MODELS, EXPERIMENTS AND COMPUTATION IN TURBULENCE
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# LES TURBULENCE MODELS. RELATION WITH STABILIZED NUMERICAL METHODS.

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## 1 INTRODUCTION

Understanding of turbulence, its quantification, prediction, simulation and control has turned into one of the most complex and important problems in science and engineering.

It is accepted that Navier-Stokes equations, used to describe the behaviour of viscous incompressible fluids, describe properly turbulent phenomena. Consequently, considering the enormous capacity of actual computers, it is possible to consider that high precision numerical simulations of the Navier-Stokes equations can solve the problem of turbulence. Unfortunately, at current rhythm of growing of computing power, the attempts of direct numerical simulation of Navier-Stokes equations have been limited to low Reynolds numbers, $\mathrm{Re}$.

This type of direct simulations are usually known by its English acronym DNS. The reason for this limited success of the DNS is explained by means of the heuristic estimator of Kolmogorov, $O\left(\mathrm{Re}^{9/4}\right)$, of the necessary degrees of freedom to simulate a flow to a certain Reynolds number. Considering the current advance of the computation technology, this estimator indicates that the possibility of using DNS for flows with high Reynolds numbers is still surely distant.

From its beginnings the attempts of simulating turbulence have been focused on models based on the average in time or in space of magnitudes involved in the problem (velocity, pressure,...) originating the models of turbulence associated with the RANS equations (Reynolds Averaged Navier-Stokes) like $k-\varepsilon$, $k-\omega$,... These models have been widely used in engineering as an alternative to the impossibility to overcome the difficulties of DNS.

The scene is even more complicated by the lack of a mathematical theory for turbulence deduced from the Navier-Stokes equations, as well as for the models used in numerical simulation. This fact explains that many of the current methods for the study of this phenomenon are often based on heuristic or empirical hypotheses.

In recent years a significant progress has been carried out in the development of new turbulence models based on the fact that not the entire range of scales of the flow is interesting for the majority of engineering applications. In this type of applications information contained in "the large scales" of the flow is enough to analyse magnitudes of interest as velocity, temperature,... Therefore, the idea that the global flow behaviour can be correctly approximated without the necessity to approximate the smaller scales correctly, is seen by many authors as a possible great advance in the modelling of turbulence. This fact has originated the design of turbulence models that describe the interaction of small scales with large scales. These models are commonly known as Large Eddy Simulation  models (from now on, LES).

Many LES models have been proposed [1] but no satisfactory mathematical justification has been still found [3].

One of the aims of this text is to show some important results in LES modelling and to identify which are main mathematical problems for the development of a complete theory. A relevant aspect of LES theory, which we will consider in our work, is the close relationship between the mathematical properties of LES models and the numerical methods used for their implementation. This relation has been pointed out by several authors, as Ferziger [2]: "In general, there is a close connection between numerical methods and the modelling approach used in flow simulation; this connection has not been sufficiently appreciated by many authors". Under the light of this observation, it will be shown how most LES models can be considered to be regularization techniques, at the continuous or discrete level. LES models are numerical schemes to solve problems as solution uniqueness, existence of a maximum principle, convergence to suitable solutions, or convergence in graph norm. In last years it is more and more common the idea in the scientific community, especially in the numerical community, that turbulence models and stabilization techniques play a very similar role. Methodologies used to simulate turbulent flows, RANS or LES approaches, are based on the same concept: unability to simulate a turbulent flow using a finite discretization in time and space. Turbulence models introduce additional information (impossible to be captured by the approximation technique used in the simulation) to obtain physically coherent solutions. On the other side, numerical methods used for the integration of partial differential equations (PDE) need to be modified in order to able to reproduce solutions that present very high localized gradients. These modifications, known as stabilization techniques, make possible to capture these sharp and localized changes of the solution. According with previous paragraphs, the following natural question appears:

*Is it possible to reinterpret stabilization methods as turbulence models?*

This question suggests a possible principle of duality between turbulence modelling and numerical stabilization. More than to share certain properties, actually, it is suggested that the numerical stabilization can be understood as turbulence. The opposite will occur if turbulence models are only necessary due to discretization limitations instead of a need for reproducing the physical behaviour of the flow. Finally: can turbulence models be understood as a component of a general stabilization method? These questions are still open. It is possible to find in the literature attempts of connecting these two fields as reported in, [23][24][25][30][31][32]. Ideas proposed in these works are still vague and they do not provide a conclusive response.

This text is organized in the following way. Section 2 introduces the notation and preliminary comments used along the text, such as the concept of suitable weak solution. Section 3 presents a mathematical definition of LES model. Then, in section 4 the phenomenon of energy cascade, narrowly related to LES modelling, is reviewed. Section 5 presents the concept of filtering in LES, seen by many authors as the paradigm of LES. It will be seen that this technique give rise to a paradox for LES models. Additionally it is proved that filtering techniques can be viewed as regularization techniques to solve the solution uniqueness. In section 6, Smagorinsky model is presented as a case of regularization based on a p-Laplacian operator and to solve the problem of solution uniqueness. In section 7 models based on multi-scale methods are checked. Section 8 introduces the stabilization technique termed Finite Calculus (FIC) developed by Oñate *et al*. [36][38][40][41] for the finite element solution of incompressible flows [41][42][43]. The stabilization terms introduced by the FIC approach provide to the numerical scheme with an intrinsic features to model flows at low and high Reynolds numbers as recent shown in [61]. Section 9 introduces the conceptual frame where to answer previous questions and to indicate which properties of the FIC method contribute to explore the relationship between numerical stabilization and turbulence. Finally section 10 presents conclusions and final remarks.

## 2 NOTATION AND PRELIMINARS

This section introduces the notation and definitions of function spaces of common use in the text. In general, the fluid domain is considered an open subset $\Omega \subset \mathbb{R}^3$ with Lipschitz regular boundary $\Gamma$.

**Definition 1 (3D torus)**

*When $\Omega = (0, 2\pi)^3$ and periodic boundary conditions are assigned, then domain is called 3D torus.*

Vectors with real or complex coefficients are denoted in bold letters. The hermitic vector norm is notated by $|\cdot|$, equivalent to $l^2$-norm. For a multi-index $\mathbf{k} \in \mathbb{Z}^d$ we defined $|\mathbf{k}|_\infty = \max_{1 \le i \le d} |k_i|$, equivalent to $l^\infty$-norm.

For $1 \le p \le +\infty$, $L^p(\Omega)$ is the $\mathbb{C}$-vectorial space of Lebesgue measurable functions, such that:

$$\begin{aligned} &\int_\Omega |f(\mathbf{x})|^p < +\infty \text{ si } 1 \le p < \infty \\ &|f(\mathbf{x})| < +\infty \text{ si } p = \infty \end{aligned} \tag{2.1}$$

Partial derivative of a function $u$ respect to a variable $\xi$ is denoted as $\partial u / \partial \xi$; in case that $u$ is one variable function, $\xi$, then $d_\xi u$. As usually $W^{m,p}(\Omega)$ designates the Sobolev space formed by functions in $L^p(\Omega)$ and such that its partial derivatives (in weak sense, if necessary) until order $m$ also belong to $L^p(\Omega)$. The function norm in $W^{m,p}(\Omega)$ will be denoted by $\|\cdot\|_{m,p}$.

### 2.1 Navier-Stokes equations

In general, it is accepted that the Navier-Stokes equations are an admissible model to describe turbulent viscous incompressible flows.

The Navier-Stokes equations for incompressible viscous flows are

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f} & \text{in } (0,\mathrm{T})\times\Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } (0,\mathrm{T})\times\Omega \\ \mathbf{u}|_\Gamma = 0 & \text{or periodic} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \end{cases} \tag{2.2}$$

where $\mathbf{u}$ is the velocity vector, $p$ is the pressure and $\mathbf{u}_0$ is the velocity at initial time and $\mathbf{f}$ is the source term, or the external forces term, and $\nu$ the kinematic viscosity. It has to be noted that that density $\rho$ is taken equal to the unit. Boundary conditions given simplify the Navier-Stokes problem but do not affect the intrinsic properties of the solution.

Consider the following vector function spaces:

$$X = \begin{cases} H_0^1(\Omega) & \text{if } \mathbf{v}|_\Gamma = 0 \\ \left\{\mathbf{v}\in H^1(\Omega), \mathbf{v} \text{ periodic}\right\} & \text{if } \mathbf{v} \text{ is periodic} \end{cases}$$

$$V = \{\mathbf{v}\in X \mid \nabla\cdot\mathbf{v} = 0\}$$

$$H = \left\{\mathbf{v}\in L^2(\Omega), \nabla\cdot\mathbf{v} = 0, \mathbf{v}\cdot n|_\Gamma = 0 \text{ or } \mathbf{v} \text{ periodic}\right\}$$

$$U = L^2(0,T;X)\cap L^\infty\left(0,T;L^2(\Omega)\right)$$

$$Q = L^1\left(0,T;L_0^2(\Omega)\right)$$

Note that these function spaces include, implicitly, boundary and initial conditions such as the incompressibility restriction, $\nabla\cdot\mathbf{v} = 0$. It is assumed that $\mathbf{u}_0$ belongs to $H$. $P_H$ denotes the orthogonal projection of $L^2(\Omega)$ in $H$.

### 2.2 Suitable weak solutions

Turbulence is a complex phenomenon difficult to describe from the physical and mathematical points of view. Since the definition of turbulence due to Leray in 1930 [4], naming *turbulent solution* to any weak solution of the Navier-Stokes equations, progress has been frustrating slow. The greatest obstacle to analyze the Navier-Stokes equations is the problem of solution uniqueness in 3-D. This question remain unanswered due to the possibility that vortex destruction, by effect of viscosity, taking place at smaller scales than Kolmogorov scale, cannot be excluded.

If weak solutions are not unique, a fundamental issue is to distinguish physically relevant solutions. A first approach to this problem is to consider the definition of suitable weak solution proposed by Scheffer [5]:

**Definition 2**

*A weak solution of Navier-Stokes* $(\mathbf{u},p)$ *is suitable if* $\mathbf{u}\in U$, $p\in Q$ *and local energetic balance*

$$\partial_t\left(\frac{1}{2}\mathbf{u}^2\right) + \nabla\cdot\left(\left(\frac{1}{2}\mathbf{u}^2 + p\right)\mathbf{u}\right) - \nu\Delta\left(\frac{1}{2}\mathbf{u}^2\right) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \le 0 \tag{2.3}$$

*is satisfied in a distributional sense.*

By analogy with nonlinear conservatives laws condition (2.3) can be interpreted as an entropy condition since its purpose is to select physically relevant solutions of (2.2). Duchon and Robert [6] have given an explicit expression for the distribution, $D(\mathbf{u})$, necessary to add to (2.3) so equality is fulfilled. For a smooth flow $D(\mathbf{u})$ is zero; but for nonregular flows $D(\mathbf{u})$ is nontrivial. Suitable solutions are those that $D(\mathbf{u}) \geq 0$, *i. e.*, if a singularity appears only solutions that dissipate energy are suitable.

## 3 LES: A DEFINITION

LES approximations are solutions of finite dimensional problems calculated by means of a computer. Thus a reasonable definition of a LES model must cover the gap between theories of modelling based on LES concepts and numerical techniques used for its implementation. In general, it exists a close connection between the numerical methods and the model used for the simulation; this connection usually is not sufficiently used. LES approximations must select physically relevant solutions of the Navier-Stokes equations after taking the limit to zero of the discretization size. Guermond and Prudhomme [7] propose the following definition:

**Definition 3**

*A succession* $\left(\mathbf{u}_\gamma, p_\gamma\right)_{\gamma>0} \in U \times Q$ *is a LES approximation of (2.2) if*

*1- There are two vectorial spaces of finite dimension* $X_\gamma \subset X$ *and* $M_\gamma \subset L^2(\Omega)$ *such that* $\mathbf{u}_\gamma \in C^1\left([0,T];X_\gamma\right)$ *and* $p_\gamma \in C^0\left([0,T];M_\gamma\right)$ *for all* $T > 0$ .

*2- The succession converges to a weak solution of (2.2). Thus* $\mathbf{u}_\gamma \to \mathbf{u}$ *and* $p_\gamma \to p$ *weakly, with* $(\mathbf{u}, p) \in U \times Q$ .

*3- Solution* $(\mathbf{u}, p)$ *is admissible, in the sense of the Definition 2*

In previous definition, two hidden parameters are important to emphasize. Since $X_\gamma$ and $M_\gamma$ are two finite spaces, it exists a discretization parameter $h$ associated with the smallest scale that can be represented in $X_\gamma$, approximately $\dim\left(X_\gamma\right) = O\left(\left(L/h\right)^3\right)$, where $L = diam(\Omega)$. In the following we introduce a parameter $\varepsilon$ associated to some regularization of the Navier-Stokes equations. The parameter $\varepsilon$ is the length scale of the smallest vortexes allowed to be activated by the nonlinear mechanisms in the flow. The parameter $\gamma$ in definition 3 is a combination of parameters $h$ and $\varepsilon$.

Construction of a LES model is split into the following three steps:

1- Construction of a Pre-LES model. This step consists in the regularization of the Navier-Stokes equations, introducing the regularization parameter $\varepsilon$. The objective of regularization is to obtain a well-posed problem for all time. Additionally, the limit of solutions of the Pre-LES model must be a weak solution of the Navier-Stokes equations when $\varepsilon \to 0$, and this solution limit must be suitable.

2- Discretization of Pre-LES model. This step introduces the mesh size $h$ and the finite dimensional spaces $X_\gamma$ and $M_\gamma$.

3- Determination of a relation between $h$ and $\varepsilon$. These parameters must be chosen so that the discrete solutions converge to an admissible solution when $h \to 0$ and $\varepsilon \to 0$

# 4 ENERGY CASCADE

This section presents a short introduction to Kolmogorov theory, since it is commonly referenced in LES. In order to understand the physical foundations of Kolmogorov theory we will review the fundamental vortex stretching mechanism. More details can be found in [8], [9] and [10].

