin \phi}{r} \\ 0 & \frac{dN_i}{ds} & -N_i \\ \frac{\eta_i/L}{(1+\eta_i\eta)} \sin \phi, & \frac{1+\eta_i\eta}{2r} \cos \phi, & 0 \\ 0 & 0 & -\frac{(1+\eta_i\eta)}{2r} \sin \phi \\ 0 & \eta_i/L & -\frac{(1+\eta_i\eta)}{2} \end{bmatrix} \quad (13)$$

To ensure singularity of the K_2 part of the matrix a single integration point is necessary (which also corresponds with the minimum order of integration required). The remainder of the stiffness matrices can be evaluated exactly but no overall singularity develops if a single integration point is used and in the subsequent examples it was found that a single point integration sufficed for all calculations, no *additional accuracy being gained by a two point integration* of the K_1 part of the matrix. Clearly an explicit form of the element matrices can be written now by simply replacing the integrals in (12) by the appropriate values of the integral evaluated of the element mid point ($\eta = 0$; $r = r_m$).

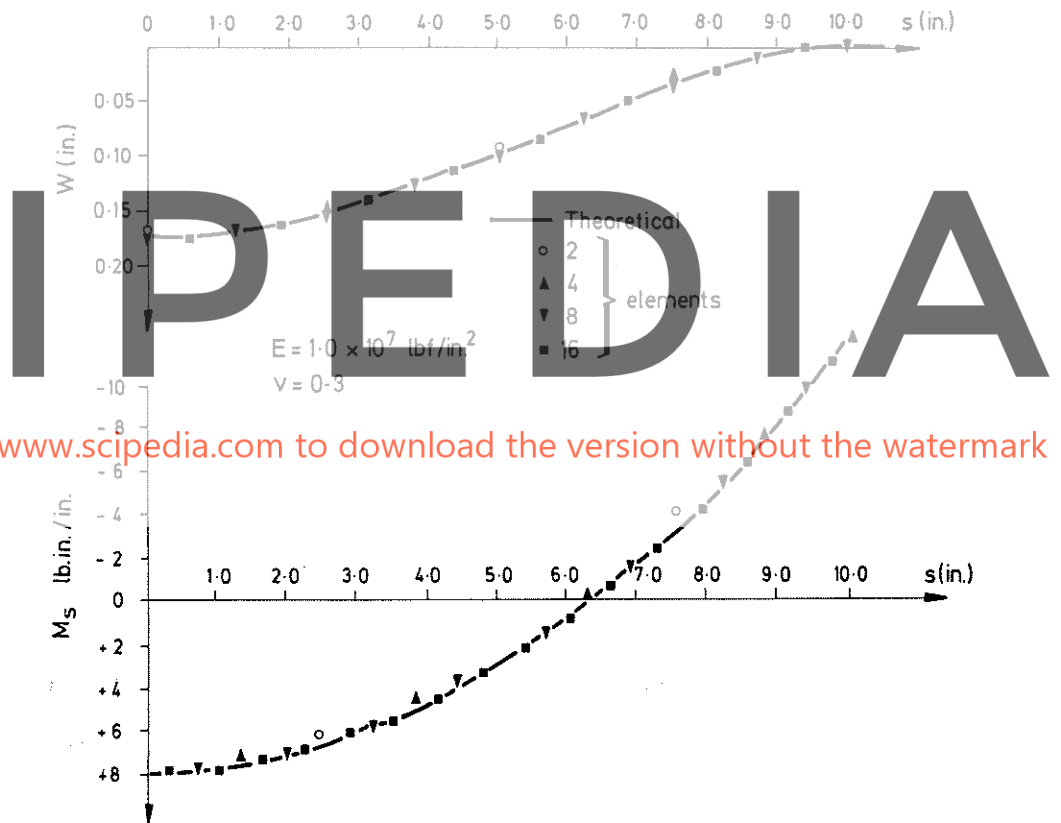
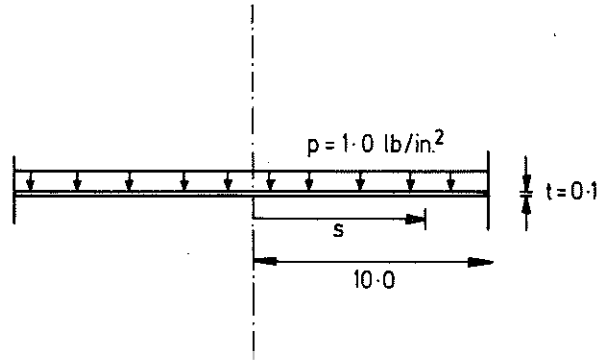


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NUMERICAL EXAMPLES

To demonstrate the effectiveness of the element several thin plate and shell solutions were analyzed.

