The value of the distant future: Discounting in random environments

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Abstract

We analyze how future costs must be balanced against present costs. This is traditionally done using an exponential function with a constant discount rate. The choice of discount rate can dramatically effect the question on what is the value of the future. This is specially critical for environmental problems such as global warming, and it has generated a controversy as to the urgency for immediate action (Stern, 2006; Nordhaus, 2007a,b). We briefly review the issue for the nonspecialist and take into account the randomness of the economic evolution by studying the discount function of three widely used processes for the dynamics of interest rates: Ornstein-Uhlenbeck, Feller and log-normal. We also outline our previous empirical survey on 14 countries over time spans ranging up to more than 300 years We estimate the parameters of one of the models studied (the Ornstein-Uhlenbeck process) and obtain the long-run discount rate for all these countries. The long-run discount obtained for stable countries (countries that have not suffered periods of destabilizing inflation) supports the low discounting rate proposed by Stern (2006) over higher rates that have been advocated by others (Nordhaus, 2007a,b).

I. Introduction

One important quantitative procedure in economics and finance is that of “discounting”. This process tries to answer a key question: How can we value the future? The discounting mechanism weights the future relative to the present and the weighting method is carried through a discount function which usually takes the form of a decreasing exponential (Samuelson, 1937). Indeed, under a steady rate of interest r, a dollar inverted today, at time $t = 0$, will yield e^{rt} at time $t > 0$. That is to say, a dollar in any future time t is worth e^{-rt} today.

In this simple example r is fixed but in practice rates are uncertain and it is not realistic to represent discounting by a deterministic function of time such as the decreasing exponential with a fixed rate and some kind of average over all possible interest rate paths must be taken. Before developing these ideas, let us remark that the problem of discounting shows its great importance not only in finance but in long-run environmental planning (Dasgupta, 2004). Indeed, and assuming again a steady rate of interest, an environmental problem that costs X to fix at time t in the future is equivalent to an investment of $e^{-rt}X$ today. Thus if r is substantial, any benefit at some distant time would justify a negligible investment now. Letting interest rates to be a proxy for economic growth, a different version of the above argument is that technologies in the future will be so powerful that they will overshadow anything we can achieve with present day technology. In this sense it is more rational to follow policies fostering economic growth than try to combat global warning now.

Remaining within long-run discounting, it is no surprise that the choice of a discount rate has vast consequences and is the object of intense debates and contradictory estimates (Arrow et al., 2013). For example, in a highly influential report on climate change commissioned by the UK government, Stern (2006) uses a discount rate of 1.4%, which on a 100 year horizon implies a present value of 25 % (meaning the future is worth 25% as much as the present). In contrast, Nordhaus (2007b) argues for a discount rate of 4%, which implies a present value of 2%, and at other times has advocated rates as high as 6% (Nordhaus, 2007a), which implies a present value of 0.3%. Stern has been widely criticized for using such a low rate (Nordhaus, 2007b,a; Dasgupta, 2006; Mendelsohn, 2006; Weitzman, 2007; Nordhaus, 2008). The choice of discount rate is probably the biggest factor influencing the debate on the urgency to respond to global warning and the issue is far from being settled. What is the right number? And is it even correct to use an exponential discount?

For environmental problems normative approaches to choosing discount rates are based on ethical grounds (Stern, 2014a,b) and assumptions about economic growth. They also depend on arguments involving the maximization of utility functions that are chosen for mathematical convenience Heal and Milner (2014). Economists present a variety of reasons for discounting, including impatience, economic growth and declining marginal utility; all of them embedded in the Ramsey formula (Ramsey, 1928), which forms the basis for standard approaches to discounting (Arrow et al., 2013).

However, as mentioned above, rates are uncertain and it is not realistic to represent discounting by deterministic functions of time such as decreasing exponentials with a fixed rate and, therefore, some kind of average over all interest rate paths must be taken. This problem is particularly severe for environmental problems, where in questions such as global warming one must consider costs and benefits 100 or more years in the future. It also occurs in finance, where discounting times are typically thirty years or less, where it has long been recognized that interest rates must be modeled as random processes Vasicek (1977); Cox, Ingersoll, and Ross (1985); Dothan (1978); Brigo and Mercurio (2006).

A more positive approach to discounting consists in figuring out how the market trades off present consumption for future consumption. For the near future one can readily find the corresponding market interest rate for money, and by making assumptions about likely inflation one can infer the market discount rate for real consumption (see, for instance, Newell and Pizer 2003, or Farmer et al. 2014). For the distant future, a practical economist engaged in the environmental debate might try to use, as the forward discount rate, the average of historical interest rates which occurred in the last 200 hundred years (2.7 % in stable countries (Farmer et al., 2014)), or take the average of Wall Street forward looking models which price bonds of maturity as long as 30 years. However, we have shown (Farmer et al., 2014) that, due to historical fluctuations of short real interest rates, the appropriate rate is considerably below these averages.

Moreover, the presence of fluctuations can dramatically alter the functional form of the discount function. If interest rates follow a geometric random walk, for example, Farmer and Geanakoplos (2009) (see also Geanakoplos et al., 2014) have shown that in some circumstances the discount function decays as a power law of the form $t^{-1/2}$. They called this hyperbolic discount because the discount factor obeys the equation of an hyperbola instead of the usual exponential function. In the large time limit a hyperbolic function is much greater than any exponentially decaying function, showing that there is no positive long run rate of interest in this case. The hyperbola assigns an infinite value to any permanent positive flow of consumption, meaning that the infinite future is infinitely valuable.

Nonetheless, anecdotal evidence suggests that long-term exponential behavior is the typical case. We have examined a variety of different processes, including more general log-normal processes, the Feller process, and the Ornstein-Uhlenbeck process (Farmer et al. 2015). We have found that the case of the simple log-normal process studied by Farmer and Geanakoplos (2009) was the only one that did not display long-term exponential behavior. All the other examples deviated from exponential behavior for short times, but, for a wide range of parameters, eventually converged to an exponential function. This suggests that, while the transient non-exponential behavior can be important for a few decades, the most important question is the long-term discount rate.

Which model is most appropriate depends on the problem under study. Thus if we deal with environmental problems we should use real rates which are nominal interest rates corrected by inflation. In Sect. IV we present the main results on an empirical survey we have recently done on real rates of 14 countries covering 87 to 318 years (see Farmer et al., 2014, and Sect. IV for details). Data clearly show that in many epochs and for all countries real rates frequently become negative, often by substantial amounts and for long periods of time. In environmental problems we are, therefore, lead to the Ornstein-Uhlenbeck process, since that model allows for negative values, while other processes, such as Feller or log-normal, exclusively deal with positive values. However, financial settings use nominal rates which usually are positive and, therefore, either the Feller or the log-normal processes are more appropriate (Brigo and Mercurio, 2006). In Sect. III we will present a summary on the key results of each model.