## 4.1 Vortex stretching mechanism

We rewrite the momentum equation in terms of the vorticity, defined as:

$$\mathbf{w} = \nabla \times \mathbf{u} = rot(\mathbf{u}) \tag{4.1}$$

Taking the rotational of the momentum equations it is obtained:

$$\partial_t \mathbf{w} + \mathbf{u} \cdot \nabla \mathbf{w} - \nu \Delta \mathbf{w} - \mathbf{D} \cdot \mathbf{w} = \nabla \times \mathbf{f} \tag{4.2}$$

Where $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right)$ is the symmetric rate of deformation tensor. Equation (4.2) resembles the momentum equation for $\mathbf{u}$, excepts for the term $\mathbf{D} \cdot \mathbf{w}$ termed vortex stretching term. This term explains some of the differences between 2-D flows and 3-D flows.

In 2D problems defined in the $(x, y)$ plane, the velocity has components $(u, v, 0)$. Therefore the vorticity has only one component different from zero, *i. e.* $\mathbf{w} = (0, 0, \omega)$. It is deduced that the vortex stretching term, $\mathbf{D} \cdot \mathbf{w}$, is identically null and it does not take part in the evolution equation for $\mathbf{w}$. In 3D problems, this term can give rise to strong potential local phenomena known as *the vortex stretching mechanism*. We recall that $\mathbf{D}$ is symmetric, therefore diagonalizable. In addition, $\mathbf{D}$ has null trace $(tr(\mathbf{D}) = \nabla \cdot \mathbf{u} = 0)$, consequently, at least one of its eigenvalue is nonnegative.

In case that $\mathbf{w}$ is aligned with an eigenvector associated to a positive eigenvalue, vorticity intensity and angular velocity increase in a way the that diffusive term, $\nu \Delta \mathbf{w}$, and the source term, $\nabla \times \mathbf{f}$, are too weak to balance this mechanism. From a physical point of view, this implies that an elemental fluid domain first will be contracted along an orthogonal direction to the vorticity, and later will be stretched according to the vorticity direction to be able to conserve angular moment (if dissipation is neglected). The vortex stretching mechanism is responsible for the local amplification of vorticity intensity, as well as, for the formation of smaller and smaller scale structures in the flow. This phenomenon implies therefore energy transfer from the greater scales to the smaller ones, process known as *the energy cascade*.

### 4.2 Fourier analysis and the energy cascade

Study of homogenous and isotropic turbulence is usually carried out in a 3D periodic domain $\Omega = (0, L)^3$ using spectral analysis.

As $\mathbf{u}(\mathbf{x}, t)$ is square integralable, the velocity field can be expressed in Fourier series

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{l}} \hat{\mathbf{u}}(\mathbf{l}, t) e^{i\mathbf{l}\cdot\mathbf{x}} \tag{4.3}$$

Where $\mathbf{l}$ are wave numbers given by $\mathbf{l} = {2\pi\mathbf{n}}/{L}, \mathbf{n} \in \mathbb{Z}^3$, and the Fourier coefficients or Fourier modes satisfy:

$$\hat{\mathbf{u}}(\mathbf{l}, t) = \frac{1}{L^3} \int_{\Omega} \mathbf{u}(\mathbf{x}, t) e^{-i\mathbf{l}\cdot\mathbf{x}} d\mathbf{x} \tag{4.4}$$

The main advantage of this decomposition is to distinguish the different scales in the flow. The associated scale to wave number $\mathbf{l}$ is defined as ${2\pi}/{|\mathbf{l}|_\infty}$. Considering the Fourier transform of the Navier-Stokes equations, we obtain the temporal evolution equation of each mode $\hat{\mathbf{u}}(\mathbf{l}, t)$ as

$$\begin{cases} d_t \hat{\mathbf{u}}(\mathbf{l}, t) = \hat{\mathbf{f}}(\mathbf{l}) - \nu |\mathbf{l}|^2 \hat{\mathbf{u}}(\mathbf{l}, t) + \dfrac{i}{L^3}\left(\mathbf{I} - \dfrac{\mathbf{l}\mathbf{l}^t}{|\mathbf{l}|^2}\right) \cdot \displaystyle\sum_{\mathbf{l}_1 + \mathbf{l}_2 = \mathbf{l}} \hat{\mathbf{u}}_1(\mathbf{l}_1, t) \cdot \mathbf{l}_2 \hat{\mathbf{u}}_2(\mathbf{l}_2, t) \\ \mathbf{l} \cdot \hat{\mathbf{u}}(\mathbf{l}, t) = 0 \end{cases} \tag{4.5}$$

Where $\mathbf{I}$ is the unit tensor, $\mathbf{I} - {\mathbf{l}\mathbf{l}^t}/{|\mathbf{l}|^2}$ is the projection in space of the free divergence field in spectral space, and $\hat{\mathbf{f}}(\mathbf{l})$ are modes of external force, assumed to be time independent. The $\hat{\mathbf{f}}(\mathbf{l})$ term provides energy to the system for wave number $\mathbf{l}$. It is assumed that this contribution is only important for large wavelengths and enough to maintain the flow in a turbulent stationary state. This term does not excite small scales of the flow directly.

The second term in(4.5), $\nu |\mathbf{l}|^2 \hat{\mathbf{u}}(\mathbf{l}, t)$, is a viscous dissipation term. Due to this term, viscous dissipation is more efficient in the small scales than in the large ones. The last term is the Fourier transform of the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$. This term allows the coupling of modes in the frequency space. This mechanism activates smaller and smaller scales in the flow.

Equation (4.5) shows clearly that any triad is coupled if one wave number is the sum of the other two. In summary, the nonlinear term causes the transference of energy from large scales, excited by external forces, to the smaller scales where the effects of viscosity are predominant. This mechanism is another explanation of the energy cascade.

Lesieur [8] gives an intuitive description of the energy cascade: the flow reaches a stationary state where the vortex stretching mechanism generates an “infinite hierarchy of vortexes’’; where vortexes absorb energy from greater vortexes in which they are contained. The process finishes in the dissipative viscous scale, where effects of fluid viscosity are dominant. The nonlinear term does not participate in the global balance of energy since:

$$\int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} d\mathbf{x} = \int_{\Omega} \mathbf{u} \cdot \nabla \left(\frac{1}{2}\mathbf{u}^2\right) d\mathbf{x} = \int_{\Omega} \nabla \cdot \left(\frac{1}{2}\mathbf{u}^2 \mathbf{u}\right) d\mathbf{x} = 0 \tag{4.6}$$

Again it is clear that the role of the nonlinear term is to redistribute energy from the larger scales to the smaller ones.

Let see how to decompose the kinetic energy of the flow based on the contributions of each wave number.

The instantaneous value of the kinetic energy of the average velocity is, using Parseval identity:

$$\frac{1}{L^3}\int_{\Omega} \frac{1}{2}\mathbf{u}^2 (\mathbf{x}, t)\, d\mathbf{x} = \sum_{\mathbf{l}} \frac{1}{2} |\hat{\mathbf{u}}(\mathbf{l}, t)|^2 = \sum_{k} \left\{ \sum_{|\mathbf{l}|_\infty = k} \frac{1}{2} |\hat{\mathbf{u}}(\mathbf{l}, t)|^2 \right\} \tag{4.7}$$

where $k = {}^{2\pi n}\!/\!_{L}, n \in \mathbb{N}$ .Then a possible decomposition of the kinetic energy is:

$$\frac{1}{2L^3} \|\hat{\mathbf{u}}(\cdot, t)\|_0^2 = \frac{2\pi}{L} \sum_{k} E(k, t) \tag{4.8}$$

where

$$E(k, t) := \frac{L}{2\pi} \sum_{|\mathbf{l}|_\infty = k} \frac{1}{2} |\hat{\mathbf{u}}(\mathbf{l}, t)|^2 \tag{4.9}$$

defines the kinetic energy associated with wave numbers$\mathbf{l}, |\mathbf{l}|_\infty = k$ , or, equivalently, associated with the length scale ${}^{2\pi}\!/\!_{k}$ .

## 5 FILTERING IN LES

The main objective of the LES technique is to modify the original Navier-Stokes equations into a new equations system which (hopefully) has a simpler numerical solution, while conserving most of the energetic properties of the original problem. The classical idea is to use a filter to separate the large scales from the small ones. Applying a filter to the Navier-Stokes equations yields a new system of equations that governs the large scales, except for a term which is still expressed in terms of the velocity of the small scales.

The description of this term is based on a procedure known as *the closure problem* which yields a new equation system where the velocity (and pressure) of large scales are the only unknowns.

This section shows how the filtering procedure associated to the closure problem leads to a paradox. Additionally we show that a suitable filter is, in fact, equivalent to a the regularization of the Navier-Stokes equations. In terms of mathematical analysis, regularization of equations solves the problem of solution uniqueness.

### 5.1 Filter operator and the closure problem

A filter operator is denoted by $\overline{(\cdot)} : w \to \overline{w}$ . Filtering can be perforemed in space or time, or both in space and time. Although many types of filters can be used [11], it is assumed here that the filter operator is linear and commutes with differentials operators (properties shared by most of filters).

Applying the filter operator to the Navier-Stokes equations leads to the following new system of equations:

$$\begin{cases} \partial_t \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} + \nabla \overline{p} - \nu \Delta \overline{\mathbf{u}} = \overline{\mathbf{f}} - \nabla \cdot \mathrm{T} & \text{in } \Omega \\ \nabla \cdot \overline{\mathbf{u}} = 0 & \text{in } \Omega \\ \overline{\mathbf{u}}|_{\Gamma} = 0 & \text{or periodic} \\ \overline{\mathbf{u}}|_{t=0} = \overline{\mathbf{u}}_0 & \end{cases} \tag{5.1}$$

where

$$\mathrm{T} = \overline{\mathbf{u} \otimes \mathbf{u}} - \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} \tag{5.2}$$

it is commonly denoted the subgrid-scale tensor.

In order to be able to solve (5.1) for $\overline{\mathbf{u}}$, without having to calculate $\mathbf{u}$, tensor $\mathrm{T}$ needs to be expressed in terms of $\overline{\mathbf{u}}$ solely.

The closure problem is therefore equivalent to find a model, $\mathrm{T}(\overline{\mathbf{u}})$, for the subgrid-scale tensor.

### 5.2 The closure paradox

It is logical to consider that if problem (5.1) can be exactly closed, *i.e.* with any extra *ad hoc* hypothesis, then the closure problem would disappear and simulation of turbulence would be physically exact. Exact closure is in fact possible. This result was proposed by Germano in [12] and [13].

**Proposition 1**

Be $\Omega$ the 3-D torus, then the exact closure of (5.1) is possible.

*Proof*

Let be $\varepsilon > 0$ a cut-off scale. We considered the following filter (from now on, filter of Helmholtz):

For a given function $\mathbf{v}$, the filtered function $\overline{\mathbf{v}}$ is defined as the solution of the following elliptical EDP

$$\overline{\mathbf{v}} - \varepsilon^2 \Delta \overline{\mathbf{v}} = \mathbf{v} \tag{5.3}$$

This is $\overline{\mathbf{v}} := \left(I - \varepsilon^2 \Delta\right)^{-1} \mathbf{v}$. Following Agmon-Douglis-Nirenberg theorem, the Helmholtz filter is continuous between $L^q(\Omega), 1 < q < +\infty$, and $W^{2,q}(\Omega)$ [14].

It is possible to prove that this filter commutes with the temporal and spatial derivatives. Then the Helmholtz filter is an acceptable filter.

Using

$$\begin{aligned} \mathbf{u} \otimes \mathbf{u} &= \left(\overline{\mathbf{u}} - \varepsilon^2 \Delta \overline{\mathbf{u}}\right) \otimes \left(\overline{\mathbf{u}} - \varepsilon^2 \Delta \overline{\mathbf{u}}\right) \\ \overline{\mathbf{u}} \otimes \overline{\mathbf{u}} &= \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} - \varepsilon^2 \Delta \overline{\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}} \end{aligned} \tag{5.4}$$

It follows

$$\begin{aligned}
\mathrm{T}_{ij} &= \overline{\left(\overline{u}_i - \varepsilon^2 \Delta \overline{u}_i\right)\left(\overline{u}_j - \varepsilon^2 \Delta \overline{u}_j\right)} - \overline{u}_i \overline{u}_j = \\
&= \overline{\overline{u}_i \overline{u}_j} - \overline{\varepsilon^2 \overline{u}_j \Delta \overline{u}_i + \overline{u}_i \Delta \overline{u}_j} + \varepsilon^4 \overline{\Delta \overline{u}_i \Delta \overline{u}_j} - \overline{u}_i \overline{u}_j = \\
&= \varepsilon^2 \overline{\Delta\left(\overline{u}_i \overline{u}_j\right) - \overline{u}_j \Delta \overline{u}_i - \overline{u}_i \Delta \overline{u}_j} + \varepsilon^4 \overline{\Delta \overline{u}_i \Delta \overline{u}_j} = \\
&= 2\varepsilon^2 \overline{\nabla \overline{u}_i \cdot \nabla \overline{u}_j} + \varepsilon^4 \overline{\Delta \overline{u}_i \Delta \overline{u}_j}
\end{aligned} \tag{5.5}$$

Eq. (5.5) is an exact closure for the subgrid-scale tensor.

#

Another way to obtain an exact closure is using smoothed functions. Let $\phi$ be a bounded and positive function in $\mathbb{R}^3$, decreasing rapidly to infinite and such that its Fourier transformed is nonnull. As an example, $\phi(\mathbf{x}) = \pi^{-3/2} e^{-|\mathbf{x}|^2}$ satisfies previous hypotheses. Then for $\varepsilon > 0$, the filter operator is defined as:

$$\overline{\mathbf{v}} := \phi_\varepsilon * \mathbf{v} \tag{5.6}$$

Denoting $F$ the Fourier transformation,

$$\forall \mathbf{u} \in L^1(\Omega), \mathbf{u} = F^{-1}\left(\frac{F(\overline{\mathbf{u}})}{F(\phi)}\right) \tag{5.7}$$

In this case the subgrid-scale tensor can also be expressed as a function of velocity of the large scales with no need of more hypotheses.

Previous results can be generalized observing that an exact closure is possible whenever the filter operator induces an isomorphism. In fact, filters (5.3) and (5.6) are isomorphisms. Intuitively it is verified that this type of operators does not remove information from the velocity field, simply deforms the field spectra.