Figure 3 shows an analysis of a circular clamped plate with various subdivisions. It is remarkable to note that very accurate results are attainable with two elements only.



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Figure 3. Bending of circular plate under uniform load. Convergence study

Figures 4 and 5 show two examples of curved shells. In the first the results are compared to the 'exact' Timoshenko⁸ and to other finite element solutions using curved element formulations.¹²⁻¹⁴ Other solutions¹⁵⁻¹⁶ are shown for comparison in Figure 5 as here exact solutions are not available.

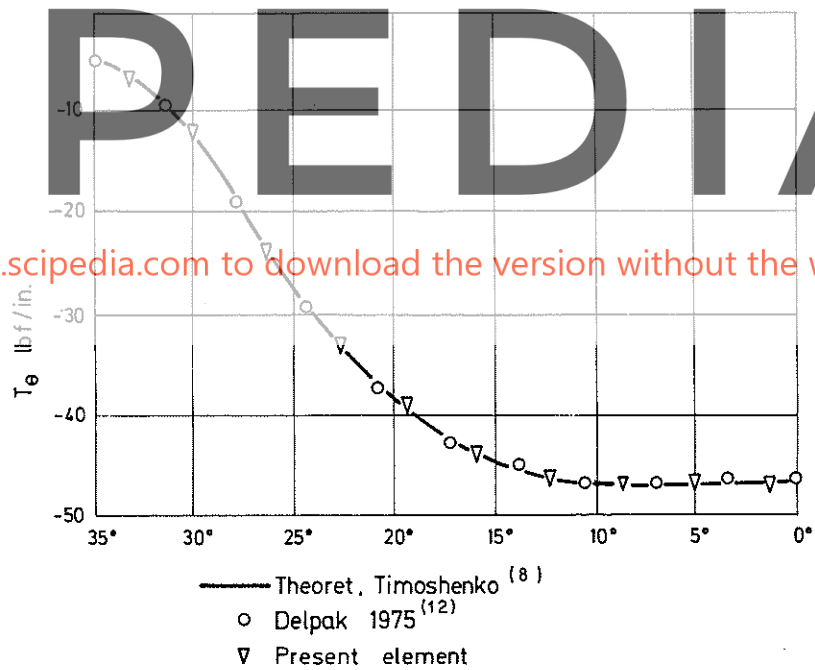
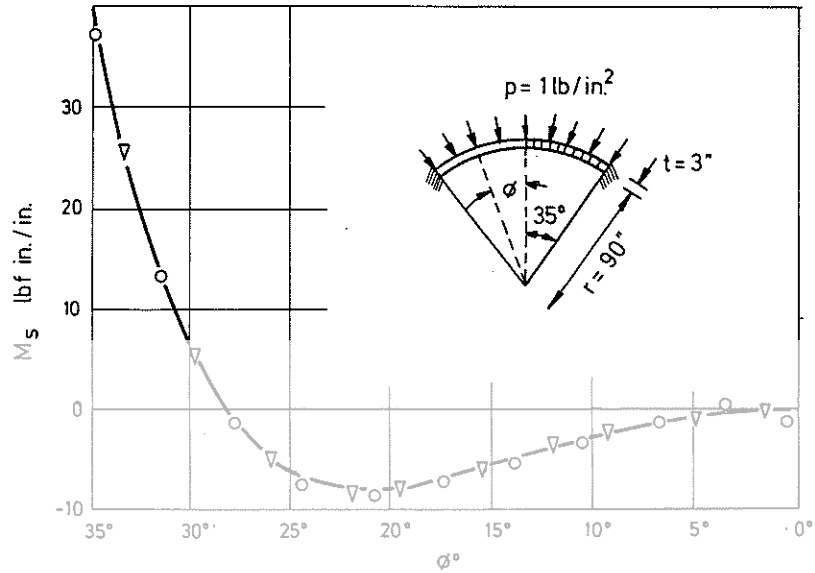
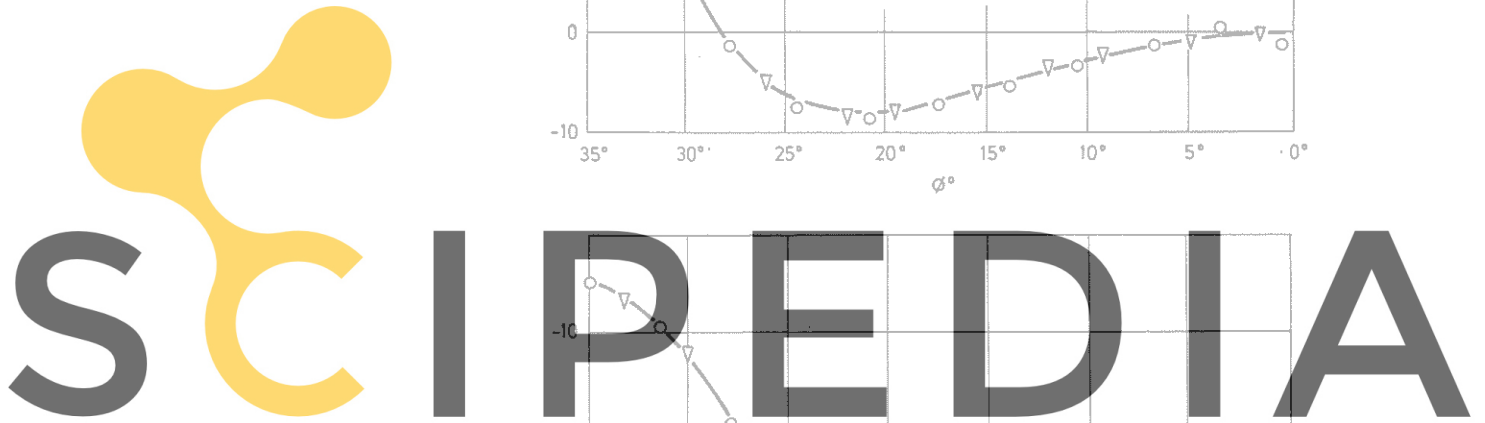