Let us finally stress three important facts. Firstly, assuming that costs and benefits can be reduced to monetary values, the discounting problem is equivalent to bond pricing. A bond is an instrument that one can purchase now that delivers a payment in the future. Similarly, to combat climate change we must spend now in order to receive environmental and economic benefits in the future. If we can quantify both the expenditure required now and the likely cost of inaction in the future, then the price of the corresponding bond gives us an indication of the discount factor. We must say, nonetheless, that there are always intangible effects that are difficult to quantify in monetary terms, and one should be suspicious of any procedure that reduces the existence of a species or a human life to a dollar value. But it is nonetheless informative to see what a purely monetary analysis implies.

Secondly, the interest rate for bonds as a function of their time to maturity is called the *yield curve*. Most bonds have a time to maturity of 30 years or less, but for environmental problems such as climate change we need to know the discount 100 years or more into the future. We don't have data on bonds of such long maturity. Thus we are faced with the problem of inferring the price of long maturity bonds from data on much shorter maturity bonds. Furthermore, the yield curve fluctuates substantially from year to year, so we need sufficient historical time series for reliable statistical inference. In order to do this we need a reasonable model for real interest rates at different maturities.

Thirdly, in addition to the factors that determine the overall level of short term rates, there is one effect influencing long term rates that must be taken into account. This is the so-called "risk aversion". The far future is less certain than the near future, so all else equal, we expect that longer term bonds bear greater risk, which should imply higher interest rates.

In the rest of this paper we will develop and summarize all these ideas concerning discounting into a consistent framework which I will try to expose in a clear and intuitive way.

II. The process of discounting

In economics the increment at a given time of the quantity of wealth, exemplified by some magnitude $M = M(t)$, is assumed to depend linearly on the quantity itself and the duration of the variation. For a continuous and instantaneous variation one then writes:

$$dM(t) \propto M(t)dt. \tag{1}$$

This is a phenomenological law based on the empirical fact that the bigger $M(t)$, the greater its variation at a given time, but also on the simplifying assumption that the increment is linear in $M(t)$ and not, for instance, quadratic. Let us incidentally note that linearity is equivalent to assuming that the interest rate, defined as the relative time derivative

$$r = \frac{1}{M(t)}\frac{dM(t)}{dt}, \tag{2}$$

is independent of $M(t)$. Note that this definition can be written as

$$r = \frac{d\ln M}{dt}, \tag{3}$$

so that the rate is the derivative of the logarithm of wealth.

In the simplest situation the growth law (1) represents a completely linear law with direct proportionality in which r is constant:

$$dM(t) = rM(t)dt, \tag{4}$$

where r is the rate and is measured in units of 1/(time). Now the growth law is readily integrated, giving

$$M(t) = e^{r(t-t_0)}M(t_0), \tag{5}$$

which yields an exponential growth connecting wealth at some earlier time t_0.

Before proceeding further we recall that the growth law (1), often in the simplest version (4), appears in numerous branches of physical and social sciences. Thus, for example, in radioactivity if $N(t)$ is the number of active nuclei at time t, the usual hypothesis is that this number decreases as

$$dN(t) = -\lambda N(t)dt,$$

where $\lambda > 0$ is the decay constant. Similar considerations apply to other situations, as they are found in chemical reactions, population dynamics, as in many other places.

In economics, discounting refers to the process of connecting wealth at different times. Specifically the discount function, which we denote by $\delta(t)$, is defined by

$$\delta(t) \equiv \frac{M(t_0)}{M(t)}, \tag{6}$$

so that $M(t_0) = \delta(t)M(t)$ in accordance with the fact that discounting specifically refers to weighting the future at some time t relative to t_0 ($t > t_0$).

In the simplest case of Eq. (5) the discount function is given by the decreasing exponential:

$$\delta(t) = e^{-r(t-t_0)}, \tag{7}$$

where $r > 0$ is the interest rate. However, as we have mentioned above, this simple form of discount, in which the interest rate is always constant, is unrealistic. A first generalization consists in assuming rates to be deterministic functions of time $r(t)$. In such a case the growth law (4) is replaced by

$$dM(t) = r(t)M(t)dt \tag{8}$$

and discount is given by

$$\delta(t) = \exp\left(-\int_{t_0}^{t} r(t')dt'\right). \tag{9}$$

Obviously if $r(t) = r$ is constant we recover the simple exponential decay of Eq. (7).

However, the assumption of rates being given by constants or by deterministic functions of time is unreasonable, at least over long periods of time. Financial interest rates are typically described as random, as the many models for stochastic interest rates appearing in the literature show (Brigo and Mercurio, 2006). Population dynamics are subject to random influences, as are chemical reactions and other physical processes where rates appear.

We therefore assume that $r(t)$ is a random function of time. This naturally means that discounting $\delta(t)$ is also random, as is clearly seen in Eq. (9). In these circumstances the effective discount function is defined as the average of $\delta(t)$:

$$D(t) = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} r(t')dt'\right)\right], \tag{10}$$

where the expectation $\mathbb{E}[\cdot]$ represents the average over all real trajectories of $r(t)$ up to time t and t_0 is an arbitrary initial time.[1] Let us note that this is formally identical to the problem of pricing bonds. The price $B(t_0|t+t_0)$ of a zero-coupon bond issued at time t_0 with unit payoff and maturing at time $t + t_0$ ($t \geq 0$) is (Brigo and Mercurio, 2006)

$$B(t_0|t+t_0) = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} n(t')dt'\right)\right],$$

where $n(t)$ is the nominal rate. The differences between these two problems is that for discounting we are interested in real interest rates $r(t)$ –which can be negative due to inflation– whereas for bond pricing we are typically interested in the nominal rate $n(t)$.

The function $r(t)$ can, in principle, be any random process. However, the simplest and most common hypothesis consists in assuming that is that rates are described by a Markovian process with continuous sample paths. That is, we assume that $r(t)$ is a diffusion process whose time evolution is governed by a stochastic differential equation of the form

$$dr = f(r)dt + g(r)dW(t), \tag{11}$$

where $f(r)$ is the drift, $g(r) > 0$ is the noise intensity and $W(t)$ is the standard Wiener process.

In terms of the cumulative process

$$x(t) = \int_{t_0}^{t} r(t')dt', \tag{12}$$

the discount function can be written as

$$D(t) = \mathbb{E}\left[e^{-x(t)}\right].$$

[1]Usually t_0 refers to the present time, which in our case and without loss of generality (see below), can be taken equal to zero, i.e., $t_0 = 0$.