For a fixed $t > 0$, the filter (5.3) induces an isomorphism between $L^\infty(0,t;H)$ and $L^\infty\left(0,t;H \cap H^2(\Omega)\right)$ and also between $L^2(0,t;V)$ and $L^2\left(0,t;V \cap H^3(\Omega)\right)$. Therefore this filter induces an isomorphism between weak solutions of (2.2) and weak solutions of (5.1).

Filtering and obtaining an exact closure do not introduce any numerical improvement. Informally speaking, since the spaces of the weak solutions are isomorphs, it is reasonable to expect that the same number of degrees of freedom will be necessary to approximate the Navier-Stokes equations as for the filtered equations. In conclusion the following paradox rises:

*Filtering and exact closure can not reduce the number of degrees of freedom..*

It is possible to conjecture that the filtering of Navier-Stokes equations is efficient if an inexact closure is used.

### 5.3 Leray Regularization

The first important result related to the filtering equations, is the proof of existence and uniqueness of the Navier-Stokes equations due to Leray [4].

Be $\Omega$ a 3-D torus. We consider a succession of smoothed functions $\{\phi_\varepsilon\}_{\varepsilon>0}$ satisfying:

$$\phi_\varepsilon \in C_0^\infty(\Omega), \quad supp(\phi_\varepsilon) \in B(0,\varepsilon), \quad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\, d\mathbf{x} = 1 \tag{5.8}$$

And the convolution product

$$\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} v(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\, d\mathbf{y} \tag{5.9}$$

Leray suggested to regularize the Navier-Stokes equations as follows:

$$\begin{cases} \partial_t \mathbf{u} + (\phi_\varepsilon * \mathbf{u})\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \phi_\varepsilon * \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} & \text{periodic} \\ \mathbf{u}|_{t=0} = \phi_\varepsilon * \mathbf{u}_0 & \end{cases} \tag{5.10}$$

Leray proved the following theorem [4]:

**Theorem 1**

For $\mathbf{u}_0 \in H, \mathbf{f} \in H,$ and $\varepsilon > 0$, (5.10) has a unique solution $C^\infty$. This solution is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$ and it exists a sub succession weakly convergent in $L^2(0,T;V)$ to a weak solution of Navier-Stokes, when $\varepsilon \to 0$.

Therefore, a moderate filtering of the convective term (and if it is necessary of the initial values and the source term) suffices to guarantee the uniqueness of a solution $C^\infty$.

Rewriting the momentum equations (5.10) as

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \phi_\varepsilon * \mathbf{f} - (\phi_\varepsilon * \mathbf{u})\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} \tag{5.11}$$

and introducing tensor $\mathrm{T}_L$ such that:

$$\nabla\cdot\mathrm{T}_L = (\phi_\varepsilon * \mathbf{u})\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} = \nabla\cdot(\mathbf{u}\otimes(\phi_\varepsilon * \mathbf{u}) - \mathbf{u}\otimes\mathbf{u}) \tag{5.12}$$

it is reasonable to interpret Leray regularization as a LES model. Equation (5.11) is in fact the same momentum equation that (5.1) except for the term $\mathrm{T}$ that is approximated by $\mathrm{T}_L$. Nevertheless, this interpretation is debatable, since the model is not invariant for a change of the coordinate system.

### 5.4 Navier-Stokes-alpha model

Considering the following equality:

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\left(\mathbf{u}^2\!/\!_2\right) \tag{5.13}$$

The Navier-Stokes problem can be rewritten as

$$\begin{cases} \partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\overline{\mathbf{u}} + \nabla\pi - \nu\Delta\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} & \text{periodic} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \end{cases} \tag{5.14}$$

where $\pi = p + \dfrac{1}{2}\mathbf{u}^2$ is the total pressure. Obviously this form of the equations has the same regularity problems as the original equations.

Following the same procedure as in the Leary regularization and introducing the following notation $\overline{\mathbf{u}} = \phi_\varepsilon * \mathbf{u}$, we obtain a regularized problem as follows:

Using the following identities

$$\begin{aligned} (\nabla\times\mathbf{u})\times\overline{\mathbf{u}} &= \overline{\mathbf{u}}\cdot\nabla\mathbf{u} - (\nabla\mathbf{u})^T\,\overline{\mathbf{u}} \\ \nabla(\mathbf{u}\cdot\overline{\mathbf{u}}) &= (\nabla\mathbf{u})^T\,\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\,\mathbf{u} \end{aligned} \tag{5.15}$$

we can rewrite (5.14) as the following equivalent system:

$$\begin{cases} \partial_t\mathbf{u} + \overline{\mathbf{u}}\cdot\nabla\mathbf{u} + (\nabla\overline{\mathbf{u}})^T\cdot\mathbf{u} + \nabla\pi' - \nu\Delta\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u},\overline{\mathbf{u}} & \text{periodic} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \end{cases} \tag{5.16}$$

where $\pi' = \pi - \mathbf{u}\times\overline{\mathbf{u}}$.

Considering again the Helmholtz filter, introduced in the proof of Proposition 1 and defined as

$$\overline{\mathbf{v}} := \left(I - \varepsilon^2\Delta\right)^{-1}\mathbf{v} \tag{5.17}$$

We can identify this LES model with the model proposed by Chen *et al.* [15], Foias, Holm y Titi [16], [17], taking $\alpha = \varepsilon$. Again the regularization of the equations gives rise to the existence and uniqueness of the solution of the regularized problem.

**Theorem 2**

Problem (5.16) with Helmholtz filter (5.17) has a unique solution $C^{\infty}, \mathbf{u}$. Solution $\mathbf{u}$ is bounded in $L^{\infty}(0,T;H) \cap L^{2}(0,T;V)$ and exists a sub succession weakly convergent in $L^{2}(0,T;V)$ to a weak solution of Navier-Stokes when $\varepsilon \to 0$.

Numerical simulations [17] have shown that the energy spectra of solution of (5.16) follows $k^{-5/3}$ the Kolmogorov law for $k < 1/\varepsilon$ and it changes to $k^{-3}$ for $k > 1/\varepsilon$.

Therefore, beyond the $\varepsilon$ scale regularization has replaced the $k^{-5/3}$ tail of spectra, which is difficult to approximate numerically, by a $k^{-3}$ tail easier to approximate.

## 6 P-LAPLACIAN MODELS

This section reviews turbulence models based on nonlinear viscosity. The Smagorinsky model, p-Laplacian regularization and models proposed by Ladyzenskaja are presented. As in the previous section it is demonstrated that these models are again regularizations of the Navier-Stokes equations, in the sense of solution uniqueness.

The section ends showing that p-Laplacians regularizations present interesting numerical properties that allow establishing $L^{\infty}$ estimators for error approximation.

### 6.1 Smagorinsky Model

This is probably the most popular LES model [18]. Smagorinsky model adds to the stress tensor a nonlinear viscous term depending on an *ad hoc* fixed small length scale.

Denoting rate of deformation tensor as $D = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{t}\right)$, the additional viscous tensor is written as

$$c_s \varepsilon^2 |\mathbf{D}| \mathbf{D} \tag{6.1}$$

Where $c_s$ is an *ad hoc* constant. Introducing the notation $\mathbf{T}(\nabla \mathbf{u}) = c_s |\mathbf{D}| \mathbf{D}$, the new perturbed Navier-Stokes equations system is

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nabla \cdot \left( \nu \nabla \mathbf{u} + \varepsilon^2 \mathbf{T}(\nabla \mathbf{u}) \right) = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u}|_{\Gamma} = 0 & \text{or periodic} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \end{cases} \tag{6.2}$$

It is common to present this method by means of the filtered Navier-Stokes equations and to interpret above system as the effect of applying the filter. The sub-grid scale tensor is modelled as $\mathrm{T} = \varepsilon^2 \mathbf{T}(\nabla \mathbf{u})$. This point of view is questionable, since unlike Leray regularization and NS-alpha model, no filter is specified and no theoretical motivation seems to justify this model.

The model constant $c_s$ is usually evaluated to reproduce the $k^{-5/3}$ law for simulating of isotropic turbulence in the 3D torus.

When solution of (6.2) is approximated, the parameter $\varepsilon$ generally is chosen equal to the mesh size, $h$, *i. e.* mixing the mathematical and computational models. Thus choice is generally the origin of confusions and problems.

While Smagorinsky turbulence model [18] is remarkable by its capacity to reproduce the $k^{-5/3}$ energy spectra [19], it has several disadvantages. One disadvantage is that the artificial dissipation does not disappear in the vicinity of contour walls (where the velocity is fixed), where it is well known that turbulence vanishes.

When $\varepsilon$ is taken equal to mesh size $h$, the Smagorinsky model induces a consistency error of order $O\left(h^2\right)$ respect to Navier-Stokes equations. This restricts the use of higher order schemes since the precision will be never higher to $O\left(h^2\right)$, also in the smooth parts of the flow.

### 6.2 Ladyzenskaja Model

Since the Navier-Stokes equations are based on Newton linearity hypothesis. Ladyzenskaja proposed in a series of articles ([20], [21]) to modify the Navier-Stokes equations for incompressible flows with high velocity gradients. She introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla \mathbf{u}), 1 \leq i,j \leq 3$ satisfying the following properties:

L1. $\mathbf{T}$ is continuous and exists $\mu \geq 1/4$ such that

$$\forall \xi \in \mathbb{R}^{3\times 3}, |\mathbf{T}(\xi)| \leq c\left(1+|\xi|^{2\mu}\right)|\xi| \tag{6.3}$$

L2. $\mathbf{T}$ fulfils the following coercitivity property

$$\forall \xi \in \mathbb{R}^{3\times 3}, \mathbf{T}(\xi) : \xi \geq c|\xi|^2\left(1+c'|\xi|^{2\mu}\right) \tag{6.4}$$

L3. $\mathbf{T}$ fulfils the following monoticity property: It exists a constant $c > 0$ such that for all solenoidal fields $\xi$ and $\eta$ in $W^{1,2+2\mu}(\Omega)$ coincident on $\Gamma$ or periodic

$$\int_\Omega (\mathbf{T}(\nabla\xi) - \mathbf{T}(\nabla\eta)) : (\nabla\xi - \nabla\eta) \geq c\int_\Omega |\nabla\xi - \nabla\eta|^2 \tag{6.5}$$

Previous conditions are satisfied in the following case:

$$\mathbf{T}(\xi) = \beta\left(|\xi|^2\right)\xi \tag{6.6}$$

where the viscous function, $\beta(\tau)$, is monotone, positive, increasing for $\tau \geq 0$ and for large values of $\tau$ the following inequality holds

$$c\tau^{\mu} \leq \beta(\tau) \leq c'\tau^{\mu} \tag{6.7}$$

With $\mu \geq 1/4$ and $c, c'$ are strictly positive constants.

The Smagorinsky model satisfies previous conditions considering $\beta(\tau) = \tau^{1/2}$. Introducing a positive constant (possibly small) $\varepsilon \geq 0$, the modified Navier-Stokes equations are:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nabla\cdot(\nu\nabla\mathbf{u} + \varepsilon\mathbf{T}(\nabla\mathbf{u})) = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u}|_\Gamma = 0 & \text{or periodic} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \end{cases} \tag{6.8}$$

The following result holds

**Theorem 3**

Assuming that the hypotheses L1, L2 and L3 hold, $\mathbf{u}_0 \in H$ and $f \in L^2\left((0,+\infty);L^2(\Omega)\right)$ then problem (6.8) has unique weak solution in $L^2\left((0,t);W^{1,2+2\mu}(\Omega)\cap V\right)\cap C^0([0,t];H)$ for all $t > 0$

Note that we have solution uniqueness for arbitrarily large times. This theorem shows that a small nonlinear viscosity contribution is in fact sufficient to stop the energy cascade. In conclusion, to perturb the Navier-Stokes equations with a term like that given in the Smagorinsky model solves the uniqueness problem.

### 6.3 p-Laplacian operator

The p-Laplacian operator is a simplified version of Smagorinsky and Ladyzenskaja models. In this section we show that this operator can be used to approximate convection dominant problems for convection-diffusion equations, and therefore it offers another mathematical interpretation of the Smagorinsky model.

Be $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) and $p \geq 2$, the p-Laplacian operator is defined as:

$$\begin{aligned} T_p : W_0^{1,p}(\Omega) &\to W_0^{1,p}(\Omega)' \\ u \quad &\to -\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) \end{aligned} \tag{6.9}$$

It is clear that $T_p$ is bounded and satisfies the following monoticity property

$$\exists\alpha > 0, \forall u,v \in W_0^{1,p}(\Omega), \quad \left\langle T_p(u) - T_p(v), u - v\right\rangle \geq \alpha\|\nabla(u-v)\|_{1,p}^p \tag{6.10}$$

where $\langle,\rangle$ denotes the dual pair. When $p = 2$, $T_p$ is simply the laplacian and for $p = 3$ $T_p$ is homologous to the Smagorinsky model

Let us consider the following convection-diffusion problem:

$$\begin{cases} \mathbf{a}\cdot\nabla u - \varepsilon\Delta u = f \\ u|_\Gamma = 0 \end{cases} \tag{6.11}$$

with $\varepsilon > 0$. In order to simplify, it is assumed that $\mathbf{a} \in H$ ($\mathbf{a}$ is a solenoidal field, with null normal trace in $\Gamma$).

It is known that the approximation of this kind of problems is complex when $h\|\mathbf{a}\|_{0,\infty}/\varepsilon$ is large, being $h$ a characteristic mesh size. The Galerkin approximation presents spurious oscillations node by node. A heuristic explanation for these oscillations is that the mesh is not fine enough so that viscous effects can damp the high gradients. This problem is also present in the numerical approximation of the Navier-Stokes equations for high Reynolds number flows.