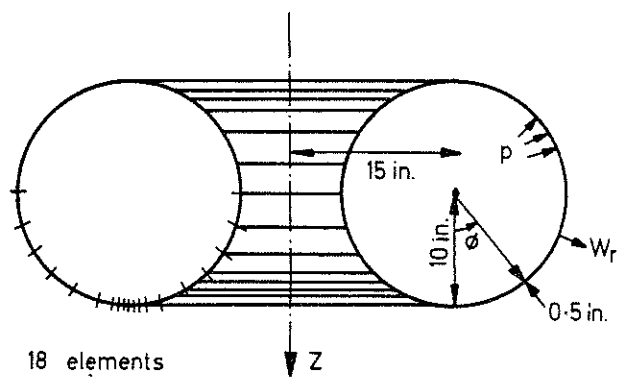
Figure 4. Spherical dome under uniform pressure. $E = 2 \times 10^6 \text{ lb/in}^2$, $\nu = \frac{1}{6}$



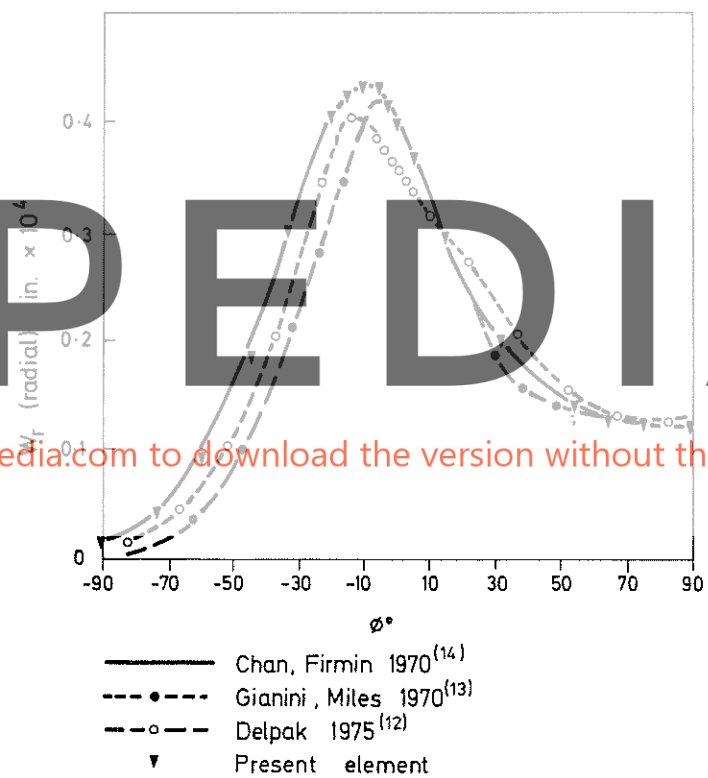
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In the above examples we note that the errors introduced by a straight line approximation to the curved shapes appear insignificant.