Therefore,

$$D(t) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} e^{-x} p(x, r, t|x_0, r_0, t_0) dx, \tag{13}$$

where $p(x, r, t|x_0, r_0, t_0)$ is the probability density function (PDF) of the bidimensional diffusion process $(x(t), r(t))$. The measure corresponding to the density p is sometimes referred to as the *data generating measure.*

From Eqs. (11)-(12) we see that $(x(t), r(t))$ is defined by the following pair of stochastic differential equations

$$\begin{aligned} dx &= r dt, \\ dr &= f(r)dt + g(r)dW(t), \end{aligned} \tag{14}$$

which implies that the joint density obeys the (forward) Fokker-Planck equation (Gardiner, 1986)

$$\frac{\partial p}{\partial t} = -r\frac{\partial p}{\partial x} - \frac{\partial}{\partial r}[f(r)p] + \frac{1}{2}\frac{\partial^2}{\partial r^2}[g^2(r)p]. \tag{15}$$

Since $x(t_0) = 0$ and $r(t_0) = 0$, the initial condition of this equation is

$$p(x, r, t_0|x_0, r_0, t_0) = \delta(x)\delta(r - r_0). \tag{16}$$

Let us incidentally note that since $f(r)$ and $g(r)$ do not depend explicitly on time, the process is time homogeneous, that is to say, invariant under time translations $(t \to t - t_0)$ and we can set $t_0 = 0$ without loss of generality.

There are two different approaches for obtaining the discount function $D(t)$. One of them, which is standard in the financial literature, is based on the backward Fokker-Planck equation and it is called the *Feynman-Kac approach* (Brigo and Mercurio, 2006). A second procedure is based on Fourier analysis (Farmer et al., 2015). Let us next succinctly present both approaches.

Feynman-Kac approach

The Feynman-Kac approach obtains a a partial differential equation for the discount function $D = D(t|r_0)$ which is based is the backward Fokker-Planck equation. This equation is called the Feynman-Kac equation and reads (Brigo and Mercurio, 2006)

$$\frac{\partial D}{\partial t} = -r_0 D + f(r_0)\frac{\partial D}{\partial r_0} + \frac{1}{2}g^2(r_0)\frac{\partial^2 D}{\partial r^2}, \tag{17}$$

with the initial condition

$$D(0|r_0) = 1. \tag{18}$$

Fourier transform approach

We have recently presented an alternative method for obtaining the discount function (Farmer et al., 2014, 2015) which turns out to be quite advantageous in linear cases and it is based on the characteristic function. The latter defined as the Fourier transform of the joint density:

$$\tilde{p}(\omega_1, \omega_2, t|r_0) = \int_{-\infty}^{\infty} e^{-i\omega_2 r} dr \int_{-\infty}^{\infty} e^{-i\omega_1 x} p(x, r, t|r_0) dx. \tag{19}$$

One of the chief advantages of working with the characteristic function is that obtaining the effective discount is straightforward. Indeed, comparison of Eq. (13),

$$D(t|r_0) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} e^{-x} p(x, r, t|r_0) dx,$$

with Eq. (19) shows that

$$D(t|r_0) = \tilde{p}\big(\omega_1 = -i, \omega_2 = 0, t|r_0\big). \tag{20}$$

Therefore, in order to obtain the discount function we only need to know the joint characteristic function of the bidimensional process (x, r).

Adding risk aversion

As we have mentioned at the end of Sect. I, the far future is less certain than the near future and we should expect that longer term discount bear greater risk, which would imply higher interest rates. In finance these risk factors are taking into account by considering the so-called "market price of risk" (Vasicek, 1977; Brigo and Mercurio, 2006).

In the context of bond pricing, if investors are risk neutral then prices can reasonably be modeled based on the data generating measure p which is the solution of the Fokker-Planck equation (15) with initial condition (16). The discount function $D(t)$ is then obtained through the Fourier transform $\tilde{p}$ or, alternatively, by solving the Feynman-Kac equation (17) with initial condition (18). This is sometimes called the *Local Expectation Hypothesis* (Cox, Ingersoll, and Ross, 1981; Gilles and Leroy, 1986). However a more general assumption is that investors are sensitive to risk, in such a case bonds can no longer be priced in this way. Instead they are priced with an artificial probability density function, p^*, usually called *risk-neutral measure*. The two measures p and p^* are related by the *market price of risk*, which is the extra return per unit risk that investors demand to bear risk. This additional return is given by a quantity $q = q(r, t)$ that in its most general form may depend on the rate r and current time t. although the most usual assumption is that $q = q(r)$ only depends on the rate (Vasicek, 1977). Following a standard procedure for bond pricing (Vasicek, 1977; Piazzesi, 2009) one takes risk into account by replacing the drift $f(r) \to f^*(r)$, where

$$f^*(r) = f(r) + g(r)q(r), \tag{21}$$

where $q(r) \geq 0$ is the market price of risk.[2] In this case the risk-neutral measure $p^*(x, r, t|r_0)$ will be given by the Fokker-Planck equation (15) with $f(r)$ replaced by $f^*(r)$, that is,

$$\frac{\partial p^*}{\partial t} = -r\frac{\partial p^*}{\partial x} - \frac{\partial}{\partial r}\Big[\big[f(r) + g(r)q(r)\big]p^*\Big] + \frac{1}{2}\frac{\partial^2}{\partial r^2}\big[g^2(r)p^*\big], \tag{22}$$

with initial condition given by Eq. (16). In an analogous way, the discount function, adjusted for risk, will now be given by the Feynman-Kac equation (17) with $f(r)$ replaced by $f^*(r)$. Using the Fourier method the discount function will be given in terms of the risk-neutral characteristic function, $\tilde{p}^*(\omega_1, \omega_2, t|r_0)$, by [cf. Eq. (20)]

$$D(t|r_0) = \tilde{p}^*\big(\omega_1 = -i, \omega_2 = 0, t|r_0\big). \tag{23}$$

[2] The form of $q(r)$ is, in principle, unknown and has to be conjectured. The simplest and habitual assumption is that $q(r) = q$ is constant. In such a case the value of q is estimated from empirical data.

III. Some models

Financial economists have developed a large number of models of interest rate processes to enable them to price bonds and other cash flows. In these models interest rates are described by positive random processes since financial interest rates never (or very rarely) go negative. Although the models could in principle be extended to arbitrary horizons, they have only been studied carefully over time horizons of up to 30 years, since bonds are rarely issued for periods longer than this.

Environmental economists are nevertheless interested in the real behavior of the economic growth over much larger horizons, in contrast to financial economists, who are typically more interested in nominal rates over shorter horizons. Their behavior is essentially different due to the fact that real rates can take on negative values. Indeed, taking nominal rates corrected by inflation as a proxy of economic growth, we have recently shown (Farmer et al., 2014) thorough an empirical study on many countries that real interest rates are negative around 25 % of the time (see Sect. IV).

To understand how discounting depends on the random process used to characterize interest rates we have studied three different models and obtained exact analytical expressions for the discount function (Farmer et al., 2015). The three models describe to varying degree a number of relevant characteristics observed in rates, while being simple enough to allow for complete analytical treatment.

The first model is based on the Ornstein-Uhlenbeck process (called Vasicek model in financial literature) which allows for negative rates and is therefore suitable for pricing environmental problems. The model has a stationary probability distribution and exhibits reversion to the mean, which means that the process tends to return to its average stationary value. The second and third models considered are given by the Feller and log-normal processes respectively. For these processes rates cannot be negative. The Feller process –also known as Cox Ingersoll and Ross (CIR) model– has reversion to the mean and a stationary probability distribution. It is one of the most popular models in finance. On the other hand, the log-normal process (Dotham model) does not have reversion to the mean and does not have a stationary distribution. Despite these shortcomings the process has also been used in the financial literature mainly because it is positive and allows for analytical treatment (Brigo and Mercurio, 2006). Let us next briefly review these models and summarize their main results.