Be $X_h \subset H_0^1(\Omega)$ a finite dimensional space with a standard interpolation property (for example, finite elements space $P_k$), *i. e.* There are constants $c > 0$ and $k > 0$ such that $\forall v \in W^{1,p}(\Omega)$

$$\inf_{v_h \in X_h} \left( \| v - v_h \|_{0,p} + h \| v - v_h \|_{1,p} \right) \leq c h^{k+1} \| v \|_{k+1,p} \tag{6.12}$$

Introducing the following notation

$$a(u,v) = (\nabla u, \nabla v), \quad b(u,v) = (\mathbf{a} \cdot \nabla u, v) \tag{6.13}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. We consider the following approximate problem

$$\varepsilon a(u_h, v_h) + b(u_h, v_h) + h^\sigma \left( T_p(u_h), u_h \right) = (f, v_h), \forall v_h \in X_h \tag{6.14}$$

with $\sigma, p \geq 2$ still to determinate. It is clear that the original problem is perturbed by a term of order $O(h^\sigma)$. In order, to conserve the optimal convergence estimators it is necessary to take $\sigma \geq k$.

**Theorem 4**

Under previous hypotheses. Given $u$ sufficiently regular,

$$\| u - u_h \|_{1,p} \leq \left( h^{\frac{1}{p-1}} + h^{\frac{k+1-\sigma}{p-1}} \right) c(u) \tag{6.15}$$

with $c(u) = c \max \left( \| u \|_{1,p}, h^k \| u \|_{k+1,p}, \| u \|_{k+1,p'}^{\frac{1}{p-1}} \right)$.

Applying previous result to the 3D problems for a second order scheme, $k = 1$, the limit case of $L^\infty$-convergence is obtained with $p = 3$ and $\sigma = k + 1 = 2$. This is formally equivalent to the Smagorinsky model.

## 7 MULTI-SCALE METHODS

In this section we describe several LES multi-scale models of common use. The multi-scale methods are related to sub-grid stabilization techniques to solve noncoercive PDE, and thus contributing to some mathematical justification of these methods.

### 7.1 Framework for multi-scale approximation

Throughout this section is assumed that we have two finite spaces to approximate velocity and pressure, respectively. In order to avoid pressure stability problems, it is assumed that both spaces fulfil the Ladyzenskaja-Babuska-Brezzi condition (LBB).

Be $P_H : X_h \to X_h$ a linear operator, named scale filter. We denote $X_h$ as the space of high resolution and $X_H = P_H(X_h)$ as the space of resolvable scales, where $H, h$

make reference to the mesh characteristic size of spaces $X_H, X_h$, respectively. It is quite common to consider $H \approx 2h$.

### 7.2 Sub-grid viscosity multi-scale methods

The robustness of methods based on artificial viscosity has generated their adaptation to the multi-scale framework.

#### 7.2.1 Sub-grid visocosity

The sub-grid viscosity models define the sub-grid scale tensor as a dissipative operator $\mathrm{T} = \nabla \cdot (\nu_t \nabla \mathbf{u})$, where the eddy/turbulent viscosity $\nu_t$, only depends on the fluctuating part of velocity. Considering a decomposition of the velocity as $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$, an option to define $\nu_t$ is to assume that the turbulent viscosity depends on the turbulent kinetic energy $e' = \frac{1}{2}\mathbf{u}'^2$ and a mixing length $l_{mix}$, thus $\nu_t \sim l_{mix} e'^{1/2}$. Numerically, this idea can be implemented using a multi-scale approach and identifying

$$\begin{aligned} \mathbf{u} &\sim \mathbf{u}_h \\ \bar{\mathbf{u}} &\sim P_H(\mathbf{u}_h) \end{aligned} \tag{7.1}$$

Then the discrete turbulent kinetic energy is

$$e'_h = \frac{1}{2}(\mathbf{u}_h - P_H(\mathbf{u}_h))^2 \tag{7.2}$$

And the mixing length $l_{mix} = H$.

The turbulent kinetic energy (TKE) model [21], is

$$\nu_t \approx cH|\mathbf{u}_h - P_H(\mathbf{u}_h)| \tag{7.3}$$

in weak form, sub-grid viscosity techniques add the following semi linear form to equation (6.8)

$$a_{sgs}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\nu_t(\mathbf{u}_h)\nabla\mathbf{u}_h, \nabla\mathbf{v}_h), \quad \mathbf{u}_h \in X_h, \mathbf{v}_h \in X_h \tag{7.4}$$

#### 7.2.2 Variational multi-scale method

An alternative proposed by Hughes *et al.* [23], [24] and [25], named LES “variational multi-scale”, is based in adding a dissipative term acting only on the unresolvable scales. Thus the sub-grid scale tensor is transformed into $\mathrm{T} = \nabla \cdot (\nu_t \nabla \mathbf{u}')'$. In the scope of the multi-scale approach, the following semi-linear form is added to the momentum equation:

$$a_{sgs}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\nu_t(\mathbf{u}_h)\nabla(\mathbf{u}_h - P_H(\mathbf{u}_h)), \nabla(\mathbf{v}_h - P_H(\mathbf{v}_h))) \tag{7.5}$$

where the eddy viscosity can be defined in several forms. Hughes *et al.* [23][24] [25] propose several definitions

$$\nu_t(\mathbf{u}_h) = cH^2 \begin{cases} |D(\mathbf{u}_h)| \\ o \\ |D(\mathbf{u}_h - P_H(\mathbf{u}_h))| \end{cases} \tag{7.6}$$

From numerical results presented in [24] and [25] cannot be drawn a clear conclusion about which is the best definition for $\nu_t$. From the numeric point of view however, the regions on which solution is smooth, the first definition leads to a consistency error of the order $O\left(H^{k+1}\right)$ in the sub-grid scale tensor, which is equal to the consistency error of the approximation method. Whereas second definition leads to an error of order $O\left(H^{2k+1}\right)$.

### 7.2.3 Discretization

Denoting by $\mathrm{I}_h$ an interpolation $L^2$-stable in $X_h$, all previous techniques can be rewritten in the following weak form of the Navier-Stokes equations:

To find $\mathbf{u}_h \in C^1(0,T;X_h)$ and $p_h \in C^0(0,T;M_h)$ such that

$$\begin{cases} (d_t\mathbf{u}_h,\mathbf{v}_h) + (\mathbf{u}_h\cdot\nabla\mathbf{u}_h,\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) + \nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) + \\ \qquad + a_{sgs}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h \\ (q_h,\nabla\cdot\mathbf{u}_h) = 0 \quad \forall q_h \in M_h \\ \mathbf{u}_h|_{t=0} = \mathrm{I}_h\mathbf{u}_0 \end{cases} \tag{7.7}$$

where $a_{sgs}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)$ is the semi linear form associated to the sub-grid scale tensor.

## 7.3 Sub-grid stabilitzation viscosity

The goal of this section is to give a partial theoretical justification of the discrete version of the LES multi-scale models. The main purpose is to show how these models are intimately related to the stabilization techniques used for non coercive partial differential equations.

**Definition 4**

*Be $A, B$ two normed spaces and $f : A \rightarrow B$ an application. The graph of $f$ is defined as a subset of $A \times B$ such that*

$$graph\ f = \left\{(x, f(x));\, x \in A\right\}$$

*The graph norm in $A$, associated to function $f$, is defined as:*

$$\|x\|_f = \|x\|_A + \|f(x)\|_B$$

*Where $\|\ \|_A$ and $\|\ \|_B$ are the norms of $A$ and $B$, respectively.*

### 7.3.1 Continuous framework

We consider the following linear problem:

For $f \in C^1([0,+\infty);L)$ and $u_o \in D(A)$, to find $u \in C^1([0,+\infty);L)$ such that

$$\begin{cases} u|_{t=0} = u_0 \\ d_t u + Au = f \end{cases} \tag{7.8}$$

Where L is a separable Hilbert space and $A : D(A) \subset L \to L$ is a linear operator. Additionally we assume that A is monotone:

$$\forall v \in D(A) \qquad (Av,v)_L \geq 0 \tag{7.9}$$

And that A is maximal

$$\forall f \in L, \quad \exists v \in D(A), \quad v + Av = f \tag{7.10}$$

As a reference example it is possible to consider $L = L^2(\mathbb{R})$ and $A = \partial_x$.

Considering $V = D(A)$ and provide $V$ with a graph norm, $\|v\|_V = \left(\|v\|_L^2 + \|Av\|_L^2\right)^{1/2}$.

It is possible to prove that the graph of $A$ is a closed space and $V$ is a space of Hilbert space with the following scalar product

$$(u,v)_L + (Au,Av)_L \tag{7.11}$$

In order to simplify the text we will assume that $V$ is a space function defined in $\Omega \subset \mathbb{R}^m, m \geq 1$.

Following the Hille-Yosida theorem ([14], [26]), problem (7.8) is well-posed and the following stability property holds

$$\begin{aligned} &\|u\|_{C^0([0,T];L)} \leq c\left(\|u_0\|_L + T\|f\|_{C^0([0,T];L)}\right) \\ &\|u\|_{C^1([0,T];L)} + \|u\|_{C^0([0,T];V)} \leq c\left(\|u_0\|_V + T\|f\|_{C^1([0,T];L)}\right) \end{aligned} \tag{7.12}$$

$A$ maximal is a fundamental property to prove that problem (7.8) is well-posed. This property can be better understood with the following proposition

**Proposition 2**

Be $E \subset F$ two Hilbert spaces with continuous and dense inclusion and be $T \in L(E;F)$ a monotone operator. The following properties are equivalent:

$$\begin{aligned} &i)\, T \text{ is maximal} \\ &ii)\ \exists c_1 > 0, c_2 > 0 \text{ such that} \\ &\forall u \in E, \quad \sup_{v \in F} \frac{(Tu,v)_F}{\|v\|_F} \geq c_1\|u\|_E - c_2\|u\|_F \end{aligned} \tag{7.13}$$

The important part in what follows is to define a discrete framework where a discrete homologue of (7.13) holds. It is well known that Galerkin techniques are not appropriate if $A$ is not coercive. In general, it is not possible to guarantee convergence in the graph norm, since the discrete homologue of (7.13) is not fulfilled uniformly respect to mesh size mesh. As a consequence, the Galerkin approximation of this equation with nonsmooth initial data produces spurious oscillations node to node.

### 7.3.2 Discrete framework

We considered the following finite dimensional spaces $X_h, X_H, X_h^H$ such that

$$V \supset X_h = X_H \oplus X_h^H \tag{7.14}$$

We assume that $X_h, X_H$ have suitable interpolation properties. It exists a dense subspace $W \subset V$ and a linear operator (interpolator) $I_H \in L(W; X_H)$ and two constants $k > 0, c > 0$ such that

$$\forall H, \forall v \in W, \quad \|v - I_H v\|_L + H\|v - I_H v\|_V \leq cH^{k+1}\|v\|_W \tag{7.15}$$

A possible interpretation of these finite dimensional spaces is to consider $X_h$ as the small scales space (great resolution), $X_H$ a large scales space (small resolution) and $X_h^H$ as a sub-grid scale, where the base functions are highly fluctuating.

Denoting by $h$ and $H$ the mesh sizes associated to $X_h$ and $X_H$ respectively, we assume that $h$ and $H$ are of same order, i.e. $c_1 h \leq H \leq c_2 h$. It is common to take $H = 2h$.

We suppose that it exists $c_i > 0$, independent of $h, H$ such that,

$$\forall v_h \in X_h, \quad \|v_h\|_V \leq \frac{c_i}{H}\|v_h\|_L \tag{7.16}$$

Indirectly, this hypothesis implies that $A$ is a first order differential operator and it can be justified considering $c_1 h \leq H \leq c_2 h$.

We define $P_H : X_h \rightarrow X_H$ as the projection of $X_h$ into $X_H$ parallel to $X_h^H$. We assume that $P_H$ is uniformly stable in the norm of $L$ with respect to $H$ and $h$.

We notice $\forall v_h \in X_h$

$$v_H = P_H v_h \qquad v_h^H = (1 - P_H) v_h \tag{7.17}$$

Let $b_h \in L\left(X_h^H, X_h^H\right)$ be such that $\forall\left(v_h^H, w_h^H\right) \in X_h^H \times X_h^H$

$$b_h\left(v_h^H, w_h^H\right) = H\int_\Omega \nabla v_h^H \cdot \nabla w_h^H dx \tag{7.18}$$

$b_h$ is a viscosity acting in sub-grid scales. This property is similar to that proposed by Hughes *et al.* [24] in model (7.6), for which the following argument can be a partial justification.

We introduce the most important hypothesis of this section, we assumed that the discrete version of (7.13) holds. More specifically, we assume that constants $c_1 > 0$ and $c_2 > 0$, independents of $H$ and $h$, exist such that

$$\forall v_h \in X_h, \quad \sup_{\phi_h \in X_h} \frac{(Av_H, \phi_h)_L}{\|\phi_h\|_L} \geq c_1 \|v_H\|_V - c_2 \|v_h\|_L \tag{7.19}$$

If $u_0 \in W$ then it can be approximated by $I_H u_0$. Thus we consider the following discrete problem:

To find $u_h \in C^1([0,+\infty); X_h)$ such that

$$\begin{cases} (d_t u_h, v_h)_L + (Au_h, v_h)_L + b_h\left(u_h^H, v_h^H\right) = (f, v_h)_L \\ \qquad u_{h|_{t=0}} = I_H u_0 \qquad \forall v_h \in X_h \end{cases} \tag{7.20}$$

This problem has solution, since it is a system of linear differential equations

**Theorem 5**

Under hypothesis (7.15), (7.16), (7.17) , (7.18) and (7.19), if $u \in C^2([0,T]; W)$, then $u_h$ satisfies the following inequalities:

$$\|u - u_h\|_{C^0([0,T];W)} \leq c_1 H^{k+1/2} \tag{7.21}$$

$$\left[\frac{1}{T}\int_0^T \|u - u_h\|_V^2\right]^{1/2} \leq c_2 H^k \tag{7.22}$$

where constants $c_1$ y $c_2$ satisfy

$$\begin{aligned} c_1 &\leq c[H + T(1+T)]^{1/2} \|u\|_{C^2([0,T];W)} \\ c_2 &\leq c[1+T]\|u\|_{C^2([0,T];W)} \end{aligned} \tag{7.23}$$

The present argument can be extended to the coercive case, when the differential operator is $A + \varepsilon D$, where $A$ is a first order differential operator and $D$ is a second order coercive operator. If $\varepsilon \in O(1)$, the Galerkin method is optimal, but if $\varepsilon$ is small, coercivity is not enough to guarantee the stability of the Galerkin method.