Figure 6 shows the use of the present element in a 'branching' solution and once again excellent comparison is attainable with alternative solutions.¹⁷

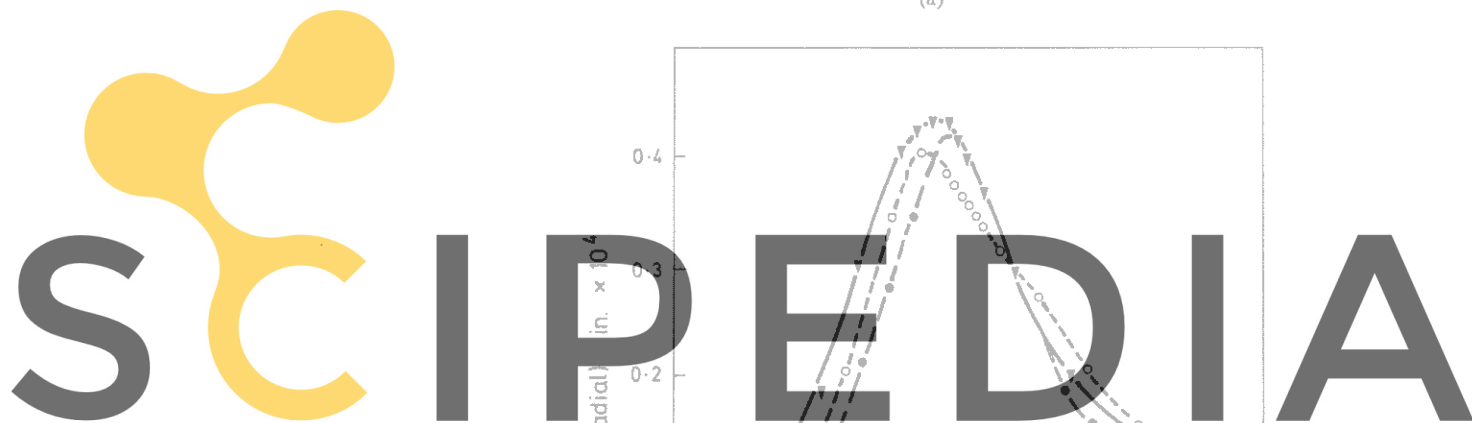


(a)

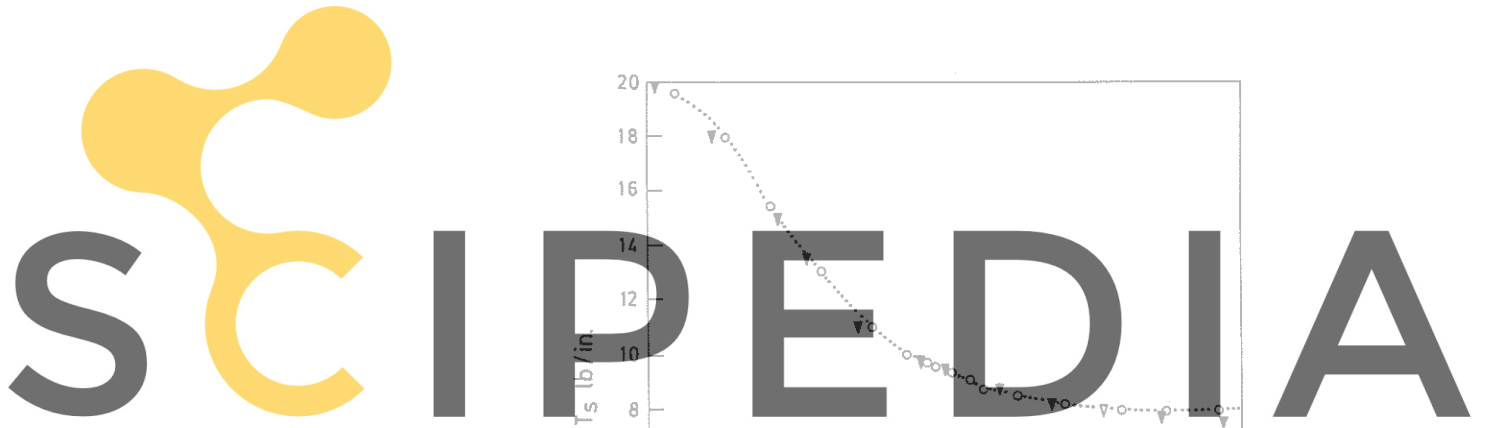
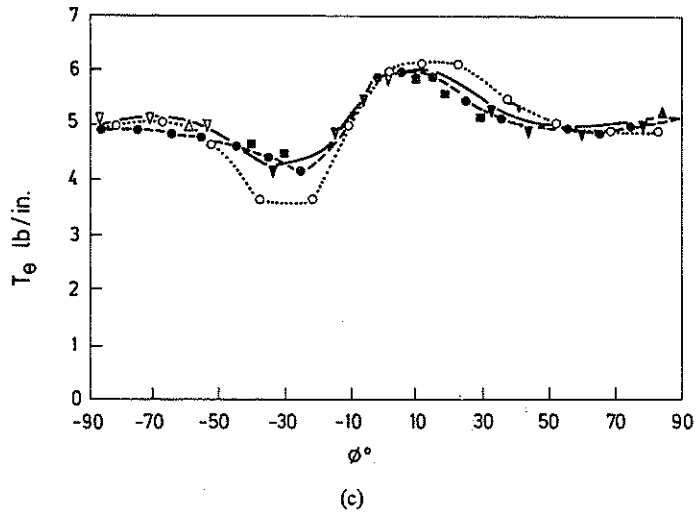