1 - Vasicek model

In this model rates are described by the Ornstein-Uhlenbeck process (Vasicek, 1977), which is characterized by linear drift and constant noise intensity

$$dr(t) = -\alpha[r(t) - m] + kdW(t), \tag{24}$$

where $r(t)$ is the rate and $W(t)$ the Wiener process. The parameter m, sometimes refereed to as "normal level," is a mean value to which rates revert, $k > 0$ is the amplitude of fluctuations, and $\alpha > 0$ is the strength of the reversion to the mean. As we will see in the next section these parameters are estimated from empirical data.

The model is Gaussian and has a stationary probability distribution (as $t \to \infty$) given by (Gardiner, 1986)

$$p_s(r) = (\alpha/\pi k^2)e^{-\alpha(r-m)^2/k^2},$$

which proves that the normal level m is the stationary mean value,

$$m = \mathbb{E}[r(t)]. \tag{25}$$

It can also be shown that the correlation function of the process, defined as the average

$$C(\tau) = \mathbb{E}\big[r(t+\tau)r(t)\big] - \big[\mathbb{E}\big[r(t)\big]\big]^2,$$

in the stationary state reads

$$C(\tau) = (k^2/2\alpha)e^{-\alpha\tau}, \tag{26}$$

which means that α^{-1} is the correlation time of the rate. Let us observe that the volatility, $\sigma^2 = C(0)$, is independent of the normal level and given by

$$\sigma^2 = k^2/2\alpha. \tag{27}$$

For this model it is possible to obtain an exact expression for the discount function $D(t)$ that reads (Farmer et al., 2015)

$$\begin{aligned} \ln D(t) &= -\frac{r_0}{\alpha}\left(1-e^{-\alpha t}\right) \\ &+ \frac{k^2}{2\alpha^3}\left[\alpha t - 2\left(1-e^{-\alpha t}\right) + \frac{1}{2}\left(1-e^{-2\alpha t}\right)\right] - m\left[t - \frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\right], \end{aligned} \tag{28}$$

where $r_0 = r(0)$ is the initial rate. Note that the exponential terms in Eq. (28) are only significant at small times, that is to say, for times smaller than the correlation time of the rate ($t < \alpha^{-1}$) but they are negligible at longer times. We thus have

$$D(t) \simeq e^{-r_\infty t}, \tag{29}$$

where

$$r_\infty = m - k^2/2\alpha^2, \tag{30}$$

is the long-run discount rate. Let us note an important fact that the long-run discount rate is smaller than the mean value of the return given by the normal level m. This reduction is quantified by the ratio k/α, which means, for instance, that long persistence (recall that this is equivalent to long correlation time, i.e., α small) or else increasing noise fluctuations (i.e., k large) decrease the long-run discount rate as compared with the average rate.

Risk aversion

As mentioned above risk aversion is taken into account by introducing the market price of risk $q(r)$ and changing drift according to Eq. (21). For the Vasicek model, in which $f(r) = -\alpha(r-m)$ and $g(r) = k$, we have

$$f^*(r) = -\alpha(r-m) + kq(r), \tag{31}$$

and taking q constant, we write

$$f^*(r) = -\alpha(r-m^*), \tag{32}$$

where

$$m^* = m + \frac{qk}{\alpha}. \tag{33}$$

Since the modified drift $f^*(r)$ has the same form that $f(r)$ we conclude that the adjusted-for-risk discount function will be given by Eq. (28) after the replacement $m \to m^*$. In particular, the adjusted long-run discount now reads [cf. Eq. (30)]

$$r_\infty^* = m + \frac{qk}{\alpha} - \frac{k^2}{2\alpha^2}. \tag{34}$$

Thus we see that the long-run discount depends on the historical rate m, but this is shifted by two terms. The first term raise the long-run rate due to the market price of risk. The second shift lowers it by an amount given by the ratio of uncertainty (as measured by k) and persistence (as measured by α). We can trivially rewrite the equation above as

$$r_\infty = m + \frac{k}{\alpha}\left(q - \frac{k}{2\alpha}\right). \tag{35}$$

This makes it clear that whether or not the overall shift in the long-run discount rate is positive or negative depends on the size of the market price of risk in relation to the ratio of the volatility parameter and the reversion rate.

It is not surprising that the market price of risk raises the long term rate, but it is not so obvious that uncertainty and persistence can lower it. Indeed for the Orstein-Uhlenbeck process it can make it arbitrarily small. For any given mean interest rate m, by varying k and α, the long-run discount rate r_∞ can take on any value less than m, including negative values, while at the same time the standard deviation σ can also be made to take on any arbitrary positive value.

It is even possible for the long-run rate to be negative. This is due to the amplification of negative real interest rates $r(t)$. Computation of the discount function involves an average over exponentials, rather than the exponential of an average. As a result, periods where interest rates are negative are amplified, and can easily dominate periods where interest rates are large and positive, even if the negative rates are rarer and weaker. It does not take many such periods to substantially reduce the long run interest rate.

To summarize, in the Vasicek model, and even taking into account risk aversion, the long-run discounting rate can be much lower than the mean, and indeed can correspond to low interest rates that are rarely observed.

2 - CIR model

In this model rates are described by the Feller process (Cox, Ingersoll, Ross, 1985). The process is a diffusion process with linear drift and linear diffusion coefficient (Feller, 1951),

$$dr(t) = -\alpha[r(t) - m]dt + k\sqrt{r(t)}dW(t), \tag{36}$$

where, as in the Ornstein-Uhlenbeck process, $m > 0$ represents the mean stationary rate and α^{-1} is the correlation time. It can be shown that the Feller process never attains negative values (Feller, 1951; Masoliver and Perelló, 2012) and it is, therefore, suitable for modeling financial nominal rates rather than real rates. The process is not Gaussian and the stationary density is given by the Gamma distribution (Farmer et al., 2015)

$$p_s(r) = \frac{(2\alpha/k^2)^\theta}{\Gamma(\theta)} r^{\theta-1} e^{-(2\alpha/k^2)r},$$

where

$$\theta = \frac{2\alpha m}{k^2} \tag{37}$$

is a positive constant that combines all the parameters of the model into a single expression. As in the Vasicek model m is the stationary mean value at which the process reverts. The stationary correlation function is also given by an exponential decreasing in time,

$$C(\tau) = (mk^2/2\alpha)e^{-\alpha\tau}.$$

Note that α^{-1} is again the correlation time but, contrary to the Vasicek model, the volatility $\sigma^2 = mk^2/2\alpha$ depends on the normal level m as well.