### 7.3.3 Examples and extension to the nonlinear case

Let $\Omega$ be a polyhedron in $\mathbb{R}^d$ and $\mathrm{T}_H$ a triangularization of $\Omega$ with simplexs.

$$\mathrm{T}_H = \cup\{K_H\} \tag{7.24}$$

Let $V$ be a vectorial functions set with values in $\mathbb{R}^m$.

Defining

$$X_H = \left\{v_H \in H^1(\Omega)^m \,\middle|\, v_H|_{K_H} \in P_1(K_H)^m, \forall K_H \in \mathrm{T}_H\right\} \tag{7.25}$$

For each triangle $K_H \in \mathrm{T}_H$ in 2D, we create four new triangles connected to the edges midpoints. In 3D, for each tetrahedron we create eight new tetrahedra as follows: In each face of the tetrahedron we connect the midpoints of two nonincident edges.

Let $h = H/2$ and $\mathrm{T}_h$ associated triangularization to previous mesh refining process. For each macro-simplex $K_H$, we define $P$ as the space of continuous functions in $K_H$ such that its restriction to a sub-simplex in $K_H$ belongs to $P_1$ and they are null in $K_H$ vertexes.

We define

$$X_h^H = \left\{ v_h^H \in H^1(\Omega)^m \left| v_h^H \right|_{K_H} \in P^m, \forall K_H \in \mathrm{T}_H \right\} \tag{7.26}$$

Considering $X_h = X_H + X_h^H$, is clear that $X_h$ can be characterized as

$$X_h = \left\{ v_H \in H^1(\Omega)^m \left| v_h \right|_{K_h} \in P_1(K_h)^m, \forall K_h \in \mathrm{T}_h \right\} \tag{7.27}$$

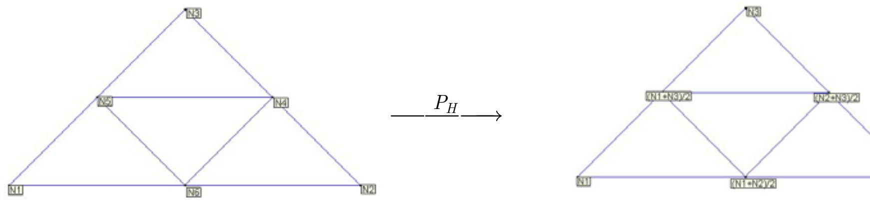


**figure 1**

figure 1 is a schematic representation of applying the projection operator $P_H : X_h \to X_H$ to a macro-element $K_H$.

It is important to emphasize that although this type of method is consistent with the exposed theory, located spurious oscillations in the neighbouring of discontinuities are still present, when the solution is discontinuous or presents shock waves. These residual oscillations are due to Gibbs phenomenon. Gibbs phenomenon is due to the succession of partial sums of the Fourier solution series which does not converge uniformly. Unless the solution is sufficiently smooth (continuity is not enough). A technique commonly used for eliminating these oscillations is adding dissipation where the solution is not smooth. Obviously the regions where the solution is not regular are unknown *a priori*, but it is logical to think that in these region holds

$$\nabla u_h^H = \nabla(u_h - P_H u_h) \approx \nabla u_h \tag{7.28}$$

Then we can introduce the following shock-capturing nonlinear term:

$$c_h\left(u_h^H; u_h, v_h\right) = c_{sc} H \int_\Omega \frac{\left|\nabla u_h^H\right|}{\left|\nabla u_h\right|} \nabla u_h \cdot \nabla v_h dx \tag{7.29}$$

Thus (7.20) is transformed to the follow new problem

$$\begin{cases} (d_t u_h, v_h)_L + (A u_h, v_h)_L + b_h\left(u_h^H, v_h^H\right) + c_h\left(u_h^H; u_h, v_h\right) = (f, v_h)_L \\ \qquad u_{h|_{t=0}} = I_H u_0 \qquad \forall v_h \in X_h \end{cases} \tag{7.30}$$

Unfortunately no theory exists to justify the use of $c_h$, but its use has been extremely efficient in practical applications. We note that if in (7.29), $\left|\nabla u_h^H\right|$ is replaced by the residual of the equation, then it can be demonstrated by means of scalar conservation laws that this term provides an estimator in $L^{\infty}$ of the approximation variable (*i.e.* a maximum principle) that guarantees the convergence to the minimum entropy solution [27][28].

In summary, we can argue the following interpretation of the terms $b_h, c_h$. The bilinear form $b_h$ is a viscous term that modifies the lack of coercivity of the original equations, while $c_h$ is a nonlinear term that accounts the Gibbs phenomenon associated to solution discontinuities.

Form $b_h$ eliminates generalized oscillations node to node produced by the lack of coercivity. Consequently $b_h$ affects only the small scales being able, solely, to attenuate the tail of the solution spectra. On the other hand, oscillations associated to Gibbs phenomenon are very localized in the neighbouring of discontinuities, which indicates that its spectral rank is wide and centred in the intermediate scales. These observations can justify using a shock-capturing stabilization term to reproduce the $k^{-5/3}$ cascade in LES. In conclusion the selection of constants for $b_h$ and $c_h$ seems an important practical issue.

## 8 FINITE CALCULUS METHOD

The Finite Calculus Method (FIC) is based in invoking the balance of fluxes in a fluid domain of finite size. This introduces naturally additional terms in the classical differential equations of momentum and mass balance of infinitesimal fluid mechanics which are function of characteristic length dimensions related to the element size in the discretized problem. The FIC terms in the modified governing equations provide the necessary stabilization to the discrete equations obtained via the standard Galerkin FEM. The FIC/FEM formulation allows to use low order finite elements (such as linear triangles and tetrahedra) with equal order approximations for the velocity and pressure variables.

The FIC/FEM formulation has proven to be very effective for the solution of a wide class of problems, such as convection-diffusion [36]-[47] and convection-diffusion-reaction [48][49][50] involving arbitrary high gradients, incompressible flow problems accounting for free surface effects, fluid-structure interaction situations [36][37][51]-[58] and quasi and fully incompressible problems in solid mechanics [59][60].

The FIC equations for incompressible flow derived in previous works of the authors [61] assumed that the dimensions of the domain where the momentum conservations law was enforced remain the same independently of the direction along which balance of momentum is imposed. As a consequence, each of the resulting FIC momentum equations contain *the same* characteristic dimensions which can be grouped in a *characteristic distance vector*. In this work, refined FIC momentum equations are derived by accepting that *the dimensions of the momentum balance domain are different for each of the momentum equations*. This introduces a *matrix form* of the characteristic distances and of the corresponding FIC terms which have better intrinsic stabilization properties.

The idea of a matrix form of the stabilization parameters is close to the element-matrix-based and element-vector-based stabilization parameters proposed in [62] where different intrinsic time parameters were defined separately for each degree of freedom of the equation system.

Stabilized FEM have been successfully used in the past to solve a wide range of fluid mechanics problems. The intrinsic dissipative properties of the stabilization terms (which can interpreted as an additional viscosity) typically suffices to yield good results for low and moderate values of the Reynolds number (*Re*). For high values of *Re* most stabilized FEM fail to provide physically meaningful results and the numerical solution is often unstable or inaccurate. The introduction of a turbulence model is mandatory in order to obtain meaningful results in these cases.

As mentioned in the previous section, the relationship between the additional dissipation introduced by the turbulence model and the intrinsic dissipative properties of stabilized FEM is an open topic which is attracting increasing attention in the CFD community. It is clear that both remedies (the turbulence model and the stabilization terms) play a similar role in the numerical solution, i.e. that of ensuring a solution which is ``physically sound'' and as accurate as possible.

It is our belief that the matrix stabilization terms introduced by the FIC/FEM formulation allow to model accurately high *Re* number flows without the need of introducing any turbulence model. The background of this belief originates in the positive experiences in the application of a very similar formulation for solving advection-diffusion and advection-diffusion-reaction problems with arbitrary sharp gradients without introducing any transverse dissipation terms [47][49]. The extension of these ideas to the Navier-Stokes equations described here provides a straightforward procedure for solving a wide class of flow problems from low to high Reynolds numbers.

We consider the stationary convection-diffusion problem in 1-D domain of length $L$. In a sub domain of length $d$ the balance of fluxes (see figure 2) is:

$$q_A - q_B = 0 \qquad (8.1)$$

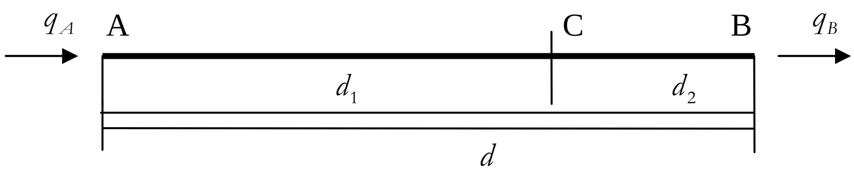


**figure 2**

where $q_A$ y $q_B$ are incoming and outcoming fluxes, respectively.

For convection-diffusion problem, flux is expressed as $q = u\phi - k\dfrac{d\phi}{dx}$.

It is possible, using Taylor series, to express fluxes in A and B based on the flux in an arbitrary interior point C of the subdomain $d$.

$$\begin{aligned} q_A &= q_C - d_1 \left.\frac{dq}{dx}\right|_C + \frac{d_1^2}{2}\left.\frac{d^2q}{dx^2}\right|_C - \mathrm{O}\left(d_1^3\right) \\ q_B &= q_C + d_2 \left.\frac{dq}{dx}\right|_C + \frac{d_2^2}{2}\left.\frac{d^2q}{dx^2}\right|_C - \mathrm{O}\left(d_2^3\right) \end{aligned} \tag{8.2}$$

Substituting (8.2) in (8.1):

$$\frac{dq}{dx} - \frac{h}{2}\frac{d^2q}{dx^2} = 0 \tag{8.3}$$

Where $h = d_1 - d_2$ and all derivatives have been evaluated in C. Considering classical infinitesimal calculus, flux balance equation is derived in infinitesimal domain, therefore equation (8.3) reduces to $\frac{dq}{dx} = 0$.

The additional FIC term in (8.3) introduces an artificial diffusivity with the corresponding stabilization effect.

For the multidimensional case the FIC formulation reads:

$$r_i - \frac{1}{2}h_{ij}\frac{\partial r_i}{\partial x_j} = 0 \quad \text{no sum in } i \tag{8.4}$$

where $i, j = 1 \div n_d$ with $n_d$ being the dimensions of space domain. In (8.4) $r_i$ denotes the balance equations along the *ith* space axis and $h_{ij}$ are the characteristic length distances.

## 8.1 FIC equations for incompressible flows

FIC equations for incompressible viscous flows, in an Eulerian coordinate system, are [40]:

### 8.1.1 Momentum conservation equation

$$r_{m_i} - \frac{1}{2}h_{ij}\frac{\partial r_{m_i}}{\partial x_j} = 0 \quad in\, \Omega \quad \text{no sum in } i \tag{8.5}$$

### 8.1.2 Mass conservation equation

$$r_d - \frac{1}{2}h_j\frac{\partial r_d}{\partial x_j} = 0 \quad in\, \Omega \tag{8.6}$$

where

$$
\begin{aligned}
& r_{m_i} = \partial_t u_i + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial s_{ij}}{\partial x_j} - f_i \\
& r_d = \frac{\partial u_i}{\partial x_j} \qquad i,\, j = 1 \div N \\
& s_{ij} = 2\nu\left(\varepsilon_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)
\end{aligned}
\tag{8.7}
$$

Where $s_{ij}$ are the deviatoric tensor of the rate of deformation, $p$ is the pressure, $\delta_{ij}$ is the delta Kronecker function and $f_i$ are the external forces per unit volume.

Boundary and initial for FIC equations FIC [36]:

$$
\begin{aligned}
& n_j \sigma_{ij} - t_i + \frac{1}{2} h_{ij} n_j r_{m_i} = 0 \quad on \quad \Gamma_t \quad \text{no sum in } i \\
& u_j = u_j^p \quad on \quad \Gamma_u \\
& u_j = u_j^0 \quad for \quad t = t_0
\end{aligned}
\tag{8.8}
$$

where $t_i$ are imposed tractions in boundary $\Gamma_t$ , $n_j$ are the components of the normal vector to the boundary and $\sigma_{ij}$ are total stress defined by

$$
\sigma_{ij} = s_{ij} - p\delta_{ij} \tag{8.9}
$$

The $h_{ij}$ and $h_j$ in previous equations, are the characteristic lengths where balance of momentum and mass is imposed. Note that for $h_{ij} = 0$ and $h_j = 0$ we recover the classical equations of infinitesimal theory.

Equations (8.5)-(8.8) are the first step to derivate a stabilized finite element method for incompressible Navier-Stokes equations. Additional terms introduced by the FIC technique are essential to handle numerical instabilities due to convection effect and incompressibility restriction. An interesting property of FIC formulation is that it allows using same interpolation order for pressure and velocity.

### 8.2 Integral stabilized forms

From the momentum equation it is possible to find [40][41]:

$$
\begin{aligned}
& \frac{\partial r_d}{\partial x_i} = \frac{h_{ii}}{2a_i}\frac{\partial r_{m_i}}{\partial x_j} \quad \text{no sum in } i \\
& a_i = \frac{2\nu}{3} + \frac{u_i h_{ii}}{2} \quad \text{no sum in } i
\end{aligned}
\tag{8.10}
$$

Substituting (8.10) in (8.6) and considering solely the terms involving derivatives of $r_{m_i}$ with respect to $x_i$ , the following alternative expression for the stabilized conservation equation is obtained.

$$r_d - \sum_{i=1}^{n_d} \tau_i \frac{\partial r_{m_i}}{\partial x_i} = 0 \tag{8.11}$$

with

$$\tau_i = \left( \frac{8\nu}{3h_{ii}h_i} + \frac{2u_i}{h_{ii}} \right)^{-1} \tag{8.12}$$

The $\tau_i$'s in (8.12) are known as intrinsic time stabilization parameters, the main interest of (8.12) resides in introducing the first derivatives in space of momentum equation into the conservation equation. These terms have good stabilization properties [42][43].