(b)

Figure 5. Toroidal shell under internal pressure. $E = 1.0 \times 10^7$ lbf/in², $\nu = 0.3$, $p = 1.0$ lbf/in²



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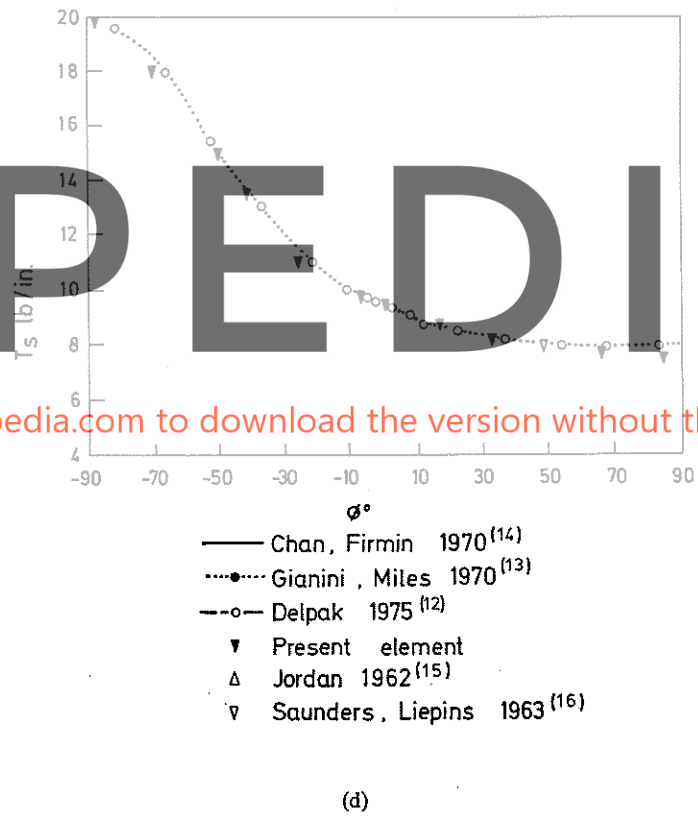
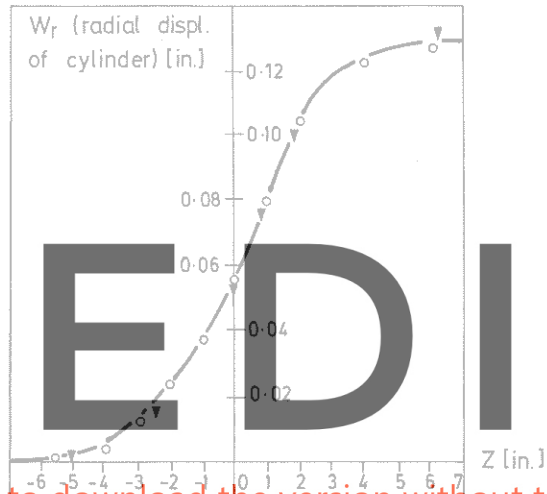
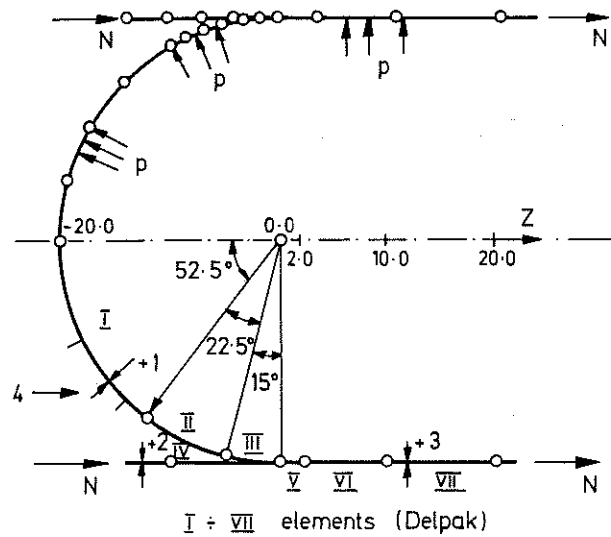