For the CIR model it is also possible to obtain the exact expression for the discount function. The result reads (Brigo and Mercurio, 2006; Farmer et al., 2015)

$$D(t) = \left[\frac{2\lambda e^{-(\lambda-\alpha)t/2}}{(\lambda+\alpha)+(\lambda-\alpha)e^{-\lambda t}}\right]^\theta \exp\left\{-\frac{2(1-e^{-\lambda t})r_0}{(\lambda+\alpha)+(\lambda-\alpha)e^{-\lambda t}}\right\}, \tag{38}$$

where r_0 is the initial rate, θ is defined in Eq. (37), and

$$\lambda = \sqrt{\alpha^2 + 2k^2}. \tag{39}$$

Notice that $\lambda > \alpha$ and the time scale represented by λ^{-1} is smaller than the correlation time α^{-1}.

As time increases (in fact, when $\lambda t \gg 1$) the effective discount (38) reduces to

$$D(t) \simeq e^{-r_\infty t} \tag{40}$$

$(t \to \infty)$, where

$$r_\infty = \frac{1}{2}(\lambda - \alpha)\theta \tag{41}$$

is the long-run discount rate of the CIR model. Substituting for Eqs. (37) and (39) this can be written as

$$r_\infty = \frac{2m}{1+\sqrt{1+2k^2/\alpha^2}}, \tag{42}$$

which clearly shows that

$$r_\infty < m.$$

Therefore, like the Vasicek model, the CIR long-run discount rate is smaller than the stationary average rate by an amount that also depends on the square of the ratio k/α. Notice that, again, either a long persistence (α small) or an increase of the noise intensity (k large) diminish the long-run discount rate.

Risk aversion

For the Feller process [cf. Eq. (36)] $f(r) = -\alpha(r-m)$ and $g(r) = k\sqrt{r}$ and the adjusted drift is

$$f^*(r) = -\alpha(r-m) + kq(r)\sqrt{r}. \tag{43}$$

For any function $q(r)$ (including a constant market price of risk) this leads to an unsolvable Fokker-Planck equation which no analytical expression for the adjusted discount and the long-run discount rate. It is, nonetheless, possible to get analytical expressions for these quantities if the market price of risk has the functional form

$$q(r) = q\sqrt{r}, \tag{44}$$

where $q \geq 0$ is a positive quantity. In such a case we may write

$$f^*(r) = -\alpha^*(r - m^*), \tag{45}$$

where

$$\alpha^* = \alpha - kq, \qquad m^* = \frac{\alpha m}{\alpha - kq}. \tag{46}$$

The adjusted drift has the same form than $f(r)$. Therefore, the adjusted discount function will be given Eq. (38) with the replacements $\alpha \to \alpha^*$ and $m \to m^*$ and the long-run discount is [cf. Eq. (42)]

$$r_\infty^* = \frac{2m^*}{1 + \sqrt{1 + 2k^2/\alpha^{*2}}}. \tag{47}$$

From the definitions of α^* and m^* we easily see that $\alpha^* \le \alpha$ and $\alpha^* m^* = \alpha m$. Hence, writing r_∞^* as

$$r_\infty^* = \frac{2\alpha^* m^*}{\alpha^* + \sqrt{\alpha^{*2} + 2k^2}},$$

we see at once that $r_\infty^* \ge r_\infty$. We therefore conclude that if the market price of risk has the special functional for given by Eq. (44), in the CIR model risk always increases the long-run discount rate regardless noise intensity and persistence.

3 - Log-normal model

In this mode rates are described by the the geometric Brownian motion (log-normal process). It can be written as

$$\frac{dr}{r} = \alpha dt + k dW(t), \tag{48}$$

where r is the interest rate, α and k are constant parameters, α may be positive or negative whereas k is always positive and $W(t)$ is the Wiener process. Equation (48) can be integrated at once yielding

$$r(t) = r_0 \exp\left\{\left(\alpha - \frac{k^2}{2}\right) t + kW(t)\right\}, \tag{49}$$

showing that $r(t)$ is never negative ($r_0 > 0$). Therefore, the log-normal model is more suited for modeling nominal interest rates in finance, which are never negative, than for modeling real rates in environmental economics. Contrary to OU and Feller processes, the log-normal process does not show reversion to the mean. Indeed, as t increases we see from Eq. (49) that the rate either diverges when $\alpha > 0$ or goes to zero if $\alpha < 0$. In an equivalent way one can also show from Eq. (49) that the mean and variance of the process are

$$\langle r(t) \rangle = r_0 e^{\alpha t}, \qquad \mathrm{Var}[r(t)] = r_0^2 e^{2\alpha t}\left(e^{k^2 t} - 1\right).$$

The discount associated with the log-normal process model was studied in 1978 (Dothan, 1978) and in finance is usually known as the Dothan model. Because it allows for analytical treatment it is one of the models used in the literature (Brigo and Mercurio, 2006). In Farmer et al. (2015) we have obtained the discounting function and discussed some of its interesting asymptotic properties. Let us here summarize the main results.

Contrary to Vasicek and CIR models where it is possible to obtain exact expressions for the discount function $D(t)$. For the log-normal case we can only obtain the exact expression of the Laplace transform,

$$\hat{D}(s) = \int_0^\infty e^{-st} D(t) dt.$$

The resulting formula –written as an integral of special functions, the Kummer function– is rather intricate and we won't write it here (see Farmer et al., 2015, for more information). From that expression we can, nonetheless, get asymptotic expressions as $t \to \infty$ for discount in real time using the so-called Tauberian theorems which relate the small s behavior of $\hat{D}(s)$ with the long-time behavior of $D(t)$ (Pitt, 1958). The final result are the following asymptotic expressions for the discount function $D(t)$ as $t \to \infty$, which, incidentally, for long-run discount is all that matters (Farmer et al., 2015)

$$D(t) \sim \begin{cases} \text{constant} & \alpha < k^2/2, \\ e^{-r_\infty t} & \alpha > k^2/2, \\ t^{-1/2} & \alpha = k^2/2. \end{cases} \tag{50}$$

The asymptotic form of the discount function thus depends on the values taken by the ratio α/k^2 between the strength of the strength of the deterministic drift α and the amplitude of fluctuations given by $k^2/2$.

(i) The case $k^2/2 > \alpha$ corresponds to strong fluctuations, where the noise intensity $k^2/2$ is greater than the drift parameter α. In this case the discount asymptotes to a constant value (see Farmer et al., 2015, for the actual value of this constant).

(ii) The case $k^2/2 < \alpha$ corresponds to mild fluctuations for which deterministic drift is stronger than noise. In such a case the discount function has the expected exponential decay (Farmer et al., 2015)

$$D(t) \sim e^{-r_\infty t}, \tag{51}$$

with a long-run rate of discount given by

$$r_\infty = \frac{1}{\delta}\left(\alpha - \frac{k^2}{2}\right), \tag{52}$$

where $0 < \delta < 1$ is a positive numerical factor which only depends on the ratio $2\alpha/k^2$ and reads

$$\delta = \psi\left(2\alpha/k^2\right) + \frac{1}{2\alpha/k^2 - 1}, \tag{53}$$

where $\psi(\cdot)$ is the digamma function.