The variational form of the stabilized equations (8.5) and (8.10) are:

$$\begin{gathered} \int_\Omega v_i \left[ r_{m_i} - \frac{1}{2} h_{ij} \frac{\partial r_{m_i}}{\partial x_j} \right] d\Omega + \int_{\Gamma_t} v_i \left( n_j \sigma_{ij} - t_i + \frac{1}{2} h_{ij} n_j r_{m_i} \right) d\Gamma = 0 \\ \int_\Omega q \left[ r_d - \sum_{i=1}^{n_d} \tau_i \frac{\partial r_{m_i}}{\partial x_i} \right] d\Omega = 0 \end{gathered} \tag{8.13}$$

where $v_i, q$ are test functions. Integrating by parts $r_{m_i}$ it is obtained:

$$\int_\Omega v_i r_{m_i} d\Omega + \int_{\Gamma_t} v_i \left( n_j \sigma_{ij} - t_i \right) d\Gamma + \int_\Omega \frac{1}{2} h_j \frac{\partial v_i}{\partial x_j} r_{m_i} d\Omega = 0 \tag{8.14}$$

$$\int_\Omega q r_d d\Omega + \int_\Omega \left[ \sum_{i=1}^{N} \tau_i \frac{\partial q}{\partial x_i} r_{m_i} \right] d\Omega - \int_\Gamma \left[ \sum_{i=1}^{N} q \tau_i n_i r_{m_i} \right] d\Gamma = 0 \tag{8.15}$$

In the following we will neglect the last term in (8.15), assuming that $r_{m_i}$ is negligible in the boundary.

Thus the moment and conservation equations, integrating as usual stress and pressure terms in $r_{m_i}$, are:

$$\begin{gathered} \int_\Omega v_i \left( \partial_t u_i + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial v_i}{\partial x_j} \left( \nu \frac{\partial u_i}{\partial x_j} - \delta_{ij} p \right) d\Omega - \int_\Omega v_i f_i d\Omega - \\ - \int_{\Gamma_t} v_i t_i d\Gamma + \int_\Omega \frac{h_{ij}}{2} \frac{\partial v_i}{\partial x_j} r_{m_i} d\Omega = 0 \quad \text{no sum in } i \end{gathered} \tag{8.16}$$

and

$$\int_\Omega q \frac{\partial u_i}{\partial x_i} d\Omega + \int_\Omega \left[ \sum_{i=1}^{n_d} \tau_i \frac{\partial q}{\partial x_i} r_{m_i} \right] d\Omega = 0 \tag{8.17}$$

In the derivation of the viscous term in equation (8.16) the following equality has been used (before integration by parts)

$$\frac{\partial s_{ij}}{\partial x_j} = 2\nu \frac{\partial \varepsilon_{ij}}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{8.18}$$

and here we have used the incompressibility condition. $\left( \frac{\partial u_i}{\partial x_i} = 0 \right)$.

### 8.3 Projections of convective term and pressure gradient

Calculation of the residual terms can be simplified if we introduce the projections of the convective term and the pressure gradient $c_i$ and $\pi_i$ , defined as:

$$\begin{aligned} c_i &= r_{m_i} - u_j \frac{\partial u_i}{\partial x_j} \\ \pi_i &= r_{m_i} - \frac{\partial p}{\partial x_i} \end{aligned} \tag{8.19}$$

It is possible to define $r_{m_i}$ as function of $c_i$ y $\pi_i$ in equations (8.16) and (8.17), respectively. Thus, $c_i$ and $\pi_i$ became two new set of variables. The integral equation system is now augmented forcing to have null residuals (in average) for the system of (8.19). This fact yields the following equation system:

$$\begin{aligned} &\int_\Omega v_i \left( \partial_t u_i + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial v_i}{\partial x_j} \left( \nu \frac{\partial u_i}{\partial x_j} - \delta_{ij} p \right) d\Omega - \int_\Omega v_i f_i d\Omega - \\ &- \int_{\Gamma_t} v_i t_i d\Gamma + \int_\Omega \frac{h_{ik}}{2} \frac{\partial v_i}{\partial x_k} \left( u_j \frac{\partial u_i}{\partial x_j} + c_i \right) d\Omega = 0 \quad \text{no sum in } i \end{aligned} \tag{8.20}$$

$$\int_\Omega q \frac{\partial u_i}{\partial x_i} d\Omega + \int_\Omega \left[ \sum_{i=1}^{n_d} \tau_i \frac{\partial q}{\partial x_i} \left( \frac{\partial p}{\partial x_i} + \pi_i \right) \right] d\Omega = 0 \tag{8.21}$$

$$\int_\Omega \delta A_i \left( u_j \frac{\partial u_i}{\partial x_j} + c_i \right) d\Omega = 0 \qquad \text{no sum in } i \tag{8.22}$$

$$\int_\Omega \delta c_i \tau_i \left( \frac{\partial p}{\partial x_i} + \pi_i \right) d\Omega = 0 \qquad \text{no sum in } i \tag{8.23}$$

with $i, j, k = 1 \div n_d$ . In equations (8.22) and (8.23) $\delta A_i$ y $\delta c_i$ are test functions.

We remark that the projections of the convective term and the pressure gradient provide consistency to the formulation, *i. e.* equations (8.20)-(8.23) have the residual form which they are null for the exact solution.

### 8.4 Finite element discretization

We choose $C^0$ continuous linear interpolations of the velocities, the pressure, the convection projections $c_i$ and the pressure gradient projections $\pi_i$ over 3-noded triangles (2D) and 4-noded tetrahedra (3D). The linear interpolations are written as

$$\begin{aligned} u_i &= N^k \overline{u}_i^k \quad , \quad p = N^k \overline{p}^k \\ c_i &= N^k \overline{c}_i^k \quad , \quad \pi_i = N^k \overline{\pi}_i^k \end{aligned} \tag{8.24}$$

Where the sum goes over the number of nodes of each element $n$ ($n$ = 3/4 for triangles/tetrahedra).

Substituting the approximations (8.24) into Eqs. (8.20)-(8.23) and choosing the Galerkin form with $\delta u_i = q = \delta c_i = \delta \pi_i = N^i$ leads to following system of discretized equations

$$\begin{aligned} \mathbf{M}\dot{\overline{\mathbf{u}}} + \mathbf{H}\overline{\mathbf{u}} - \mathbf{G}\overline{\mathbf{p}} + \mathbf{C}\overline{\mathbf{c}} &= \mathbf{f} \\ \mathbf{G}^{\mathbf{T}}\overline{\mathbf{u}} + \hat{\mathbf{L}}\overline{\mathbf{p}} + \mathbf{Q}\overline{\boldsymbol{\pi}} &= \mathbf{0} \\ \mathbf{C}\overline{\mathbf{u}} + \mathbf{M}\overline{\mathbf{c}} &= \mathbf{0} \\ \mathbf{Q}^{\mathbf{T}}\overline{\mathbf{p}} + \hat{\mathbf{M}}\overline{\boldsymbol{\pi}} &= \mathbf{0} \end{aligned} \tag{8.25}$$

where

$$\mathbf{H} = \mathbf{A} + \mathbf{K} + \hat{\mathbf{K}} \tag{8.26}$$

If we denote the node indexes with subscripts $a$, $b$ and the space indices with sub-scripts $i$, $j$, the element contributions to the components of the arrays involved in these equations are ($i$,$j$ = 1,3 for 3D problems)

$$\begin{aligned} & M_{ij}^{ab} = \left( \int_{\Omega_e} \rho N^a N^b d\Omega \right) \delta_{ij} \ , \ A_{ij}^{ab} = \left( \int_{\Omega_e} \rho N^a \left( \mathbf{u}^{\mathbf{T}} \nabla N^b \right) d\Omega \right) \delta_{ij} \\ & K_{ij}^{ab} = \left( \int_{\Omega_e} \mu \nabla^T N^a \nabla N^b d\Omega \right) \delta_{ij} \ , \ \nabla = \left[ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right]^T \\ & C_i^{ab} = \int_{\Omega_e} h_{ij} \frac{\partial N^a}{\partial x_j} N^b d\Omega, \ \mathbf{C} = \left[ C_1, C_2, C_3 \right]^T \\ & \hat{K}_{ij}^{ab} = \left( \frac{1}{2} \int_{\Omega^e} h_{ij} \frac{\partial N^a}{\partial x_j} \left( \rho \mathbf{u}^T \nabla N^b \right) d\Omega \right) \delta_{ij}, \quad G_i^{ab} = \int_{\Omega^e} \frac{\partial N^a}{\partial x_i} N^b d\Omega \end{aligned} \tag{8.27}$$

$$
\begin{aligned}
&\hat{L}^{ab} = \int_{\Omega_e} \left(\nabla^{\mathrm{T}} N^a\right)[\tau]\nabla N^b d\Omega\,,\ [\tau]=\begin{bmatrix}\tau_1 & 0 & 0\\ 0 & \tau_2 & 0\\ 0 & 0 & \tau_3\end{bmatrix}\\
&\mathbf{Q}=\left[Q_1,Q_2,Q_3\right],\ Q_i^{ab} = \int_{\Omega_e} \tau_i \frac{\partial N^a}{\partial x_i} N^b d\Omega \quad \text{no sum in } i\\
&\hat{\mathbf{C}}=\left[\hat{C}_1,\hat{C}_2,\hat{C}_3\right],\ \hat{C}_i^{ab} = \int_{\Omega_e} \rho^2 N^a \left(\mathbf{u}^{\mathbf{T}}\nabla N^b\right) d\Omega\\
&\hat{M}_{ij}^{ab} = \frac{\tau_i}{\rho} M_{ij}^{ab}\,,\quad f_i^a = \int_{\Omega_e} N^a f_i d\Omega + \int_{\Gamma_e} N^a t_i d\Gamma
\end{aligned}
\tag{8.28}
$$

It is understood that all the arrays are matrices (except **f** which is a vector) whose components are obtained by grouping together the left indices in the previous expressions (*a* and possibly *i*) and the right indices (*b* and possibly *j*).

Note that the stabilization matrix $\hat{\mathbf{K}}$ in Eq. (8.26) adds additional orthotropic diffusivity terms of value $\rho\, {}^{h_{ij}u_l}\!/_2$ .

The overall stabilization terms introduced by the FIC formulation above presented have the intrinsic capacity to ensure physically sound numerical solutions for a wide spectrum of Reynolds numbers without the need of introducing additional turbulence modelling terms. This interesting property is validated in the solution of the example presented in a next section.

### 8.4.1 Transient solution scheme

The solution in time of the system of Eq. (8.25) can be written in general form as

$$
\begin{aligned}
\mathbf{M}\frac{1}{\Delta t}\left(\overline{\mathbf{u}}^{n+1}-\overline{\mathbf{u}}^{n}\right)+\mathbf{H}^{n+\theta}\overline{\mathbf{u}}^{n+\theta}-\mathbf{G}\overline{\mathbf{p}}^{n+\theta}+\mathbf{C}^{n+\theta}\overline{\mathbf{c}}^{n+\theta} &= \mathbf{f}^{n+\theta}\\
\mathbf{G}^{\mathbf{T}}\overline{\mathbf{u}}^{n+\theta}+\hat{\mathbf{L}}^{n+\theta}\overline{\mathbf{p}}^{n+\theta}+\mathbf{Q}\overline{\boldsymbol{\pi}}^{n+\theta} &= \mathbf{0}\\
\mathbf{C}\overline{\mathbf{u}}^{n+\theta}+\mathbf{M}\overline{\mathbf{c}}^{n+\theta} &= \mathbf{0}\\
\mathbf{Q}^{\mathbf{T}}\overline{\mathbf{p}}^{n+\theta}+\hat{\mathbf{M}}^{n+\theta}\overline{\boldsymbol{\pi}}^{n+\theta} &= \mathbf{0}
\end{aligned}
\tag{8.29}
$$

Where $\mathbf{H}^{n+\theta}=\mathbf{H}^{n+\theta}\left(\mathbf{u}^{n+\theta}\right)$, etc and the parameter $\theta\in[0,1]$. The direct monolithic solution of Eqs. (8.29) is possible using an adequate iterative scheme [41]. However, in our work we have used the fractional step method described in [41].

## 8.5 Computation of the characteristic distances

The computation of the stabilization parameters is a crucial issue as they affect both the stability and accuracy of the numerical solution. The different procedures to compute the stabilization parameters are typically based on the study of simplified forms of the stabilized equations. Contributions to this topic are reported in [38][41][61][62][63]. Despite the relevance of the problem there still lacks a general method to compute the stabilization parameters for all the range of flow situations.

Recent work of the authors in the application of the FIC/FEM formulation to convection-diffusion problems with sharp arbitrary gradients [47][49] has shown that the stabilizing FIC terms take the form of a simple orthotropic diffusion if the balance equation is written in the principal curvature directions of the solution. Excellent results were reported in [47][49] by computing first the characteristic length distances along the principal curvature directions, followed by a standard transformation of the distances to global axes. The resulting stabilized finite element equations capture the high gradient zones in the vicinity of the domain edges (boundary layers) as well as the sharp gradients appearing randomly in the interior of the domain [47][49] [61]. The FIC/FEM thus reproduces the best features of both the so called transverse (cross-wind) dissipation or shock capturing methods [65] [66].

The numerical computations are simplified without apparent loss of accuracy if the main principal curvature direction of the solution at each element point is approximated by the direction of the gradient vector at the element center. The second principal direction (for 2D problems) is taken in the orthogonal direction to the gradient. For linear triangles and quadrilaterals these directions are assumed to be constant within the element [47][49] [61].

Above simple scheme has been extended in this work for the computation of the characteristic distances $h_{ij}$ for the momentum equations. As for the length parameters $h_i$ in the mass conservation equation, the simplest assumption $h_i = h_{ii}$ has been taken. Details of the algorithm for computing $h_{ij}$ are given next (the method is explained for 2D problems although it is readily extendible to 3D problems).