Figure 5. (continued from the previous page)



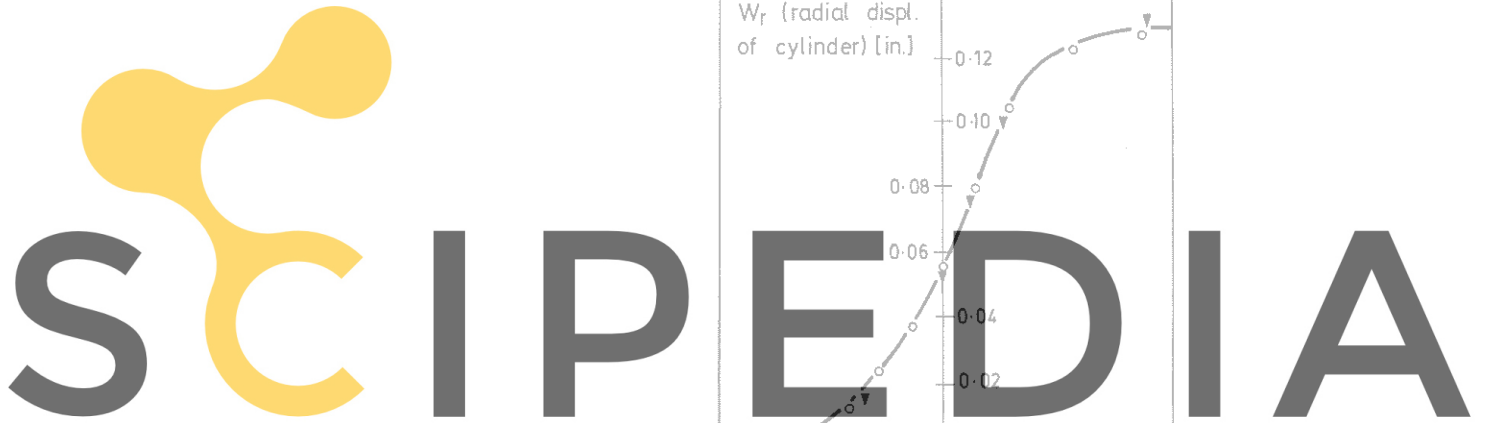
— Analytical, Kraus 1967⁽¹⁷⁾
 ○ Delpak 1975⁽¹²⁾
 ▼ Present element

Figure 6. Branching shell. $E = 1.0 \cdot 10^6$, $\nu = 0.3$, $N = 5000.0$ lbf/in, $p = 1000.0$ lbf/in², $t_1 = 0.4$ in, $t_2 = 0.5$ in, $t_3 = 0.3$ in

The final example is one of application to a more realistic problem of a water tank where the convergence of the solution is indicated by a two mesh analysis (Figure 7).

CONCLUSIONS

The element described in this paper provides not only an illustration of the effectiveness of the 'reduced integration' principle but results in probably the simplest accurate element available