Let us write Eq. (51) in a more suggestive form. Indeed, from Eq. (49) we see that

$$\mathbb{E}\left[\ln \frac{r(t)}{r_0}\right] = \left(\alpha - \frac{k^2}{2}\right) t,$$

and with the help of Eq. (51) we write Eq. (51) as

$$D(t) \sim \exp\left\{-\frac{1}{\delta}\mathbb{E}\left[\ln \frac{r(t)}{r_0}\right]\right\}, \tag{54}$$

($t \to \infty$ and $k^2/2 < \alpha$). Note that the average $\mathbb{E}[\ln r(t)/r_0]$ is what a practitioner would take as an estimate of the discount rate up to time t within the log-normal model. Since $0 < \delta < 1$, the analytical result (54) shows that the actual long-run rate of the model is a fraction of the average rate. We have shown elsewhere that the long-run discount rate is at most 73 % of the average rate (Farmer et al., 2015). In this way when $2\alpha/k^2 > 1$ the log-normal model follows a similar pattern to that of the OU and Feller models: In all of them the long-run rate is smaller than the average rate. This general statement is in fact a direct consequence of Jenesen's inequality, which states that the average of a convex function is greater than or equal to the function of the average; that is, $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$. Assuming f to be the decreasing exponential and X the cumulative process $x(t)$ defined in Eq. (12), it follows immediately that the long-run rate r_∞ must be always less than or equal to the average rate. Nonetheless, our procedure quantifies the difference among averages (Farmer et al., 2015).

(iii) The critical case $\alpha = k^2/2$, in which deterministic motion and fluctuations are balanced, leads to the hyperbolic discount function as obtained by Farmer and Geanakoplos (2009). The hyperbolic $D(t)$ is substantially greater than any exponential decaying function, showing that there is no long-run rate of interest in this case. In fact the long-run rate of interest is 0, but that does not convey as precise information as saying $D(t)$ is approximately $k/\sqrt{t}$ for all large t. Since the sum (i.e., the integral) of all these $D(t)$ is infinite, such $D(t)$ assign infinite value to any permanent positive flow of consumption: the infinite future is infinitely valuable.

Risk aversion

Let us very briefly comment on the inclusion of risk aversion in the Dothan model. For the log-normal process $f(r) = \alpha r$ and $g(r) = kr$ and

$$f^*(r) = [\alpha + kq(r)]r.$$

Assuming a constant market price of risk, $q(r) = q \geq 0$, we have

$$f^*(r) = \alpha^* r, \qquad \alpha^* = \alpha + q.$$

Again $f^*(r)$ has the same form than $f(r)$ and all previous results will apply with the replacement $\alpha \to \alpha^*$.

IV. Empirical study

To understand how discounting depends on the random process used to characterize interest rates, we have collected data for nominal interest rates and inflation for fourteen countries over spans of time ranging from 87 to 318 years. The countries in our sample are: Argentina (ARG, 1864-1960), Australia (AUS, 1861-2012), Chile (CHL, 1925-2012), Germany (DEU, 1820-2012), Denmark (DNK, 1821- 2012), Spain (ESP, 1821-2012), United Kingdom (GBR, 1694-2012), Italy (ITA, 1861-2012), Japan (JPN, 1921-2012) (see Table I), Netherlands (NLD, 1813-2012), Sweden (SWE, 1868-2012), the United States (USA, 1820-2012), and South Africa (ZAF, 1920-2012). Some examples are plotted in Figure 1. Since all but two of our nominal interest rate processes are for ten year government bonds, which pay out over a ten year period, we smooth inflation rates with a ten year moving average, and subtract the annualized inflation index from the annualized nominal rate to compute the real interest rate.

Real rates are nominal rates corrected by inflation. Nominal rates $n(t)$ are determined by IG rates constructed from the 10 year Government Bond Yield $y(t|\tau)$ with τ= 10 years.[3] Nominal rates are then estimated by

$$n(t) \sim y(t|\tau).$$

The inflation rate is estimated through the Consumer Price Index (CPI) as

$$i(t) \sim \frac{1}{\tau} \ln\big[I(t+\tau)/I(t)\big],$$

[3] If $B(t|t+\tau)$ is the government bond maturing at time $t+\tau$ with unit maturity, $B(t|t) = 1$, then the yield $y(t\tau)$ is defined as $y(t|\tau) = -\ln B(t|t+\tau)/\tau$, so that $B(t|t+\tau) = e^{-\tau y(t|\tau)}$.

	Country	Consumer Price Index	Bond Yields	from	to	records
1	Italy	CPITAM annual from 12/31/1861 quarterly from 12/31/1919	IGITA10 quarterly	12/31/1861	09/30/2012	565
2	Chile	CPCHLM quarterly	IDCHLM quarterly	03/31/1925	09/30/2012	312
3	Canada	CPCANM quarterly	IGCAN10 quarterly	12/31/1913	09/30/2012	357
4	Germany	CPDEUM annual from 12/31/1820 quarterly from 12/31/1869	IGDEU10 quarterly	12/31/1820	09/30/2012	729
5	Spain	CPESPM annual from 12/31/1821 quarterly from 12/31/1920	IGESP10 quarterly	12/31/1821	09/30/2012	709
6	Argentina	CPARGM annual from 12/31/1864 quarterly from 12/31/1932	IGARGM quarterly	12/31/1864	03/31/1960	342
7	Netherlands	CPNLDM annual	IGNLD10D annual	12/31/1813	12/31/2012	189
8	Japan	CPJPNM quarterly	IGJPN10D quarterly	12/31/1921	12/31/2012	325
9	Australia	CPAUSM annual from 12/31/1861 quarterly 12/31/1991	IGAUS10 quarterly	12/31/1861	09/30/2012	564
10	Denmark	CPDNKM annual from 12/31/1821 quarterly from 12/31/1914	IGDNK10 quarterly	12/31/1821	09/30/2012	725
11	South Africa	CPZAFM quarterly	IGZAF10 quarterly	12/31/1920	09/30/2012	329
12	Sweden	CPSWEM annual	IGSWE10 annual	12/31/1868	09/30/2012	135
13	United Kingdom	CPGBRM annual	IDGBRD* annual	12/31/1694	12/31/2012	309
14	United States	CPUSAM annual	TRUSG10M annual	12/31/1820	10/30/2012	183

Table I List of the data analyzed. Notes (i) Chile: we have taken the Discount (ID) rate since the Government Bond Yield data was not available. (ii) Germany: From 06/30/1915 to 03/31/1916 IGDEU is empty and the previous available record has been repeated . (iii) Spain: From 07/31/1936 to 12/31/1940 no records available. 07/31/1936 is empty and the previous available record has been repeated . (iv) Netherlands: 2/31/1945 is empty and the previous available record has been repeated : (v) Japan: From 12/31/1946 to 09/30/1948 is empty and the previous available record has been repeated .