For the *i*-th momentum balance equation and every time step:

1. A coordinate system $\vec{\xi}_1^{\,i}, \vec{\xi}_2^{\,i}$ is defined at each element point such that $\vec{\xi}_1^{\,i}$ is aligned with the gradient of $u_i$ ($\vec{\xi}_1^{\,i} = \nabla u_i$) and $\vec{\xi}_2^{\,i}$ is orthogonal to $\vec{\xi}_1^{\,i}$ in anticlockwise sense (Figure 3). The angle that $\vec{\xi}_1^{\,i}$ forms with the global $x_1$ is defined as $\alpha_i$. Recall that upper and lower index *i* denotes the *i*-th momentum equation.
2. The element characteristic distances $l_{i1}$ and $l_{i2}$ are defined as the maximum projections of the element sides along the $\vec{\xi}_1^{\,i}$ and $\vec{\xi}_2^{\,i}$ axes, respectively (Figure 4).
3. The characteristic distances $h_{i1}$ and $h_{i2}$ are computed as

$$\begin{Bmatrix} h_{i1} \\ h_{i2} \end{Bmatrix} = \begin{bmatrix} c_i & -s_i \\ s_i & c_i \end{bmatrix} \begin{Bmatrix} \bar{h}_{i1} \\ \bar{h}_{i2} \end{Bmatrix} \quad , i = 1,2 \tag{8.30}$$

with $c_i = \cos\alpha_i, s_i = \sin\alpha_i$ and the local distances $\bar{h}_{i1}$ and $\bar{h}_{i2}$ are

$$\bar{h}_{ij} = \left( \coth \bar{\gamma}_{ij} - \frac{1}{\bar{\gamma}_{ij}} \right) l_{ij} \ , \ \bar{\gamma}_{ij} = \frac{\bar{u}_j l_{ij}}{2\mu} \quad , j = 1,2 \tag{8.31}$$

where $\overline{u}_1$ and $\overline{u}_2$ are the components of the velocity vector along the local axes $\vec{\xi}_1^i$ and $\vec{\xi}_2^i$, respectively (Figure 3).

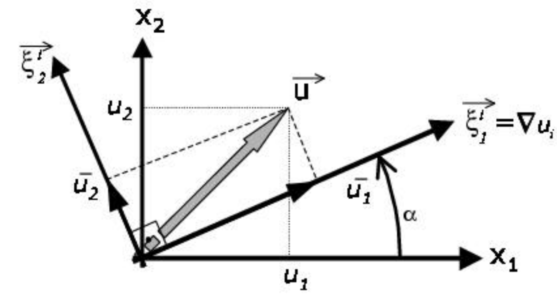


**figure 3: Local coordinate system**

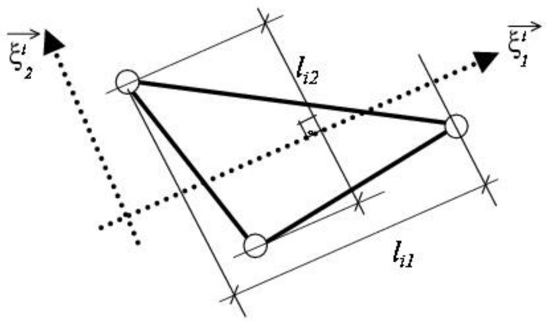


**figure 4: Characteristic distances in a triangular element**

### 8.6 Example: Flow past a cylinder at *Re* = 1000

This example shows the ability of FIC formulation to model flows at high Reynolds numbers. Others examples of this type are reported in [61].

Figure 5 shows the geometry for the analysis of the flow past a cylinder of unit diameter *D*. A unit horizontal velocity is prescribed at the inlet boundary and at the two horizontal walls. Zero pressure is prescribed at the outlet boundary. The dimensions of the analysis domain are 36x27 units. The origin of the coordinate system has been sampled at the center of the cylinder located at a distance of 13.1 units from the inlet wall. Zero velocity is prescribed at the cylinder wall. The kinematic viscosity is $\nu = 0.001$. Figure 6 shows the mesh of 91316 linear triangular elements used for the computation.

The problem has been analyzed for a value of the horizontal velocity at the entry of $u_1 = 1$ giving a Reynolds number of Re =1000. Figures 7 and 8 respectively show the velocity modulus contours and the velocity vectors for t=100 secs.

Figure 9 shows the oscillations of the horizontal velocity at the point A with coordinates (6.7, -1.02). The Strouhal number computed from the shedding frequency *n* as $S = {}^{nD}\!\big/_{|\mathbf{u}|}$ is S=0.2103. This number compares very well with the experimental result available in the literature.

Figure 9 and 10 finally shows the oscillations of the horizontal velocity at point A and the trajectories of a substance over a band of 2.45 units at the inlet.

It is a well known fact that for *Re* > 300 the flow past a cylinder exhibits 3D features. In [64] results from 2D and 3D computation were compared for *Re* = 300 and 800. While 3D features were observed even at *Re* = 300 and more so at *Re* = 800, there were no large discrepancies between the global flow parameters (such as drag, lift and Strouhal number) obtained from 2D and 3D computations. These conclusions justify the results of the 2D computations presented in here.

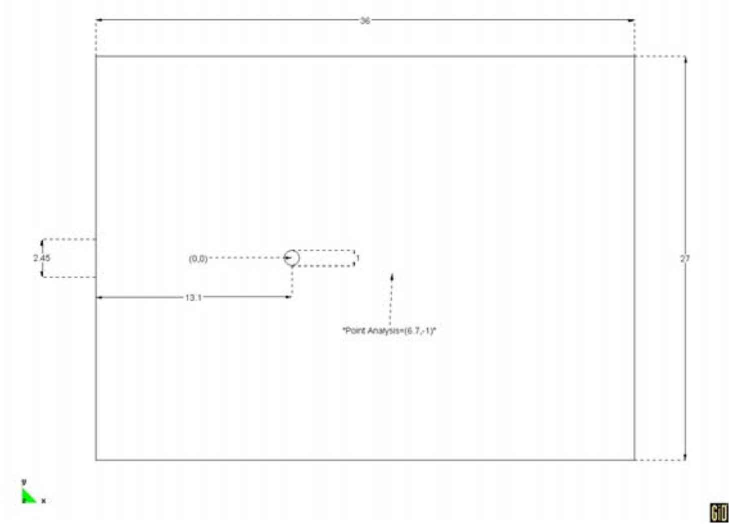


**figure 5: Flow past a cylinder of unit diameter. Analysis domain and boundary conditions**

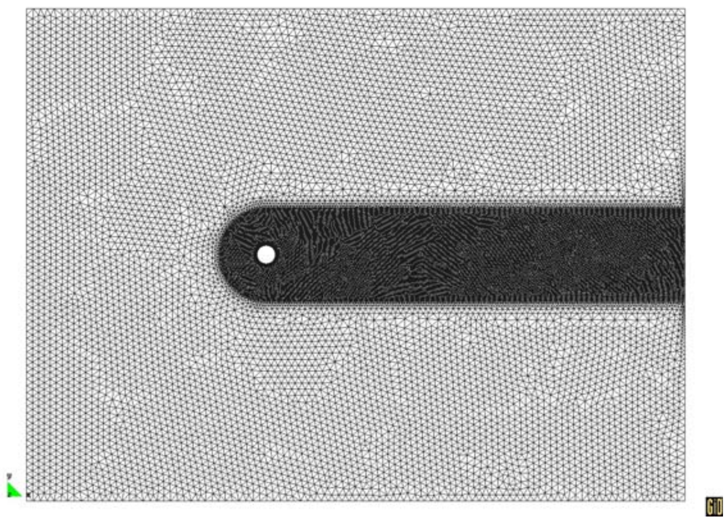

**figure 6: Flow past a cylinder. Mesh of 91316 three-noded triangles used for computations**

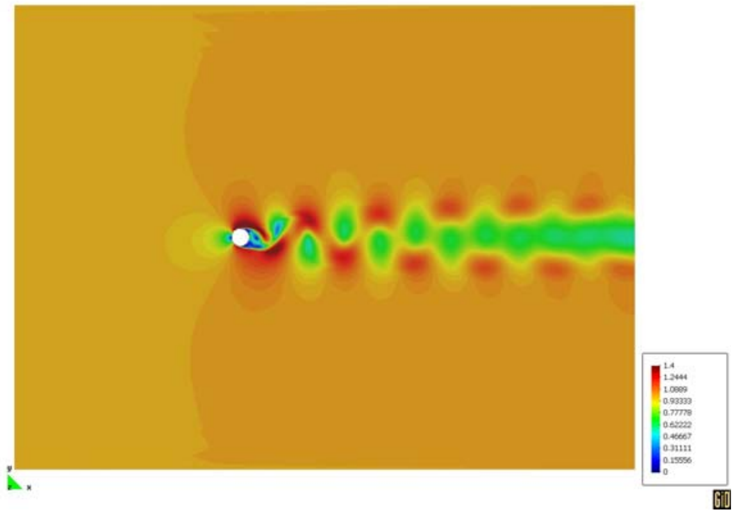


**figure 7: Flow past a cylinder, Contour of the velocity modulus for t = 100 secs.**

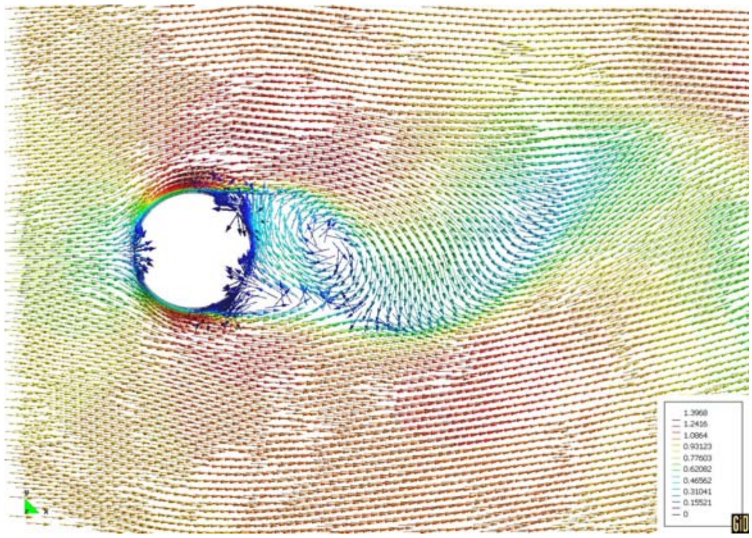


**figure 8: Flow past a cylinder, Velocity vectors for t = 100 secs.**

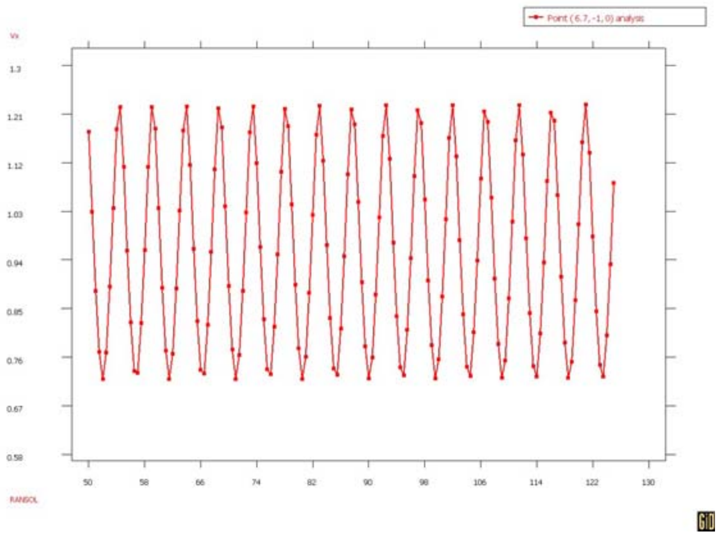


**figure 9: Flow past a cylinder. Oscillations with time of the horizontal velocity at point A**

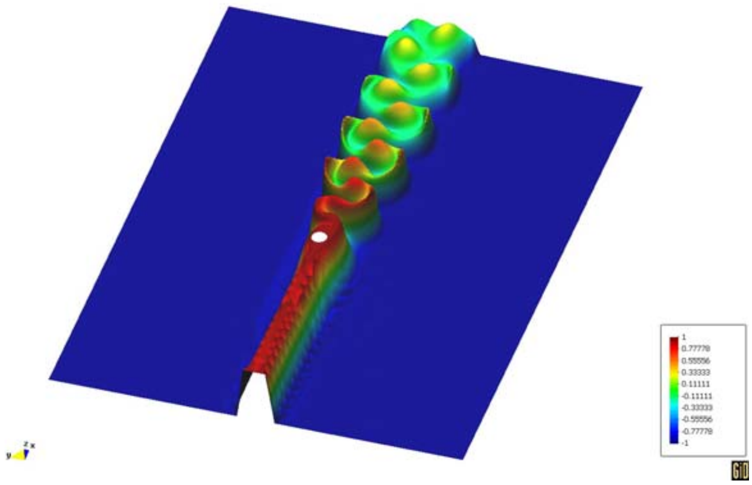

**figure 10: Trajectories of substance over a band of 2.45 units at the entry transported across the flow for t = 100 secs.**

## 9 CONCEPTUAL FRAME FOR THE "DUALITY" PRINCIPLE

This section settles down the basic definitions to formalize a theoretical frame where to formulate the "duality" principle between numerical stabilization and turbulence models. As advanced in the introduction, there exists among the scientific community [23][30][31], the tendency to consider that turbulence models can be interpreted as a part of a stabilized numerical method, and inversely, the stabilized methods can reproduce turbulent solutions with no need of additional numerical dissipation in the formulation.

### 9.1 Preliminaries

Let us consider a stabilized numerical method, for example the FIC method described in the previous section [40]. We can list the desirable properties of a method to be able to correctly integrate Navier-Stokes equations:

#### 9.1.1 Properties

1- *Select physical coherent solutions:* It is essential that the obtained solutions have physical meaning. Solutions must behave according to physics laws.
2- *Stability:* Method must be independent from space-time discretization.
3- *Convergence:* A refinement of the discretization parameters should have, associated a succession of approximations converging to a unique solution of the problem (in the limit, to the exact solution).
4- *Consistency:* The exact solution must be a solution of the stabilized formulation.
5- *Transition:* The method must be able to reproduce transition to turbulent flow in parts of the domain.
6- *Regularization:* Additional stabilization terms must regularize the Navier-Stokes equations in order to get a well-posed problem for all times, while preserving the uniqueness of the solution.