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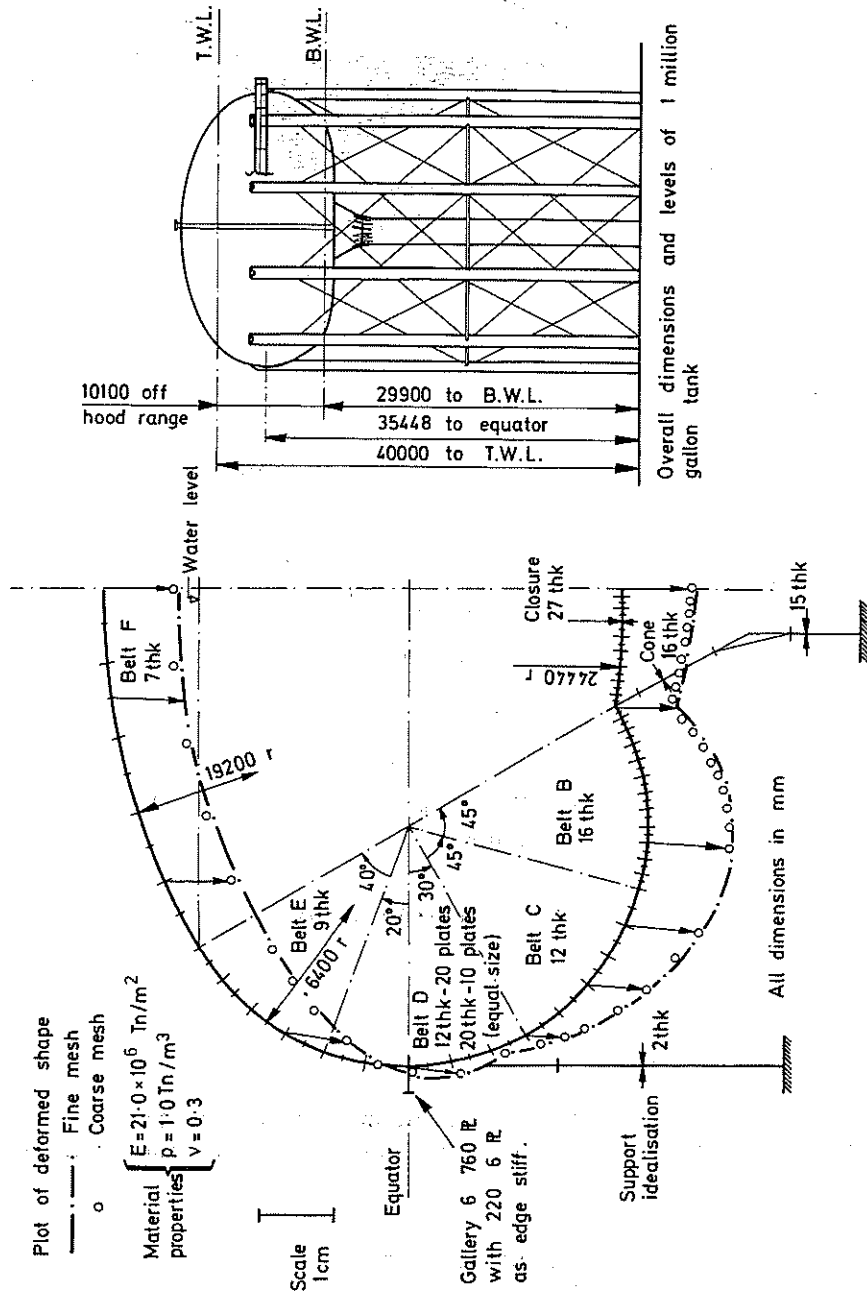


Figure 7(a). A water tank. The structure and finite element idealization. (Two meshes are used, one half of division indicated 40 and 80 element respectively)