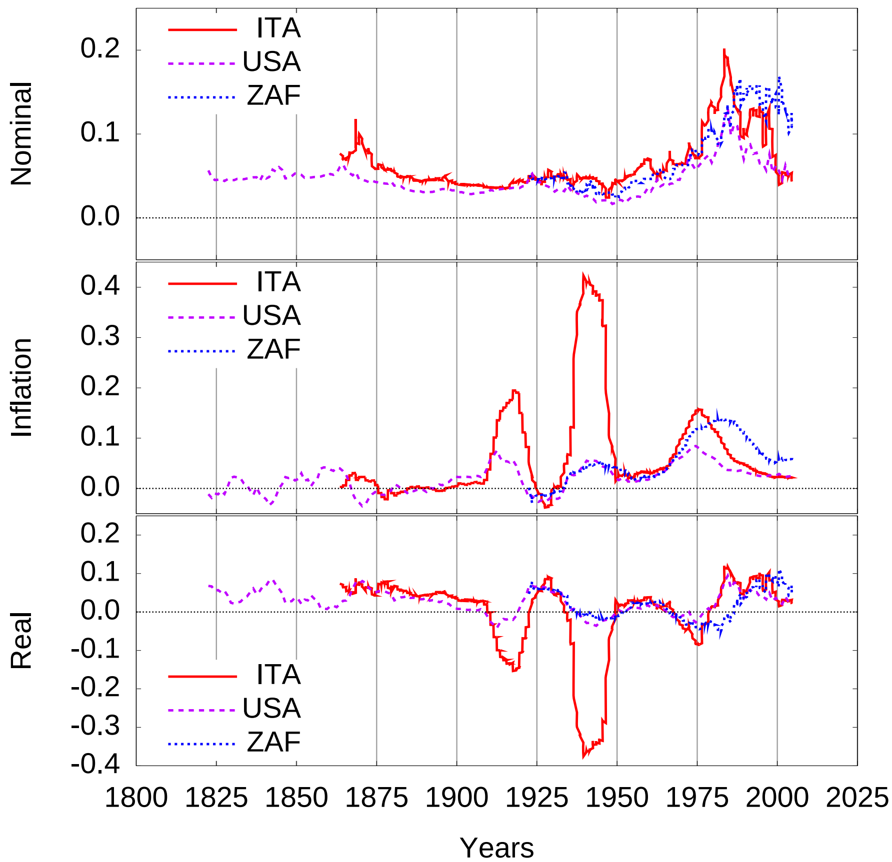

Figure 1. (Color online) Real interest rates display large fluctuations and negative rates are not uncommon. We show nominal interest rates (top), inflation (middle), and real interest rates (bottom) for Italy (ITA), United States (USA) and South Africa (ZAF).

where $I(t)$ is the aggregated inflation up to time t, and τ =10 years.[4] Finally, real interest rates $r(t)$ are defined via Fisher's procedure, subtracting realized inflation from nominal interest rates

$$r(t) = n(t) - i(t).$$

A striking feature observed in many epochs for all countries is that real interest rates frequently become negative, often by substantial amounts and for long periods of time (see Table II). On average, real interest rates are negative one quarter of the time. This makes the CIR and Dothan models less interesting for modeling real interest rates, as well as many other models which assume that interest rates are positive (Brigo and Mercurio, 2006) . We, therefore, confine the empirical work to the Vasicek model. We also assume the local expectation hypothesis for which we live in a risk neutral world and the market price of risk is zero. This is obviously a first approximation, specially for long-run discounting.[5]

[4]The relation between $I(t)$ and the Consumer Price Index (CPI) is $I(t+\tau) = I(t)\prod_{j=0}^{\tau-1}\big[1 + C(t+j)\big]$, where $C(t)$ is the time series of the empirical CPI. The instantaneous rate of inflation $i(t)$ is, therefore, estimated by the quantity $i(t+\tau)$ which written in terms of the CPI reads $i(t) \sim i(t+\tau) = (1/\tau)\sum_{j=0}^{\tau-1}\ln\big[1 + C(t+j)\big]$.

[5]Adding risk, after assuming a constant market price of risk, is under current investigation.

Country	Neg RI	m	Min	Max	k	Min	Max	α	r_∞
Italy	28% (40y)	−0.3	−9.1	5.6	6.9	0.8	10.1	0.22	−5.4
Chile	56% (43y)	−6.8	−20.2	12.0	25.2	5.6	44.1	0.40	−26
Canada	22% (20y)	**2.9**	0.1	6	**2.3**	1.1	2.0	0.26	**2.5**
Germany	14% (25y)	−10.7	−51.0	4.0	33.9	0.9	61.4	0.20	−160
Spain	25% (45y)	5.7	−0.5	13.5	2.9	1.2	3.6	0.06	−6.4
Argentina	20% (17y)	**2.4**	−2.9	6.8	**6.2**	2.8	6.7	0.39	**1.1**
Netherlands	17% (33y)	**3.2**	0.8	5.4	**1.6**	0.8	2.2	0.14	**2.4**
Japan	33% (26y)	−2.2	−7.8	4.0	9.7	1.1	13.2	0.24	−10
Australia	23% (33y)	**2.6**	−0.7	4.9	**2.3**	0.7	2.8	0.19	**1.9**
Denmark	18% (33y)	**3.2**	1.5	4.3	**2.3**	1.1	2.9	0.23	**2.7**
South Africa	43% (36y)	**1.8**	−2.2	5.5	**2.5**	1.2	2.0	0.21	**1.1**
Sweden	28% (38y)	**2.3**	−0.3	3.9	**2.5**	0.6	3.4	0.25	**1.9**
United Kingdom	14% (45y)	**3.3**	1.4	4.3	**1.9**	1.0	2.4	0.19	**2.8**
United States	31% (36y)	**2.6**	1.0	4.0	**1.8**	1.2	2.1	0.18	**2.1**
Stable countries	23% (33y)	**2.7**	−0.14	5.0	**2.6**	1.04	2.94	0.23	**2.1**
Unstable counntries	31% (36y)	**−2.9**	17.7	1.8	**16**	1.9	26.5	0.22	**−42**

Table II Parameter estimation of the Vasicek model (using real rates) in yearly units. Notes (i) "Neg RI" gives the percentage of time and the total number of years in which real interest rates are negative. (ii) The columns m, k (in %) and α are estimates taking each country time series; r_∞ (in %) is evaluated from Eq. (30). (iii) The Min and Max columns illustrate the robustness of the estimation procedure by providing the minimum and the maximum value of parameter estimation on four equal length data blocks. (iv) α is estimated by fitting the empirical correlation function to an exponential (cf. Eq. (26)) after using the whole data block. (v) Countries in boldface are those considered historically more stable with positive long-run rates $r_\infty > 0$.