From previous properties, 2-, 3- and 4- are common properties of any numerical method. Next, definitions for the stability and convergence are introduced.

### 9.2 Stability

We consider the following problem, without lost of generality

$$\begin{aligned} \partial_t \mathbf{b} + (\mathbf{a}\cdot\nabla)\mathbf{b} - k\Delta\mathbf{b} + s\mathbf{b} = \mathbf{f} \qquad & in \quad \Omega, t>0 \\ \mathbf{b} = \mathbf{g} \qquad & in \quad \partial\Omega, t>0 \\ \mathbf{b} = \mathbf{b}^0 \qquad & in \quad \Omega, t=0 \end{aligned} \tag{9.1}$$

where $\mathbf{a}$ is velocity vector satisfying $\nabla\cdot\mathbf{a}=0$, $k$ is the diffusivity.

For the momentum equations:

$$\begin{aligned} \mathbf{a}=\mathbf{b}=\mathbf{u} \qquad & s=0 \\ k=\nu \qquad & \mathbf{f}=\mathbf{f}-\nabla p \end{aligned} \tag{9.2}$$

**Definition 5**

*For problem* (9.1),

*A numerical method* $\mathbf{M}$ *is stable if every discrete approximation* $\mathbf{b}_h$ *of problem* (9.1), *in the discrete version, satisfies*

$$\|\mathbf{b}_h\| \le F_M\left(\mathbf{f},\mathbf{g},\mathbf{b}^0,k,s\right) \tag{9.3}$$

*where* $F_M\left(\mathbf{f},\mathbf{g},\mathbf{b}^0,k,s\right)$ *is a bounded function, depending solely on the problem data and method* $\mathbf{M}$.

In the previous definition the considered norm is the one associated with the approximation spaces of (9.1). For example, we consider the following theorem as an example (with same notation):

**Theorem 6**

For the steady state form of problem (9.1) $\left(\partial_t \mathbf{b} = 0\right)$. If

$$\left.\begin{array}{l} \mathbf{f} \in L^p(\Omega) \quad 2 \le n_d < p \\ \tilde{\mathbf{g}} \text{ extension of } \mathbf{g} \in W^{1,p}(\Omega) \\ \mathrm{M} \quad \text{satisfying DMP} \end{array}\right\} \Rightarrow \left\|\mathbf{b}_h\right\|_{L^\infty(\Omega)} \le \left\|\mathbf{g}\right\|_{L^\infty(\partial\Omega)} + \left\|\mathbf{f}\right\|_{L^p(\Omega)} \tag{9.4}$$

where DMP is the Discrete Maximum Principle [66], $n_d$ is spatial dimension.

According to previous theorem, under certain regularity hypotheses of problem data and if method holds, DMP then it is stable.

With the notation introduced in Definition 5, from result (9.4) it follows that

$$F\left(\mathbf{f}, \mathbf{g},\right) = \left\|\mathbf{g}\right\|_{L^\infty(\Omega)} + \left\|\mathbf{f}\right\|_{L^p(\Omega)} \tag{9.5}$$

In summary, a numerical method is stable if it is possible to bound approximations based on problem data.

### 9.3 Convergence

Convergence is one of main properties to be requested to a numerical method.

For the numerical approximation of the Navier-Stokes equations it is necessary to discretize the domain $\Omega$ in time and space.

If successively discretizations are “refined" it is desirable that the associated succession of solutions converges (in a sense to determine) to the exact solution of the problem. The following definition formalizes the concept of convergence [68].

**Definition 6**

*For problem* (9.1).

*Let* M *be a numerical method.* M *converges if*

$$\left\|\mathbf{b}_h - \mathbf{b}\right\|_X \xrightarrow[\substack{\Delta t \to 0 \\ \Delta x \to 0}]{} 0 \tag{9.6}$$

*or equivalently*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \,|\, \begin{array}{l} \Delta t < \delta \\ \Delta x < \delta \end{array} \quad \Rightarrow \quad \left\|\mathbf{b}_h - \mathbf{b}\right\|_X < \varepsilon \tag{9.7}$$

where $\Delta t$ is temporal discetization size, $\Delta x$ is a characteristic spatial mesh size and $h = f\left(\Delta t, \Delta x\right)$ is a characteristic size of the space-time discretization and $\mathbf{b}$ is the exact solution.

In previous definition $X$ denotes a generic normed function space, as for example, $X = L^{\infty}(\Omega), L^{2}(\Omega), \ldots$. This space defines the type of convergence to the exact solution.

We emphasize that the previous definition also considers convergence that do not come from a space norm, as the case of weak convergence. The weak convergence is defined by a dual pair in the function space $X$ [68].

### 9.4 Suitable solutions

It is possible that the approximation process of the governing equations yields solutions that do not respect the physical/real behaviour of the flow as, for example, not conserving the mass.

For this reason it is important to define and to verify if the numerical approximations is physically consistent.

In summary, a discretization process can relax conditions over numerical approximations, being able to violate physics laws. The definition of suitable weak solution is now recalled, with the aim to unify all definitions referring to the conceptual frame.

If weak solutions are not unique, a fundamental issue is to distinguish the physically relevant solutions.

**Definition 7**

*A weak solution* $(\mathbf{u}, p)$ *of the Navier-Stokes equations* is *suitable if* $\mathbf{u} \in U$, $p \in Q$ *and the local energy balance*

$$\partial_t \left(\frac{1}{2}\mathbf{u}^2\right) + \nabla \cdot \left(\left(\frac{1}{2}\mathbf{u}^2 + p\right)\mathbf{u}\right) - \nu\Delta\left(\frac{1}{2}\mathbf{u}^2\right) + \nu\left(\nabla\mathbf{u}\right)^2 - \mathbf{f}\cdot\mathbf{u} \leq 0 \tag{9.8}$$

*is satisfied in a distributional sense.*

It is important to remark the relation between suitability and stability concepts, sometime confusingly used in the literature. Stability is associated to the numerical method while suitability is associated to the numerical solution.

To show the existing relation between these two concepts we consider convection dominant problems, *i. e*. problems where the Péclet number, $Pe$, defined as:

$$Pe = \frac{1}{2}\frac{|\mathbf{a}|\Delta x}{k} \tag{9.9}$$

is much greater than 1. For these problems it is known that numerical solutions obtained using standard integration methods, like Galerkin-FEM, yield generalized oscillations in the entire domain [67]. Oscillating solutions lack all physical behaviour, and therefore they are not suitable according to (9.8). However, the numerical method can still be stable because bounding solutions based on problem data could continue being possible. Let us consider a stabilization method that limits the maximum and minimum solutions for the nodal values. Let us suppose that this limitation ensures that the solution norm is bounded based on problem data. According to (9.3), the method would be stable, but the obtained approximations would possibly lack all of physical meaning. Hence, they would not be suitable.

Then, if we want to obtain non-oscillating solutions it is necessary to modify the numerical method in order to guarantee suitable solutions. Additional terms will have effects on stability method.

Theorem 6 demands that the numerical method satisfies the DMP. This property is necessary to prove the bounding of solutions and thus the stability of the method. The DMP is as well a property with a clear physical meaning, and justifies that the obtained solutions are suitable.

In conclusion, stability of a method and suitability of its solutions are closed related but sometimes this relationship is not clear enough.

Additionally, this example shows other two important features of stabilized methods: The relation between stabilization and the physical behaviour of the problem.

To request a method to be stable is less restrictive than to request the method to provide stable and suitable solutions. The SUPG method can be stable, but its solutions can be non suitable. Localized oscillations close to discontinuities is as well known problem of this method [66]. These oscillations are responsible for the non suitability of the solutions. To obtain suitable solutions (equivalently to remove localised oscillations) it is necessary to introduce shock-capturing terms [42], [66]. Schematically the following relation can be established:

$$\text{Stability} \longrightarrow \text{SUPG}$$

$$\left.\begin{array}{c}\text{Stability}\\ +\\ \text{Suitability}\end{array}\right\}\ \text{SUPG+Shock-Capturing}$$

When $\Delta x \to 0$, then $Pe < 1$ and therefore stabilization is not needed. In the limit case that $\Delta x \propto \lambda_k$, where $\lambda_k$ is the Kolmogorov scale, numerical stabilization must be inactive and be equivalent to a DNS.

These features invite to consider the similar role of numerical stabilization and turbulence models.

In numerical approximations of turbulent flows is common to obtain stable solutions but with no physical meaning, unless a turbulence model is used. The turbulence model complements the numerical method in order to get suitable solutions.

### 9.5 Regularization

Pre-LES models [7] (previous filtering of Navier-Stokes equations to discretization) can, in almost all cases, be understood as a regularization of the Navier-Stokes equations.

These pre-LES models solve the uniqueness problem and yield a well-posed problem for all times.

Thus, any numerical method that can replace a turbulence model must also regularize the equations.

Pre-LES models, based on the partial or total filtering of the Navier-Stokes equations, introduce additional terms that guarantee solution uniqueness. This is the case, for instance, of the sub-grid scale tensor $\mathrm{T} = \overline{\mathbf{u} \otimes \mathbf{u}} - \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}$. The following question rises:

*Are non-linear stabilization terms introduced by a numerical method such as FIC an (implicit) model of the sub-grid scale tensor?*

The answer to this question is still pending although it hints a possible way to answer the main question formulated in the introduction: *It is possible to reinterpret stabilized and suitable methods as turbulence models?.*

### 9.6 Turbulent solutions

When velocity increases, flow becomes more and more complex until it reaches the turbulent regime. Physical and mathematical description of that phenomenon is an active research field. It is accepted that transition to turbulent flow is a concatenation of bifurcations (Hopf bifurcations)[9].

As explained previously in the paper, the dynamics of turbulent flows is chaotic and its determinist description is extremely complex, if not impossible.

We saw in 9.4, that for a Péclet number much greater than 1 it is necessary to stabilize the numerical scheme independently of the flow regime. Péclet number will be much greater than 1 for turbulent flows, due to high velocity and mesh size. Therefore the use of a stabilized method will be indispensable in this case. Then:

*Will stabilized solutions be suitable?*

Section 2.2 already suggest that in general the answer is no. Since the discretization resolution is greater than Kolmogorov scale, small scales can not be represented and, consequently, the numerical results for the flow dynamics will not be physically consistent.

Let us consider the following decomposition. Let $\mathbf{u}$ be the exact solution of Navier-Stokes problem (in turbulent regime). Then:

$$\mathbf{u} = \mathbf{u}_h + \mathbf{e}_h \tag{9.10}$$

Where $\mathbf{u}_h \in C^1\left(0,T;X_h\right)$, $X_h \subset X$, $\dim\left(X_h\right) < \infty$ and $\mathbf{e}_h$ is the approximation error.

If $\mathbf{u}_h$ is a suitable and physically consistent solution, then $\mathbf{e}_h$ is due to numerical discretization. However, if $\mathbf{u}_h$ is no suitable, then $\mathbf{e}_h$ must contain physically relevant information that $\mathbf{u}_h$ is not able to capture. Then

$$\mathbf{e}_h = \mathbf{e}_h^t + \mathbf{e}_h^d \tag{9.11}$$

Where $\mathbf{e}_h^t$ represents the “turbulent error” contribution, containing physically relevant information, and $\mathbf{e}_h^d$ is the intrinsic error due to the numerical approximation.

It can be deduced from the previous decomposition that turbulent solutions are function of the flow regime and the numerical method used for the approximation. Based on the goodness of the numerical method used, the solution will be turbulent or non-turbulent. According to the actual methodology for the simulation of turbulent flows, $\mathbf{e}_h^t$ is modelled using some turbulence model.

## 10 CONCLUSIONS

It has been verified that filtering, although widely accepted as a paradigm in LES, can lead to a paradox. In fact exact closure is possible, *i.e.*, the sub-grid scale tensor can be expressed exactly in terms of the filtered velocity. Nevertheless exact filtering does not entail any benefit since space of solutions are isomorphs. Colloquially speaking, the number of degrees of freedom necessary to represent a solution of the Navier-Stokes equations would be the same to represent a solution of the filtered Navier-Stokes equations.

This result throws in the doubt on the convenience of total filtering, as it is commonly done.

Under the light of the exposed results, a LES model must satisfy the two following criteria:

- A LES model must be a technique to transform Navier-Stokes equations into a well-posed set of PDE.
- A LES model must select physically relevant solutions (suitable weak solutions).

Among other the admissible techniques we have analyzed those which regularize the convective term partially (Leray regularization and NS-alpha model).

Other admissible techniques are those that consist in adding a nonlinear viscosity to momentum equations (Smagorinsky model and Ladyzenskaja model). Contrary to that it is commonly done in the literature these types of models do not need a filter to be justified.

With respect to second criteria, the regularization technique, based in the Leray model, seems to be a good candidate. This is, the solution of the Navier-Stokes obtained as the succession limit of the regularized solutions that fulfils the energetic inequality (2.3), when the limit of the Galerkin approximation of non regularized equations cannot fulfil it.

We have tried to mathematically justify some multi-scale LES models. It has been verified that these numerical techniques are closely related to the stabilization techniques used for non coercive PDE.

Under the light of ideas and definitions exposed in Section 10, it is possible to reformulate the questions formulated in the introduction in a more rigorous and clear way. With a defined conceptual frame it is possible to establish the subjects of interest and to identify the possible connections among them. Thus the question considered in the introduction

*Is it possible to reinterpret stabilization methods as turbulence models?*

It can be re-expressed within this conceptual frame in the following way:

*For a given discretization, are stabilized methods able to capture* $\mathbf{e}_h^t$ *?*

Definitions of stability, convergence, regularity and admissibility have been introduced. These definitions are fundamental to understand relations and limitations of numerical methods used to approximate the Navier-Stokes problem. In this way, these definitions establish a consistent scene where we can formulate the conjectured principle of "duality" between stabilization and turbulence.

In addition, the relation between stabilization and physical behaviour of the problem has been shown. The possibility of connecting these two fields: stabilization and physics, can contribute with new ideas for calculating the stabilization parameters in a consistently and effective manner.

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