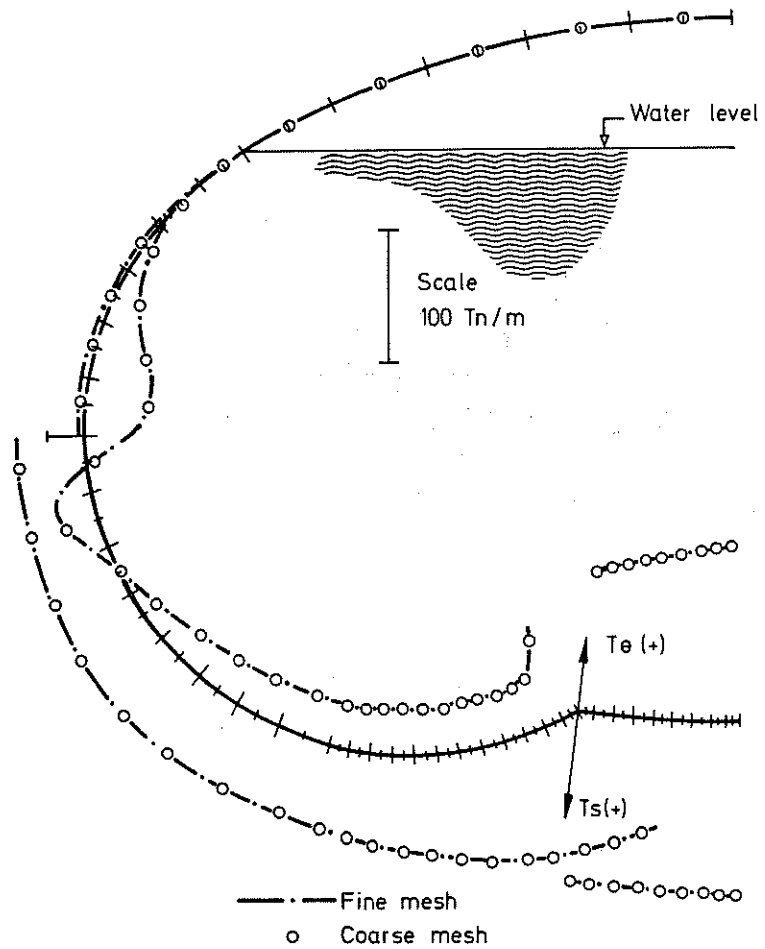


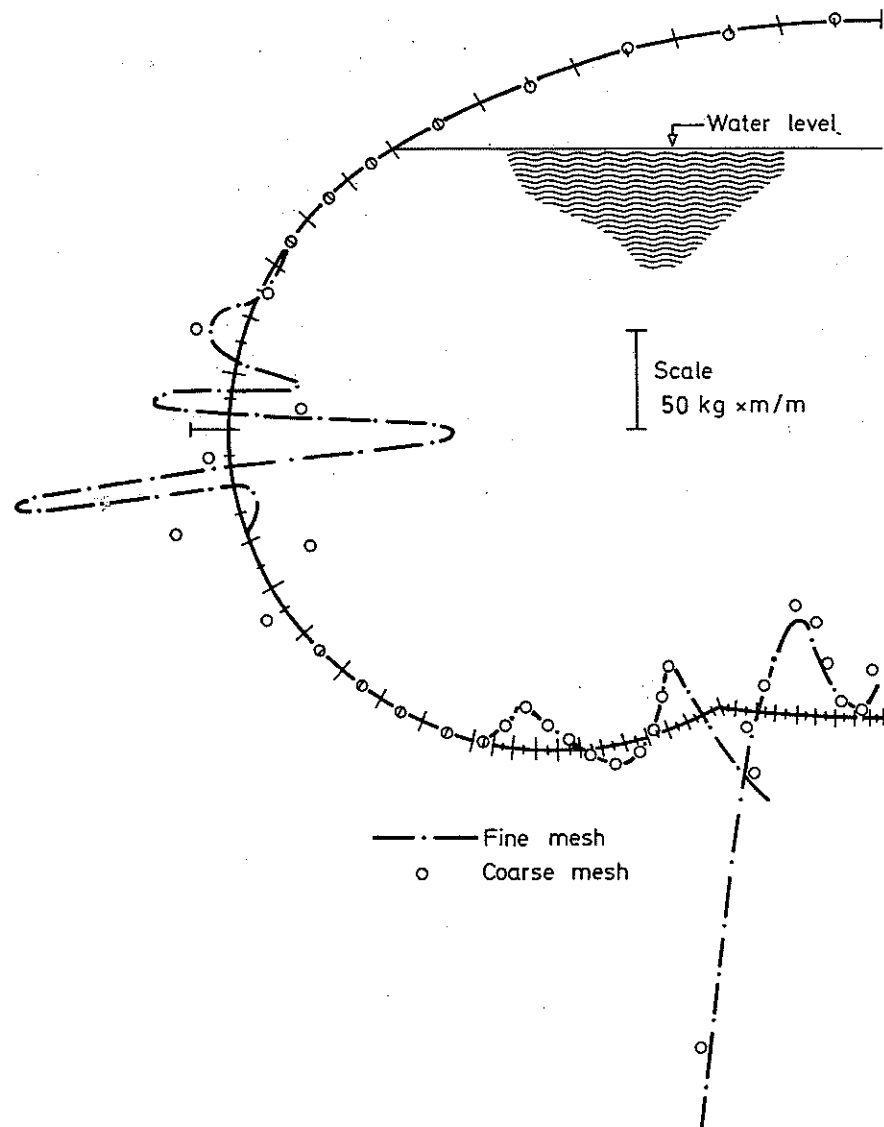
Figure 7(b). Plot of membrane force resultants

for axisymmetric shell analysis. Its algebraic form is extremely simple and extensions for dealing with large deformation, stability or non-axisymmetric load distribution can follow a routine pattern.

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Figure 7(c). Plot of bending moment (M_s) distribution

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