We estimate the parameters m, α and k of the Vasicek model to each of the data series. The parameter m is easily estimated because it is the stationary mean value of the rate [cf. Eq. (25)]

$$m = \mathbb{E}[r(t)].$$

The estimation of parameters α and k is based on the correlation function of the Ornstein-Uhlenbeck process. Thus from Eq. (26) we have

$$C(t-t') = \frac{k^2}{2\alpha} e^{-\alpha|t-t'|}.$$

Evaluating then the empirical correlation from data and fitting it by an exponential we estimate α (measured in year units) for each country. The third and last parameter, k, is obtained from the empirical standard deviation $\sigma^2 = \mathbb{E}[|r(t) - m|^2]$, which for the Vasicek model is given by Eq. (27). That is,

$$k = \sigma\sqrt{2\alpha}.$$

The resulting parameters for all countries are listed in Table II along with its maximum and minimum value for each country.

Once the parameters of the Vasicek model have been estimated, the long-run discount rate is readily evaluated from Eq. (30),

$$r_\infty = m - k^2/2\alpha^2.$$

The fourteen countries divide into two very clear groups. Nine countries, with relative stable (and positive) real interest rates, have long run positive rates (boldface type in Table II). The

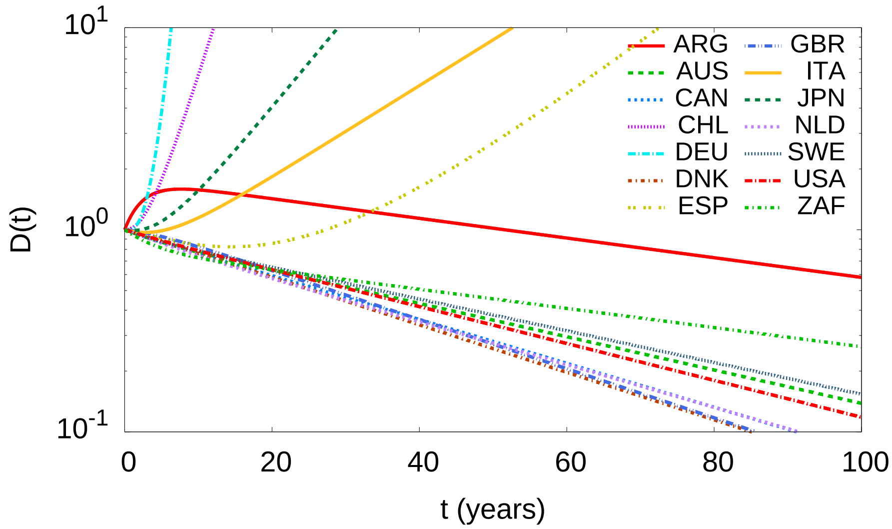

Figure 2. (Color online) The discount function $D(t)$, Eq. (28), as a function of time for the Ornstein-Uhlenbeck model for the fourteen countries in our sample (note that the vertical axis is in a logarithmic scale, we are thus plotting $\ln D(t)$ as a function of t). In order to calculate $D(t)$ we have taken the parameters reported in Table II and use them in Eq. (28). Notice that $D(t)$ quickly reaches its long-run exponential behavior. The long-run rates of the unstable countries vary dramatically, while most of the stable countries are fairly similar.

average historical rate for these nine countries is $\overline{m} = 2.7\%$ while the average long-run rate is $\overline{r}_\infty = 2.1\%$ which is on average 29% lower than $\overline{m}$. Five countries with less stable behavior have long-run negative rates and an exponentially increasing discount. (It may not be a coincidence that all five have experienced fascist governments). In four cases the average rate m is negative due to at least one period of runaway inflation; the exception is Spain, which has a (highly positive) mean real interest rate, but still has a long-run negative rate.

In Fig. 2 we show the exact discount function $D(t)$ given by Eq. (28) for all countries as a function of time, illustrating the dramatic difference between the two groups. In most cases the behavior is monotonic; however, it can also be non-monotonic, as illustrated by Argentina, which initially increases and then decreases.

V. Concluding remarks

Financial economists have developed a large number of models of interest rate processes to enable them to price bonds and other cash flows. In these models interest rates are described by positive random processes since financial interest rates –the nominal rates– never (or very rarely) go negative. Although the models could in principle be extended to arbitrary horizons, they have only been studied carefully over time horizons of up to 30 years, since bonds are rarely issued for periods longer than this.

On the other hand, environmental economists are interested in the real behavior of the economic growth over much larger horizons, in contrast to financial economists, who are typically more interested in nominal rates over shorter horizons. Their behavior is essentially different due to the fact that real rates can take on negative values. Taking nominal rates corrected by inflation as a proxy of economic growth, we have seen from an empirical survey on 14 countries that real interest rates are negative more than 25 % of the time.

To understand how discounting depends on the random process used to characterize interest rates we have studied three different models and obtained exact analytical expressions for the discount function. The three models describe to varying degrees a number of relevant characteristics observed in rates, while being simple enough to allow for complete analytical treatment.

In the first model rates are represented by the Ornstein-Uhlenbeck process (Vasicek model) which allows for negative rates and is therefore suitable for pricing environmental problems. The model has a stationary probability distribution and exhibits reversion to the mean, which means that the process tends to recur to its average stationary value.

In the second and third models considered rates are represented by the Feller and log-normal processes respectively. For these processes rates cannot be negative. The Feller process (CIR model) has reversion to the mean and a stationary probability distribution constituting one of the most popular models in finance. On the other hand, the log-normal process (Dotham model) has no reversion to the mean nor a stationary distribution.

We have carried out the empirical study of real rates and bearing in mind that real interest rates may be negative we have thus used the Vasicek model. When we estimate the parameters of the Vasicek model –that is m, α and k, assuming no risk aversion– of the nine countries which never faced destabilizing inflation, we find an average historical rate $\overline{m} = 2.7$ % whereas, due to fluctuations, the long-run discounting has an average of $\overline{r}_\infty = 2.1$ %, which is around 22 % smaller than the historical average represented by $\overline{m}$. Let us incidentally note that our value of 2.1 % is closer to Stern's estimate (1.4 %) than that of Nordhhaus (4 %).

It is also worth mentioning the case of the United Kingdom where the historical rate over more than 300 years is 3.3 % while the long-run discount rate is $r_\infty = 2.8$ % (see Table II). This long-run discount is very close to the one recently obtained by Giglio, Maggiori and Stroebel (2015) who, using data on housing markets in the United Kingdom during 2004–2013 and Singapore during 1995–2013, have estimated an annual discount rate of 2.6 % for payments more than 100 years in the future.

We finish with the following reflection aimed at environmental concerns and with which we had finished one of our papers on the problem (Farmer et al., 2014): "Real interest rates are typically closely related to economic growth, and economic downturns are a reality. The great depression lasted for 15 years, and the fall of Rome triggered a depression in western Europe that lasted almost a thousand years. In light of our results here, arguments that we should wait to act on global warming because future economic growth will easily solve the problem should be viewed with some skepticism. When we plan for the future we should always bear in mind that sustained economic downturns may visit us again, as they have in the past".

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